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Lacunary series with random gaps

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Acknowledgement

To my supervisor Professor Berkes, to my parents, to Professor Silver, to my aunt and uncle, to my grandpa and last but not the least, to Vasja Bojanic. I am sure they all understand very well indeed why their names are listed in this document.

Abstract

It is well known that a concept of independence provides a fruitful ground for results in Probability Theory. These include, but are definitely not restricted to, various standard and functional laws of the iterated logarithm and strong approximation of empirical processes. On another note, theory of trigonometric series with random amplitudes is almost complete but not much is known if the randomness lies within the frequency itself, especially in the case of integer-valued harmonics. Lastly, there are numerous extremely difficult and essentially hopeless problems in deterministic mathematics which become explicitly solvable upon randomization.

Contribution of this thesis is three-fold, namely in each one of the directions stated above.

Having in mind both applications to Number Theory and Mathematical Analysis as our main agenda, we introduce an auxiliary probability space on which we define all the necessary randomness. Our main idea is to think of Number-Theoretic and Mathematical Analysis objects of interest as phenomena on another space of actual interest (which is often the interval (0,1) equipped with Lebesgue measure and Borel sigma-field) and then obtain results with probability one on the auxiliary space. In other words we solve, with probability one, open problems in other fields of Mathematics.

First paper establishes the functional Strassen law of the iterated logarithm for the partial sums of periodic functions of dependent random variables. We discover that the limit set is a scaled Strassen set and that the limit is not constant almost everywhere which is very different from the case of independent variables. We obtain numerous Strassen-style corollaries which allow precise asymptotics of very complicated objects including upper densities of certain sets which are very powerful results in the class of laws of the iterated logarithm.

Second paper uses a similar model of an increasing random walk introduced by Schatte in the early 1980s. Here we study the asymptotics of the empirical distribution function and discover that the limit set in the corresponding functional law of the iterated logarithm is the unit ball of the corresponding Reproducing Kernel Hilbert Space. This powerful result has many important corollaries, namely on the probabilistic front, in one line argument, it recovers the entire i.i.d theory developed by Finkelstein. Moreover, on the Number Theory and Mathematical Analysis front, we recognize that the quantity we computed gives us hands-on asymptotics for star discrepancy and Lp discrepancy of a huge class of increasing sequences with probability one.

Needless to say, these results for fixed sequence are way beyond the scope of deterministic mathematics. Third paper is about trigonometric series with random frequencies. Here we use a different model of the frequency-domain randomness. We extend old results of Erdos and discover very surprising limit distributions, say a sum of independent mixed-normal and Cauchy or some infinitely divisible distribution, to name just a few. We deduce that it very much matters how close together are the intervals on which consecutive frequencies are defined, and distinguish between cases of small, large and intermediate gaps. Note that in the case of intermediate gaps that pure normal limit is also possible.

Introduction

This thesis contains three chapters that in spite of the fact that they share author and ideas, are essentially quite disjoint. We, without any further delay, move straight into the corresponding descriptions.

Chapter 1:

It is common knowledge that the assumption of independence is the most fertile ground for results in Probability Theory. These include, among countless others, Central Limit Theorems and Laws of the Iterated Logarithm, which in turn, can almost be thought of as "signatures of independence" in the underlying structure.

It is then clear why (especially in the early days of Probability, but today as well) discoveries of suchlike behaviour in heavily dependent structures were (are) quite remarkable. We refer the reader to the ice breaking result of Salem and Zygmund [51]; i.e. to their Central Limit Theorem for the lacunary trigonometric system

$$(\sin 2\pi n_k x)_{k\in\mathbb{N}}; \quad n_{k+1}/n_k \geq q > 1.$$

The corresponding Law of the Iterated Logarithm was proved by Erdős and Gál [21].

For completeness, we point out that $(\sin 2\pi n_k x)_{k\in\mathbb{N}}$ are random variables (in fact, very dependent ones) on the probability space $((0,1),\mathcal{B},\lambda)$ where the notation is self-explained. Lacunarity was weakened by Erdős [20] and Berkes [3] showing the existence of (random) sequences $(n_k)_{k\in\mathbb{N}}$ with $n_{k+1} - n_k \to \infty$ (with any prescribed velocity) such that the Central Limit Theorem for the trigonometric system $(\cos 2\pi n_k x)_{k\in\mathbb{N}}$ holds with probability one on the n_k-space. The question of existence of n_k's with bounded gaps for which the Central Limit Theorem holds, posed by Berkes, was answered by Bobkov and Götze, but it took almost 30 years to get there, see [12]. More results in this direction were obtained by Fukuyama, see for example [26].

These 3 papers (Berkes [3], Bobkov and Götze [12] and Fukuyama [26]) use random constructions of different nature for their n_k's. We asked ourselves, can a Central Limit Theorem with mean 0 and variance 1 be achieved in n_k's were to be continuous random variables instead of integers? We propose a uniform independent bounded gap model, basically a hybrid between the constructions of Berkes [3] and Bobkov and Götze [12]. The answer is no, the corresponding limit turns out to be an interesting mixed Gaussian, for details see Theorem 9.

The problem of limit classification is still open, more results are given in Fukuyama [26]. The conjecture is that any L^2 function could be the limit, thought of embedded in the variance of the Gaussian that is.

Chapter 2:

Laws of the Iterated Logarithm have been discovered by Hartman and Wintner [31] and in a different form by Kolmogorov [36]; where this last result is still a fundamental reference. Much more fundamental are the so-called Functional Laws of the Iterated Logarithm, introduced by Strassen in [56]. This result implies the classical Law of the Iterated Logarithm via simple one line observation. A version of the Strassen's result that is of major interest to us was proved by Major [39].

In this chapter we first encounter the brilliant ideas of P. Schatte, that allow us to turn dependent structure of partial sums into independent one to which the result of Major can be applied. We thus obtain a (Strassen-type) of a Functional Law of the Iterated Logarithm and a Weighted Law of the Iterated Logarithm; with several powerful à-la-Strassen Corollaries, see Strassen [56] and our corresponding results. For more details on the classical Weighted Law of the Iterated Logarithm see [7], [16] and [24].

Chapter 3:

This chapter is possibly the most important one in this thesis, namely it is the one to justify its very title.

The fundamental difference from the other two chapters is that this one is application oriented.

We remind the reader of an open problem in Number Theory; to find the exact value of

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}}\, D_N^*(\{n_k x\}) \qquad (*)$$

where D_N^* is the star discrepancy of the sequence $(n_k x)$.

The solution to this problem is known for a very restricted class of sequences of integers $(n_k)_{k\in\mathbb{N}}$; for some more details see [25], [27], [44], say.

We remind the reader of the fact that the set of all increasing sequences of integers can be (bijectively) identified with the interval (0, 1). Moreover it turns out that, almost any increasing sequences of integers must satisfy $n_k \mid k \to 2$ as $k \to \infty$.

Now, if we are to substitute n_k by $S_k = X_1 + \cdots + X_k$; $(X_n)_{n\in\mathbb{N}}$ a sequence of independent and identically distributed and absolutely continuous random variables the story is different.

On one hand, we admit: yes, S_k's are then themselves absolutely continuous random variables and could not possibly be integers. However, they are a very good way of simulating the linear growth; and if you want, we can easily (by the Strong Law of Large Numbers) construct the X_k's in a way that $\frac{S_k}{k} \to 2$ as $k \to \infty$.

This "philosophical shift" is not novel. For example, the result of Carleson [15] gives necessary and sufficient conditions for the almost everywhere convergence of the series $\sum_{k=1}^{\infty} c_k \sin 2\pi kx$. But, solving the same problem for n_k instead of k for other functions f seems to be a formidable task. Nevertheless, Schatte comes to the rescue, for details see [6]; where Berkes and Weber extend Carleson's result to a much larger class of functions for almost all such sequences, without imposing any additional constraints other than those of Carleson himself.

The value of the expression $(*)$ is a function $Ax^{1/2}$ which we shall speak of later. Unfortunately, not much is known about it, simulations suggest continuity but nowhere-differentiability; however formalising these claims is a challenge for days to come. The actual result is Corollary 1* and all the other results of ours in this chapter are to be thought of as groundwork for this one.

Chapter 1

Central Limit Theorems for Trigonometric Systems with Random Frequencies

1.1 Introduction

For convenience and completeness we shall start by quoting and discussing some classical and some new results in the field of random trigonometric systems.

Theorem 1 (Salem and Zygmund, 1947). *Let* $(n_k)_{k\in\mathbb{N}}$ *be a sequence of positive integers satisfying the Hadamard gap condition*

$$n_{k+1}/n_k \geq q > 1. \tag{1.1}$$

Then the trigonometric system $(\sin 2\pi n_k x)_{k\geq 1}$ *obeys the central limit theorem; i.e.*

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} \sin 2\pi n_k x \leq t\sqrt{N/2} \right\} =$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du \tag{1.2}$$

where λ *denotes the Lebesgue measure.*

Furthermore, we also have:

Theorem 2 (Erdős and Gál, 1955). *Under the Hadamard gap condition* (1.1) *we have*

$$\limsup_{N\to\infty}(N\log\log N)^{-1/2}\sum_{k=1}^{N}\sin 2\pi n_k x = 1 \quad \textit{for almost every } x. \tag{1.3}$$

These two early results are rather remarkable. Namely, thought of as a sequence of random variables on $((0,\,1),\mathcal{B},\lambda)$ (here $\mathcal{B}$ is simply the Borel σ-algebra of subsets of $(0,\,1)$); the trigonometric system $(\sin 2\pi n_k x)_{k\geq 1}$ is anything but a sequence of independent random variables; basic trigonometry actually reveals the nature of its heavy dependence!

Nevertheless, Theorems 1 and 2 above reveal the striking nature of the Hadamard trigonometric system; $(\sin 2\pi n_k x)_{k\geq 1}$ behaves like a sequence of independent random variables, since it satisfies the Law of the Iterated Logarithm and the Central Limit Theorem in the most classical sense ($N(0,\,1)$ limit in the Central Limit Theorem and 1 as the constant in the Law of the Iterated Logarithm).

Efforts have been made at the time to relax the Hadamard gap condition while maintaining the illustrated remarkable properties of the corresponding trigonometric system. The first result of this sort is as follows:

Theorem 3 (Erdős, 1962). *The Central Limit Theorem* (1.2) *remains valid if we substitute the Hadamard gap condition with*

$$n_{k+1}/n_k \geq 1 + c_k k^{-1/2}; \quad c_k \to \infty \quad \textit{as} \quad k\to\infty. \tag{1.4}$$

Moreover, this result is sharp in the sense that for all $C > 0$ there exists a sequence $(n_k)_{k\in\mathbb{N}}$ satisfying $n_{k+1}/n_k \geq 1 + Ck^{-1/2}$; $k \geq k_0$ such that the Central

Limit Theorem (1.2) is false.

The complementary Law of the Iterated Logarithm was proved by Takahashi, see [57].

For sequences $(n_k)_{k\in\mathbb{N}}$ growing slower than the speed defined in (1.4), the asymptotic behaviour of the partial sums of $\sin 2\pi n_k x$ depends sensitively on the number theoretic properties of $(n_k)_{k\in\mathbb{N}}$ and deciding the validity of the Central Limit Theorem is generally a very difficult problem.

Here are some results in this direction:

Theorem 4 (Salem and Zygmund, 1954)**.** *There exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ with*

$$n_{k+1} - n_k = O(\log k) \tag{1.5}$$

such that the Central Limit Theorem (1.2) *and the Law of the Iterated Logarithm* (1.3) *are both valid.*

It took another quarter of a century until the following strong result, which almost completed the theory. It reads as follows:

Theorem 5 (Berkes, 1979)**.** *Let $\omega(k)$ be any function satisfying $\omega(k) \to \infty$ as $k \to \infty$. Then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers satisfying the gap condition*

$$n_{k+1} - n_k = O(\omega(k)) \tag{1.6}$$

such that both Central Limit Theorem and Law of the Iterated Logarithm ((1.2) *and* (1.3) *respectively*) *are satisfied.*

Remarkable as it is, this result, as pointed out by Berkes, left the following question open: Is it possible to have a sequence of integers $(n_k)_{k\in\mathbb{N}}$ with bounded gaps; i.e.,

$$n_{k+1} - n_k = O(1)$$

such that the Central Limit Theorem (1.2) still holds?

The question remained open for nearly 30 years. The answer is in the negative and it is provided by the following result:

Theorem 6 (Bobkov and Götze, 2007)**.** *Let $\{X_n\}_{n=1}^{\infty}$ be an orthonormal system in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that in probability*

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \to 0 \quad \textit{as} \quad n \to \infty. \tag{1.7}$$

Given an increasing sequence of indices $\tau = \{n_k\}_{k=1}^{\infty}$, assume that $S_N \Rightarrow \xi$ weakly in distribution, for some random variable ξ.

Then

$$\mathbb{E}\xi^2 \leq \Lambda - \operatorname{den}(\tau). \tag{1.8}$$

Here, we use the notation

$$\operatorname{den}(\tau) = \limsup_{N\to\infty} N/n_N$$

for the upper density of the sequence τ in the row of all natural numbers. In particular, if

$$\sup_k \, [n_{k+1} - n_k] < +\infty;$$

this quantity is positive; so ξ cannot be standard normal.

However, it has been shown very recently that normal limits are still possible for sequences $(n_k)_{k\in\mathbb{N}}$ with bounded gaps, the variance of the limit shall always be strictly less than 1 but it can be made as close to 1 as desired.

We are now ready to state the result:

Theorem 7 (Fukuyama, 2011)**.** *Fukuyama introduces the following notation:*

$$\left|\left\{x \in [0,1] \,\middle|\, \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \cos 2\pi n_k x \leq t\right\}\right| \to n_{0,\,1/4}(-\infty, t] \tag{1.9}$$

to denote the convergence in distribution to $N(0,\,1/4)$ *limit and the following one (to represent earlier results of Bobkov and Götze, see* [12]*)*

$$\left|\left\{x \in [0,1] \,\middle|\, \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \cos 2\pi n_k x \leq t\right\}\right| \to N_{0,\,\varrho^2}(-\infty, t] \tag{1.10}$$

where $\varrho^2(x) = 1/2 - 1/2d - 1/d^2 \sum_{n=1}^{d-1} (d-n) \cos 2\pi n x$ $(d = 2, 3, \dots)$ *while the corresponding measure is*

$$n_{0,\varrho^2}(A) = \int_0^1 n_{0,\varrho^2(x)}(A)dx; \quad A \in \mathcal{B}(\mathbb{R})$$

to denote the convergence in distribution to the mixed-Gaussian limit.

Now, let $\{a_n\}_{n\geq 1}$ *be a sequence of real numbers satisfying* $\sum_{n=1}^{\infty} |a_n| \leq 1/12$*. Then there exists a sequence* $(n_k)_{k\geq 1}$ *of positive integers satisfying* $1 \leq n_{k+1} - n_k \leq 9$ *and a Central Limit Theorem* (1.10) *holds for*

$$\varrho^2(x) = 1/4 + \sum_{n=1}^{\infty} a_n \cos 2\pi n x.$$

Also, there exists a sequence satisfying $1 \leq n_{k+1} - n_k \leq 5$ *and a Central Limit Theorem* (1.9)*.*

Common feature of constructions of sequences $(n_k)_{k\in\mathbb{N}}$ in proofs of Theorem 4, Theorem 5 and Theorem 7 is that they are all random. This indicates further that trigonometric series with random frequencies have remarkable properties. We will now take a closer look at these three constructions.

1. Construction in Theorem 4

The sequence used will be a sequence of heads within the sequence of heads within the infinite sequence of heads and tails generated by repeated tossing of a fair coin.

If we denote by n_k the sequence of heads, then along this sequence the Central Limit Theorem holds.

Moreover, by the Erdős–Rényi "pure heads" theorem we have

$$n_{k+1} - n_k = O(\log k).$$

with probability one.

2. Construction in Theorem 5

Berkes starts off by reducing the problem to the one where the function $\omega(k)$ satisfies the following four properties:

(i) $\omega(k)$ is positive,

(ii) $\omega(k)$ is non-decreasing,

(iii) $\omega(k)$ is integer-valued,

(iv) $\omega(k+1) \leq 2\omega(k)$,

and proceeds by introducing the following sequence of sets:

$$U_1 = \{j:\ 1 \le j \le \omega(1)\},$$

$$U_2 = \{j:\ \omega(1) < j \le \omega(1) + \omega(2)\}, \dots,$$

$$U_k = \{j:\ \omega(1) + \cdots + \omega(k-1) < j \le \omega(1) + \cdots + \omega(r)\}, \dots .$$

Then the $(n_k)_{k\in\mathbb{N}}$ are chosen to be independent random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ in a way that n_j is uniformly distributed on U_j; for all $j \in \mathbb{N}$.

Last, but definitely not the least, is the following spectacularly complicated construction due to Fukuyama.

3. Construction(s) in Theorem 7

There are two results which are proven in Theorem 7, Fukuyama classifies two different types of limits, pure and mixed Gaussian. The elaborate body of construction eventually (and we shall indicate where exactly) branches into 2 parts; each being used to obtain its own class of limits. We shall not test the reader's patience any further:

Let $a_0 = 1/4$ and $\varepsilon_n \in \{-1, 1\}$ according to $a_n = \varepsilon_n |a_n|$. Define quantities $\ell(v, \varepsilon)$ and $g(v, \varepsilon)$ as follows:

$$\big(\ell(0,+1), g(0,+1)\big) = (4,\ 0),$$

$$\big(\ell(1,+1), g(1,+1)\big) = (6,\ 1), \quad \big(\ell(1,-1), g(1,-1)\big) = (2,\ 1),$$

$$\big(\ell(2,+1), g(2,+1)\big) = (8,\ 2), \quad \big(\ell(2,-1), g(2,-1)\big) = (4,\ 2),$$

$$\big(\ell(v,\varepsilon), g(v,\varepsilon)\big) = \begin{cases} (6m, m) & \text{if } (v,\varepsilon) = (3m, \pm 1), \quad m \geq 1, \\ (6m+2, m+1) & \text{if } (v,\varepsilon) = (3m+1, \pm 1), \quad m \geq 1, \\ (6m+4, m+2) & \text{if } (v,\varepsilon) = (3m+2, \pm 1), \quad m \geq 1. \end{cases} \tag{1.11}$$

We must also quote the following result in order to justify and explain the notation we shall use later.

Theorem 8 (Lemma 1, Fukuyama [26])**.** *Assume*

$$\sum_{n=1}^{\infty} \frac{2|a_n|\ell(n,\varepsilon_n)}{\varrho(n,\varepsilon_n)} \leq 1 \tag{1.12}$$

and put

$$\mu = \frac{\ell(0,+1)}{\left(1 - \sum_{n=1}^{\infty} \frac{2|a_n|\ell(n,\varepsilon_n)}{\varrho(n,\varepsilon_n)} + \ell(0,+1) \sum_{n=1}^{\infty} \frac{2a_n}{\varrho(n,\varepsilon_n)}\right)}.$$

Then there exists a sequence $\{v_k\}$ *of non-negative integers such that*

$$v_k = O(\log k), \tag{1.13}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \ell\big(v_k, \varepsilon_{v_k}\big) = \mu \tag{1.14}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \varepsilon_{v_k} \varrho(v_k, \varepsilon_{v_k}) \cos 2\pi v_k x =$$

$$= 2\mu\big(\varrho^2(x) - 1/4\big) \text{ for almost every } x. \tag{1.15}$$

Now let $\{Y_j\}$ be a sequence of i.i.d. random variables taking values ± 1 with probability 1/2.

Fukuyama then defines related sequence $\{\widetilde{Y}_j\}$ as follows:

If $(v_k)_{k\in\mathbb{N}}$ is a sequence satisfying all the requirements of Theorem 8, then let $\Lambda_0 = 0$, $\Lambda_n = \sum_{k=1}^{n} \ell(v_k, \varepsilon_{v_k})$ $(n = 1, 2, \dots)$ $(\widetilde{Y}_j)$'s will be defined block-wise as follows:

$$\widetilde{Y}_{\Lambda_{n-1}+1}, \dots, \widetilde{Y}_{\Lambda_{n-1}+\ell(v_n,\varepsilon_{v_n})} = \widetilde{Y}_{\Lambda_n} \quad \text{for } n = 1, 2, \dots .$$

To relax the heavy notation Fukuyama drops some of the indices; namely in what follows Λ_{n-1}, v_n and ε_n shall be replaced by Λ, v and ε respectively.

Now, if $v \in \{0, 1, 2\}$ we put

$$\big(\widetilde{Y}_{\Lambda+1}, \dots, \widetilde{Y}_{\Lambda+\ell(v,\varepsilon)}\big) \ \text{ equal to}$$
$$\begin{cases} \big(Y_{\Lambda+1}, Y_{\Lambda+1}, Y_{\Lambda+3}, -Y_{\Lambda+3}\big) & \text{if } (v,\varepsilon) = (0,+1), \\ \big(Y_{\Lambda+1}, Y_{\Lambda+1}, Y_{\Lambda+3}, -Y_{\Lambda+3}, Y_{\Lambda+5}, Y_{\Lambda+5}\big) & \text{if } (v,\varepsilon) = (1,+1), \\ \big(Y_{\Lambda+1}, -Y_{\Lambda+1}\big) & \text{if } (v,\varepsilon) = (1,-1), \\ \big(Y_{\Lambda+1}, Y_{\Lambda+2}, Y_{\Lambda+1}, Y_{\Lambda+2}, Y_{\Lambda+5}, -Y_{\Lambda+5}, Y_{\Lambda+7}, Y_{\Lambda+7}\big) & \text{if } (v,\varepsilon) = (2,-1), \\ \big(Y_{\Lambda+1}, Y_{\Lambda+2}, -Y_{\Lambda+1}, -Y_{\Lambda+2}\big) & \text{if } (v,\varepsilon) = (2,-1). \end{cases}$$

If $v = 3m$ $(m \in \mathbb{N})$, we define

$$\widetilde{Y}_{\Lambda+3j-1} = \varepsilon\widetilde{Y}_{\Lambda+3m+3j+1} = Y_{\Lambda+3j+1} \quad (j = 0, 1, \dots, m-1),$$

$$\widetilde{Y}_{\Lambda+3j+2} = (-1)^j\widetilde{Y}_{\Lambda+3j+3} = \widetilde{Y}_{\Lambda+3j+2} \quad (j = 0, 1, \dots, 2m-1).$$

If $v = 3m+1$ $(m \in \mathbb{N})$, we define:

$$\widetilde{Y}_{\Lambda+3j+1} = \varepsilon\widetilde{Y}_{\Lambda+3m+3j+2} = Y_{\Lambda+3j+1} \quad (j = 0, 1, \dots, m),$$

$$\widetilde{Y}_{\Lambda+3j-2} = (-1)^j\widetilde{Y}_{\Lambda+3j+2} = Y_{\Lambda+3j+2} \quad (j = 0, 1, \dots, m-1),$$

$$\widetilde{Y}_{\Lambda+3j+3} = (-1)^j\widetilde{Y}_{\Lambda+3j+4} = Y_{\Lambda+3j+3} \quad (j = m, m+1, \dots, 2m-1).$$

If, however, $v = 3m + 2$ ($m \in \mathbb{N}$) we define

$$\widetilde{Y}_{\Lambda+3j+1} = \varepsilon \widetilde{Y}_{\Lambda+3m+3j+3} = Y_{\Lambda+3j+1} \quad (j = 0, 1, \ldots, m),$$

$$\widetilde{Y}_{\Lambda+3m+2} = \varepsilon \widetilde{Y}_{\Lambda+6m+4} = Y_{\Lambda+3m+2},$$

$$\widetilde{Y}_{\Lambda+3j+2} = (-1)^j \widetilde{Y}_{\Lambda+3j+3} = Y_{\Lambda+3j+2} \quad (j = 0, 1, \ldots, m-1),$$

$$\widetilde{Y}_{\Lambda+3j+4} = (-1)^j \widetilde{Y}_{\Lambda+3j+5} = Y_{\Lambda+3j+4} \quad (j = m, m+1, \ldots, 2m-1).$$

Finally we identify our sequence $(n_j)_{j\in\mathbb{N}}$ with the set $\{k \in \mathbb{N} : \widetilde{Y}_k = 1\}$. This defines the corresponding sequence(s).

The "branching point" of the argument is as follows: If we want a pure Gaussian limit, we put $v_k \equiv 0$ and $\varepsilon_{v_k} \equiv +1$. Otherwise we get a mixed Gaussian limit distribution.

1.2 Result

We will now state and prove our result. Instead of integers our random frequencies are now uniformly distributed continuous random variables on disjoint intervals of equal length. The limit is a different mixed-Gaussian. Without further delay we proceed as follows:

Theorem 9 (Berkes and Rašeta)**.** *Let $S_1, S_2, \ldots$ be a sequence of independent random variables on some space $(\Omega, \mathcal{A}, \mathbb{P})$ with $S_k \sim \mathcal{U}[20k-20, 20k-10]$, $k \in \mathbb{N}$.*

Furthermore, we introduce the probability measure μ on $(-\infty, +\infty)$ by

$$\mu(A) = \frac{1}{\pi} \int_A \left(\frac{\sin x}{x} \right)^2 dx \quad \forall A \text{ in the Borel } \sigma\text{-field.} \tag{1.16}$$

Then:

$$\frac{\sum_{k=1}^{N} \sin S_k x}{\sqrt{N/2}} \xrightarrow{d} X \quad \mathbb{P} - a.s.$$

where the characteristic function of X is given by

$$\phi_X(\lambda) = \int_{-\infty}^{+\infty} \exp\left(-\frac{\lambda^2}{2}\left(1 - \left(\frac{\sin 5x}{5x}\right)^2\right)\right) d\mu(x). \tag{1.17}$$

Proof. Define

$$\varphi_k(x) = \sin S_k x - \mathbb{E}^{\mathbb{P}}(\sin S_k x) = \sin S_k x - \left(\frac{\sin 5x}{5x}\right) \sin(20k - 15)x$$

by basic algebra.

We now claim that

$$\frac{\sum_{k=1}^{N} \varphi_k(x)}{\sqrt{N/2}} \xrightarrow{d} X \Longrightarrow \frac{\sum_{k=1}^{N} \sin S_k x}{\sqrt{N/2}} \xrightarrow{d} X \text{ for almost every } \omega. \tag{1.18}$$

Recall the basic trigonometric identity

$$\sin\varphi + \sin(\varphi + \alpha) + \sin(\varphi + 2\alpha) + \cdots + \sin(\varphi + n\alpha) =$$
$$= \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \sin\left(\varphi + \frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}. \tag{1.19}$$

(1.19) applied to our case clearly yields:

$$\sum_{k=1}^{N} \sin(20k - 15)x = \sum_{k=1}^{N} \sin\left((-15x) + (20x)k\right) =$$
$$= \frac{\sin\frac{(N+1)\cdot 20x}{2} \cdot \sin\left(-15x + \frac{N \cdot 20x}{2}\right)}{\sin\frac{20x}{2}} =$$

$$= \frac{\sin 10(N+1)x \sin(10N-15)x}{\sin 10x}, \tag{1.20}$$

whence it follows that

$$\frac{\sum_{k=1}^{N} \sin S_k x}{\sqrt{N/2}} = \frac{\sum_{k=1}^{N} \varphi_k(x)}{\sqrt{N/2}} +$$

$$+ \left(\frac{\sin 5x}{5x}\right) \cdot \frac{1}{\sqrt{N/2}} \cdot \frac{\sin 10(N+1)x \sin(20N-15)x}{\sin 10x} \tag{1.21}$$

with the second summand on the RHS tends to 0 for almost all x with respect to measure φ. This is because μ and the Lebesgue measure are equivalent and all countable sets have Lebesgue measure 0. Hence, trivially, the second summand therefore tends to 0 φ in probability and whence (1.18) follows from Fubini's Theorem and Slutsky's Lemma applied to (1.21).

We have now reduced the problem to dealing with random variables with $\mathbb{P}$-expectation 0; the convenience of such an approach shall become clear later on. Now let us introduce

$$T_N = \frac{1}{\sqrt{N/2}} \sum_{k=1}^{N} \varphi_k(x).$$

The heart of our argument lies in the following two claims:

(i) $T_{N^3} \xrightarrow{d} X$, $\mathbb{P}$-almost surely where the characteristic function of X is given by (1.17).

(ii) We claim that (i) is actually sufficient, namely that $T_{N^3} \xrightarrow{d} X$ ($\mathbb{P}$-a.s.) $\Rightarrow$ $T_N \xrightarrow{g} X$ ($\mathbb{P}$-a.s.).

We focus on (ii) first. Partition $\mathbb{N}$ in the following way:

$$\forall M \in \mathbb{N}\ \exists N \in \mathbb{N} \ \text{ with } \ N^3 < M \leq (N+1)^3.$$

We then write

$$T_M = T_{N^3(M)} + \big(T_M - T_{N^3(M)}\big)$$

where M and N are as above.

We introduce $\Pi_M = T_m - T_{N^3(M)}$. Our strategy will be to show that

$$(\mathbb{E}^{\mathbb{P}}\Pi_M^2)^{1/2} \to 0 \Rightarrow \Pi_M \xrightarrow{L^2} 0 \Rightarrow \Pi_M \xrightarrow{\mathbb{P}} 0$$

and assuming (i), (ii) shall follow by Slutsky's lemma and Fubini's theorem.

To this end we have

$$\begin{aligned} T_M - T_{N^3(M)} &= \frac{1}{\sqrt{M/2}}\sum_{k=1}^{M}\varphi_k(x) - \frac{1}{\sqrt{N^3/2}}\sum_{k=1}^{N^3}\varphi_k(x) = \\ &= \left\{\frac{1}{\sqrt{M/2}}\sum_{k=1}^{M}\varphi_k(x) - \frac{1}{\sqrt{N^3/2}}\sum_{k=1}^{M}\varphi_k(x) + \right. \\ &\quad \left. + \frac{1}{\sqrt{N^3/2}}\sum_{k=1}^{M}\varphi_k(x) - \frac{1}{\sqrt{N^3/2}}\sum_{k=1}^{N^3}\varphi_k(x)\right\} = \\ &= \left\{\left(\frac{1}{\sqrt{M/2}} - \frac{1}{\sqrt{N^3/2}}\right)\sum_{k=1}^{M}\varphi_k(x) + \frac{1}{\sqrt{N^3/2}}\sum_{k=N^3+1}^{M}\varphi_k(x)\right\}. \end{aligned}$$

For simplicity introduce

$$a(x) := \left(\frac{1}{\sqrt{M/2}} - \frac{1}{\sqrt{N^3/2}}\right)\sum_{k=1}^{M}\varphi_k(x),$$

$$b(x) := 1/\sqrt{N^3/2}\cdot\sum_{k=N^3+1}^{M}\varphi_k(x).$$

Clearly $(a+b)^2 \le 2(a^2+b^2)\ \forall a,b \in \mathbb{R}$ and hence

$$\left(\int_{-\infty}^{+\infty}\big(a(x)+b(x)\big)^2\frac{1}{\pi}\left(\frac{\sin x}{x}\right)^2 dx\right)^{1/2} \le$$

$$\leq \left(\int_{-\infty}^{+\infty} 2\big(a^2(x) + b^2(x)\big) \frac{1}{\pi} \left(\frac{\sin x}{x}\right)^2 dx \right)^{1/2}.$$

Elementary algebra yields

$$a^2(x) = \left(\frac{1}{\sqrt{M/2}} - \frac{1}{\sqrt{N^3/2}} \right)^2 \left\{ \sum_{k=1}^{M} \varphi_k^2(x) + 2 \sum_{i \neq j} \varphi_i(x)\varphi_j(x) \right\}.$$

It is known that if $|\alpha| > 4$, then

$$\int_{-\infty}^{+\infty} \cos \alpha x \left(\frac{\sin x}{x}\right)^2 dx = 0. \tag{1.22}$$

We now claim that $\{\varphi_k(x)\}_{k \in \mathbb{N}}$ are orthogonal, i.e.

$$\int_{-\infty}^{+\infty} \varphi_k(x)\varphi_\ell(x) d\mu(x) \alpha \delta_{k\ell}; \text{ where } \delta \text{ is the Kronecker's symbol.}$$

We proceed as follows:

$$\int_{-\infty}^{+\infty} \sin S_k x \sin S_\ell x d\mu(x) =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \cos(S_k - S_\ell) x d\mu(x) - \frac{1}{2} \int_{-\infty}^{+\infty} \cos(S_k + S_\ell) x d\mu(x). \tag{1.23}$$

Recall that $S_n \sim \mathcal{U}[20n - 20,\ 20n - 10]$ by construction. This trivially implies that

$$|S_k - S_\ell| \geq 10 > 4 \text{ for all } k \neq \ell \text{ and}$$

$$|S_k + S_\ell| = S_k + S_\ell \geq S_\ell > 10 > 4.$$

It then follows that both integrals on the RHS of (1.23) vanish and so

$$\int_{-\infty}^{+\infty} \sin S_k x \sin S_\ell x d\mu(x) = 0 \quad \forall k \neq \ell. \tag{1.24}$$

The orthogonality of $\{\varphi_k(x)\}_{k\geq 1}$ follows from (1.24) by same tedious algebra and Fubini's theorem.

But then:

$$\int_{-\infty}^{+\infty} 2a^2(x)dy(x) =$$

$$= 2\int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{M/2}}-\frac{1}{\sqrt{N^3/2}}\right)^2\left\{\sum_{k=1}^{M}\varphi_k^2(x)+\sum_{i\neq j}\varphi_i(x)\varphi_j(x)\right\}d\mu(x) =$$

$$= 2\left(\frac{1}{\sqrt{M/2}}-\frac{1}{\sqrt{N^3/2}}\right)^2\left\{\sum_{k=1}^{M}\int_{-\infty}^{+\infty}\varphi_k^2(x)d\mu(x)+\sum_{i\neq j}\int_{-\infty}^{+\infty}\varphi_i(x)\varphi_j(x)d\mu(x)\right\} =$$

$$= (\text{by the orthogonality of } \{\varphi_k(x)\}_{k\geq 1}) =$$

$$= 2\left(\frac{1}{\sqrt{M/2}}-\frac{1}{\sqrt{N^3/2}}\right)^2\sum_{k=1}^{M}\int_{-\infty}^{+\infty}\varphi_k^2(x)d\mu(x) \leq$$

$$\leq 8M\left(\frac{1}{\sqrt{M/2}}-\frac{1}{\sqrt{N^3/2}}\right)^2 (|\varphi_k(x)|\leq 2)\,.$$

Recall that $N^3 \leq M \leq (N+1)^3$.

$$\text{Simple algebra shows that the last quantity is at most } 96/N. \tag{1.25}$$

An identical computation shows that

$$\int_{-\infty}^{+\infty} 2b^2(x)d\mu(x) \leq 96/N \tag{1.26}$$

whence it follows that

$$\left(\int_{-\infty}^{+\infty} 2\big(a^2(x)+b^2(x)\big)d\mu(x)\right)^{1/2} \leq 8\sqrt{3}/\sqrt{N}$$

and so

$$(\mathbb{E}^{\mu}\Pi_M^2)^{1/2} \leq 8\sqrt{3}/\sqrt{N} \leq 8\sqrt{3}/(M^{1/3}-1)^{1/2} \to 0 \text{ as } M \to \infty.$$

Thus (ii) holds and proving (i) is a task we focus on in order to complete the proof.

The characteristic function of the corresponding partial sum is

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty} \exp\left(\frac{i\lambda}{\sqrt{N/2}} \sum_{k=1}^{N} \varphi_k(x)\right) d\mu(x) =$$

$$= \int_{-\infty}^{+\infty} \prod_{k=1}^{N} \exp\left(\frac{i\lambda}{\sqrt{N/2}} \varphi_k(x)\right) d\mu(x). \tag{1.27}$$

Basic complex analysis gives us

$$\exp(z) = (1+z)\exp\bigl(z^2/2 + o(z^2)\bigr) \quad \text{for} \quad z \to 0. \tag{1.28}$$

Since $|\varphi_k(x)| \leq 2$ for all $k \in \mathbb{N}$ it follows by (1.28) that

$$\exp\left(\frac{i\lambda}{\sqrt{N/2}}\varphi_k(x)\right) = \left(1 + \frac{i\lambda}{\sqrt{N/2}}\right)\left(\frac{-\lambda^2\varphi_k^2(x)}{N} + o\left(\frac{2\lambda^2\varphi_k^2(x)}{N}\right)\right).$$

Observe that $\bigl(\varphi_k^2(x)\bigr)_{k\geq 1}$ is itself a sequence of independent random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, for any fixed $x \in \mathbb{R}$.

Trivially, $\bigl|\varphi_k^2(x) - \mathbb{E}^{\mathbb{P}}\varphi_k^2(x)\bigr| \leq 8$ and so

$$\bigl(\varphi_k^2(x) - \mathbb{E}^{\mathbb{P}}\varphi_k^2(x)\bigr)^4 \leq 4096.$$

Thus, by the Strong Law of Large Numbers and Fubini's theorem it follows that

$$\frac{1}{N}\sum_{k=1}^{N}\bigl(\varphi_k^2(x) - \mathbb{E}^{\mathbb{P}}\varphi_k^2(x)\bigr) \overset{\mu\text{-a.e.}}{\longrightarrow} 0.$$

But:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\varphi_k^2(x) &= \mathbb{E}^{\mathbb{P}}\big((\sin S_k x - \mathbb{E}^{\mathbb{P}}\sin S_k x)^2\big) = \\ &= \mathbb{E}^{\mathbb{P}}\sin^2 S_k x - (\mathbb{E}^{\mathbb{P}}\sin S_k x)^2 = \\ &= \frac{1}{2} - \frac{1}{2}\mathbb{E}^{\mathbb{P}}\cos^2 S_k x - \frac{1}{2}\left(\frac{\sin 5x}{5x}\right)^2 + \frac{1}{2}\left(\frac{\sin 5x}{5x}\right)^2 \cdot \cos(40k-30)x;\end{aligned}$$

upon some basic algebra. Hence it follows that

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N}\varphi_k^2(x) = \frac{1}{2} - \frac{1}{2}\cdot\left(\frac{\sin 5x}{5x}\right)^2 - \\ - \frac{1}{2}\left(\frac{\sin 10x}{10x}\right)\cdot\frac{1}{N}\sum_{k=1}^{N}\cos(40k-30)x + \\ + \frac{1}{2}\left(\frac{\sin 5x}{5x}\right)^2\cdot\frac{1}{N}\sum_{k=1}^{N}\cos(40k-30)x \qquad (1.29)\end{aligned}$$

whence arguing exactly as before we finally deduce that

$$\frac{1}{N}\sum_{k=1}^{N}\varphi_k^2(x) \xrightarrow{\mu\text{-a.s.}} \frac{1}{2}\left(1 - \left(\frac{\sin 5x}{5x}\right)^2\right).$$

Simple algebra shows that

$$\begin{aligned}\prod_{k=1}^{N}\exp\left(\frac{-\lambda^2\varphi_k^2(x)}{N} + o\left(\frac{2\lambda^2\varphi_k^2(x)}{N}\right)\right) = \\ = \exp\left(\frac{\lambda^2}{N}\sum_{k=1}^{N}\varphi_k^2(x)(-1+o(1))\right) = \\ = \exp\left(-(1+o(1))\frac{\lambda^2}{N}\sum_{k=1}^{N}\varphi_k^2(x)\right)\end{aligned}$$

and thus our characteristic function reads

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty}\prod_{k=1}^{N}\left(1 + \frac{i\lambda\varphi_k(x)}{\sqrt{N/2}}\right)\exp\left(-(1+o(1))\frac{\lambda^2}{N}\sum_{k=1}^{N}\varphi_k^2(x)\right)d\mu(x).$$

More simple algebra coupled with Dominated Convergence Theorem shows that

$$\phi_{T_N}(\lambda) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} \left(1 + \frac{i\lambda\varphi_k(x)}{\sqrt{N/2}}\right) \exp\bigl(-\lambda^2 g(x)\bigr) d\mu(x) + o(1)$$

where, for brevity, we introduced

$$g(x) = \frac{1}{2}\left(1 - \left(\frac{\sin 5x}{5x}\right)^2\right). \tag{1.30}$$

So we will be done provided we can show that

$$\int_{-\infty}^{+\infty} \prod_{k=1}^{N^3} \left(1 + \frac{i\lambda}{\sqrt{N^3/2}}\varphi_k(x)\right) \exp\bigl(-\lambda^2 g(x)\bigr) d\mu(x) \xrightarrow{\mathbb{P}\text{-a.s.}} \int_{-\infty}^{+\infty} \exp\bigl(-\lambda^2 g(x)\bigr) d\mu(x);$$

since the limit function is continuous at $\lambda = 0$.

Define

$$\Gamma_N = \int_{-\infty}^{+\infty} \left[\prod_{k=1}^{N^3} \left(1 + \frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}} - 1\right)\right] \exp\bigl(-\lambda^2 g(x)\bigr) d\mu(x).$$

Thus, it will be sufficient to show that $\Gamma_N \xrightarrow{\mathbb{P}\text{-a.s.}} 0$; and this will trivially follow provided we can show that $|\Gamma_N| \xrightarrow{\mathbb{P}\text{-a.s.}} 0$; where $|z|$ is the modulus of the complex number z.

Let $\Theta_n := |\Gamma_n|$. Beppo-Levy's theorem says:

$$\sum_{n\in\mathbb{N}} \mathbb{E}\Theta_n^2 < \infty \Rightarrow \Gamma_n \to 0; \quad \mathbb{P}\text{-almost surely.}$$

We shall therefore focus on showing that

$$\sum_{N\in\mathbb{N}} \mathbb{E}(\Gamma_N \overline{\Gamma}_N) < \infty;$$

upon which the proof will be complete.

To this end we have:

$$\mathbb{E}^{\mathbb{P}}\Gamma_N\overline{\Gamma}_N = \int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\left[\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right]\left[\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right]\cdot$$
$$\cdot\exp\bigl(-\lambda^2 g(x)\bigr)\exp\bigl(-\lambda^2 g(y)\bigr)d\mu(x)d\mu(y)d\mathbb{P}(\omega).$$

For brevity introduce:

$$A_N(x,y,\omega) := \left[\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right]\left[\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right];\ A_N(x,y,\omega)\in\mathbb{C}.$$

Define

$$B_N(x,y,\omega) := \mathrm{Re}\bigl(A_N(x,y,\omega)\bigr) \quad \text{and}$$
$$C_N(x,y,\omega) := \mathrm{Im}\bigl(A_N(x,y,\omega)\bigr);$$

so that we can write the above as

$$\mathbb{E}^{\mathbb{P}}\Gamma_N\overline{\Gamma}_N = \int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} B_N(x,y,\omega)\exp\bigl(-\lambda^2 g(x)\bigr)\exp\bigl(-\lambda^2 g(y)\bigr)d\mu(x)d\mu(y)d\mathbb{P}(\omega)+$$
$$+\,i\cdot\int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} C_N(x,y,\omega)\exp\bigl(-\lambda^2 g(x)\bigr)\exp(-\lambda^2 g(y))d\mu(x)d\mu(y)d\mathbb{P}(\omega).$$

Clearly

$$\bigl|B_N(x,y,\omega)\bigr| \le \bigl|A_N(x,y,\omega)\bigr| =$$
$$= \left|\left[\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right]\cdot\left[\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right]\right| =$$
$$= \left|\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right|\cdot\left|\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right|.$$

Using a bold bound $|z_1 - z_2| \le |z_1| + |z_2|$ and many times the relation $|z_1 z_2| = |z_1|\,|z_2|$ we get

$$\left|\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right| \le 1+\prod_{k=1}^{N^3}\left|1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right| =$$
$$= 1+\prod_{k=1}^{N^3}\left(\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)\left(1-\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)\right)^{1/2} =$$
$$= 1+\prod_{k=1}^{N^3}\left(1+\frac{2\lambda^2\varphi_k^2(x)}{N^3}\right)^{1/2}. \tag{1.31}$$

But we know that $1+x \le e^x$ so that the bound in the above is $1+\exp(4\lambda^2)$; using again $|\varphi_k(x)| \le 2$ for all $x \in \mathbb{R}$ and all k in $\mathbb{N}$. Similarly,

$$\left|\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right| \le 1+\exp(4\lambda^2). \tag{1.32}$$

Thus $|B_N(x,y,\omega)| \le \big(1+\exp(4\lambda^2)\big)^2$ and in the identical fashion we get that

$$\big|C_N(x,y,\omega)\big| \le \big(1+\exp(4\lambda^2)\big)^2,$$

too. Since $|\sin x/x| \le 1$ for all $x \in \mathbb{R}$ we also see that $g(x) \ge 0$ for all x, which, coupled with the above, easily yields:

$$\int_\Omega\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |B_N(x,y,\omega)| \exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big)d\mu(x)d\mu(y)d\mathbb{P}(\omega) \le$$
$$\le \int_\Omega\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\big(1+\exp(4\lambda^2)\big)^2 d\mu(x)d\mu(y)d\mathbb{P}(\omega) = \big(1+\exp(4\lambda^2)\big)^2 < \infty.$$

Similarly,

$$\int_\Omega\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |C_N(x,y,\omega)| \exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big)d\mu(x)d\mu(y)d\mathbb{P}(\omega) \le$$

$$\leq \int_{\Omega}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\big(1+\exp(4\lambda^2)\big)^2 d\mu(x)d\mu(y)d\mathbb{P}(\omega) = \big(1+\exp(4\lambda^2)\big)^2 < \infty.$$

Putting all this together one can see that Fubini's theorem can be applied to yield:

$$\int_{\Omega}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_N(x,y,\omega)\exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big)d\mu(x)d\mu(y)d\mathbb{P}(\omega) =$$

$$= \int_{\Omega}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} B_N(x,y,\omega)\exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big)d\mu(x)d\mu(y)d\mathbb{P}(\omega)+$$

$$+\, i\int_{\Omega}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_N(x,y,\omega)\exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big)d\mu(x)d\mu(y)d\mathbb{P}(\omega) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbb{E}^{\mathbb{P}}\left[\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right]\left[\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right]\cdot$$

$$\cdot\exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big)d\mu(x)d\mu(y).$$

Part of the above integral under the $\mathbb{P}$-expectation is:

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)-1\right]\left[\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1\right] =$$

$$= \mathbb{E}^{\mathbb{P}}\prod_{k=1}^{N^3}\left(1+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}\right)\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}\right)-1 =$$

$$= \mathbb{E}^{\mathbb{P}}\prod_{k=1}^{N^3}\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}+\frac{2\lambda^2}{N^3}\varphi_k(x)\varphi_k(y)\right)-1.$$

But, via grouping independent quantities, one can see that, for all but fixed x and y in $\mathbb{R}^2$ we have that

$$\left(1-\frac{i\lambda\varphi_k(y)}{\sqrt{N^3/2}}+\frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}}+\frac{2\lambda^2}{N^3}\varphi_k(x)\varphi_k(y)\right)_{k\geq 1}$$

is itself a sequence of independent random variables as $(\Omega, \mathcal{A}, \mathbb{P})$, so that the above expression equals

$$\prod_{k=1}^{N^3} \mathbb{E}^{\mathbb{P}} \left(1 - \frac{\lambda\varphi_k(y)}{\sqrt{N^3/2}} + \frac{i\lambda\varphi_k(x)}{\sqrt{N^3/2}} + \frac{2\lambda^2}{N^3}\varphi_k(x)\varphi_k(y)\right) - 1 =$$
$$= \prod_{k=1}^{N^3} \left(1 - \frac{i\lambda\mathbb{E}^{\mathbb{P}}\varphi_k(y)}{\sqrt{N^3/2}} + \frac{i\lambda\mathbb{E}^{\mathbb{P}}\varphi_k(y)}{\sqrt{N^3/2}} + \frac{2\lambda^2}{N^3}\mathbb{E}^{\mathbb{P}}\varphi_k(x)\varphi_k(y)\right) - 1 =$$
$$= \prod_{k=1}^{N^3} \left(1 + \frac{2\lambda^2}{N^3}\mathbb{E}^{\mathbb{P}}\varphi_k(x)\varphi_k(y)\right) - 1.$$

Introduce, for brevity, $\Psi_k(x,y) = \mathbb{E}^{\mathbb{P}}\varphi_k(x)\varphi_k(y)$. Then our expression of interest $\mathbb{E}^{\mathbb{P}}\Gamma_N\overline{\Gamma}_N$ reads

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \prod_{k=1}^{N^3} \left(1 + \frac{2\lambda^2}{N^3}\Psi_k(x,y)\right) \exp\bigl(-\lambda^2 g(x)\bigr)\exp\bigl(-\lambda^2 g(y)\bigr) d\mu(x)d\mu(y) -$$
$$- \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\bigl(-\lambda^2 g(x)\bigr)\exp\bigl(-\lambda^2 g(y)\bigr) d\mu(x)d\mu(y).$$

We know that

$$1 + x = \exp\bigl(x + O(x^2)\bigr) \quad \text{for} \quad |x| \le 1 \quad \text{i.e.}$$
$$|\log(1+x) - x| \le Cx^2 \quad \text{for all} \quad |x| \le 1 \quad \text{for some } C \in \mathbb{R}^+.$$

Note that $|\Psi_k(x,y)| \le 4 \Rightarrow$ for all N large enough

$$\left|\frac{2\lambda^2}{N^3}\Psi_k(x,y)\right| \le 1; \quad \forall k \in \mathbb{N};$$

$\Rightarrow$ for all $N \in \mathbb{N}$ large enough

$$\left|\log\left(1 + \frac{2\lambda^2\Psi_k(x,y)}{N}\right) - \frac{2\lambda^2\Psi_k(x,y)}{N}\right| \le C\lambda^4/N^2 \tag{1.33}$$

which easily yields that

$$\sum_{k=1}^{N}\left|\log\left(1 + \frac{2\lambda^2\Psi_k(x,y)}{N}\right) - \frac{2\lambda^2\Psi_k(x,y)}{N}\right| \le \frac{C\lambda^4}{N}.$$

However, the above also implies that

$$\left|\log \prod_{k=1}^{N}\left(1+\frac{2\lambda^2\Psi_k(x,y)}{N}\right)-\sum_{k=1}^{N}\frac{2\lambda^2\Psi_k(x,y)}{N}\right|\leq \frac{C\lambda^4}{N}.$$

For brevity introduce yet another quantity

$$G_N(x,y):=\sum_{k=1}^{N}\frac{2\lambda^2\Psi_k(x,y)}{N}.$$

Then it is clear that

$$\prod_{k=1}^{N}\left(1+\frac{2\lambda^2\Psi_k(x,y)}{N}\right)\leq \exp\left(G_N(x,y)+\frac{C\lambda^4}{N}\right).$$

For $|x|\leq 1/5$ say

$$\exp(x)\leq 1+\frac{5}{4}x. \tag{1.34}$$

Similar ideas yield that there exist $\alpha,\beta\in\mathbb{R}^1$ such that

$$\prod_{k=1}^{N}\left(1+\frac{2\lambda^2\Psi_k(x,y)}{N}\right)\leq 1+\frac{\alpha}{N}+\beta|G_N(x,y)|. \tag{1.35}$$

Observe that our measure μ is σ-finite so that Tonelli's theorem applied to $G_N^2(x,y)$ yields:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_N^2(x,y)d\mu(x)d\mu(y)=\int_{(-\infty,+\infty)^2} G_N^2(x,y)d(\mu\bigotimes\mu)=$$
$$=\int_{(-\infty,+\infty)^2} G_N^2(\lambda)d(\mu\bigotimes\mu)(z).$$

Trivially, by the very definition of the product measure, $\big((-\infty,+\infty)^2,\,\mathcal{B}(\mathbb{R}^2),\mu\bigotimes\mu\big)$ is itself a probability space so that

$$\int_{(-\infty,+\infty)^2} G_N^2(z)d(\mu\bigotimes\mu)(z)=\mathbb{E}^{\mu\bigotimes\mu}(G_N^2)=\mathbb{E}^{\mu\bigotimes\mu}(|G_N|^2)\geq$$

$$\geq \left(\mathbb{E}^{\mu \bigotimes \mu}|G_N|\right)^2 \text{ (by Jensen's inequality)}$$

whence

$$\left(\mathbb{E}^{\mu \bigotimes \mu}|G_N|\right)^2 \leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_N^2(x,y)d\mu(x)d\mu(y). \tag{1.36}$$

Recall that

$$G_N(x,y) = \sum_{k=1}^{N} \frac{2\lambda^2}{N}\Psi_k(x,y)$$

so that

$$\begin{aligned} G_N^2(x,y) &= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{4\lambda^4}{N^2}\Psi_k(x,y)\Psi_\ell(x,y) = \\ &= \frac{4\lambda^4}{N^2}\sum_{i=1}^{N}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_i^2(x,y)d\mu(x)d\mu(y) + \\ &\quad + \frac{4\lambda^4}{N^2}\sum_{k\neq\ell}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_k(x,y)\Psi_\ell(x,y)d\mu(x)d\mu(y). \end{aligned}$$

Arguing exactly as before one can deduce that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_k(x,y)\Psi_\ell(x,y)d\mu(x)d\mu(y) = 0; \text{ whenever } k \neq \ell.$$

Using $\Psi_k^2(x,y) \leq 16$ for all x, y and k we get:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_N^2(x,y)d\mu(x)d\mu(y) \leq 64\lambda^4/N$$

and whence it follows that

$$\begin{aligned} \frac{64\lambda^4}{N} &\geq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_N^2(x,y)d\mu(x)d\mu(y) \geq \\ &\geq \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |G_N(x,y)|d\mu(x)d\mu(y)\right)^2; \text{ i.e.} \end{aligned}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |G_N(x,y)| d\mu(x) d\mu(y) \le 8\lambda^2/\sqrt{N}. \tag{1.37}$$

It follows that we can bound our expression of interest in the following way:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\Gamma_N\overline{\Gamma}_N \le & \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Big(1+\frac{\alpha}{N^3}+\beta|G_N^3(x,y)|\Big)\exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big) d\mu(x) d\mu(y) - \\
& - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big) d\mu(x) d\mu(y) = \\
= & \frac{\alpha}{N^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big) d\mu(x) d\mu(y) + \\
& + \beta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \big|G_{N^3}(x,y)\big| \exp\big(-\lambda^2 g(x)\big)\exp\big(-\lambda^2 g(y)\big) d\mu(x) d\mu(y) \le \\
\le & \frac{\alpha}{N^3} + \beta\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \big|G_{N^3}(x,y)\big| d\mu(x) d\mu(y) \le \quad (\text{since } g(t) \ge 0 \text{ for all } t \in \mathbb{R}) \\
\le & \frac{\alpha}{N^3} + \frac{8\beta\lambda^2}{N^{3/2}}.
\end{aligned} \tag{1.38}$$

Thus

$$\mathbb{E}^{\mathbb{P}}\Gamma_N\overline{\Gamma}_N \le \gamma_\lambda / N^{3/2} \tag{1.39}$$

for some constant γ_λ. Finally we have

$$\sum_{N\in\mathbb{N}} \mathbb{E}^{\mathbb{P}}\Gamma_N\overline{\Gamma}_N < \infty$$

and the proof is complete. □

Chapter 2

Limit Theorems for the Schatte Model

2.1 Introduction

In this chapter we shall be dealing with a particular structure of weakly dependent random variables, namely the remarkable construction of Peter Schatte from the 1980's. More formally, the underlying sequence of random variables $(X_j)_{j\in\mathbb{N}}$ will be i.i.d. with X_1 absolutely continuous. We shall establish the Strassen-type Law of the Iterated Logarithm together with a Weighted Law of the Iterated Logarithm, both for functions of $S_n x = (X_1 + \cdots + X_n)x$, under mild conditions on f. In particular, we discover that the limits in the above are not constants as in the classical theory, but remarkable functions of x.

Again, for completeness of the exposition, we shall remind the reader of some classical results, introduce some newer ones, hence setting up the framework for those of our own.

2.2 Classical and Strassen-type of Laws of the Iterated Logarithm

Theorem 10 (Hartman and Wintner, 1941). *Let* $(X_k)_{k\geq 1}$ *be a sequence of independent and identically distributed random variables, with* $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$.

Then, with probability one,

$$\limsup_{n\to\infty} \frac{S_n}{(2n\log\log n)^{1/2}} = 1. \tag{2.1}$$

The following result is definitely an absolute classic:

Theorem 11 (Kolmogorov, 1929). *Let* $(X_j)_{j\in\mathbb{N}}$ *be a sequence of independent, zero-mean but not necessarily identically distributed random variables.*

Furthermore, let $S_n = X_1 + \cdots + X_n$ *and assume that* $\mathbb{E}X_j^2 < \infty$ $\forall j \in \mathbb{N}$ *with* $\mathbb{V}\mathrm{ar}\, S_n \to \infty$ *as* $n \to \infty$*. Introduce, for brevity,* $A_n = \mathbb{V}\mathrm{ar}\, S_n$*. Then if there exists a sequence of constants* M_k *such that* $|X_k| \leq M_k$ *almost surely, and*

$$M_n = o\left(\frac{A_n}{(\log\log A_n^2)^{1/2}}\right),$$

then

$$\limsup_{n\to\infty} \frac{S_n}{(2A_n^2\log\log A_n^2)^{1/2}} = 1; \quad \textit{almost surely.} \tag{2.2}$$

It took some time for another fundamental breakthrough. The following result is absolutely astonishing:

Theorem 12 (Strassen, 1964). *Let* $(X_n)_{n\geq 1}$ *be a sequence of i.i.d. zero-mean and unit variance random variables. Let* $S_n = X_1 + \cdots + X_n$ *and define* $(\eta_n)_{n\geq 1}$ *to be a sequence of continuous functions on* $[0,1]$ *via linearly interpolating* $(2n\log\log n)^{-1/2}S_i$ *at* i/n.

Then, with probability 1*, the set of limit points of the sequence* $(\eta_n)_{n\geq 3}$ *with respect to the uniform topology coincides with the set of absolutely continuous functions* x *on* $[0,1]$ *such that*

$$x(0) = 0 \quad \text{and} \quad \int_0^1 \dot{x}^2 dt \leq 1. \tag{2.3}$$

It became a standard in probability theory to call this set K.

There is no better set of words to comment on this result than "raw power".

For example, recovering the Hartman–Wintner's Law of the Iterated Logarithm from this result is astonishingly easy:

For $a \leq b$, $a, b \in [0,1]$

$$|x(a) - x(b)| = \left| \int_a^b \dot{x}^2 dt \right| \leq \left(\int_a^b dt \int_a^b \dot{x}^2 dt \right)^{1/2} \leq \sqrt{b-a} \text{ for any } x \in K. \tag{2.4}$$

With $a = 0$, $b = 1$ we see that

$$\sup_{x\in k} x(1) = 1 \quad \text{and the supremum is attained at } x = t.$$

But this means that

$$\mathbb{P}\left\{ \limsup_{n\to\infty} (2n \log\log n)^{-1/2} S_n = 1 \right\} = 1$$

and we are done!

Via calculus of variations Strassen obtains several remarkable corollaries of his result. To give the reader a flavour of it, we state a few of those:

Theorem 13 (Strassen, 1964)**.** *Let* $(X_j)_{j\in\mathbb{N}}$ *and* S_n *be as before. Let* $a \geq 1$ *be a* *real number. Then*

$$\mathbb{P}\left\{\limsup_{n\to\infty} n^{-1-a/2}(2\log\log n)^{-a/2}\sum_{i=1}^{n}|S_i|^a = \frac{2(a+2)^{\frac{a}{2}-1}}{\int_0^1 \frac{dt}{\sqrt{1-t^a}} a^{a/2}}\right\} = 1, \qquad (2.5)$$

and

Theorem 14 (Strassen, 1964)**.** *The set-up is as before. Suppose we want to determine the relative frequency of the events*

$$S_n > (1-\varepsilon)(2n\log\log n)^{1/2}.$$

Let $c \in [0,1]$ *and set*

$$c_i = \begin{cases} 1 & \text{if } S_i > c(2i\log\log i)^{1/2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{P}\left\{\limsup_{n\to\infty}\frac{1}{n}\sum_{i=3}^{n} c_i = 1-\exp\left\{-4\left(\frac{1}{c^{-2}}-1\right)\right\}\right\} = 1. \qquad (2.6)$$

This reveals a surprising result. Namely set $c = \frac{1}{2}$ in (2.6) to learn that, with probability one, for infinitely many $n \in \mathbb{N}$ the percentage of times $i \le n$ when $S_i > \frac{1}{2}(2i\log\log i)^{1/2}$ exceeds 99.999, but only for finitely many n exceeds 99.9999.

In one of our results, we shall need the following result:

Theorem 15 (Major, 1977)**.** *Let* $(X_n)_{n\ge1}$ *be a sequence of independent random variables with* $\mathbb{E}X_i = 0$*;* $B_n = \mathbb{E}(S_n^2) < \infty\ \forall n \ge 1$ *and* $B_n \to \infty$ *where* $S_n = \sum_{i=1}^{n} X_i$*. Let* $(M_n)_{n\ge1}$ *be a sequence of real numbers s.t.*

$$M_n^2 = o(B_n/\log\log B_n)$$

and M_n is the almost sure bound on X_n. The process $S(t)$, $t \geq 0$ is defined by setting $S(B_n) = S_n$ and it will be linear on $[B_n, B_{n+1}]$, $n \geq 0$. Then, $S_n(t) = S(B_n t)(2B_n \log\log B_n)^{-1/2}$ is relatively compact in $C[0,1]$ and the set of its limit points agrees with the formerly introduced Strassen set K.

2.3 Laws of the Iterated Logarithm with Weights

Although the question itself seems natural, it took many years for it to be posted: "What happens if we introduce weights in the Law of the Iterated Logarithm?"

Theorem 16 (Chow and Teicher, 1973)**.** *If $\{X_n : n \geq 1\}$ are independent and identically distributed random variables with*

$$\mathbb{E}X_1 = 0, \quad \mathbb{E}X_1^2 = 1$$

and $(A_n)_{n\geq 1}$ is a sequence of real constants satisfying:

(i) $$\frac{a_n^2}{\sum_{j=1}^{n} a_j^2} \leq \frac{C}{n}, \qquad n \geq 1,$$

(ii) $$\sum_{j=1}^{n} a_j^2 \to \infty$$

for some $Cin(0,\infty)$, then

$$\mathbb{P}\left\{ \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} a_j X_j}{\left(2\sum_{j=1}^{n} a_j^2 \log\log \sum_{j=1}^{n} a_j^2\right)^{1/2}} = 1 \right\} = 1. \tag{2.7}$$

The next result is more general and is important because of the technique of Skorohod representation it uses but it is not sharper condition-wise.

Theorem 17 (Fisher, 1992). *Let K be the Strassen set. $(X_j)_{j\in\mathbb{N}}$ will, again, be a sequence of i.i.d. zero-mean and unit variance random variables.*

Let $A_n^2 = \sum_{j=1}^{n} a_j^2$ and define the random function S by linearly interpolating S_n on $[A_n^2, A_{n+1}^2]$. Moreover, define a sequence of functions $(U_n)_{n\geq 1}$ by

$$U_n(t) = (2A_n^2 \log\log A_n^2) S(A_n^2 t).$$

If $A_n^2 \to \infty$ and $a_n^2/A_n^2 = O(1/n)$ then, with probability one, $\{U_n : n \geq 1\}$ is relatively compact and the set of its limit points coincides with K. This now, as in Strassen's case, implies the corresponding law of the Iterated Logarithm.

Using similar ideas to those of Fisher the following result can be obtained:

Theorem 18 (Berkes and Weber, 2007). *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. zero-mean and finite variance random variables.*

If $\mathbb{E}X_1^2 \log_+ |X_1| < \infty$ and

$$A_n^2 >> n, \quad a_n = O(A_n n^{-\gamma})$$

for some $\gamma > 0$, then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_n X_n}{\sqrt{2A_n^2 \log\log A_n^2}} = \|X_1\|_2 \tag{2.8}$$

with probability one.

2.4 Laws of the Iterated Logarithm with Non-Constant Limits

So far we have been in the "standard" framework. Patient reader shall (soon enough) discover that in our results non-constant limits appear. (Un)fortunately, this is not the first time suchlike behavior was established in the history of mathematics, as the following results show:

Theorem 19 (Erdős and Fortet, 1949). *Let $f(t) = \cos 2\pi t + \cos 4\pi t$ and define $n_k = 2^k - 1$. Then, for almost every t*

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} f(n_k t)}{(2n \log\log n)^{1/2}} = |\cos 2\pi t|^{1/2}, \tag{2.9}$$

which clearly is not a constant.

Here is another example:

Theorem 20 (Weiss, 1959). *Let $(\phi_n(x))_{n\geq 1}$ be a uniformly bounded orthonormal system of real-valued functions on the interval $[0,1]$. Then there exists a subsequence $\{\phi_{n_k}(x)\}_{k\geq 1}$ and a real-valued function $f(x)$, $\int_0^1 f^2(x)dx = 1$; $0 \leq f(x) \leq B$, where B is the uniform bound as $\{\phi_n(x)\}_{n\geq 1}$; such that for any arbitrary sequence $\{a_k\}$ of real numbers satisfying*

$$A_N = (a_1^2 + a_2^2 + \cdots + a_N^2)^{1/2} \to \infty \quad \textit{as} \quad N \to \infty,$$

$$M_N = o\big(A_N (\log\log A_N)^{-1/2}\big), \quad \textit{where}$$

$$M_N = \max_{1\leq k\leq N} |a_k|$$

we have

$$\limsup_{n\to\infty} \frac{S_n(x)}{(2A_n^2 \log\log An^2)^{1/2}} = f(x) \tag{2.10}$$

where $S_n(x) = \sum_{j=1}^{n} a_j \phi_{k_j}(x)$.

2.5 The Schatte's Infrastructure

In this subsection we shall introduce various results of P. Schatte from the 1980's that will be a base for building our tools in what follows.

Theorem 21 (Schatte, 1984)**.** *Let* $(X_j)_{j\in\mathbb{N}}$ *be a sequence of independent and identically distributed random variables. Let* $Y_n = \sum_{i=1}^{n} X_i \pmod 1$*, where moreover we assume* $0 \leq X_n < 1$ *for all* $n \in \mathbb{N}$.

Let $p_n(x)$ *denote the density of* Y_n. *Then the following assertions are equivalent:*

(a) *Density* $p_m(x)$ *is bounded for some* m.

(b) $\sup_{0\leq x<1} |p_n(x) - 1| \to 0$ *as* $n \to \infty$.

(c) $\sup_{0\leq x<1} |p_n(x) - 1| \leq C\omega^n$*, where* C *and* $\omega < 1$ *are real constants.*

Condition (a) is fulfilled in at least 2 situations, namely if the X_i have bounded density or $p_m(x) \in L^p$ for some $p > 1$ and some m. For the later see Ibragimov and Linnik [32], page 128.

Theorem 22 (Schatte, 1988)**.** *Assume the three random vectors* $X = (X_1, \ldots, X_r)$, U *and* $(W_1, \ldots, W_s)$ *are independent and let* $W = f(X)$ *for some measurable function* f. *If* U *is uniformly distributed, then the two random vectors*

$$X \quad \text{and} \quad \big(\{W + U + W_1\}, \{W + U + W_2\}, \ldots, \{W + U + W_s\}\big)$$

are also independent.

Here, and elsewhere in this section, $\{x\}$ shall stand for the fractional part of real number x.

The above remarkable result has an easy, but equally remarkable consequence.

Theorem 23 (Schatte, 1988)**.** *Let W and U be independent random variables, with U uniformly distributed. Then $\{W + U\}$ is independent of W.*

Theorem 24 (Schatte, 1988)**.** *Let X be a random variable, with distribution function $F(x)$, where*

$$\sup_{0 \leq x < 1} |F(x) - x| \leq \varepsilon.$$

Furthermore, let U be a uniformly distributed random variable that is indpendent of X. Then there exists a uniformly distributed random variable V such that

(i) $|V - X| \leq \varepsilon$,

(ii) $V = f(U, X)$ *where f is measurable.*

We point out that if X is continuous, U is not necessary in the construction of V, namely it suffices to take

$$V = F(X),$$

i.e. to “put” X into its own distribution function.

2.6 Results

Before we finally start to talk about our results we need a bit more patience from our reader to complete the set-up.

In what follows, $(X_n)_{n\geq 1}$ will be as in the Schatte model, thus a sequence of independent identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Moreover, we demand that X_1 is bounded with bounded density.

Furthermore, let f be a periodic function with period 1, Hölder α-continuous with

$$\int_0^1 f(x)dx = 0,$$

$$\int_0^1 f^2(x)dx = 1$$

for some positive α.

Let U be a uniformly distributed random variable independent of the underlying sequence $(X_n)_{n\geq 1}$.

Define a positive real-valued function as follows:

$$A_x := 1 + 2\sum_{g=1}^{\infty} \mathbb{E}^{\mathbb{P}} f(U)f(U + S_g x) \tag{2.11}$$

where, as before, S_n stands for

$$X_1 + \cdots + X_n.$$

We are now ready to begin:

Theorem 25 (Rašeta). *Let $(X_j)_{j\in\mathbb{N}}$, f and A_x be as described above. For any $x \in \mathbb{R}$ define the sequence $(\Gamma_n^x)_{n\in\mathbb{N}}$ of functions on $[0,1]$ by*

$$\Gamma_n^x(0) = 0, \quad \Gamma_n^x(k/n) = (2n\log\log n)^{-1/2}\sum_{j=1}^{k} f(S_j x) \quad (k = 0,\ldots,n)$$

and $\Gamma_n^x(t)$ *is linear on* $[k/n, (k+1)/n]$, *with* $k \in \{0, \dots, n-1\}$. *Then,* $\mathbb{P}$*-almost surely,* $(\Gamma_n^x)_{n\in\mathbb{N}}$ *is relatively compact in* $C[0,1]$ *for almost all* x *and the set of its limit points coincides with the scaled Strassen set*

$$K = \Bigg\{ y(t) : y \text{ is absolutely continuous in } [0,1],\ y(0) = 0 \quad \text{and} \quad \int_0^1 (\dot{y}(t))^2 dt \le A_x^{1/2} \Bigg\}. \tag{2.12}$$

Proof. We will start with some lemmas.

Lemma 1. *Let* $(X_j)_{j\in\mathbb{N}}$ *be a sequence of random variables chosen according to the Schatte model.*

Define a sequence of sets as follows:

$I_1 := \{1, 2, \dots, \beta\}$

$I_2 := \{p_1, p_1+1, \dots, p_1+\beta_1\}$ *where* $p_1 \ge \beta + \ell + 2$

$\vdots$

$I_n := \{p_{n-1}, p_{n-1}+1, \dots, p_{n-1}+\beta_{n-1}\}$ *where* $p_{n-1} \ge p_{n-2} + \beta_{n-2} + \ell + 2$

$\vdots$

for some $\ell \in \mathbb{N}$. *Fix* $x \in \mathbb{R}\setminus\{0\}$. *Then there exists a sequence* $\delta_1^x, \delta_2^x, \dots$ *of random variables satisfying:*

(i) $|\delta_n^x| \le C_x e^{-\lambda_x \ell}$ $\forall n \in \mathbb{N}$, *where* λ_x *and* C_x *are some positive constants that depend on* x *only.*

(ii) *The random variables*

$$\sum_{i\in I_1} f(S_i x),\ \sum_{i\in I_2} f(S_i x - \delta_1^x), \dots, \sum_{i\in I_n} f(S_i x - \delta_{n-1}^x)$$

are independent.

Proof. We shall construct inductively a sequence $(\delta_n^x)_{n\in\mathbb{N}}$ satisfying:

(a) $|\delta_n^x| \le C_x e^{-\lambda_x \ell}$ for all $n \in \mathbb{N}$,

(b) $\sum_{i\in I_n} f(S_i x - \delta_{n-1}{}^x)$ is independent of

$$\left(\sum_{i\in I_1} f(S_i x), \dots, \sum_{i\in I_{n-1}} f(S_i x - \delta_{n-2}{}^x)\right)$$

for all $n \ge 2$.

This sequence clearly satisfies the conditions (i) and (ii) above, and thus the proof will be complete.

Define

$$\delta_1^x := \left\{(S_{\beta+\ell} - S_\beta)x\right\} - F_{\{(S_{\beta+\ell}-S_\beta)x\}}\left(\left\{(S_{\beta+\ell} - S_\beta)x\right\}\right) \tag{2.13}$$

where, as before, $\{x\}$ stands for the fractional part of the real number x and $F_X(X)$ means putting random variable X into its own distribution function, whence defining a new random variable.

By Theorem of Schatte we know that if X is a continuous random variable taking values in $[0, 1)$, then

$$|X - F_X(X)| \le \sup_{0\le\xi\le1} |\mathbb{P}(X \le \xi) - \xi|. \tag{2.14}$$

By the very definition, X_j are all of bounded density and whence absolutely continuous, hence continuous. Thus, by Theorem of Schatte we have

$$|\delta_1^x| \le \sup_{0\le\xi\le1} \left|\mathbb{P}\left(\left\{(X_{\beta+1} + \dots + X_{\beta+\ell})x\right\} \le \xi\right) - \xi\right|.$$

Using the fact that

$$\{a+b\} = \big\{\{a\}+\{b\}\big\} \quad \text{for all } a,b\in\mathbb{R} \tag{2.15}$$

coupled with the fact that the $(X_j)_{j\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, we have that

$$\begin{aligned}\big\{(X_{\beta+1}+\cdots+X_{\beta+\ell})x\big\} &\stackrel{d}{=} \big\{(X_1+\cdots+X_\ell)x\big\}\\ &\stackrel{d}{=} \big\{\{X_1x\}+\cdots+\{X_\ell x\}\big\}.\end{aligned}$$

It thus trivially follows that

$$\mathbb{P}\left(\big\{(X_{\beta+1}+\cdots+X_{\beta+\ell})x\big\}\le\xi\right) =$$

$$= \mathbb{P}\left(\big\{\{X_1x\}+\cdots+\{X_\ell x\}\big\}\le\xi\right),$$

i.e. that

$$\sup_{0\le\xi\le1}\Big|\mathbb{P}\left(\big\{(X_{\beta+1}+\cdots+X_{\beta+\ell})x\big\}\le\xi\right)-\xi\Big| =$$

$$= \sup_{0\le\xi\le1}\Big|\mathbb{P}\left(\big\{\{X_1x\}+\cdots+\{X_\ell x\}\big\}\le\xi\right)-\xi\Big|,$$

i.e. finally that

$$|\delta_1^x| \le \sup_{0\le\xi\le1}\Big|\mathbb{P}\left(\big\{\{X_1x\}+\cdots+\{X_\ell x\}\big\}\le\xi\right)-\xi\Big|.$$

Trivially, X_j's are bounded, whence for each x $\{X_jx\}$ is itself absolutely continuous having bounded density.

But then Theorem 21 of Schatte applies directly, with $m=1$, to give:

$$|\delta_1^x| \le C_x e^{-\lambda_x \ell}. \tag{2.16}$$

Furthermore:

$$\{S_{p_1}x-\delta_1^x\} = \big\{S_{p_1}x-\{(S_{\beta+\ell}-S_\beta)x\}+F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{(S_{\beta+\ell}-S_\beta)x\}\big)\big\} =$$

$$= \big\{S_{p_1}x-(S_{\beta+\ell}-S_\beta)x+F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{(S_{\beta+\ell}-S_\beta)x\}\big)\big\} =$$

$$= \big\{S_{p_1}x-(X_{\beta+1}+\cdots+X_{\beta+\ell})x+F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{S_{\beta+\ell}-S_\beta)x)\big\} =$$

$$= \big\{(X_1+\cdots+X_\beta)x+(X_{\beta+\ell+1}+\cdots+X_{p_1})x+$$

$$+ F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{(S_{\beta+\ell}-S_\beta)x\}\big)\big\}.$$

Similarly,

$$\{S_{p_1+1}x-\delta_1^x\} = \big\{(X_1+\cdots+X_\beta)x+(X_{\beta+\ell+1}+\cdots+X_{p_1+1})x+$$

$$+ F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{(S_{\beta+\ell}-S_\beta)x\}\big)\big\}$$

$$\vdots$$

$$\{S_{p_1+\beta_1}x-\delta_1^x\} = \big\{(X_1+\cdots+X_\beta)x+(X_{\beta+\ell+1}+\cdots+X_{\beta_1+p_1})x+$$

$$+ F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{(S_{\beta+\ell}-S_\beta)x\}\big)\big\}.$$

Define:

$$X = (X_1x, X_2x, \ldots, X_\beta x),$$

$$W = f(X) = X_1x+\cdots+X_\beta x,$$

$$U = F_{\{(S_{\beta+\ell}-S_\beta)x\}}\big(\{(S_{\beta+\ell}-S_\beta)x\}\big),$$

$$\big(W_1^x,\ldots,W_{p_1+\beta_1}^{\;x}\big) = \big((X_{\beta+\ell+1}+\cdots+X_{p_1})x,\ldots,(X_{\beta+\ell+1}+\cdots+X_{p_1+\beta_1})x\big).$$

Observe the following three simple but crucial facts:

- Indices that appear in X take values in the set $\{1,\dots,\beta\}$.
- Indices that appear in U take values in the set $\{\beta+1,\dots,\beta+\ell\}$.
- Indices that appear in W_j's take values in the set $\{\beta+\ell+1,\dots,\beta_1+p_1\}$.

Thus, indices that appear in X, U and W_j's come from disjoint sets.

Since the underlying random variables $(X_j)_{j\in\mathbb{N}}$ are independent it then follows directly from Theorem of Schatte that the 2 random vectors X and

$$\big(\{W+U+W_1^x\},\dots,\{W+U+W_{p_1+\beta_1}^{\ x}\}\big)$$

are independent.

But this means (precisely!) that

$$(X_1x,\dots,X_\beta x) \ \text{ and } \ \big(\{S_{p_1}x-\delta_1 x\},\dots,\{S_{p_1+\beta_1}x-\delta_1 x\}\big)$$

are independent random vectors. Thus, trivially,

$$\sum_{j\in I_1} f\big(\{S_j x\}\big) \perp\!\!\!\perp \sum_{j\in I_2} f\big(\{S_j x-\delta_1^x\}\big).$$

However, $f(\{y\})=f(y)$ for all y so that we finally have

$$\sum_{j\in I_1} f(S_j x) \perp\!\!\!\perp \sum_{j\in I_2} f(S_j x-\delta_1^x). \tag{2.17}$$

Now suppose we have established one result up to index n. Consider the $n+1$-situation:

Define:

$$\begin{aligned}\delta_n{}^x := &\big\{\big(S_{p_{n-1}+\beta_{n-1}+\ell}-S_{p_{n-1}+\beta_{n-1}}\big)x\big\}-\\ &-F_{\{(S_{p_{n-1}+\beta_{n-1}+\ell}-S_{p_{n-1}+\beta_{n-1}})x\}}\big(\big\{(S_{p_{n-1}+\beta_{n-1}+\ell}-S_{p_{n-1}+\beta_{n-1}})x\big\}\big).\end{aligned}$$

Exactly as before:

$$\big\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\big\} \overset{d}{=} \big\{(X_1 + \cdots + X_\ell)x\big\}$$

and whence it follows that

$$|\delta_n^x| \leq C_x e^{-\lambda_x \ell}.$$

Tedious but identical algebra as for $n = 1$ yields:

$$\begin{aligned}
&\{S_{p_n}x - \delta_n^x\} = \\
&= \big\{(X_1x + \cdots + X_{p_{n-1}+\beta_{n-1}}x) + (X_{p_{n-1}+\beta_{n-1}+\ell+1}x + \cdots + X_{p_n}x) + \\
&\qquad + F_{\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\}}\big(\big\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\big\}\big)\big\}, \\
&\{S_{p_n+1}x - \delta_n^x\} = \\
&= \big\{(X_1x + \cdots + X_{p_{n-1}+\beta_{n-1}}x) + (X_{p_{n-1}+\beta_{n-1}+\ell+1}x + \cdots + X_{p_n+1}x) + \\
&\qquad + F_{\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\}}\big(\big\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\big\}\big)\big\}, \\
&\vdots \\
&\vdots \\
&\{S_{p_n+\beta_n}x - \delta_n^x\} = \\
&= \big\{(X_1x + \cdots + X_{p_{n-1}+\beta_{n-1}}x) + (X_{p_{n-1}+\beta_{n-1}+\ell+1}x + \cdots + X_{p_n+\beta_n}x) + \\
&\qquad + F_{\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\}}\big(\big\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\big\}\big)\big\}.
\end{aligned}$$

Define the following 3 random vectors:

$$\begin{aligned}
X &= \big(X_1x, X_2x, \ldots, X_{p_{n-1}+\beta_{n-1}}x, \delta_1^x, \ldots, \delta_{n-1}^x\big), \\
U &= F_{\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\}}\big(\big\{(S_{p_{n-1}+\beta_{n-1}+\ell} - S_{p_{n-1}+\beta_{n-1}})x\big\}\big), \\
\big(W_1^x, \ldots, W_{p_n+\beta_n}^x\big) &= \big(X_{p_{n-1}+\beta_{n-1}+\ell+1}x + \cdots + X_{p_n}x, \ldots,
\end{aligned}$$

$$\ldots, X_{p_{n-1}+\beta_{n-1}+\ell+1}x + \cdots + X_{p_n+\beta_n}x\big).$$

Moreover, let $W = X_1x + \cdots + X_{p_{n-1}+\beta_{n-1}}x$.

As before, we observe three very simple but crucial facts:

- Indices that appear in X take values in the set $\{1, \ldots, p_{n-1} + \beta_{n-1}\}$.
- Indices that appear in U take values in the set $\{p_{n-1} + \beta_{n-1}, \ldots, p_{n-1} + + \beta_{n-1} + \ell\}$.
- Indices that appear in W_j's take values in the set $\{p_{n-1} + \beta_{n-1} + \ell + 1, \ldots \ldots, p_{n-1} + \beta_n\}$.

It follows, exactly as before, that X, U and $(W_1^x, \ldots, W_{p_n+\beta_n}^{\ x})$ are 3 independent random vectors. But then, exactly as before:

$$\big(X_1x, \ldots, X_{p_{n-1}+\beta_{n-1}}x, \delta_1^x, \ldots, \delta_{n-1}^{\ x}\big) \perp\!\!\!\perp \big(\{S_{p_n}x - \delta_n^x\}, \ldots, \{S_{p_n+\beta_n}x - \delta_n^x\}\big),$$

and whence, using periodicity of f,

$$\sum_{i\in I_{n+1}} f(S_ix - \delta_n^x) \perp\!\!\!\perp \left(\sum_{i\in I_1} f(S_ix), \sum_{i\in I_2} f(S_ix - \delta_1^x), \sum_{i\in I_n} f(S_ix - \delta_{n-1}^{\ x})\right). \qquad (2.18)$$

Thus, by induction, the result holds for all $n \in \mathbb{N}$ and the proof is complete.

Now, put

$$\widetilde{m}_k = \sum_{j=1}^{k} \lfloor j^{1/2} \rfloor, \quad \widehat{m}_k = \sum_{j=1}^{k} \lfloor j^{1/4} \rfloor \qquad (2.19)$$

($\lfloor x \rfloor$ stands for the integer part of the real number x) and let

$$m_k = \widetilde{m}_k + \widehat{m}_k\,. \qquad (2.20)$$

Define 2 sequences $T_1, T_2, \ldots$ and $T_1^*, T_2^*, \ldots$ of random variables by

$$T_k := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \big(f(S_jx - \Delta_{k-1}^{\ x}) - \mathbb{E}f(S_jx - \Delta_{k-1}^{\ x})\big)\,, \qquad (2.21)$$

$$T_k^* := \sum_{j=m_{k-1}+\lfloor\sqrt{k}\rfloor+1}^{m_k} \left(f(S_j x - \Pi_{k-1}^{\ x}) - \mathbb{E}f(S_j x - \Pi_{k-1}^{\ x})\right) \tag{2.22}$$

and choose the variables $(\Delta_k^x)_{k\in\mathbb{N}}$, $(\Pi_k^x)_{k\in\mathbb{N}}$ so that

(i) $\Delta_0^x = 0$; $|\Delta_k^x| \leq C_x e^{-\lambda_x \lfloor\sqrt[4]{k}\rfloor}$;

$(T_k)_{k\in\mathbb{N}}$ is a sequence of independent random variables.

(ii) $\Pi_0^x = 0$; $|\Pi_k^x| \leq C_x e^{-\lambda_x \lfloor\sqrt{k}\rfloor}$;

$(T_k^*)_{k\in\mathbb{N}}$ is a sequence of independent random variables.

Note that this choice is possible by Lemma 1.

We now prove the following:

Lemma 2.

$$\sum_{k=1}^{n} \mathbb{V}\mathrm{ar}\,(T_k) \sim A_x \widetilde{m}_n;$$
$$\sum_{k=1}^{n} \mathbb{V}\mathrm{ar}\,(T_k^*) \sim A_x \widehat{m}_n.$$

Proof. Some basic algebra yields:

$$\begin{aligned}\mathbb{V}\mathrm{ar}\,(T_k) = &\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E}f^2(S_j x - \Delta_{k-1}^{\ x}) + \\ &+ 2\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} \mathbb{E}f(S_\ell x - \Delta_{k-1}^{\ x}) f(S_{\ell+\rho} x - \Delta_{k-1}^{\ x}) - \\ &- \left[\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E}f(S_j x - \Delta_{k-1}^{\ x})\right]^2.\end{aligned}$$

For simplicity we define

$$L_x^{(k)} := \left(\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E} f(S_j x - \Delta_{k-1}^{\,x}) \right)^2 . \tag{2.23}$$

Observe that

$$\begin{aligned} \left| f(S_j x) - f(S_j x - \Delta_{k-1}^{\,x}) \right| &\leq 2C \left| S_j x - \Delta_{k-1}^{\,x} - S_j x \right|^{\alpha} = \\ &= 2C |\Delta_{k-1}^{\,x}|^{\alpha} \leq 2C C_x e^{-\alpha\lambda_x \lfloor \sqrt[4]{k-1} \rfloor}; \end{aligned}$$

by the Hölder-α-continuity of f and the very construction of Δ_k's.

Furthermore:

$$\begin{aligned} \left| \mathbb{E} f(S_j x) \right| &= \left| \mathbb{E} f(\{S_j x\}) \right| \text{ (since } f \text{ is periodic with period 1)} = \\ &= \left| \mathbb{E} f(\{S_j x\}) - 0 \right| = \left| \mathbb{E} f(\{S_j x\}) - \int_0^1 f(\xi) d\xi \right| = \\ &= \left| \mathbb{E} f(\{S_j x\}) - \mathbb{E} f\big(F_{\{S_j x\}}(\{S_j x\})\big) \right| \leq \end{aligned}$$

(since $F_X(x)$ is always uniformly distributed)

$$\begin{aligned} &\leq 2C\mathbb{E} \left| \{S_j x\} - F_{\{S_j x\}}(\{S_j x\}) \right|^{\alpha} \leq \\ &\leq 2C C_x^{\alpha} e^{-\alpha\lambda_x} j; \end{aligned}$$

using the same Schatte-type arguments as in the proof of Lemma 1.

Putting all these things together yields:

$$L_x^{(k)} \leq 16 C^2 C_x^{2\alpha} \lfloor \sqrt{k} \rfloor^2 e^{-2\alpha\lambda_x \lfloor \sqrt[4]{k-1} \rfloor}$$

using very bold bounds indeed.

Moreover,

$$\mathbb{E}f^2(S_jx - \Delta_{k-1}^{\ x}) = 1 + \gamma_j^x + \varepsilon_j^x; \quad \text{where}$$

$$\gamma_j^x = \mathbb{E}f^2(S_jx - \Delta_{k-1}^{\ x}) - \mathbb{E}f^2(S_jx), \tag{2.24}$$

$$\varepsilon_j^x = \mathbb{E}f^2(\{S_jx\}) - \mathbb{E}f^2\big(F_{\{S_jx\}}(\{S_jx\})\big) \tag{2.25}$$

since f is periodic with period 1 and $\int\limits_0^1 f^2(\xi)d\xi = 1$. Since f is continuous and periodic, it is clearly bounded and call this bound M.

Applying the same reasoning as before, it is easy to see that

$$|\gamma_j^x| \leq 4MCC_x^{\ \alpha}e^{-\alpha\lambda_x\lfloor \sqrt[4]{k-1}\rfloor} \tag{2.26}$$

and

$$|\varepsilon_j^x| \leq 4MCC_x^{\ \alpha}e^{-\lambda_x\alpha_j}. \tag{2.27}$$

Define, for brevity,

$$\Lambda_x^{(k)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \gamma_j^x \tag{2.28}$$

and

$$O_x^{(k)} := \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \varepsilon_j^x \tag{2.29}$$

whence it follows that

$$\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \mathbb{E}f^2(S_jx - \Delta_{k-1}^{\ x}) = \lfloor\sqrt{k}\rfloor + \Lambda_x^{(k)} + O_x^{(k)}$$

where

$$|\Lambda_x^{(k)}| \leq 4MC_x^{\alpha}C\lfloor\sqrt{k}\rfloor e^{-\alpha\lambda_x\lfloor \sqrt[4]{k-1}\rfloor} \tag{2.30}$$

and

$$|O_x^{(k)}| \leq \frac{4MCC_x^{\alpha}}{1-e^{-\alpha\lambda_x}}e^{-\alpha\lambda_x(m_{k-1}+1)}\big(1-e^{-\alpha\lambda_x\lfloor\sqrt{k}\rfloor}\big) \tag{2.31}$$

again using, almost embarrassingly, bold bounds which turn out to be more than sufficient for our purposes.

We now turn to the most interesting part of this argument, namely to the "cross-term contribution".

Define:

$$\begin{aligned}
e_\ell^x &:= \mathbb{E} f(S_\ell x - \Delta_{k-1}^{\ x}) f(S_{\ell+\rho} x - \Delta_{k-1}^{\ x}) - \mathbb{E} f(S_\ell x) f(S_{\ell+\rho} x - \Delta_{k-1}^{\ x}),\\
g_\ell^x &:= \mathbb{E} f(S_\ell x) f(S_{\ell+\rho} x - \Delta_{k-1}^{\ x}) - \mathbb{E} f(S_\ell x) f(S_{\ell+\rho} x),\\
h_\ell^x &:= \mathbb{E} f(S_\ell x) f(S_{\ell+\rho} x) - \mathbb{E} f\big(F_{\{S_\ell x\}}(\{S_\ell x\})\big) f(S_{\ell+\rho} x),\\
i_\ell^x &:= \mathbb{E} f\big(F_{\{S_\ell x\}}(\{S_\ell x\})\big) f(S_\ell x + T_\rho^{\ell,x}) -\\
&\qquad - \mathbb{E} f\big(F_{\{S_\ell x\}}(\{S_\ell x\})\big) f\big(F_{\{S_\ell x\}}(\{S_\ell x\}) + T_\rho^{\ell,x}\big)\\
&\qquad (\text{where } \ T_\rho^{\ell,x} = (X_{\ell+1} + \cdots + X_{\ell+\rho})x),\\
c_\rho^{\ell,x} &:= \mathbb{E} f\big(F_{\{S_\ell x\}}(\{S_\ell x\})\big) f\big(F_{\{S_\rho x\}}(\{S_\rho x\}) + T_\rho^{\ell,x}\big) \qquad (2.32)
\end{aligned}$$

Then, arguing exactly as before one obtains the following inequalities:

$$\begin{aligned}
|e_\ell^x| &\le 2MC \cdot C_x^\alpha e^{-\alpha\lambda_x \lfloor \sqrt[4]{k-1} \rfloor},\\
|g_\ell^x| &\le 2MC \cdot C_x^\alpha e^{-\alpha\lambda_x \lfloor \sqrt[4]{k-1} \rfloor},\\
|h_\ell^x| &\le 2MC \cdot C_x^\alpha e^{-\alpha\lambda_x \ell},\\
|i_\ell^x| &\le 2MC \cdot C_x^\alpha e^{-\alpha\lambda_x \ell}.
\end{aligned} \qquad (2.33)$$

For brevity define

$$E_x^{(k)} := 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} e_\ell^x \qquad (2.34)$$

we have the following chain of inequalities:

$$
\begin{aligned}
|E_x^{(k)}| &\le 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} |e_\ell^x| \le \\
&\le 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} 2MCC_x^\alpha e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor} = \\
&= 4MC\cdot C_x^\alpha e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor} \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} (\lfloor\sqrt{k}\rfloor-\rho) = \\
&= 4MC\cdot C_x^\alpha e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor}\big((\lfloor\sqrt{k}\rfloor-1)+(\lfloor\sqrt{k}\rfloor-2)+(\lfloor\sqrt{k}\rfloor-(\lfloor\sqrt{k}\rfloor-1)\big) = \\
&= 4MC\cdot C_x^\alpha e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor}(1+2+\cdots+(\lfloor\sqrt{k}\rfloor-1)) = \\
&= 2MC\cdot C_x^\alpha\lfloor\sqrt{k}\rfloor(\lfloor\sqrt{k}\rfloor-1)e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor}.)
\end{aligned}
\tag{2.35}
$$

Along the same lines, define

$$
G_x^{(k)} := 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+\lfloor\sqrt{k}\rfloor}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} g_\ell^x. \tag{2.36}
$$

Exactly as above one obtains

$$
|G_x^{(k)}| \le 2MC\cdot C_x^\alpha\lfloor\sqrt{k}\rfloor(\lfloor\sqrt{k}\rfloor-1)e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor}. \tag{2.37}
$$

Somewhat heavier algebra is needed to obtain the bounds for the absolute values of the following two quantities:

$$
\begin{gathered}
P_x^{(k)} := 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} h_\ell^x \\
\text{and} \\
Q_x^{(k)} := 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} i_\ell^x\,.
\end{gathered}
\tag{2.38}
$$

It turns out that

$$\max\big(|P_x^{(k)}|,\,|Q_x^{(k)}|\big) \leq$$
$$\leq \frac{4MC\cdot C_x^{\alpha}}{1-e^{-\alpha\lambda_x}}e^{-\alpha\lambda_x(m_{k-1}+1)}\cdot$$
$$\cdot\left((\lfloor\sqrt{k}\rfloor-1)-\frac{e^{\alpha\lambda_x}}{e^{\alpha\lambda_x}-1}\big(e^{-\alpha\lambda_x}-e^{-\alpha\lambda_x\lfloor\sqrt{k}\rfloor}\big)\right). \tag{2.39}$$

The term $c_\rho^{\ell,x}$ needs some special attention.

Recall that

$$c_\rho^{\ell,x} = \mathbb{E}f\big(F_{\{S_\ell x\}}(\{S_\ell x\})\big)f\big(F_{\{S_\ell x\}}(\{S_\ell x\})+T_\rho^{\ell,x}\big).$$

Let us make several easy but far reaching observations:

- $F_{\{S_\ell x\}}(\{S_\ell x\})$ is uniformly distributed for all $\ell\in\mathbb{N}$.
- $F_{\{S_\ell x\}}(\{S_\ell x\})$ is independent of $T_\rho^{\ell,x}$ since they are made of disjoint indices associated to independent random variables.
- $T_\rho^{\ell,x}\overset{d}{=}S_\rho x$; by the very definition of the Schatte structure.

It follows that $c_\rho^{\ell,x}$ is actually an ℓ-independent quantity. We can thus rewrite it as follows:

$$c_\rho^{\ell,x} = c_\rho^x = \mathbb{E}f(U)f(U+S_\rho x)$$

for U uniform and

$$U \perp\!\!\!\perp \sigma(X_n : n\geq 1).$$

Using the standard machinery of stationarity we see that the cumulative contribution of c_g^x's shall take the following form:

$$2\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1}\ \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} c_\rho^{\ell,x} =$$

$$= 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \sum_{\ell=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\rho} c_\rho^x =$$

$$= 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} (\lfloor\sqrt{k}\rfloor - \rho) c_\rho^x = 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \lfloor\sqrt{k}\rfloor c_\rho^x - 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho c_\rho^x =$$

$$= 2\lfloor\sqrt{k}\rfloor \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} c_\rho^x - 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho c_\rho^x =$$

$$= 2\lfloor\sqrt{k}\rfloor \left(\sum_{\rho=1}^{\infty} c_\rho^x - \sum_{\rho=\lfloor\sqrt{k}\rfloor}^{\infty} c_\rho^x \right) - 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho c_\rho^x =$$

$$= \lfloor\sqrt{k}\rfloor \cdot 2 \sum_{\rho=1}^{\infty} c_\rho^x - 2\lfloor\sqrt{k}\rfloor \sum_{\rho=\lfloor\sqrt{k}\rfloor}^{\infty} c_\rho^x - 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho c_\rho^x .$$

Thus we have:

$$B_n^x = \mathbb{V}\mathrm{ar}\,(T_1 + T_2 + \cdots + T_n) =$$

$$= \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}\,(T_k) \quad \text{(by independence)}$$

$$= \sum_{k=1}^{3} \mathbb{V}\mathrm{ar}\,(T_k) + \sum_{k=4}^{n} \mathbb{V}\mathrm{ar}\,(T_k) =$$

$$= D_1^x + \sum_{k=4}^{n} \big(\lfloor\sqrt{k}\rfloor + \Lambda_x^{(k)} + O_x^{(k)} + E_x^{(x)} + G_x^{(k)} + P_x^{(k)} + Q_x^{(k)} \big) +$$

$$+ \lfloor\sqrt{k}\rfloor \cdot 2 \sum_{g=1}^{\infty} c_\rho^x - 2 \sum \lfloor\sqrt{k}\rfloor \sum_{\rho=\lfloor\sqrt{k}\rfloor}^{\infty} c_\rho^x - 2 \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho c_\rho^x - L_x^{(k)} =$$

$$= \widetilde{m}_n \left(\frac{D_1^x}{\widetilde{m}_n} + \frac{\widetilde{m}_n - \widetilde{m}_3}{\widetilde{m}_n} \left(1 + 2 \sum_{\rho=1}^{\infty} c_\rho^x \right) + \frac{\sum_{k=4}^{n} \Lambda_x^{(k)}}{\widetilde{m}_n} + \right.$$

$$+\frac{\sum_{k=4}^{n} O_x^{(k)}}{\widetilde{m}_n}+\frac{\sum_{k=4}^{n} E_x^{(k)}}{\widetilde{m}_n}+\frac{\sum_{k=4}^{n} P_x^{(k)}}{\widetilde{m}_n}+$$

$$+\frac{\sum_{k=4}^{n} Q_x^{(k)}}{\widetilde{m}_n}-\frac{2\sum_{k=4}^{n}\lfloor\sqrt{k}\rfloor \sum_{\rho=\lfloor\sqrt{k}\rfloor}^{\infty} c_\rho^x}{\widetilde{m}_n}-$$

$$\left.-2\cdot\frac{\sum_{k=4}^{n}\sum_{j=1}^{\lfloor\sqrt{k}\rfloor-1}\rho c_\rho^x}{\widetilde{m}_n}-\frac{\sum_{k=4}^{n} L_x^{(k)}}{\widetilde{m}_n}\right).$$

Observe, for example, the following:

$$\begin{aligned}\left|\frac{\sum_{k=4}^{n}\Lambda_x^{(k)}}{\widetilde{m}_n}\right| &\leq \frac{1}{\widetilde{m}_n}\sum_{k=4}^{n}|\Lambda_x^{(k)}| \leq \\ &\leq \frac{1}{\widetilde{m}_n}\sum_{k=4}^{n} 4MC\cdot C_x^\alpha\lfloor\sqrt{k}\rfloor e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor} \leq \\ &\leq \frac{1}{\widetilde{m}_n}4MC\cdot C_x^\alpha\sum_{k=1}^{\infty}\lfloor\sqrt{k}\rfloor e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor}.\end{aligned}$$

But, for all k large enough

$$\lfloor\sqrt{k}\rfloor e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor} \leq \frac{a_1^x}{k^2},$$

and whence $\sum_{k=1}^{\infty}\lfloor\sqrt{k}\rfloor e^{-\alpha\lambda_x\lfloor\sqrt[4]{k-1}\rfloor}$ converges so that

$$\frac{1}{\widetilde{m}_n}\sum_{k=4}^{n}\Lambda_x^{(k)} \quad \text{converges to} \quad 0. \tag{2.40}$$

Similarly after(at times) tedious algebra one can deduce that

$$\bullet \quad \frac{\sum_{k=4}^{n} O_x^{(k)}}{\widetilde{m}_n} \to 0; \quad \frac{\sum_{k=4}^{n} P_x^{(k)}}{\widetilde{m}_n} \to 0;$$
$$\bullet \quad \frac{\sum_{k=4}^{n} E_x^{(k)}}{\widetilde{m}_n} \to 0; \quad \frac{\sum_{k=4}^{n} Q_x^{(k)}}{\widetilde{m}_n} \to 0. \tag{2.41}$$

The analysis of c_ρ^x-related quantities needs more care. We proceed as follows:

$$c_\rho^x = \mathbb{E}f(U)f(U+S_\rho x) = \mathbb{E}f(U)f(\{U+S_\rho x\}) =$$
$$= \mathbb{E}f(U)f\big(\{U+\{S_\rho x\}\}\big) = \mathbb{E}f(U)f(U+\{S_\rho x\}) =$$
$$(\text{using } \{x+y\} = \{x+\{y\}\}) =$$
$$= \mathbb{E}f(U)f(U+\{S_\rho x\}) - \mathbb{E}f(U)f\big(U+F_{\{S_\rho x\}}(\{S_\rho x\})\big) +$$
$$+ \mathbb{E}f(U)f\big(U+F_{\{S_\rho x\}}(\{S_\rho x\})\big).$$

Since $U \perp\!\!\!\perp \sigma(X_j : j \in \mathbb{N})$ we have trivially that

$$U \perp\!\!\!\perp F_{\{S_\rho x\}}(\{S_g x\}) \Rightarrow \big\{U + F_{\{S_\rho x\}}(\{S_\rho x\})\big\} \perp\!\!\!\perp U$$

as a direct consequence of Theorem of Schatte because $F_{\{S_\rho x\}}(\{S_\rho x\})$ is itself uniformly distributed!

Whence it immediately follows that

$$c_\rho^x = \mathbb{E}f(U)f(U+S_\rho x) = \mathbb{E}f(U)f(U+\{S_\rho x\}) -$$
$$- \mathbb{E}f(U)f\big(U+F_{\{S_\rho x\}}(\{S_\rho x\})\big).$$

Thus:

$$|c_\rho^x| = \big|\mathbb{E}f(U)f(U+\{S_\rho x\}) - \mathbb{E}f(U)f(U+F_{\{S_\rho x\}}(\{S_\rho x\}))\big| \leq$$

$$\leq \mathbb{E}\big|f(U)(f(U+\{S_\rho x\}) - f(U+F_{\{S_\rho x\}}(\{S_\rho x\}))\big| \leq$$

$$\leq M\mathbb{E}\big|U+\{S_\rho x\} - U - F_{\{S_\rho x\}}(\{S_\rho x\})\big|^\alpha \cdot C \leq$$

$$\leq MC\,C_x^\alpha e^{-\alpha\lambda_x\rho};$$

using the exact same Schatte-type arguments as before.

It is then a routine to see that

$$\sum_{\rho=\lfloor\sqrt{k}\rfloor}^{\infty} c_\rho^x \leq \frac{MC\,C_x^\alpha e^{-\alpha\lambda_x\lfloor\sqrt{k}\rfloor}}{1-e^{-\alpha\lambda_x}} \tag{2.42}$$

and that

$$\sum_{k=4}^{n}\lfloor\sqrt{k}\rfloor \sum_{\rho=\lfloor\sqrt{k}\rfloor}^{\infty} |c_\rho^x| \leq$$

$$\leq \frac{MCC_x^\alpha}{1-e^{-\alpha\lambda_x}}\sum_{k=1}^{\infty}\lfloor\sqrt{k}\rfloor e^{-\alpha\lambda_x\lfloor\sqrt{k}\rfloor};$$

and this sum clearly converges.

We now turn our attention to the term

$$2\sum_{k=4}^{n}\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1}\rho c_\rho^x.$$

As before, it is easy to see that the absolute value of the above cannot exceed

$$2MCC_x^\alpha\sum_{k=4}^{n}\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1}\rho e^{-\alpha\lambda_x\rho}.$$

Define

$$h(\rho) = \rho e^{-\alpha\lambda_x g} \Rightarrow h'(\rho) = e^{-\alpha\lambda_x\rho}(1-\rho\alpha\lambda_x).$$

Thus if $\rho \leq 1/\alpha\lambda_x$ h will be increasing and it shall be decreasing otherwise.

We can now split the sum as follows:

$$2MCC_x^\alpha \sum_{k=4}^{n} \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x\rho} \leq$$

$$\leq 2MCC_x^\alpha \sum_{k=4}^{n} \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x\rho} =$$

$$= 2MCC_x^\alpha \sum_{k=1}^{(2+\lfloor 1/\alpha\lambda_x\rfloor)^2} \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x\rho} +$$

$$+ 2MCC_x^\alpha \sum_{k=1+(2+\lfloor 1/\alpha\lambda_x\rfloor)^2} \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x\rho}.$$

Now:

$$\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x\rho} = \sum_{\rho=1}^{\lfloor 1/\alpha\lambda_x\rfloor-1} \rho e^{-\alpha\lambda_x\rho} +$$

$$+ \lfloor 1/\alpha\lambda_x\rfloor e^{-\alpha\lambda_x\lfloor 1/\alpha\lambda_x\rfloor} + \sum_{\rho=1+\lfloor 1/\alpha\lambda_x\rfloor}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x\rho} =$$

$$= (\text{upon introducing dummy but friendlier index } j) =$$

$$\sum_{j=1}^{\lfloor 1/\alpha\lambda_x\rfloor-1} j e^{-\alpha\lambda_x j} + \lfloor 1/\alpha\lambda_x\rfloor e^{-\alpha\lambda_x\lfloor 1/\alpha\lambda_x\rfloor} + \sum_{j=1+\lfloor 1/\alpha\lambda_x\rfloor}^{\lfloor\sqrt{k}\rfloor-1} j e^{-\alpha\lambda_x j}.$$

For $j \leq \lfloor 1/\alpha\lambda_x\rfloor - 1$ $h(j)$ will be increasing.

Let us now observe the following:

$$\int_j^{j+1} \xi e^{-\alpha\lambda_x\xi} d\xi \geq \int_j^{j+1} \min_{\xi\in[j,j+1]} \xi e^{-\alpha\lambda_x\xi} d\xi =$$

(since the function is increasing)

$$= j e^{-\alpha\lambda_x j}.$$

Similarly, if $j \geq \lfloor 1/\alpha\lambda_x \rfloor + 1$

$$\int_{j=1}^{j} \xi e^{-\alpha\lambda_x \xi} d\xi \geq j e^{-\alpha\lambda_x j}.$$

Putting all this together we can see that

$$\begin{aligned}
\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x \rho} \leq & \sum_{j=1}^{\lfloor 1/\alpha\lambda_x \rfloor - 1} \int_{j}^{j+1} \xi e^{-\alpha\lambda_x \xi} d\xi + \\
& + \lfloor 1/\alpha\lambda_x \rfloor e^{-\alpha\lambda_x \lfloor 1/\alpha\lambda_x \rfloor} + \\
& + \sum_{j=1+\lfloor 1/\alpha\lambda_x \rfloor}^{\lfloor\sqrt{k}\rfloor-1} \int_{j-1}^{j} \xi e^{-\alpha\lambda_x \xi} d\xi = \\
= & \int_{1}^{\lfloor 1/\alpha\lambda_x \rfloor} \xi e^{-\alpha\lambda_x \xi} d\xi + \lfloor 1/\alpha\lambda_x \rfloor e^{-\alpha\lambda_x \lfloor 1/\alpha\lambda_x \rfloor} + \\
& + \int_{\lfloor 1/\alpha\lambda_x \rfloor}^{\lfloor\sqrt{k}\rfloor-1} \xi e^{-\alpha\lambda_x \xi} d\xi = \\
= & \int_{1}^{\lfloor\sqrt{k}\rfloor-1} \xi e^{-\alpha\lambda_x \xi} d\xi + \lfloor 1/\alpha\lambda_x \rfloor e^{-\alpha\lambda_x \lfloor 1/\alpha\lambda_x \rfloor}
\end{aligned}$$

whence, upon some tedious algebra we see that

$$\begin{aligned}
\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x \rho} \leq e^{-\alpha\lambda_x} & \left(\frac{1}{\alpha\lambda_x} + \frac{1}{(\alpha\lambda_x)^2} \right) - \\
& - e^{-\alpha\lambda_x(\lfloor\sqrt{k}\rfloor-1)} \left(\frac{\lfloor\sqrt{k}\rfloor - 1}{\alpha\lambda_x} + \left(\frac{1}{\alpha\lambda_x} \right)^2 \right) + \\
& + \lfloor 1/\alpha\lambda_x \rfloor e^{-\alpha\lambda_x \lfloor 1/\alpha\lambda_x \rfloor}.
\end{aligned}$$

For brevity, define

$$a_2^x := 2MCC_x^\alpha \sum_{k=1}^{(2+\lfloor 1/\alpha\lambda_x \rfloor)^2} \sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1} \rho e^{-\alpha\lambda_x \rho}. \tag{2.43}$$

We then have:

$$2MCC_x^{\alpha}\sum_{k=4}^{n}\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1}\rho e^{-\alpha\lambda x\rho} =$$

$$= a_2^x + 2MCC_x^{\alpha}\sum_{k=1+(2+\lfloor 1/\alpha\lambda_x\rfloor)^2}^{n}\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1}\rho e^{-\alpha\lambda_x\rho} \leq$$

$$\leq a_2^x + 2MCC_x^{\alpha}\sum_{k=1+(2+\lfloor 1/\alpha\lambda_x\rfloor)^2}^{n}\Bigg\{e^{-\alpha\lambda_x}\left(\frac{1}{\alpha\lambda_x}+\frac{1}{(\alpha\lambda_x)^2}\right) - e^{-\alpha\lambda_x(\lfloor\sqrt{k}\rfloor-1)}\cdot$$
$$\cdot\left(\frac{\lfloor\sqrt{k}\rfloor-1}{\alpha\lambda_x}+\left(\frac{1}{\alpha\lambda_x}\right)^2\right) + \lfloor 1/\alpha\lambda_x\rfloor e^{-\alpha\lambda_x\lfloor 1/\alpha\lambda_x\rfloor}\Bigg\}.$$

Yet again, for brevity, we define

$$a_3^x := e^{-\alpha\lambda_x}\left(\frac{1}{\alpha\lambda_x}+\left(\frac{1}{\alpha\lambda_x}\right)^2\right) +$$
$$+ \lfloor 1/\alpha\lambda_x\rfloor e^{-\alpha\lambda_x\lfloor 1/\alpha\lambda_x\rfloor}.$$

Then the above complex expression takes a slightly friendlier form:

$$a_2^x + 2MCC_x^{\alpha} n a_3^x + \sum_{k=1}^{n} e^{-\lambda_x(\lfloor\sqrt{k}\rfloor-1)}\left(\frac{\lfloor\sqrt{k}\rfloor-1}{\alpha\lambda_x}+\left(\frac{1}{\alpha\lambda_x}\right)^2\right).$$

It is now clear, since $\frac{3}{2}\widetilde{m}_n \sim n^{3/2}$ that

$$\frac{1}{\widetilde{m}_n}\sum_{k=4}^{n}\sum_{\rho=1}^{\lfloor\sqrt{k}\rfloor-1}\rho c_\rho^x \to 0 \quad \text{as} \quad n\to\infty. \tag{2.44}$$

Moreover, the series $\sum_{\rho=1}^{\infty} c_\rho^x$ converges absolutely. Putting all this together and recalling the very definition of A_x we deduce that

$$\mathbb{V}\text{ar}\left(\sum_{j=1}^{n} T_j\right) \sim A_x\widetilde{m}_n\,. \tag{2.45}$$

In an identical fashion one can deduce that

$$\mathbb{V}\text{ar}\left(\sum_{j=1}^{n} T_j^*\right) \sim A_x \widehat{m}_n \tag{2.46}$$

and the proof is complete. □

We are now ready to prove our theorem. We shall apply the result of Major both for T_k (long blocks) and T_k^* (short blocks).

Put

$$B_n = B_n^x = \sum_{k=1}^{n} \mathbb{V}\text{ar}\,(T_k), \qquad M_n = 2M\sqrt{n}.$$

It then follows directly from Lemma 2 that

$$M_n^2 = o(B_n/\log\log B_n). \tag{2.47}$$

Define the sequence $(\Psi_n^x)_{n\in\mathbb{N}}$ of random functions on $[0,1]$ such that

$$\begin{aligned} &\Psi_n^x(0) = 0; \\ &\Psi_n^x(B_k^x/B_n^x) = (2B_n^x \log\log B_n^x)^{-1/2} \sum_{j=1}^{k} T_j; \quad \text{for } k = 0, 1, \ldots, n, \end{aligned} \tag{2.48}$$

and demand Ψ_n^x is linear on $\left[B_k^x/B_n^x,\, B_{k+1}^x/B_n^x\right]$; $k = 0, 1, \ldots, n-1$. Then by Major's result it follows that $(\Psi_n^x)_{n\in\mathbb{N}}$ is, $\mathbb{P}$-almost surely, relatively compact in $C[0,1]$, and the set of its limit points agrees with the Strassen set.

Similarly, let

$$D_n^x = \sum_{k=1}^{n} \mathbb{V}\text{ar}\,(T_k^*)$$

and define another sequence of random functions $(\zeta_n^x)_{n\in\mathbb{N}}$ by

$$\begin{aligned} &\zeta_n^x(0) = 0; \\ &\zeta_n^x(D_k^x/D_n^x) = (2D_n^x \log\log D_n^x)^{-1/2} \sum_{j=1}^{k} T_j^*; \quad \text{for } k = 0, 1, \ldots, n, \end{aligned} \tag{2.49}$$

and demand ζ_n^x linear on $\left[D_k^x/D_n^x,\ D_{k+1}^{\ x}/D_n^x\right]$ for $k \in \{0, \dots, n-1\}$.

Again, by Major's result it follows that, $\mathbb{P}$-almost surely, $(\zeta_n^x)_{n\in\mathbb{N}}$ is relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set.

Define the following quantities:

$$a_k^x := \sum_{j=1}^{k} \sum_{\ell=m_{j-1}+1}^{m_{j-1}+\lfloor\sqrt{j}\rfloor} \big(f(S_\ell x) - f(S_\ell x - \Delta_{j-1}^{\ x})\big),$$

$$b_k^x := \sum_{j=1}^{k} \sum_{\ell=m_{j-1}+1}^{m_{j-1}+\lfloor\sqrt{j}\rfloor} \big(f(S_\ell x - \Delta_{j-1}^{\ x}) - \mathbb{E}f(S_\ell x - \Delta_{j-1}^{\ x})\big) = \sum_{j=1}^{k} T_j,$$

$$c_k^x := \sum_{j=1}^{k} \sum_{\ell=m_{j-1}+1}^{m_{j-1}+\lfloor\sqrt{j}\rfloor} \mathbb{E}f(S_\ell x - \Delta_{j-1}^{\ x}),$$

$$d_k^x := \sum_{j=1}^{k} \sum_{\ell=m_{j-1}+\lfloor\sqrt{j}\rfloor+1}^{m_j} \big(f(S_\ell x) - f(S_\ell x - \Pi_{j-1}^{\ x})\big),$$

$$p_k^x := \sum_{j=1}^{k} \sum_{\ell=m_{j-1}+\lfloor\sqrt{j}\rfloor+1}^{m_j} \big(f(S_\ell x - \Pi_{j-1}^{\ x}) - \mathbb{E}f(S_\ell x - \Pi_{j-1}^{\ x})\big) = \sum_{j=1}^{k} T_j^*,$$

$$q_k^x := \sum_{j=1}^{k} \sum_{\ell=m_{j-1}+\lfloor\sqrt{j}\rfloor+1}^{m_j} \mathbb{E}f(S_\ell x - \Pi_{j-1}^{\ x})\big), \tag{2.50}$$

where Δ_k's and Π_k's are exactly as before.

Observe that

$$a_k^x + b_k^x + c_k^x + d_k^x + p_k^x + q_k^x = \sum_{j=1}^{m_k} f(S_j x).$$

Define another sequence of random functions $\big(\Phi_n^x(t)\big)_{n\geq 1}$ by:

$$\begin{aligned} &\Phi_n^x(0) = 0 \quad \forall n \in \mathbb{N}; \\ &\Phi_n^x\big(B_k^x/B_n^x\big) = \sum_{j=1}^{m_k} f(S_j x)/(2B_n^x \log\log B_n^x)^{1/2} \end{aligned} \tag{2.51}$$

for $k \in \{0, \dots, n\}$ and Φ_n^x is linear on $\left[B_k^x/B_n^x,\ B_{k+1}^x/B_n^x\right]$ for $k \in \{0, \dots, n-1\}$.

Let $\|\cdot\|$ be the sup-norm on $C[0,1]$. Observe that

$$\left\|\Psi_n^x - \Phi_n^x\right\| = \sup_{0 \le t \le 1} \left|\Psi_n^x(t) - \Phi_n^x(t)\right| =$$

$$= \max_{0 \le t \le 1} \left|\Psi_n^x(t) - \Phi_n^x(t)\right| =$$

(since $[0,1]$ is a compact set and difference of 2 continuous functions is itself

continuous)

$$= \max_{0 \le k \le n-1} \max_{\frac{B_k^x}{B_n^x} \le t \le \frac{B_{k+1}^x}{B_n^x}} \left|\Psi_n^x(t) - \Phi_n^x(t)\right| =$$

$$= \max_{0 \le k \le n-1} \max\left(\left|\Psi_n^x\left(\frac{B_k^x}{B_n^x}\right) - \Phi_n^x\left(\frac{B_k^x}{B_n^x}\right)\right|, \right.$$

$$\left. \left|\Psi_n^x\left(\frac{B_{k+1}^x}{B_n^x}\right) - \Phi_n^x\left(\frac{B_{k+1}^x}{B_n^x}\right)\right| \right) =$$

$$\left(\text{since both } \Psi_n^x \text{ and } \Phi_n^x \text{ are linear on } \left[\frac{B_k^x}{B_n^x}, \frac{B_{k+1}^x}{B_n^x}\right]\right)$$

$$= \max_{0 \le k \le n} \left|\Phi_n^x\left(\frac{B_k^x}{B_n^x}\right) - \Phi_n^x\left(\frac{B_k^x}{B_n^x}\right)\right| =$$

$$= \max_{0 \le k \le n} \left|\frac{a_k^x + b_k^x + c_k^x + d_k^x + p_k^x + q_k^x}{(2B_n^x \log\log B_n^x)^{1/2}} - \frac{b_k^x}{(2B_n^x \log\log B_n^x)^{1/2}}\right| =$$

$$= \max_{0 \le k \le n} \left|\frac{a_k^x + c_k^x + d_k^x + p_k^x + q_k^x}{(2B_n^x \log\log B_n^x)^{1/2}}\right| \le$$

$$\le \frac{1}{(2B_n^x \log\log B_n^x)^{1/2}} \left(\max_{0 \le k \le n} |a_k^x| + \max_{0 \le k \le n} |c_k^x| + \right.$$

$$\left. + \max_{0 \le k \le n} |d_k^x| + \max_{0 \le k \le n} |p_k^x| + \max_{0 \le k \le n} |q_k^x| \right). \tag{2.52}$$

Using the same Schatte-type ideas to which the reader was heavily exposed to in the proof of Lemma 2 one obtains the following bounds:

$$
\begin{aligned}
|a_k^x| &\le 2C \cdot C_x^\alpha \sum_{j=1}^{\infty} \lfloor \sqrt{j} \rfloor e^{-\alpha\lambda_x \lfloor \sqrt[4]{j-1} \rfloor} := M_1^x, \\
|c_k^x| &\le 2C \cdot C_x^\alpha \sum_{j=1}^{\infty} \lfloor \sqrt{j} \rfloor e^{-\alpha\lambda_x \lfloor \sqrt[4]{j-1} \rfloor} + \\
&\quad + 2C \cdot C_x^\alpha \sum_{j=1}^{\infty} \lfloor \sqrt{j} \rfloor e^{-\alpha\lambda_x \lfloor \sqrt{j-1} \rfloor} := M_2^x, \\
|d_k^x| &\le 2C \cdot C_x^\alpha \sum_{j=1}^{\infty} \lfloor \sqrt[4]{j} \rfloor e^{-\alpha\lambda_x \lfloor \sqrt{j-1} \rfloor} := M_3^x, \\
|p_k^x| &= \left| (2D_n^x \log\log D_n^x)^{1/2} \zeta_n^x \left(\frac{D_k^x}{D_n^x} \right) \right| = \\
&= (2D_n^x \log\log D_n^x)^{1/2} \left| \zeta_n^x \left(\frac{D_k^x}{D_n^x} \right) \right| \le \\
&\le M_4^x \cdot (2D_n^x \log\log D_n^x)^{1/2} \ \text{ for some } M_4^x \text{ because relative} \\
&\quad \text{compactness of } (\zeta_n^x)_{n\in\mathbb{N}} \text{ implies its uniform boundedness.}
\end{aligned}
\tag{2.53}
$$

Finally, as before

$$
|q_k^x| \le M_5^x; \quad \text{for some } M_5^x.
$$

However, Lemma 2 tells us that

$$
\begin{aligned}
D_n^x &\sim A_x \widehat{m}_n \sim \frac{4}{5} A_x n^{5/4}, \qquad \text{while} \\
B_n^x &\sim A_x \widetilde{m}_n \sim \frac{2}{3} A_x n^{3/2}.
\end{aligned}
\tag{2.54}
$$

Putting these facts together yields that

$$
\|\Psi_n^x - \Phi_n^x\| \to 0; \quad \mathbb{P}\text{-almost surely.} \tag{2.55}
$$

Thus it clearly follows that $(\Phi_n^x)_{n\in\mathbb{N}}$ is relatively compact in $C[0,1]$ with probability 1 and the set of its limit points agrees with the Strassen set.

Let us now introduce the following sequence of random functions:

$$\Theta_n^x(0) = 0 \quad \text{for all } n \in \mathbb{N},$$
$$\Theta_n^x\left(\frac{B_k^x}{B_n^x}\right) = \sum_{j=1}^{m_k} f(S_jx)/(2A_x m_n \log\log m_n)^{1/2}; \quad k \in \{0,\dots,n\} \tag{2.56}$$

and Θ_n^x is linear on $[B_k^x/B_n^x,\ B_{k+1}{}^x/B_n^x]$, $k \in \{0,\dots,n-1\}$.

Then we have:

$$\left\|\Phi_n^x - \Theta_n^x\right\| = \text{(arguing exactly as before)} =$$
$$= \max_{0\le k\le n}\left|\frac{\sum_{j=1}^{m_k} f(S_jx)}{(2B_n^x \log\log B_n^x)^{1/2}} - \frac{\sum_{j=1}^{m_k} f(S_jx)}{(2A_x m_n \log\log m_n)^{1/2}}\right| \le$$
$$\le \sup_{t\in[0,1]} |\Phi_n^x(t)| \cdot \left|1 - \frac{(2B_n^x \log\log B_n^x)^{1/2}}{(2A_x m_n \log\log m_n)^{1/2}}\right|. \tag{2.57}$$

However, $(\Phi_n^x)_{n\ge1}$ is, $\mathbb{P}$-almost surely, uniformly bounded. Moreover, from Lemma 2 we know that

$$B_n^x \log\log B_n^x \sim A_x m_n \log\log m_n \tag{2.58}$$

It follows that, $\mathbb{P}$-almost surely, $(\Theta_n^x)_{n\ge1}$ is relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set.

We shall now define another sequence of random functions:

$$\xi_n^x(0) = 0 \quad \text{for all} \quad n \in \mathbb{N},$$
$$\xi_n^x\left(\frac{m_k}{m_n}\right) = \sum_{j=1}^{m_k} f(S_jx)/(2A_x m_n \log\log m_n)^{1/2}; \quad k \in \{0,\dots,n\} \tag{2.59}$$

and ξ_n^x is linear on $\left[\frac{m_k}{m_n}, \frac{m_{k+1}}{m_n}\right]$; $k \in \{0, \ldots, n-1\}$.

We claim, surprise-surprise, that $(\xi_n^x)_{n\in\mathbb{N}}$ is itself, $\mathbb{P}$-almost surely, relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set.

In order to prove the above claim, let us define the following map:

$$T_n : [0, 1] \to [0, 1], \quad T_n \text{ maps } \left[\frac{m_k}{m_n}, \frac{m_{k+1}}{m_n}\right]$$

$$\text{to } \left[\frac{B_k^x}{B_n^x}, \frac{B_{k+1}^x}{B_n^x}\right] \text{ in a linear way, with}$$

$$T_n\left(\frac{m_k}{m_n}\right) = \frac{B_k^x}{B_n^x}. \tag{2.60}$$

It is easily seen that $\xi_n^x(t) = \Theta_n^x(T_n(t))$. Thus:

$$\left\|\Theta_n^x - \xi_n^x\right\| = \max_{0\le t\le 1}\left|\Theta_n^x(t) - \Theta_n^x(T_n(t))\right|.$$

However, $\mathbb{P}$-almost surely, $(\Theta_n^x)_{n\in\mathbb{N}}$ is equicontinuous and hence it will be sufficient to show that

$$\max_{0\le t\le 1}\left|T_n(t) - t\right| \to 0 \quad \text{as} \quad n \to \infty.$$

Using the same ideas as before one can see that:

$$\max_{0\le t\le 1}|T_n(t) - t| = \max_{0\le k\le n}\left|\frac{B_k^x}{B_n^x} - \frac{m_k}{m_n}\right|. \tag{2.61}$$

Recall that $B_n^x \sim A_x m_n$. Standard $\varepsilon - N$ argument shows that the quantity on the RHS tends to 0 as $n \to \infty$.

Thus, $(\xi_n^x)_{n\in\mathbb{N}}$ is itself, $\mathbb{P}$-almost surely, relatively compact in $C[0, 1]$ and the set of its limit points agrees with the Strassen set.

In order to complete our proof we shall have to introduce one last sequence of random functions:

$$\theta_n^x(0) = 0 \quad \text{for all} \quad n \in \mathbb{N},$$

$$\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right) = \sum_{j=1}^{\ell} f(S_j x)/(2A_x n \log\log n)^{1/2} \tag{2.62}$$

$\ell \in \{0, \ldots, m_{p(n)}\}$ and θ_n^x is linear on $\left[\frac{\ell}{m_{p(n)}}, \frac{\ell+1}{m_{p(n)}}\right]$; $\ell \in \{0, \ldots, n-1\}$.

Here $(p(n))_{n\in\mathbb{N}}$ is a sequence of integers defined implicitly via inequalities:

$$m_{p(n)} \le n < m_{p(n)+1}.$$

We proceed by showing that

$$\left\|\Gamma_n^x - \theta_n^x\right\| \to 0; \quad \mathbb{P}\text{-almost surely.}$$

As before,

$$\left\|\Gamma_n^x - \theta_n^x\right\| = \max_{0\le\ell\le m_{p(n)-1}} \max_{\frac{\ell}{m_{p(n)}}\le t\le\frac{\ell+1}{m_{p(n)}}} \left|\Gamma_n^x(t) - \theta_n^x(t)\right|.$$

It is essential we establish the connection between the interpolation points of random functions θ_n^x and Γ_n^x.

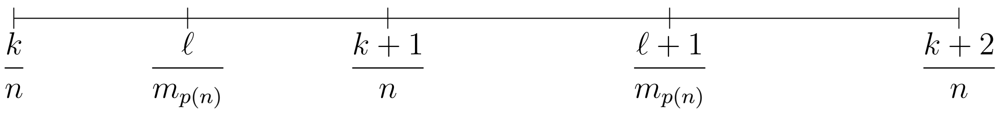
For this purpose define

$$k := \max\left\{\overline{k} : \frac{\overline{k}}{n} \le \frac{\ell}{m_{p(n)}}\right\}. \tag{2.63}$$

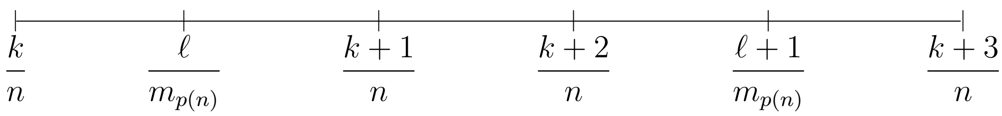
Then one of the following has to hold (we include pictures to illustrate our reasoning and make our point clearer):

(i)

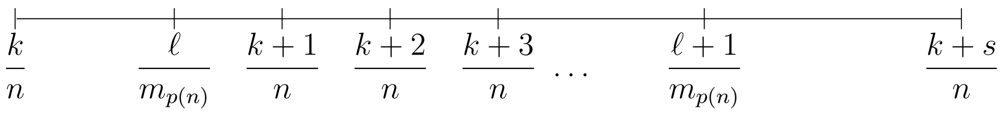
(ii)



(iii)



(iv)



where $s \geq 4$.

We start by showing that cases (i) and (iv) are impossible

Suppose (i) were true. Then

$$\frac{k+1}{n} - \frac{k}{n} > \frac{\ell+1}{m_{p(n)}} - \frac{\ell}{m_{p(n)}} \Rightarrow m_{p(n)} > n \qquad \# \quad (2.64)$$

since

$$m_{p(n)} \leq n < m_{p(n)+1}.$$

Now suppose (iv) were possible. Then

$$\frac{k+3}{n} - \frac{k+1}{n} < \frac{\ell+1}{m_{p(n)}} - \frac{\ell}{m_{p(n)}} \Rightarrow 2 < \frac{n}{m_{p(n)}} \qquad \# \quad (2.65)$$

since $m_{p(n)} \sim n$ and so the above would fail for all n large enough.

Thus, if we take n large enough, only (i) and (iii) are possible.

We shall deal with these two cases separately. Firstly, let us suppose (i) holds; then arguing exactly as before we would have:

$$\max_{\frac{\ell}{m_{p(n)}} \leq t \leq \frac{\ell+1}{m_{p(n)}}} \left|\Gamma_n^x(t) - \theta_n^x(t)\right| \leq$$

$$\leq \max\Bigg(\left|\Gamma_n^x\left(\frac{k}{n}\right) - \theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right|, \left|\Gamma_n^x\left(\frac{k+1}{n}\right) - \theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right|,$$

$$\left|\Gamma_n^x\left(\frac{k+1}{n}\right) - \theta_n^x\left(\frac{\ell+1}{m_{p(n)}}\right)\right|, \left|\Gamma_n^x\left(\frac{k+2}{n}\right) - \theta_n^x\left(\frac{\ell+1}{m_{p(n)}}\right)\right|\Bigg). \qquad (2.66)$$

For convenience we choose to examine the second quantity in the above in detail, others are dealt with in an identical fashion.

Notice that

$$\frac{\ell}{m_{p(n)}} \leq \frac{k+1}{n} \Rightarrow \ell \leq \frac{k+1}{n} \cdot m_{p(n)} \leq k+1 \cdot \frac{n}{n} = k+1,$$

so, $\ell \leq k+1$.

Whence:

$$\left|\Gamma_n^x\left(\frac{k+1}{n}\right) - \theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right| =$$

$$= \left| \frac{\sum_{j=1}^{k+1} f(S_j x)}{(2A_x n \log\log n)^{1/2}} - \frac{\sum_{j=1}^{\ell} f(S_j x)}{(2A_x n \log\log n)^{1/2}} \right| \leq$$

$$\leq \frac{1}{(2A_x n \log\log n)^{1/2}} \sum_{j=\ell+1}^{k+1} |f(S_j x)| \leq$$

$$\leq \frac{M(k-\ell+1)}{(2A_x n \log\log n)^{1/2}}\,.$$

However, we also know that

$$\frac{k}{n} \leq \frac{\ell}{m_{p(n)}} \Rightarrow k \leq \frac{\ell \cdot n}{m_{p(n)}} \Rightarrow$$

$$\Rightarrow k - \ell + 1 \leq \frac{\ell}{m_{p(n)}}\big(n - m_{p(n)}\big) + 1.$$

Thus:

$$\left|\Gamma_n^x\left(\frac{k+1}{n}\right) - \theta_n\left(\frac{\ell}{m_{p(n)}}\right)\right| \leq$$

$$\leq \frac{M}{(2A_x n \log\log n)^{1/2}} \left(\frac{\ell}{m_{p(n)}}\big(n - m_{p(n)}\big) + 1\right).$$

Similarly, we obtain the following inequalities:

$$\left|\Gamma_n^x\left(\frac{k}{n}\right) - \theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right| \leq \frac{M}{(2A_x n \log\log n)^{1/2}} \left(\frac{\ell}{m_{p(n)}}\big(n - m_{p(n)}\big) + 2\right),$$

$$\left|\Gamma_n^x\left(\frac{k+1}{n}\right)-\theta_n^x\left(\frac{\ell+1}{n}\right)\right|\leq$$

$$\leq\frac{M}{(2A_x n\log\log n)^{1/2}}\left(\frac{\ell}{m_{p(n)}}(n-m_{p(n)})+2\right)\tag{2.67}$$

and finally,

$$\left|\Gamma_n^x\left(\frac{k+2}{n}\right)-\theta_n^x\left(\frac{\ell+1}{n}\right)\right|\leq$$

$$\leq\frac{M}{(2A_x n\log\log n)^{1/2}}\left(\frac{\ell}{m_{p(n)}}(n-m_{p(n)})+3\right)\tag{2.68}$$

whence, under the assumption (i), we have:

$$\max_{\frac{\ell}{m_{p(n)}}\leq t\leq\frac{\ell+1}{m_{p(n)}}}\left|\Gamma_n^x(t)-\theta_n^x(t)\right|\leq$$

$$\leq\frac{M}{(2A_x n\log\log n)^{1/2}}\left(\frac{\ell}{m_{p(n)}}(n-m_{p(n)})+3\right).\tag{2.69}$$

Now suppose (iii) holds.

As before, one has the following:

$$\max_{\frac{\ell}{m_{p(n)}}\leq t\leq\frac{\ell+1}{m_{p(n)}}}\left|\Gamma_n^x(t)-\theta_n^x(t)\right|\leq$$

$$\leq\max\left(\left|\Gamma_n^x\left(\frac{k}{n}\right)-\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right|,\left|\Gamma_n^x\left(\frac{k+1}{n}\right)-\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right|,\right.$$

$$\left|\Gamma_n^x\left(\frac{k+1}{n}\right)-\theta_n^x\left(\frac{\ell+1}{m_{p(n)}}\right)\right|,\left|\Gamma_n^x\left(\frac{k+2}{n}\right)-\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right)\right|,$$

$$\left.\left|\Gamma_n^x\left(\frac{k+2}{n}\right)-\theta_n^x\left(\frac{\ell+1}{m_{p(n)}}\right)\right|,\left|\Gamma_n^x\left(\frac{k+3}{n}\right)-\theta_n^x\left(\frac{\ell+1}{n}\right)\right|\right).\tag{2.70}$$

Exactly as before one can deduce that

$$\max_{\frac{\ell}{m_{p(n)}}\leq t\leq\frac{\ell+1}{m_{p(n)}}}\left|\Gamma_n^x(t)-\theta_n^x(t)\right|\leq$$

$$\leq \frac{M}{(2A_x n \log\log n)^{1/2}} \left(\frac{\ell}{m_{p(n)}} \left(n - m_{p(n)}\right) + 4 \right).$$

Finally we have, putting everything together:

$$\begin{aligned}
&\left\| \Gamma_n^x - \theta_n^x \right\| \leq \\
&\leq \max_{0 \leq \ell \leq m_{p(n)}-1} \frac{M}{(2A_x n \log\log n)^{1/2}} \left(\frac{\ell}{m_{p(n)}} \left(n - m_{p(n)}\right) + 4 \right) \leq \\
&\leq \frac{M}{(2A_x n \log\log n)^{1/2}} \left(n - m_{p(n)} + 4\right) \leq \\
&\leq \frac{M}{(2A_x n \log\log n)^{1/2}} \left(m_{p(n)+1} - m_{p(n)} + 4\right) \leq \\
&\leq \frac{3M(p_{(n)+1})^{1/2}}{(2A_x n \log\log n)^{1/2}} \quad \text{for all } n \text{ large enough.}
\end{aligned} \tag{2.71}$$

However, recall that

$$\begin{aligned}
m_{p(n)} &\sim n, \\
m_{p(n)} &\sim \frac{2}{3} p(n)^{3/2}.
\end{aligned} \tag{2.72}$$

Hence it immediately follows that

$$\begin{aligned}
&{p_{(n)}}^{1/2} \sim C n^{1/3} \Longrightarrow \\
&\Longrightarrow \frac{3M(p_{(n)+1})^{1/2}}{(2A_x n \log\log n)^{1/2}} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{2.73}$$

Whence in order to show, $\mathbb{P}$-almost surely, that $(\Gamma_n^x)_{n\in\mathbb{N}}$ is relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set, it will be sufficient to show that $(\theta_n^x)_{n\in\mathbb{N}}$ satisfies the required properties.

To this end we shall focus our attention on showing that

$$\left\| {\xi_{p(n)}}^x - \theta_n^x \right\| \to 0 \ \ \mathbb{P}\text{-almost surely.}$$

Clearly, due to the very nature of $(p(n))_{n\in\mathbb{N}}$ $\left(\xi_{p(n)}^{\ x}\right)_{n\in\mathbb{N}}$ inherits all the necessary properties from $(\xi_n^x)_{n\in\mathbb{N}}$ with $\mathbb{P}$-probability one.

As before,

$$\left\|\theta_n^x - \xi_{p(n)}^{\ x}\right\| =$$

$$= \max_{0\le k\le p(n)-1} \max_{\frac{m_k}{m_{p(n)}}\le t\le \frac{m_{k+1}}{m_{p(n)}}} \left|\xi_{p(n)}^{\ x}(t) - \theta_n^x(t)\right| .$$

Again it comes as no surprise that

$$\max_{\frac{m_k}{m_{p(n)}}\le t\le \frac{m_{k+1}}{m_{p(n)}}} \left|\xi_{p(n)}^{\ x}(t) - \theta_n^x(t)\right| \le$$

$$\le \left(\max_{m_k\le \ell\le m_{k+1}} \left|\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right) - \xi_{p(n)}^{\ x}\left(\frac{m_k}{m_{p(n)}}\right)\right| ,\right.$$

$$\left. \max_{m_k\le 0\le m_{k+1}} \left|\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right) - \xi_{p(n)}^{\ x}\left(\frac{m_{k+1}}{m_{p(n)}}\right)\right|\right).$$

Clearly the following holds:

$$\left|\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right) - \xi_{p(n)}^{\ x}\left(\frac{m_k}{m_{p(n)}}\right)\right| \le$$

$$\le \left|\frac{\sum_{j=1}^{m_k} f(S_j x)}{(2A_x m_{p(n)} \log\log m_{p(n)})^{1/2}}\right| \cdot \left|1 - \frac{(2A_x m_{p(n)} \log\log m_{p(n)})^{1/2}}{(2A_x n \log\log n)^{1/2}}\right| +$$

$$+ \frac{M(\ell - m_{k-1} + 1)}{(2A_x n \log\log n)^{1/2}} . \tag{2.74}$$

However, $\left(\xi_{p(n)}^{\ x}\right)$ is uniformly bounded $\Rightarrow$

$$\Rightarrow \max_{m_k\le \ell\le m_{k+1}} \left|\theta_n^x\left(\frac{\ell}{m_{p(n)}}\right) - \xi_{p(n)}^{\ x}\left(\frac{m_k}{m_{p(n)}}\right)\right| \le$$

$$\leq M^x \cdot \left| 1 - \frac{(2A_x m_{p(n)} \log\log m_{p(n)})^{1/2}}{(2A_x n \log\log n)^{1/2}} \right| +$$

$$+ \frac{M}{(2A_x n \log\log n)^{1/2}} \left(\lfloor (k+1)^{1/2} \rfloor + \lfloor (k+1)^{1/4} \rfloor \right)$$

for some M^x.

Using identical methods we deduce that

$$\begin{aligned}
&\left\| \xi_{p(n)}^{\,x} - \theta_n^x \right\| \leq \\
&\leq M^x \cdot \left| 1 - \frac{(2A_x m_{p(n)} \log\log m_{p(n)})^{1/2}}{(2A_x n \log\log n)^{1/2}} \right| + \\
&\quad + \frac{M}{(2A_x n \log\log n)^{1/2}} \max_{0 \leq k \leq p(n)-1} \left(\lfloor (k+1)^{1/2} \rfloor + \lfloor (k+1)^{1/4} \rfloor \right) \leq \\
&\leq M^x \cdot \left| 1 - \frac{(2A_x m_{p(n)} \log\log m_{p(n)})^{1/2}}{(2A_x n \log\log n)^{1/2}} \right| + \\
&\quad + \frac{M}{(2A_x n \log\log n)^{1/2}} 2p(n)^{1/2}.
\end{aligned} \tag{2.75}$$

Using the $p(n)$-asymptotics we discussed before we deduce that

$$\left\| \xi_{p(n)}^{\,x} - \theta_n^x \right\| \to 0 \quad \text{as} \quad n \to \infty; \ \mathbb{P}\text{-almost surely.} \tag{2.76}$$

Thus $(\theta_n^x)_{n \in \mathbb{N}}$ is, $\mathbb{P}$-almost surely, relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set.

The proof is now complete. □

As in Stassen's work, see Theorems 13 and 14 in the Introduction of this very chapter, we have several immediate consequences:

Corollary 1. *Assume the Schatte model set-up. Then:*

(a) *$\mathbb{P}$-almost surely,*

$$\limsup_{n\to\infty}(2n\log\log n)^{-1/2}\sum_{k=1}^{n} f(S_k x) = A_x^{1/2} \tag{2.77}$$

for almost every x.

(b) *Let $T_k = \sum_{j=1}^{k} f(S_j x)$ and $a \geq 1$ be some real number.*

Then, $\mathbb{P}$-almost surely,

$$\limsup_{n\to\infty} n^{-1-a/2}(2\log\log n)^{-a/2}\sum_{k=1}^{n}|T_k|^a =$$

$$= \frac{\int\limits_0^1 \frac{dt}{(1-t^a)^{1/2}} \cdot a^{a/2}}{2(a+2)^{\frac{a}{2}-1}} A_x^{a/2} \tag{2.78}$$

for almost every x.

(c) *Let T_k be as in* (b) *and let $\#\{-\}$ stand for the number of elements of the set $\{-\}$. Then, $\mathbb{P}$-almost surely*

$$\limsup_{n\to\infty}\frac{1}{n}\#\left\{k \leq n :\ (2k\log\log k)^{-1/2}T_k \geq cA_x^{1/2}\right\} =$$

$$= 1 - \exp\left(-4\left(\frac{1}{c^2}-1\right)\right) \quad \textit{for almost every } x;$$

$$\textit{where } c \textit{ is any number in } [0,\,1]. \tag{2.79}$$

Let us now state and prove our next result. It is a version of the Weighted Law of the Iterated Logarithm.

Theorem 26 (Rašeta)**.** *Let us inherit the entire set-up and notation from our previous result, namely Theorem 25.*

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of reals satisfying the following conditions:

(i) $|a_n| \geq n^{\varepsilon}$ *where* ε *satisfies:* $(1+2\varepsilon)\left(\frac{1}{2}-\gamma\right) \geq \frac{1}{2}$.

(ii) $\max\limits_{1\leq k\leq n} |a_k| = o\left(\left(\sum_{k=1}^{n} a_k^2\right)^{\gamma}\right)$; $\gamma \in (0, 1/2)$.

Define $E_n^x = \mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} a_k f(S_k x)\right)$ *and assume there exists a positive function of* x*, called* $\theta(x)$*; with*

$$E_n^x \geq \theta(x) \sum_{k=1}^{n} a_k^2 \quad \textit{for all } x \textit{ and all } n \in \mathbb{N}.$$

Then,

$\mathbb{P}$*-almost surely*

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k f(S_k x)}{(2E_n^x \log\log E_n^x)^{1/2}} = 1 \tag{2.80}$$

for almost every x*.*

Proof. Our philosophy of proof has not changed much; namely the idea is as follows:

(a) introduce long and short blocks;

(b) use the long blocks to get the asymptotics on n;

(c) force this asymptotics in the original problem by (pardon my French) performing brutal murder of the short block contribution, whence the Weighted Law of the Iterated Logarithm shall follow from its long block counterpart.

Let α, β be two positive real numbers with $\alpha > \beta$.

Let the k^{th} long block have length $\lfloor k^\alpha \rfloor$ and the k^{th} short block length $\lfloor k^\beta \rfloor$.

Analogously to our previous proof we define

$$m_n = \sum_{k=1}^{n} \big(\lfloor k^\alpha \rfloor + \lfloor k^\beta \rfloor \big). \tag{2.81}$$

We will establish several conditions which, if met simultaneously, would be enough to complete the proof.

In order to repeat essentially identical computations we shall group these in one place and demonstrate that the assumptions (i) and (ii) from the very statement of our result provide a reasonable umbrella for our sufficient conditions.

This may sound a bit orthodox; but I believe it is more honest than working backwards trying to impress the reader with observations that seem to miraculously come out of "thin air".

Our first idea is to establish the result along a subsequence $(m_n)_{n\in\mathbb{N}}$ and to find a condition that shall ensure this will complete the proof.

As before, define implicitly the sequence $(p(n))_{n\in\mathbb{N}}$ of integers using the following inequalities:

$$m_{p(n)} \le n < m_{p(n)+1}; \quad n \in \mathbb{N}. \tag{2.82}$$

Suppose now that our result has been established along the subsequence $(m_n)_{n\in\mathbb{N}}$. Nature of the sequence $(p(n))_{n\in\mathbb{N}}$ is clearly such that then the result would have been established along $(m_{p(n)})_{n\in\mathbb{N}}$ too.

But then we have:

$$\overline{\lim_{n\to\infty}} \frac{\sum_{k=1}^{n} a_k f(S_k x)}{(2E_n^x \log\log E_n^x)^{1/2}} =$$

$$= \overline{\lim_{n\to\infty}} \frac{\sum_{k=1}^{m_{p(n)}} a_k f(S_k x) + \sum_{k=m_{p(n)+1}}^{n} a_k f(S_k x)}{(2E_n x \log\log E_n x)^{1/2}} =$$

$$= \overline{\lim_{n\to\infty}} \left\{ \left(\frac{2E_{m_{p(n)}}^{\ x} \log\log E_{m_{p(n)}}^{\ x}}{2E_n^x \log\log E_n^x} \right)^{1/2} \cdot \frac{\sum_{k=1}^{m_{p(n)}} a_k f(S_k x)}{(2E_{m_{p(n)}}^{\ x} \log\log E_{m_{p(n)}}^{\ x})^{1/2}} + \right.$$

$$\left. + \frac{\sum_{k=m_{p(n)+1}}^{n} a_k f(S_k x)}{(2E_n^x \log\log E_n^x)^{1/2}} \right\}, \tag{2.83}$$

whence it shall be sufficient to establish the following two relations:

$$\frac{E_{m_{p(n)}}^{\ x}}{E_n^x} \to 1 \quad \text{as} \quad n \to \infty$$

and (2.84)

$$\frac{1}{(2E_n^x \log\log E_n^x)^{1/2}} \sum_{k=m_{p(n)+1}}^{n} a_k f(S_k x) \to 0.$$

We look at the second condition first due to its simplicity:

$$\left| \frac{1}{(2E_n^x \log\log E_n^x)^{1/2}} \sum_{k=m_{p(n)+1}}^{n} a_k f(S_k x) \right| \le$$

$$\leq \frac{M \cdot \max_{1\leq k\leq n} |a_k|(n - m_{p(n)})}{(2E_n^x \log\log E_n^x)^{1/2}} \leq$$

(M is as before a bound on f)

$$\leq \frac{2M \max_{1\leq k\leq n} |a_k|(p(n)+1)^\alpha}{(2E_n^x \log\log E_n^x)^{1/2}} . \tag{2.85}$$

We shall summarize and enumerate this condition by:

$$(2) \qquad \frac{p(n)^\alpha \max_{1\leq k\leq n} |a_k|}{(2E_n^x \log\log E_n^x)^{1/2}} \to 0. \tag{2.86}$$

Let us now return to the first one.

Observe that

$$E_n^x = \sum_{k=1}^{n} a_k^2 \mathbb{E} f^2(S_k x) + \sum_{\substack{k\neq \ell \\ k,\ell\in\{1,\dots,n\}}} a_k a_\ell \mathbb{E} f(S_\ell x) -$$

$$- \left(\sum_{k=1}^{n} a_k \mathbb{E} f(S_k x)\right)^2,$$

and so trivially

$$E_{m_{p(n)}}^{\ x} = \sum_{k=1}^{m_{p(n)}} a_k^2 \mathbb{E} f^2(S_k x) +$$

$$+ \sum_{\substack{k\neq \ell \\ k,\ell\in\{1,\dots,m_{p(n)}\}}} a_k a_\ell \mathbb{E} f(S_k x) f(S_\ell x) - \left(\sum_{k=1}^{m_{p(n)}} a_k \mathbb{E} f(S_k x)\right)^2 .$$

Our job is identical to showing that

$$\frac{E_n^x - E_{m_{p(n)}}^{\ x}}{E_n^x} \to 0 \quad \text{as} \quad n\to\infty, \quad \text{i.e.} \tag{2.87}$$

$$\frac{1}{E_n^x}\Bigg\{\sum_{k=m_{p(n)+1}}^{n} a_k^2 \mathbb{E} f^2(S_k x) + \sum_{\substack{k\neq\ell \\ k,\ell\in\{m_{p(n)+1},\dots,n\}}} a_k a_\ell \mathbb{E} f(S_k x) f(S_\ell x) -$$

$$- \left(\sum_{k=1}^{n} a_k \mathbb{E} f(S_k x)\right)^2 + \left(\sum_{k=1}^{m_{p(n)}} a_k \mathbb{E} f(S_k x)\right)^2 \Bigg\}.$$

Firstly:

$$\sum_{k=m_{p(n)+1}}^{n} a_k^2 \mathbb{E} f^2(S_k x) \leq M^2 \max_{1\leq k\leq n} a_k^2 (n - m_{p(n)})$$

$$\leq 2M^2 \max_{1\leq k\leq n} a_k^2 (p(n)+1)^\alpha.$$

Bounding cross terms is somewhat more delicate:

$$\sum_{\substack{k\neq\ell \\ k,\ell\in\{m_{p(n)+1},\dots,n\}}} a_k a_\ell \mathbb{E} f(S_k x) f(S_\ell x) =$$

$$= 2 \sum_{\rho=1}^{n-m_{p(n)}} \sum_{k=m_{p(n)+1}}^{n-\rho} a_k a_{k+\rho} \mathbb{E} f(S_k x) f(S_{k+\rho} x) =$$

$$= 2 \sum_{\rho=1}^{n-m_{p(n)}} \sum_{k=m_{p(n)+1}}^{n-\rho} a_k a_{k+\rho} \mathbb{E} f(S_k x) f(S_k x + T_k^\rho x),$$

$$\big(T_k^\rho = X_{k+1} + \cdots + X_{k+\rho}\big) =$$

$$= 2 \sum_{\rho=1}^{n-m_{p(n)}} \sum_{k=m_{p(n)+1}}^{n-g} a_k a_{k+\rho} \Big[\mathbb{E} f(S_k x) f(S_k x + T_k^\rho x) -$$

$$- \mathbb{E} f(S_k x) f\big(S_k x + F_{\{T_k^\rho x\}}(\{T_k^\rho x\})\big)\Big] +$$

$$+ 2 \sum_{\rho=1}^{n-m_{p(n)}} \sum_{k=m_{p(n)+1}}^{n-\rho} \mathbb{E} f(S_k x) f\big(S_k x + F_{\{T_k^\rho x\}}(\{T_k^\rho x\})\big) \cdot a_k a_{k+\rho}.$$

However, by Schatte-type arguments we deduce easily that the last sum vanishes. Whence the upper bound on the cross terms is $\sim \max_{1\le k\le n} a_k^2 \cdot p(n)^\alpha$. Thus our first condition becomes

$$\frac{\max_{1\le k\le n} a_k^2 \cdot p(n)^\alpha}{E_n^x} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.88}$$

We now focus on establishing sufficient conditions for the Weighted Law of the Iterated Logarithm along the subsequence $(m_n)_{n\in\mathbb{N}}$.

Observe that

$$\begin{aligned}
\sum_{k=1}^{m_n} a_k f(S_k x) = & \sum_{k=1}^{n} \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor k^\alpha \rfloor} a_j f(S_j x - \Delta_{k-1}^{x}) + \\
& + \sum_{k=1}^{n} \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor k^\alpha \rfloor} a_j (f(S_j x) - f\big(S_j x - \Delta_{k-1}^{x}\big) + \\
& + \sum_{k=1}^{n} \sum_{j=m_{k-1}+\lfloor k^\alpha \rfloor+1}^{m_k} a_j f\big(S_j x - \Pi_{k-1}^{x}\big) + \\
& + \sum_{k=1}^{n} \sum_{j=m_{k-1}+\lfloor k^\alpha \rfloor+1}^{m_k} a_j \big(f(S_j x) - f(S_j x - \Pi_{k-1}^{x})\big),
\end{aligned}$$

where the two sequences of random variables $(\Delta_k^x)_{k\in\mathbb{N}}$ and $(\Pi_k^x)_{k\in\mathbb{N}}$ have an identical meaning and purpose as before.

As pointed out before, the idea is that the only contribution should come from the long blocks, i.e. that the asymptotics is to be directly inherited from long blocks.

The fundamental difference (and the unfortunate truth) here is that the long block asymptotics is much more vague than $m_n A_x$; where A_x is our previously encountered function. Simply, $a_k a_{k+\ell}$ is now a quantity that does not depend on ℓ only and whence the beautiful stationarity-type argument breaks down. This in turn forced us to put super-bold bounds on the short blocks (versus obtaining the result along short blocks too, and disposing of it by using $\widehat{m}_n/m_n \to 0$ as $n \to \infty$).

Lifting this ban shall be a part of our future work; for the time being we must demand the following three conditions:

$$\frac{1}{\left(E^x_{m_n} \log\log E^x_{m_n}\right)^{1/2}} \sum_{k=1}^{n} \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor k^\alpha \rfloor} a_j \big(f(S_j x) - f(S_j x - \Delta^x_{k-1})\big) \to 0,$$

$$\frac{1}{\left(E^x_{m_n} \log\log E^x_{m_n}\right)^{1/2}} \sum_{k=1}^{n} \sum_{j=m_{k-1}+\lfloor k^\alpha \rfloor+1}^{m_k} a_j \big(f(S_j x) - f(S_j x - \Pi^x_{k-1})\big) \to 0,$$

$$\frac{1}{\left(E^x_{m_n} \log\log E^x_{m_n}\right)^{1/2}} \sum_{k=1}^{n} \sum_{j=m_{k-1}+\lfloor k^\alpha \rfloor+1}^{m_k} a_j f(S_j x - \Pi^x_{k-1}) \to 0. \tag{2.89}$$

Again arguing as what can be called "usual" by now one can obtain the following conditions that ensure the validity of the relations listed above:

$$\frac{\max\limits_{1 \le k \le m_n} |a_k|}{\left(2E^x_{m_n} \log\log E^x_{m_n}\right)^{1/2}} \to 0 \quad \text{as} \quad n \to \infty; \tag{2.90}$$

$$\frac{n^{1+\beta} \max\limits_{1 \le k \le m_n} |a_k|}{\left(E^x_{m_n} \log\log E^x_{m_n}\right)^{1/2}} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.91}$$

For brevity, we now define:

$$B^x_n := \mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor k^\alpha \rfloor} a_j f(S_j x - \Delta^x_{k-1})\right)$$

and (2.92)

$$D^x_n := \mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} \sum_{j=m_{k-1}+\lfloor k^\alpha \rfloor+1}^{m_k} a_j f(S_j x - \Pi^x_{k-1})\right).$$

Now, we want to get the Weighted Law of the Iterated Logarithm for long blocks.

Using the famous result of Kolmogorov (see Theorem 11) we find the following two conditions:

$$B_n^x \to \infty \quad \text{as} \quad n \to \infty,$$

$$\max_{1 \le k \le m_n} |a_k| \cdot n^\alpha = o\left(\left(\frac{B_n^x}{\log\log B_n^x}\right)^{1/2}\right). \tag{2.93}$$

Finally we want the block asymptotics to be the right asymptotics; whence we must demand:

$$\frac{B_n^x}{E_{m_n}^x} \to 1 \quad \text{as} \quad n \to \infty. \tag{2.94}$$

This last condition turns out to be best dealt with upon a further split into the following two conditions:

$$\frac{B_n^x + D_n^x - E_{m_n}^x}{E_{m_n}^x} \to 0 \quad \text{and} \quad \frac{D_n^x}{E_{m_n}^x} \to 0 \quad \text{as} \quad n \to \infty.$$

Reader has been overexposed to the Schatte-machinery already, and must trust us that, upon some heavy algebra, one can obtain a more compact list of conditions that is easier to put under an umbrella:

$$\begin{aligned} &\text{(i)} && (1+2\varepsilon)\left(\frac{1}{2}-\gamma\right) > \alpha/(\alpha+1), \\ &\text{(ii)} && (1+\alpha)(1+2\varepsilon)\left(\frac{1}{2}-\gamma\right) \ge 1+\beta, \\ &\text{(iii)} && (1+\alpha)(1+2\varepsilon)\left(\frac{1}{2}-\gamma-\delta\right) > \alpha \end{aligned} \tag{2.95}$$

for some $\delta > 0$; as small as you may please. But then it is immediately clear that it shall suffice to have

$$(1+2\varepsilon)\left(\frac{1}{2}-\gamma\right) \ge 1/2 \quad // \tag{2.96}$$

The interplay between ε and γ is a rather interesting one; values of γ close to 1/2 correspond on strong assumptions on ε while small values of γ yield mild ε-assumption. Making this condition more Kolmogorovian remains a challenge. $\square$

Chapter 3

The Schatte Model as a Tool in Analysis and Number Theory

3.1 Introduction

Up to now we have been locked within the fortress of Probability; one could think of previous results in the context of limit theorems for dependent random variables, a certain form of "l'art pour l'art", which is a 19^{th} slogan for "art for art's sake".

However, it is now time for us to use the Schatte model to get insight into other fields of mathematics, where deterministic methods either fail altogether or yield, however complicated, not very general results to say the least.

We shall now illustrate our point with several examples. The following is a very famous result with a proof that takes almost superhumane efforts to comprehend.

3.2 Schatte's Structure in Analysis

Theorem 27 (Carleson, 1966). *Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of reals. Then the series $\sum_{k=1}^{\infty} c_k \sin 2\pi kx$ converges almost everywhere if and only if $\sum_{k\in\mathbb{N}} c_k^2 < \infty$.*

Now, under mild conditions on f, suppose we are to ask a similar question, namely:

Suppose $\sum_{k=1}^{\infty} c_k^2 < \infty$; does it follow that $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges almost everywhere, where $(n_k)_{k\in\mathbb{N}}$ is some subsequence of $\mathbb{N}$?

For non-random $(n_k)_{k\in\mathbb{N}}$ this is generally false (for more details see Nikishin [41]).

Finding the precise almost everywhere convergence criteria for $\sum_{k=1}^{\infty} c_k f(n_k x)$ and for the existence of $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(n_k x)$ seems to be mission impossible in analysis for almost 100 years since the problem was first raised by Khinchin. For details see [34]. To illustrate how deep in the dark analysis is concerning this problem let us simply mention that the problem is unsolved even for $n_k = k!$

Now let us forget about n_k's and let us return to the Schatte's S_k's. The methods of ours, unfortunately, restrict us to the domain of those random frequencies that are not accumulating point-asses; i.e. we are restricted to the domain of absolute continuity. Nevertheless, think of S_k's as a "simulation" of the integral n_k's. As we know, almost all increasing sequences of integers are of linear growth, moreover almost all of them satisfy $\frac{n_k}{k} \to 2$ as $k \to \infty$.

This essential feature can easily be realised within our framework, since the S_k's can trivially be chosen in a way to have suchlike asymptotics. Thus, although strictly formally speaking, we do not really solve the problem in hand, the method shall surely provide us with a pretty good idea of what is going on. These probabilistic arguments shall therefore provide us with our very own "quantum of solace", due to failure of deterministic methods.

To illustrate the power of randomness let us state the following result:

Theorem 28 (Berkes and Weber, 2009)**.** *Assume that the underlying random variables* $(X_n)_{n\in\mathbb{N}}$ *are those of Schatte, defined on some probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*. Moreover, let* f *satisfy all the conditions imposed in the previous chapter.*

Then, $\mathbb{P}$*-almost surely*

$$\sum_{k=1}^{\infty} c_k f(S_k x) \quad \text{converges for almost every } x$$

$$\text{provided that} \quad \sum_{k\in\mathbb{N}} c_k^2 < \infty.$$

It is clear now that dropping determinism on the frequency puts Carleson's result on steroids, mild randomisation extends his theory to a vast class of functions!

Before we are ready to finish off our story we must remind the reader of some of the underlying concepts.

3.3 Classical Results on Discrepancies

Definition 1. A bounded sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers is said to be equidistributed on an interval $[a, b]$ if for any subinterval $[c, d]$ of $[a, b]$ we have

$$\lim_{N\to\infty} \frac{\big|\{s_1, \dots, s_n\} \cap [c, d]\big|}{n} \to \frac{d-c}{b-a} \tag{3.1}$$

(here the notation $\big|\{s_1, \dots, s_n\} \cap [c, d]\big|$ denotes the number of elements out of first n elements of the sequence that are between c and d).

Definition 2. We define discrepancy $D(N)$ of a sequence $\{S_1, S_2, \dots\}$ with respect to the interval $[a, b]$ as

$$D(N) = \sup_{a\le c\le d\le b} \left| \frac{\big|\{S_1, \dots, S_N\} \cap [c, d]\big|}{N} - \frac{d-c}{b-a} \right| \tag{3.2}$$

It is clear that the sequence is equidistributed if the discrepancy $D(N) \to 0$ as $N \to \infty$.

Definition 3. Sequence $(a_n)_{n\in\mathbb{N}}$ is said to be equidistributed mod 1 or uniformly equidistributed mod 1 if $(\{a_n\})_{n\in\mathbb{N}}$ is equidistributed w.r.t. the interval [0, 1].

(Here and elsewhere $\{x\}$ shall still stand for the fractional part of a real number x.)

Definition 4. Given a point set $p = (x_n)_{n=0}^{N-1}$ in the s-dimensional unit cube $I = [0, 1)^s$; the star discrepancy is defined as

$$D_N{}^*(P) \equiv \sup_{J\in Y*} D(J, P)$$

where the local discrepancy is defined by

$$D(J, P) = \left| \frac{\text{number of } x_n \in J}{N} - \text{Vol}(J) \right| ,$$

where $\text{Vol}(J)$ is the content of J, and $Y*$ is the class of s-dimensional subintervals J of I of the form

$$J \equiv \prod_{i=1}^{s} [0,\, u_i) \quad 0 \le u_i \le 1 \;\text{ for } 1 \le i \le s. \tag{3.3}$$

Definition 5. Let $d, n \in \mathbb{N}$. For $0 < p < \infty$ and the point set $\{t_1, \ldots, t_n\} \subseteq [0,\, 1)^d$ we define the L_p-star discrepancy by the L_p-norm of the discrepancy function; namely:

$$\text{disc}_p^*(t_1, \ldots, t_n) =$$

$$\left(\int\limits_{[0,1]^d} \left| \lambda^d([0, x]) - \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{[0,x]}(t_i) \right|^p dx \right)^{1/p}$$

where $\lambda^d([0,x])$ is the d-dimensional Lebesgue measure of the box:

$$[0,x] \equiv \{y \in [0,1]^d : \ 0 \leq y_i \leq x_i : \ i = 1, \ldots, d\}. \qquad (3.4)$$

We will now explore some classical and some more recent results in this field:

Theorem 29 (Weyl, 1916). *Let α be an irrational number. Then the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed* mod 1*, where $\{x\}$ stands for the fractional part of a real number x. Moreover; if p is a polynomial with at least* 1 *irrational coefficient (other than the constant term), then the sequence $(p(n))_{n\in\mathbb{N}}$ is uniformly distributed* mod 1*.*

This is another result of Weyl's; for details see [37].

Furthermore, $(\log n)_{n\in\mathbb{N}}$ is not uniformly distributed mod 1. For details, see, yet again, [37].

The following is a famous result of Analytic Number Theory.

Theorem 30 (Vinogradov, 1935). *Let α be an irrational number. Then $(p_\alpha)_{p\in P}$ ($P \equiv$ set of primes) is equidistributed* mod 1*.*

We now move to so-called "metric theorems", namely the results that describe the behaviour of some parametrized sequence for almost all values of suchlike parameter.

We start with the following result:

Theorem 31 (Bernstein, 1911). *For any sequence of distinct integers $(b_n)_{n\in\mathbb{N}}$; the sequence $(b_n\alpha)_{n\in\mathbb{N}}$ is equidistributed* mod 1 *for almost all α.*

More parametrisations have been studied, for example let us mention the following result:

Theorem 32 (Koksma, 1937). *The sequence* $(\alpha^n)_{n\in\mathbb{N}}$ *is equidistributed* mod 1 *for almost all values of* $\alpha > 1$.

It is interesting to point out that whether $(e^n)_{n\in\mathbb{N}}$ or $(\pi^n)_{n\in\mathbb{N}}$ are equidistributed mod 1 is still unknown!

However, it is known that the sequence $(\alpha^n)_{n\in\mathbb{N}}$ is not equidistributed mod 1 if α is a so-called PV number. A PV (Pisot–Vijayaraghavan) number is a real algebraic number, strictly larger than 1 that has its all Galois conjugates bounded (again strictly) by 1 in absolute value.

We shall now move on to some more contemporary results for which our results will turn out to be either a logical continuation of, or they are somewhat similar in nature.

We start with the following result:

Theorem 33 (Philipp, 1975). *Let* $(n_k)_{k\in\mathbb{N}}$ *be a lacunary sequence of integers, that is* $\exists q > 1$ *with*

$$n_{k+1}/n_k > q \quad \textit{for all} \quad k \in \mathbb{N}.$$

Let D_N *denote the discrepancy of the sequence* $(n_k x)_{k\in\mathbb{N}}$. *Then, for almost every* x,

$$32^{-1/2} \leq \limsup_{N\to\infty} \frac{N D_N(x)}{\sqrt{N \log\log N}} \leq C \tag{3.5}$$

where

$$C \leq 166 + 664(q^{1/2} - 1)^{-1}. \tag{3.6}$$

Philipp conjectured that

$$C^2 \leq 2 \sup_I \limsup_{N\to\infty} N^{-1} \int_0^1 \Big(\sum_{k\leq N} \mathbb{1}_I\big(\{n_k x\}\big) - |I| \Big)^2 dx, \tag{3.7}$$

where the supremum is taken over all intervals of the form $I = [\alpha, \beta)$; $0 \leq \alpha < \beta \leq 1$; $\mathbb{1}_I$ will stand for the indicator of I, while $|I|$ will stand for the length of I.

The bound on C^2 suggested by the RHS of (3.7) did not come out of thin air, it is an educated guess Philipp made based on his previous work. Namely in his paper [43] Philipp computed the value of the corresponding lim sup for $\eta_k = 2^k$. The value turns out to be constant almost everywhere that equals the RHS of (3.7); where one should put 2^k for η_k. However, many years later, the question of value of the lim sup is still open; except for some special classes of sequences η_k.

The following result was the first one in this direction:

Theorem 34 (Fukuyama, 2008). *Let Σ_θ be the* lim sup *in the case $\eta_k = \theta^k$. Then*

$$\Sigma_\theta = \begin{cases} 1/2 & \text{if } \theta^c \text{ is irrational for all } c \in \mathbb{N}, \\ \sqrt{42/2} & \text{if } \theta = 2, \\ \dfrac{\sqrt{(\theta+1)\theta(\theta-2)}}{2\sqrt{(\theta-1)^3}} & \text{if } \theta \geq 4 \text{ is an even integer}, \\ \dfrac{\sqrt{\theta+1}}{2\sqrt{\theta-1}} & \text{if } \theta \geq 3 \text{ is an odd integer}. \end{cases} \tag{3.8}$$

Although rather complicated, the lim sup in the above is still a constant.

The following result is the first one with a non-constant almost everywhere lim sup in this theory:

Theorem 35 (Aistleitner, 2010). *Define a lacunary sequence $(\eta_k)_{k\in\mathbb{N}}$ as follows:*

$$\eta_k = \begin{cases} 2^{k^2} & \text{if } k \equiv 1 \pmod 4, \\ 2^{(k-1)^2+1} - 1 & \text{if } k \equiv 2 \pmod 4, \\ 2^{k^2+k} & \text{if } k \equiv 3 \pmod 4, \\ 2^{(k-1)^2+(k-1)+1} - 2 & \text{if } k \equiv 0 \pmod 4. \end{cases} \tag{3.9}$$

Then

$$\limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} = \Psi(x)$$

$\big(D_N$ *is the discrepancy of* $(\eta_k x)_{k\in\mathbb{N}}\big)$*; where*

$$\Psi(x) = \begin{cases} 3/4\sqrt{2} & \text{for } 0 \le x \le 3/8, \\ \sqrt{2(1-x)x - x/2} & \text{for } 3/8 \le x \le 7/16, \\ \sqrt{49/128 - x/4} & \text{for } 7/16 \le x \le 1/2, \\ \Psi(1-x) & \text{for } 1/2 < x \le 1. \end{cases} \tag{3.10}$$

For the picture of $\Psi(x)$ see Figure 3.1

Last but not the least result we shall mention here is as follows:

Theorem 36 (Fukuyama and Miyamoto, 2011). *Let $(\eta_k)_{k\in\mathbb{N}}$ be a sequence of integers.*

We first introduce the following quantities:

$$\sum\{\eta_k x\} = \limsup_{N\to\infty} \frac{ND_N\{\eta_k x\}}{\sqrt{2N \log\log N}} \tag{3.11}$$

and

$$\sum\nolimits^*\{\eta_k x\} = \limsup_{N\to\infty} \frac{N{D_N}^*\{\eta_k x\}}{\sqrt{2N \log\log N}} \tag{3.12}$$

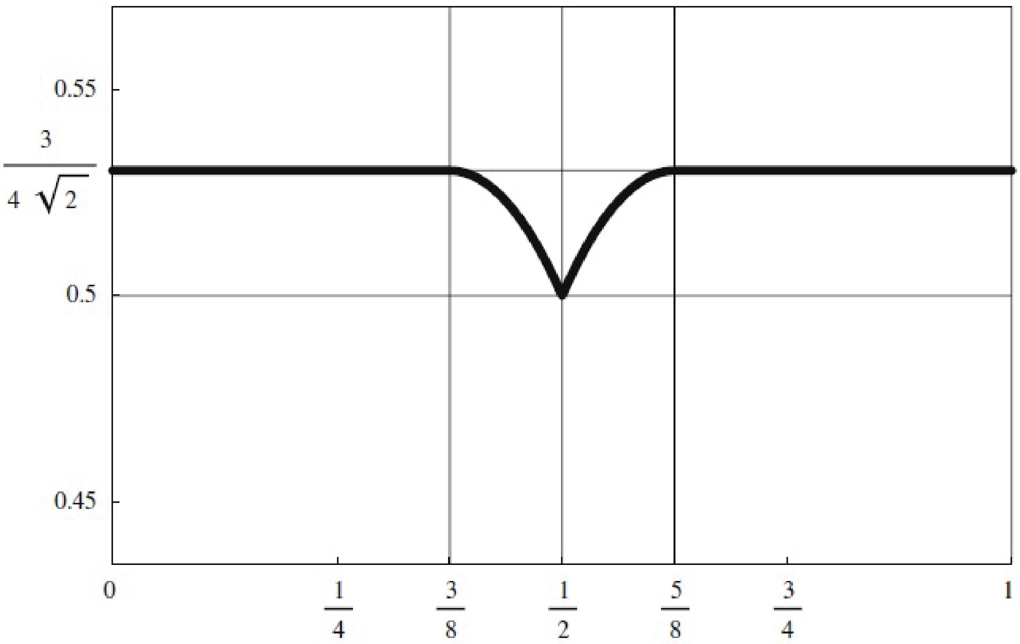


Figure 3.1: graph of $\Psi(x)$

where $D_N\{\eta_k x\}$ *and* $D_N^*\{\eta_k x\}$ *stand for discrepancy and star discrepancy, respectively, of the sequence* $\{\eta_k x\}$*.*

Split $\mathbb{R}$ *into two parts; namely:*

$$\text{Those } \theta \in \mathbb{R} \text{ with } \theta^r \notin \mathbb{Q} \text{ for any } r \in \mathbb{N} \tag{3.13}$$

and those $\theta \in \mathbb{R}$ *such that there exist* p, q *and* $r \in \mathbb{N}$ *with*

$$\begin{gathered} r = \min\{n \in \mathbb{N} : \ \theta^n \in \mathbb{Q}\}; \quad grd(p,q) = 1; \\ \theta = \sqrt[r]{p/q}. \end{gathered} \tag{3.14}$$

If (3.13) *holds, define* $\Sigma_\theta = 1/2$. *If* (3.14) *holds, define* Σ_θ *as follows:*

$$\Sigma_\theta = \begin{cases} \sqrt{(pq+1)(pq-1)}\,/2 & p,q \ \textit{odd}, \\ \sqrt{(p+1)p(p-2)/(p-1)^3}\,/2 & p \geq 4, \ \textit{peven}, \ q=1, \\ \sqrt{42/9} & p=2, \quad q=1, \\ \sqrt{22/9} & p=5, \quad q=2. \end{cases} \tag{3.15}$$

Then, for almost every x *one has the following:*

$$\Sigma\big\{(\theta^k-1)x\big\} = \Sigma_\theta, \tag{3.16}$$

$$\Sigma^*\big\{(\theta^k-1)x\big\} = \Sigma^*_\theta(x), \tag{3.17}$$

where $\Sigma^*_\theta(x)$ *is a continuous function on the torus.*

Moreover, if (3.13) *holds, then*

$$\Sigma^*_\theta(x) = \Sigma_\theta(x) = 1/2. \tag{3.18}$$

Furthermore, if (3.18) *holds and either one of these three conditions holds:*

(i) p *and* q *are both odd,*

(ii) $q = 1$,

(iii) $p = 5$, $q = 2$,

then $\Sigma^*_\theta(x)$ *is not constant and*

$$\Sigma^*_\theta(x) < \Sigma_\theta \quad \textit{except for finitely many} \ \ x. \tag{3.19}$$

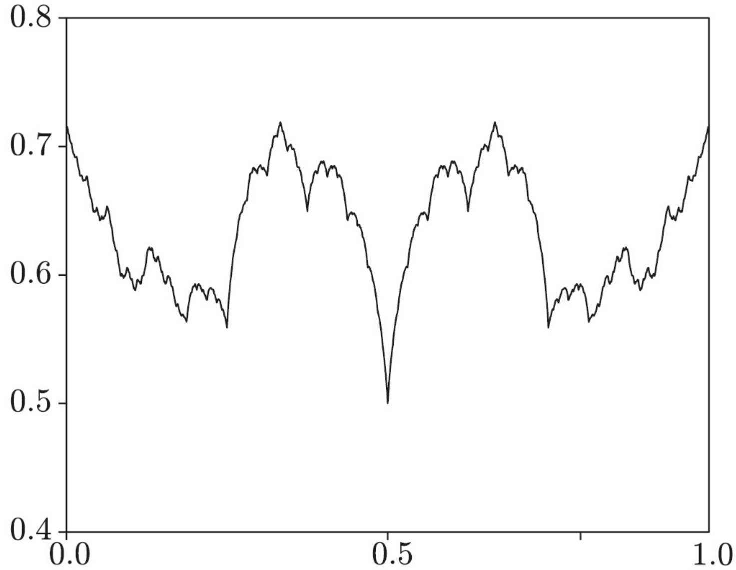


Figure 3.2: lim sup function for $n_k = 2^k - 1$.

For the graph of $\Sigma_2^*(x)$ see 3.2.

This indicates irregular behaviour of $\Sigma_2^*(x)$ when θ is a power root of integers.

Before we move on to our results we have some more set-up to do and one more directly relevant result to mention.

3.4 Reproducing Kernel Hilbert Spaces and the Result of Finkelstein

Definition 6. Let X be an arbitrary set and H a Hilbert space of complex valued functions on X. We say that H is a Reproducing Kernel Hilbert Space if the linear map $L_x : f \to f(x)$ from H to the complex numbers is continuous for all x.

Now, by the Riesz representation theorem this implies that for all x in X there exists a unique element K_x of H with the property that

$$f(x) = L_x(f) = \langle f, k_x \rangle \quad \text{for all} \quad f \in H. \tag{3.20}$$

The function k_x is called the point-evaluation function at the point x. Since H is a space of functions, k_x is itself a function that has X as its domain, and whence can be written by $k_x(y)$. The family of functions $k_x(y)$ may be embedded into a single function $K: X \times X \to \mathbb{C}$ be defining

$$K(x, y) := \overline{K_x(y)}. \tag{3.21}$$

This function is known as the reproducing kernel of the Hilbert space H and its uniqueness is ensured via Riesz representation theorem.

For many further details on the reproducing kernel Hilbert spaces see Nachman [40], Berliner and Thomas [8] and Oodaira [42].

We shall now state the following result which will turn out to be a special case of that of our own:

Theorem 37 (Finkelstein, 1971)**.** *Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Moreover, let X_1 be uniformly distributed on $[0, 1]$. For fixed $\omega \in \Omega$ and $x \in [0, 1]$ define $F_n(x, \omega)$ to be the empirical distribution of the X_i at stage n; that is $nF_n(x, \omega)$ is the number of $X_1(\omega), X_2(\omega), \ldots, X_n(\omega)$ which are smaller than x.*

Define

$$G_n(x, \omega) := \frac{nF_n(x, \omega) - nF(x)}{(2n \log \log n)^{1/2}} \tag{3.22}$$

where $F(x)$ stands for the distribution function of X_1.

Now define K to be the set of all elements f of $C[0,1]$ fulfilling the following three conditions:

(i) $f(0) = f(1) = 0$,

(ii) f is absolutely continuous with respect to Lebesgue measure,

(iii) $\int\limits_0^1 (\dot{f}(x))^2 dx \leq 1$,

where $\dot{f}$ stands for the derivative of f with respect to Lebesgue measure. Then, there exists $\Omega_0 \in \mathcal{A}$; $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$, the sequence

$$\big(G_n(\omega, \cdot)\big)_{n \geq 3}$$

is relatively compact in $C[0,\, 1]$ and the set of its limit points is K.

3.5 Results

We now, without further delay, state our first result.

Theorem 38 (Berkes and Rašeta)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Furthermore, let X_1 be bounded with bounded density. Then there exists a set $G \subseteq \mathbb{R}$ with Lebesgue measure 0 such that, $\mathbb{P}$-almost surely, the sequence of functions*

$$\alpha_N(t,x) = \sqrt{\frac{N}{2\log\log n}}\big(F_N(t,x) - t\big) \quad 0 \leq t \leq 1, \quad N = 1, 2, \ldots \tag{3.23}$$

is relatively compact in the Skorohod space $D[0,1]$ for all fixed $x \notin G$ and its class of limit functions is identical with the unit ball B_Γ of the reproducing kernel Hilbert *space determined by the covariance function*

$$\Gamma(s,s') = \mathbb{E}g_s(U)g_{s'}(U) + \sum_{\varrho=1}^{\infty} \mathbb{E}g_s(U)g_{s'}(U+S_\varrho x) +$$

$$+ \sum_{\varrho=1}^{\infty} \mathbb{E}g_{s'}(U)g_s(U+S_\varrho x). \tag{3.24}$$

Here, U is the uniformly distributed random variable independent of the sequence $(X_n)_{n\in\mathbb{N}}$; $g_s = \mathbb{1}_{(0,s)} - s$ is the centered indicator function of the interval $(0,s)$ and

$$F_N(t,x) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{(-\infty,x)}(S_k x) \tag{3.25}$$

will stand for the empirical distribution of the sample

$$\{S_1 x\}, \{S_2 x\}, \ldots, \{S_N x\} \qquad (S_n = X_1 + \cdots + X_n)$$

and $\mathbb{1}_{(a,b)}$ will be the indicator function of the interval (a,b), extended with period 1.

Proof. Without loss of generality we can assume $x = 1$. We follow the classical argument of Finkelstein [23], proving first a finite-dimensional Law of the Iterated Logarithm for the values of the function $\alpha_N(t,1)$ in (3.23) restricted to a finite subset $\{t_1,\ldots,t_r\}$ of [0, 1] and then to show the relative compactness of α_N in the Skorohod topology.

Let $0 = t_0 < t_1 < \cdots < t_r = 1$ and put

$$\mathbf{Y}_k\big(f_{(0,t_1)}(S_k), f_{(t_1,t_2)}(S_k), \ldots, f_{(t_{r-1},t_r)}(S_k)\big)$$

where $f(a,b) = \mathbb{1}_{(a,b)} - (b-a)$.

We will start with the following lemma.

Lemma 1*. *With probability* 1 *(w.r.t.* $\mathbb{P}$*), the class of limit points of the sequence*

$$\Big\{(2N\log\log N)^{-1/2}\sum_{k=1}^{N}\boldsymbol{Y}_k,\ \ N=1,2,\dots\Big\}$$

in $\mathbb{R}^{r+1}$ *is the ellipsoid*

$$\Big\{(x_1,\dots,x_{r+1})\cdot\sum_{i,j=1}^{r+1}\Gamma(t_i,t_j)x_ix_j\le 1\Big\}.$$

This lemma can be proved by a blocking argument very similar to that of the proof of Theorem 25, except that instead of the result of Major [39] we use the coupling inequality of Berthet and Mason, see [10], page 155.

Put $\Psi(n)=\sup\limits_{0\le t\le 1}\big|\mathbb{P}(S_n\le t)-t\big|$ and note that by Theorem 1 of Schatte [54] we have

$$\Psi(n)\le Ce^{-\lambda n},\quad n\ge 1, \tag{3.26}$$

for some constants $C,\lambda>0$.

Lemma 2*. *Let* $f=\mathbb{1}_{(a,b)}-(b-a)$ *for some* $0\le a<b\le 1$*. Then*

$$\mathbb{E}\Big(\sum_{k=M+1}^{M+N}f(S_k)\Big)^2\le C\|f\|N \tag{3.27}$$

for any $M\ge 0$*,* $N\ge 1$ *where* $\|f\|=\Big(\int_0^1 f^2(x)dx\Big)^{1/2}$ *and* C *is an absolute constant. The conclusion remains valid if* f *is a Lipschitz function with*

$$\int_0^1 f(x)dx=0.$$

Proof. In what follows, C denotes positive constants, possibly different at different places. We first show that

$$\big|\mathbb{E}f(S_k)f(S_\ell)\big|\le C\Psi(\ell-k)\|f\|\qquad (k<\ell). \tag{3.28}$$

Indeed, by Schatte [53] there exists a random variable Δ with $|\Delta| \leq \Psi(\ell - k)$ such that $S_\ell - \Delta$ is a uniform random variable independent of S_k. Hence

$$\mathbb{E}f(S_\ell - \Delta) = \int_0^1 f(x)dx = 0$$

and thus

$$\mathbb{E}f(S_k)f(S_\ell - \Delta) = \mathbb{E}f(S_k)\mathbb{E}f(S_\ell - \Delta) = 0. \tag{3.29}$$

On the other hand,

$$\begin{aligned}
&\big|\mathbb{E}f(S_k)f(S_\ell) - \mathbb{E}f(S_k)f(S_\ell - \Delta)\big| \leq \\
&\leq \mathbb{E}\big(|f(S_k)|\ |f(S_\ell) - f(S_\ell - \Delta)|\big) \leq \\
&\leq \big(\mathbb{E}f^2(S_k)\big)^{1/2}\big(\mathbb{E}|f(S_\ell) - f(S_\ell - \Delta)|^2\big)^{1/2}.
\end{aligned} \tag{3.30}$$

Since X_1 has a bounded density, by Theorem 1 of Schatte [54] the density φ_n of S_n exists for all $n \geq 1$ and satisfies $\varphi_n \to 1$ uniformly on $[0, 1]$. Thus

$$\mathbb{P}\{S_n \in I\} \leq C|I| \qquad (n \geq 1) \tag{3.31}$$

for some constant $C > 0$; and whence we get

$$\mathbb{E}f^2(S_k) \leq C\|f\|^2. \tag{3.32}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}\big|f(S_\ell) - f(S_\ell - \Delta)\big|^2 = \\
&= \mathbb{E}\big|\mathbb{1}_{(a,b)}(S_\ell) - \mathbb{1}_{(a,b)}(S_\ell - \Delta)\big|^2.
\end{aligned} \tag{3.33}$$

The expression on the right-hand side differs from 0 only if one of S_ℓ or $S_\ell - \Delta$ is inside (a,b) and the other is outside the interval. In this case S_ℓ is closer to the boundary of (a,b) than $|\Delta|$, and since $|\Delta| \le \Psi(\ell - k)$; the probability of this event is at most $C\Psi(\ell-k)$ by (3.31). Thus (3.33) yields

$$\mathbb{E}\big|f(S_\ell) - f(S_\ell - \Delta)\big|^2 \le C\Psi(\ell - k) \tag{3.34}$$

which, together with (3.30) to (3.33), gives

$$\big|\mathbb{E}(f(S_k)f(S_\ell)) - \mathbb{E}f(S_k)f(S_\ell - \Delta)\big| \le C\Psi(\ell-k).$$

Thus using (3.29) we get (3.28).

Now by (3.28)

$$\Big|\sum_{M+1\le k<\ell\le M+N} \mathbb{E}f(S_k)f(S_\ell)\Big| \le$$

$$\le CN\|f\| \sum_{\ell\ge 1} \ell^{-1} \le CN\|f\|$$

which, together with (3.32) completes the proof of Lemma 2*. □

For Lipschitz functions f the argument is similar.

Lemma 3*. *Let* $f = \mathbb{1}_{(a,b)} - (b-a)$ *for some* $0 \le a < b \le 1$*. Then for any* $M \ge 0$*,* $N \ge 1$*, real* $t \ge 1$ *and* $\|f\| \ge N^{-1/4}$ *we have*

$$\mathbb{P}\Big\{\Big|\sum_{k=M+1}^{M+N} f(S_k)\Big| \ge t\|f\|^{1/4}(N\log\log N)^{1/2}\Big\} \le$$

$$\le \exp\big(-Ct\|f\|^{-1/2}\log\log n\big) + t^{-2}N^{-1}. \tag{3.35}$$

Proof. We divide the interval $[M+1, M+N]$ into subintervals $I_1,\dots,I_L$; with $L \sim N19/20$; where each interval I_u contains $\sim N1/20$ terms.

We set

$$\sum_{K=M+1}^{M+N} f(S_k) = \eta_1 + \cdots + \eta_L$$

where

$$\eta_p = \sum_{k \in I_p} f(S_k).$$

We deal with the sums $\sum \eta_{2j}$ and $\sum \eta_{2j+1}$ separately. Since there is a separation $\sim N^{1/20}$ between the even block sums η_{2j}, we can apply Lemma 4.3 of [6] to get

$$\eta_{2j} = \eta_{2j}^* + \eta_{2j}^{**}$$

where

$$\begin{aligned} \eta_{2j}^* &= \sum_{k \in I_{2j}} f(S_k - \Delta_j), \\ \eta_{2j}^{**} &= \sum_{k \in I_{2j}} \big(f(S_k) - f(S_k - \Delta_j)\big) \end{aligned} \tag{3.36}$$

where the Δ_j are the random variables with

$$|\Delta_j| \le \Psi(N^{1/20}) \le N^{-10}$$

and the random variables η_{2j}^*, $j = 1, 2, \dots$ are independent. Relation (3.34) in the proof of Lemma 2* shows that the L_2 norm of each summand in η_{2j}^{**} is at most $C\Psi(N^{1/20}) \le CN^{-10}$ and thus for $\|f\| \ge N^{-1/4}$ we have

$$\|\eta_{2j}^{**}\| \le CN^{-9} \le C\|f\|N^{-8}. \tag{3.37}$$

Thus

$$\Big\|\sum \eta_{2j}^{**}\Big\| \le C\|f\|N^{-7}$$

and therefore by Markov inequality

$$\mathbb{P}\left(\Big|\sum \eta_{2j}^{**}\Big| \ge t\|f\|^{1/4}(N \log\log N)^{1/2}\right) \le$$

$$\le Ct^{-2}\|f\|^{-1/2}(N\log\log N)^{-1}\|f\|^2 N^{-1/14} \le$$

$$\le t^2 N^{-1}. \tag{3.38}$$

Now let $|\lambda| = O(N^{-1/16})$; then

$$|\lambda\eta^*_{2j}| \le C|\lambda|N^{1/20} \le 1/2 \quad \text{for} \quad N \ge N_0$$

and thus using $e^x \le 2 + x + x^2$ for $\|x\| \le 1/2$ we get, using $\mathbb{E}\eta^*_{2j} = 0$;

$$\mathbb{E}\left(\exp\left(\lambda\sum_j \eta^*_{2j}\right)\right) = \prod_j \mathbb{E}\big(e^{\lambda\eta^*_{ij}}\big) \le$$

$$\le \prod_j \mathbb{E}\big(1 + \lambda\eta^*_{2j} + \lambda^2{\eta^*_{2j}}^2\big) =$$

$$= \prod_j \big(1 + \lambda^2\mathbb{E}{\eta^*_{2j}}^2\big) \le \exp\left(\lambda^2\sum_j \mathbb{E}{\eta^*_{2j}}^2\right). \tag{3.39}$$

By Lemma 2*

$$\|\eta_{2j}\| \le C\|f\|^{1/2}N^{1/40}$$

which, together with (3.37) and the Minkowski's inequality, implies

$$\big\|\eta^*_{2j}\big\| \le C\|f\|^{1/2}N^{1/40}$$

and thus the last expression in (3.39) cannot exceed

$$\exp\left(\lambda^2 C\|f\|\sum_j N^{1/20}\right) \le \exp\big(\lambda^2 C\|f\|N\big).$$

Thus choosing

$$\lambda = (\log\log N/N)^{1/2}\|f\|^{-3/4}$$

(note that by $\|f\| \geq N^{-1/4}$ we have $|\lambda| = O(N^{-1/6})$) and thus using Markov's inequality we get

$$\mathbb{P}\Bigg\{\Bigg|\sum_j \eta_{2j}^*\Bigg| \geq t\|f\|^{1/4}(N\log\log N)^{1/2}\Bigg\} \leq$$

$$\leq \exp\Big\{-\lambda t\|f\|^{1/4}(N\log\log N)^{1/2} + \lambda^2 C\|f\|N\Big\} =$$

$$= \exp\Big(-\|f\|^{-1/2}t\log\log N + C\|f\|^{-1/2}\log\log N\Big) \leq$$

$$\leq \exp\big(-C'\|f\|^{-1/2}t\log\log N\big) \tag{3.40}$$

completing the proof of Lemma 3*. □

Now, with Lemma 3* in hand, the relative compactness of the sequence α_n in the $D[0,1]$ topology can be proved by a dyadic chaining argument, similar to the proof of Proposition 3.3.2 in Philipp [45].

Now observe that if X_1 is uniformly distributed on (0, 1), the $\{S_k x\}$ are independent uniformly distributed random variables (Schatte-type arguments).

Moreover, $\Gamma(s,s')$ reduces to the covariance function $s(1-s')(s<s')$ of the Brownian bridge. In this case the limit set in Theorem 38 reduces to the set

$$K = \Bigg\{y(t):\ y \text{ is absolutely continuous on } [0,\,1],$$

$$y(0) = y(1) = 0;\ \int_0^1 \dot{y}^2(t)dt \leq 1\Bigg\}$$

which is the result of Finkelstein, see Theorem 37 and [23].

Furthermore, we point out that

$$\sup_{0\leq t\leq 1}\big|F_N(t,x) - t\big|$$

is the star discrepancy $D_N^*(\{\eta_k x\})$ of the sequence $\{S, x\}, \ldots, \{S_N x\}$, while

$$\int_0^1 \left|F_N(t,x) - t\right|^p dt$$

is the L_p discrepancy $D_N^{(p)}\big(\{\eta_k x\}\big)$ of the same sequence.

From these two observations and our Theorem 38 we get the following two results:

Corollary 1*. *Assume all the notation used previously. Then:*

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N^*\big(\{\eta_k x\}\big) = \sup_{y\in B_\Gamma} \|y\|_\infty \tag{3.41}$$

$\mathbb{P}$-*almost surely for almost every* x.

Corollary 2*. *Assume all the notation used previously. Then:*

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N^{(p)}\big(\{\eta_k x\}\big) = \sup_{y\in B_\Gamma} \|y\|_p, \quad p \geq 1, \tag{3.42}$$

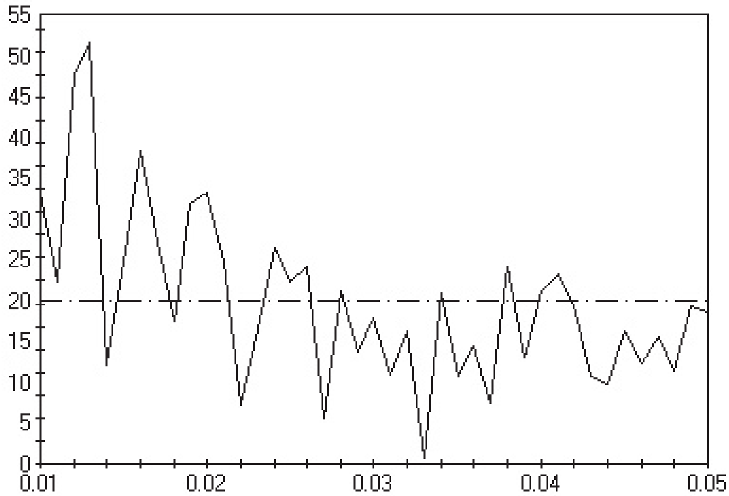
$\mathbb{P}$-*almost surely for almost every* x.

We observe that, although we are not dealing with integers, our results are essentially the "next best thing" when it comes to shedding some light on a difficult conjecture of Philipp; see page 96 of this dissertation. The result on the star discrepancy is of different philosophy to those of Fukuyama and Aistleitner; it can be thought of as a "simulation of a general case"; certainly not dealing with only specific/restricted class of functions; whence being some kind of complement of their work.

The value of the star discrepancy is $A_x^{1/2}$. This function turns out to have remarkable properties. Simulation suggests it is continuous but nowhere differentiable and, likely, unbounded at zero, too.

Formalizing these statements shall be a part of our future work, so far we came essentially empty-handed.

For the picture of this important function see below:



□

□

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