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Inflation market models

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Acknowledgements

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Overview

A quite general class of models is the Jarrow-Yildirim model which extends the Heath-Jarrow-Morton term structure model to the inflation setting. Following Mercurio we formulate inflation market models which are similar to the well known LIBOR market models of interest rates. These models can be viewed as special cases of the general Jarrow-Yildirim model. The aim of this thesis is to give a detailed description of inflation market models, including theoretical aspects as well as practical ones like calibration and Monte Carlo valuation of general inflation-linked derivatives.

In the first chapter we introduce the consumer price index and talk about common inflation-linked financial instruments like inflation-linked bonds or swaps. The second chapter is divided into two parts. Part one discusses the general Jarrow-Yildirim inflation model, while part two describes several specific (low-dimensional) models suitable for practical purposes. In chapter three we first examine historical data and use this to choose a specific model. We then tackle the question of how to calibrate the model parameters using current market data. Finally we discuss how to use Monte Carlo simulation in this framework.

Contents

Remarks on notations

Financial instruments

We sometimes consider similar financial products in different markets. Quantities like prices or rates related to such products always contain a subscript denoting the market to which they belong.

n	...	nominal interest rate related,
R	...	real interest rate related,
I	...	inflation related.

For instance the price of a nominal zero coupon bond is denoted by $P_n(t, T)$, while $f_R(t, T)$ denotes the instantaneous forward rate in the "real" worlds. In case the index is omitted, we always refere to nominal quantities. Some financial instruments incure with an additional superscript index. They are then part of a market model and the index is linked to the final payment date, e.g. F_n^i denotes the nominal forward interest rate for the time interval $[T_{i-1}, T_i]$.

Brownian motions

Throughout this work we use several Brownian motions. For ease of reading we use different letters in different contexts,

$B(t)$	...	Brownian motions under the real-world measure (respectively before a measure change),
$W(t)$	...	Brownian motions under a risk-neutral measure,
$Z(t)$	...	Brownian motions used for market models.

Brownian motions are often used with a superscript. This superscript refers to the measure under which they are Brownian motions (e.g. W^T denotes a Brownian under the measure Q_n^T).

Day count convention

We often use time differences (most of the time denoted by δ), e.g. $\delta = T_i - T_{i-1}$. Potential time adjustments due to underlying day count conventions are assumed to be included.

Correlations

Instantaneous correlations are denoted by ρ. The subscript n, R, I represents the market. If there is a superscript, the quantity is referring to correlations of a market model (e.g. $\rho_{n,I}^{i,j}$ denotes the correlation between a nominal forward rate F_n^i and an inflation forward CPI $\mathcal{I}^j$).

Other notation guidelines

We denote a transpose with a superscript T. Notice that above we denoted Brownian motions under the T-forwad measure with the same superscript. However, this will lead to no confusion, since we won't need to use transposes of Brownian motions and also it should be clear out of context, where a transpose is to be used.

Index of notations

The index $k \in \{n, R\}$ stands for nominal or real.

$r_k(t)$	instantaneous short rate
$f_k(t,T)$	instantaneous forward rate for time T
$P_k(t,T)$	price of a zero coupon bond with maturity T
$y_k(t,T)$	continuously compounded annualized yield ($y_k(t,T) = -\frac{\ln(P_k(t,T))}{T-t}$)
$F_k(t,S,T)$	simple compounded annualized forward rate for $[S,T]$
$F_k^i(t)$	$F_k(t,T_{i-1},T_i)$
$f_k(t,S,T)$	continuously compounded annualized forward rate for $[S,T]$
$I(t)$	value of the consumer price index
$\mathcal{I}(t,T)$	forward consumer price index value for time T
$\mathcal{I}^i(t)$	$\mathcal{I}(t,T_i)$ or sometimes $\mathcal{I}(t,T_{2i})$
$F_I(t,S,T)$	simple compounded annualized forward inflation rate
$F_I^i(t)$	$F_I(t,T_{i-1},T_i)$ or sometimes $F_I(t,T_{2(i-1)},T_{2i})$
$Y_i(t)$	$\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}$
$\mathbb{E}[X]$	expectation of X
$\mathbb{V}[X]$	variance of X
$(\mathcal{F}_t)_{0\le t}$	a filtration
$\mathbb{E}^Q[X \vert \mathcal{F}_t]$	conditional expectation of X given $\mathcal{F}_t$ under the probability measure Q
$N(\mu,\sigma^2)$	a normal distribution with expectation μ and variance σ^2 (standard deviation σ)

1. Inflation markets

1.1. Introduction

In recent years the volumes traded in inflation-linked markets have been increasing continuously. While inflation-linked bonds have been issued for some time, inflation derivatives have become increasingly popular in the last years. The amount traded in US inflation derivatives had an enormous boost in the last 7 years as can be seen in figure 1.1.

Figure 1.1.: Trading volume in inflation-linked derivatives [30]

With the increasing volume of those markets also the complexity of the traded inflation-linked products has increased. Therefore sophisticated inflation models are necessary to price these products. This thesis gives an overview of the models proposed and used in the area of pricing inflation-linked derivatives.

1.2. Inflation and consumer price indices

Inflation is referred to as the rate of change of the average level of prices (see Burda and Wyplosz [8], p. 8). In practice already measuring average price levels is a problem by itself. A lot of literature is reviewing questions about which products to consider and how to weigh them in a basket of goods, as well as how weights and goods have to be adapted over the years, how to include the fact that certain products will always get cheaper over time, ... Regardless of the exact calculation method the final measurement is a consumer price index (CPI) picturing an average price level and through its change inflation. Although the calculation of those indices is very interesting, we shall not concern us with the details. For our purpose we consider a given CPI index that we later model as a stochastic process.

One can find several CPIs measuring a countries (or regions) inflation. Some of the most-used CPIs for inflation-linked products can be found in table 1.1. The most used CPI index in the euro area is the HICP ex tobacco, the developement of which can be found in figure 1.2.

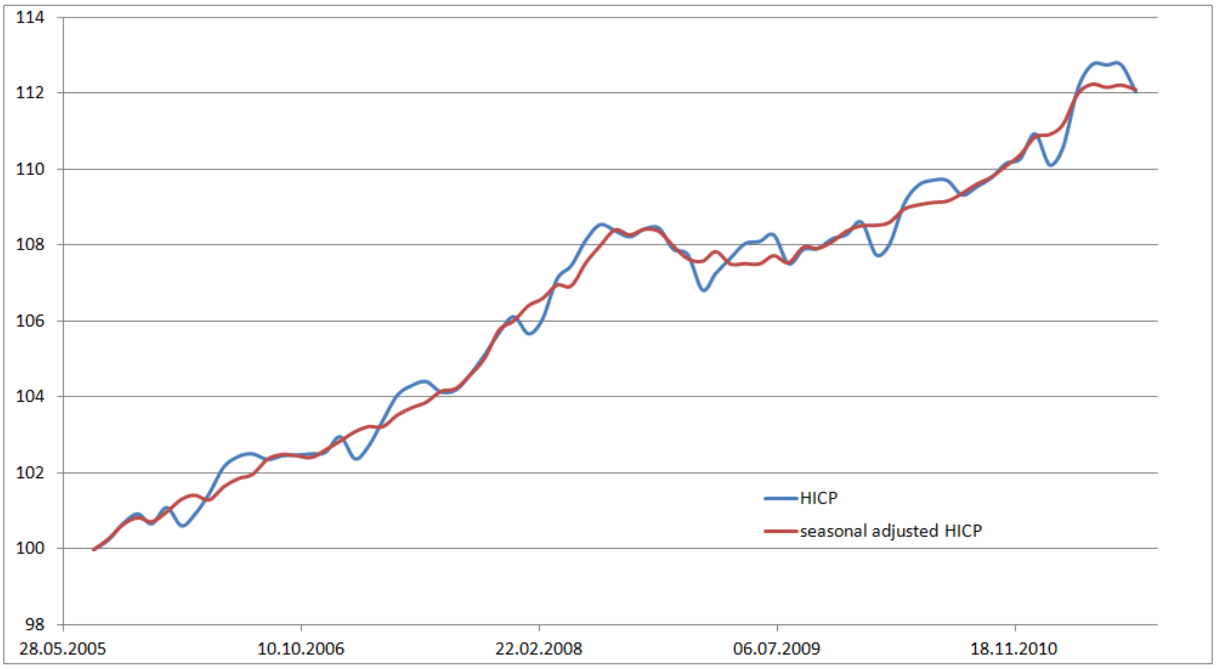


Figure 1.2.: Eurozone HICP Ex Tobacco
SOURCE: Bloomberg L.P. (2006), <CPTFEMU Index>: Jul. 05 to Jul. 11, retrieved 23 Sep. 2011

Before we introduce the most standard inflation-linked products we take a closer look at the underlying CPIs. One problem is that CPIs are only published monthly and lag behind. E.g. the march value of the euro HICP is only published in June. However, for modelling purposes we don't bother with this lag, since for cashflow calculations of derivatives it is always the lagged CPI used and therefore we simply choose to model the lagged CPI index instead of the actual one. Also because the CPI index is only published monthly, to get values for dates in between publication dates one has to interpolate CPI values. This is typically done either using the value at the last publication date before or by linear interpolation. An overview of those features for the most common indices can be found in table 1.1.

Index name	CPI for All Urban Consumers	HICP Ex Tobacco	United Kingdom retail prices index
Country	USA	EU	UK
Bloomberg Ticker	CPURNSA Index	CPTFEMU Index	UKRPI Index
Lag	3 month	3 month	2 month
Interpolation	linear	no	no

Table 1.1.: Inflation indices details

1.2.1. Seasonality

Another problem with inflation indices is seasonality. Because of certain periodic effects there are patterns in the CPI developement. They can partly be explained by effects like christmas shopping, holidays or seasons of the year. To model a CPI one has to account for this. Introducing this directly in a model is rather complicated. Therefore one estimates the seasonal effects from historical data and models the adjusted CPI (which results from subtracting the seasonal effects from the CPI). To estimate the seasonal effects one can resort to classical time series methods (see e.g. Hamilton [17]). One approach is to estimate (additive or multiplicative) effects by a linear regression out of historical data. A second approach is the use of ARIMA models as e.g. done by the United States Bureau of Labor Statistics [35]. For simplicity we stick with the first approach, simply applying linear regression on historical data. The results of such a seasonal detrending can be seen in figure 1.2.

1.3. Inflation-linked products

1.3.1. Inflation-linked bonds

Historically the first inflation-linked products were inflation-linked bonds (ILBs) issued by some governments. France and Italy were one of the first countries to issue such bonds. Also the US ILBs, called treasury inflation-protected securities (TIPS), and Great Britain and German ILBs can be found frequently in todays markets. We now describe the structure of these inflation-linked products.

As mentioned before inflation-linked products are linked to a CPI (the time t value is denoted by $I(t)$). The idea behind ILBs is not to fix the nominal amount at a certain level like it is the case with nominal fixed coupon bonds, but to link the nominal amount to the developement of a CPI index and therefore to inflation. Let us consider an ILB issued at time T_0 with maturity T. At the time of issuance the nominal amount of the ILB shall be 1. At time $t \leq T$ prices measured by the CPI index $I(t)$ will have changed. Following the idea behind ILBs the nominal should now be $\frac{I(t)}{I(T_0)}$. At maturity the paid nominal amount is then $\frac{I(T)}{I(T_0)}$. If the nominal amount is adjusted as above the bond holder got rid off inflation risk. Therefore ILBs can be viewed as a classical bond investment without inflation risk. The definition of a inflation-linked zero coupon bond is as follows.

Definition 1.1: *An inflation-linked zero coupon bond with issuance date T_0 and maturity T is a bond paying $\frac{I(T)}{I(T_0)}$ at T. We denote the price of this bond at time t by $P_{ILB}(t, T_0, T)$.*

By classical theory of arbitrage-free pricing (see section A.2) assuming a riskless nominal asset B_n with price

$$B_n(t) = \exp\left\{\int_0^t r_n(s)\,\mathrm{d}s\right\},$$

where $r_n(t)$ is the riskless nominal short rate, the price of such a zero coupon ILB is given by

$$P_{ILB}(t, T_0, T) = \mathbb{E}^Q\left[\exp\left\{-\int_t^T r_n(s)\,\mathrm{d}s\right\}\frac{I(T)}{I(T_0)}\middle|\mathcal{F}_t\right]. \tag{1.1}$$

$\mathbb{E}^Q$ denotes the expectation under a risk-neutral measure and $\mathcal{F}_t$ - the σ-algebra generated by the underlying processes up to time t.

In markets basically all ILBs issued are not zero coupon ILBs but fixed coupon ILBs. Let us consider payment dates $T_0 < T_1 \leq \cdots \leq T_N$ with typically annual (or semianual) differences. At each payment date a real fixed rate bond will pay a certain coupon rate p (e.g. 3 %) of the current nominal amount. Hence at time $T_i, 1 \leq i \leq N$ there will be a payment of

$$p \cdot \frac{I(T_i)}{I(T_0)},$$

and at maturity T_N additionally the nominal amount of $\frac{I(T_N)}{I(T_0)}$ will be payed back. In analogy to nominal fixed rate bonds a real fixed rate bond can be viewed as a combination of several zero coupon ILBs. Assuming there exist prices for fixed coupon ILBs for all maturities T one can as in the case of nominal bonds bootstrap the prices of zero coupon ILBs (see Hull [25]).

Remark: In reality the redemption of ILBs (e.g. TIPS) is floored at 1, therefore guaranteeing a payment as high as the original nominal amount. Quoted market prices of ILBs include the value of this option. However, in most markets, e.g. US and euro markets, the value of this option is neglible, since central banks basically guarantee no long-lasting deflation (negative inflation). Hence markets have been ignoring these included options which has the big advantage of allowing for simple valuation of real fixed rate bonds given a real yield curve. Rigorous treatment would require to calculate included option values when using market data, but since the value of these options is neglible, in practice this is mostly ignored. Formulas for valuation of those included options can be found in e.g. Henrard [21].

In (1.1) we see that the time t price of a zero coupon ILB depends also on the issuance date T_0 (i.e. the CPI value at that time) of the bond. We could consider a zero coupon

ILB with the same maturity T and a different issuance date $\tilde{T}_0$. For $t \geq \max\{T_0, \tilde{T}_0\}$ we have the following relationship

$$
\begin{aligned}
P_{ILB}(t, T_0, T) &= \mathbb{E}^Q \left[\exp\left\{ -\int_t^T r_n(s)\,\mathrm{d}s \right\} \frac{I(T)}{I(T_0)} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}^Q \left[\exp\left\{ -\int_t^T r_n(s)\,\mathrm{d}s \right\} \frac{I(T)}{I(\tilde{T}_0)} \frac{I(\tilde{T}_0)}{I(T_0)} \middle| \mathcal{F}_t \right] \\
&= \frac{I(\tilde{T}_0)}{I(T_0)} P_{ILB}(t, \tilde{T}_0, T).
\end{aligned}
$$

Given the history of the CPI index and the price of one zero coupon ILB we can therefore calculate zero coupon ILB prices for arbitrary issuance dates (and the same maturity). We could e.g. choose a zero coupon ILB issued at T_0, where $I(T_0) = 1$. This zero coupon ILB has a payoff of $I(T)$ at maturity T. We make the following definition.

Definition 1.2: *A real zero coupon bond with maturity T (also called real T-bond) is a bond paying $I(T)$ at time T. We denote its price by $P_{ILB}(t, T)$. Especially $P_{ILB}(T, T) = I(T)$.*

Now consider

$$
P_R(t, T) := \frac{P_{ILB}(t, T)}{I(t)} = \mathbb{E}^Q \left[\exp\left\{ -\int_t^T r_n(s)\,\mathrm{d}s \right\} \frac{I(T)}{I(t)} | \mathcal{F}_t \right] = P_{ILB}(t, t, T), \tag{1.2}
$$

which satisfies $P_R(T, T) = 1$. It can be interpreted as the price of a real zero coupon bond in terms of CPI units. By the last equation it can be also interpreted as the price of a zero coupon ILB issued at time t. One has to be careful not to interprete this as the price of a tradable asset since for different t $P_R(t, T)$ gives the price of different zero coupon ILBs (issued at different dates).

We can consider a fictional world, where the currency is CPI units. This world is often referred to as the real interest rate world. In this world $P_R(t, T)$ is the equivalent of a classical nominal zero coupon bond price (since it is paying one CPI unit at time T). We can use classical interest rate theory and consider forward rates, yields or other instruments in this fictional world. E.g. the real yield can be defined as

$$
y_R(t, T) = -\frac{1}{T - t} \ln\left(P_R(t, T)\right)
$$

and this allows us to calculate a yield curve out of given real bond prices. This interpretation has been used by practicioners and a market for real interest instruments developed. Instruments are priced using real yields and then converted to nominal prices by multiplying with the CPI value $I(t)$ (a consequence of (1.2)). The US yield curves for the nominal and real euro market of 23 Sep. 2011, bootstrapped out of nominal bonds, ILBs and the underlying CPI can be seen in figure 1.3. This allows the interpretation of nominal and real interest rate markets as a domestic and foreign market linked together with the CPI

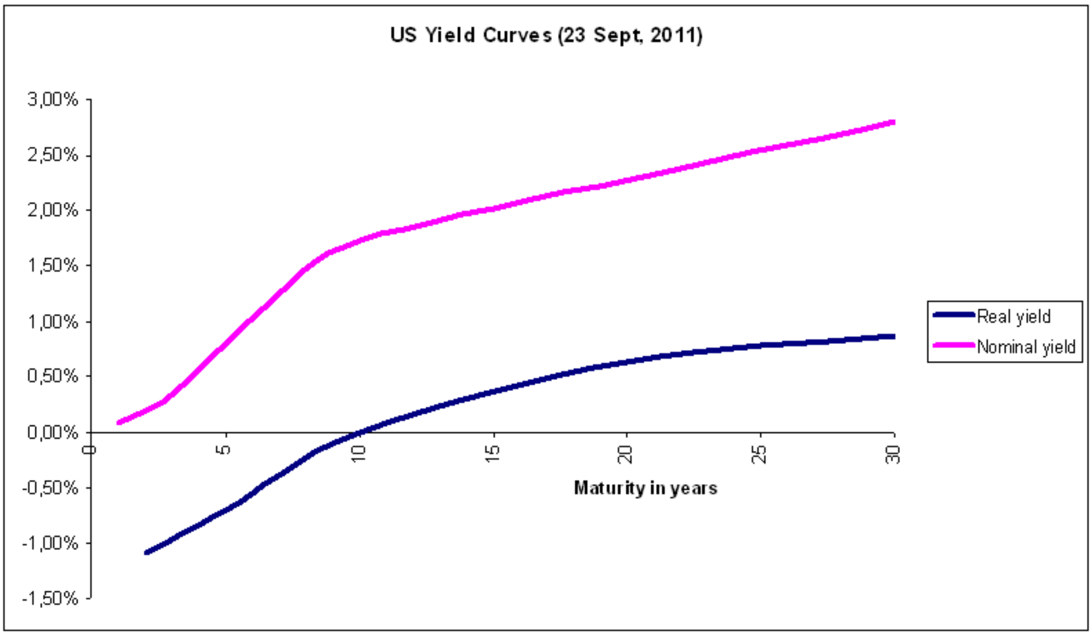


Figure 1.3.: US nominal and real yields
SOURCE: Bloomberg L.P. (2006), <USGG> and <USGGT> retrieved 23 Sep. 2011

taking the part of an exchange rate. This is called the foreign-currency analogy and we use this analogy later for developing a model allowing us to value more complex derivatives.

Remark: Note that real rates are sometimes negative. Negative nominal interest rates would theoretically allow for arbitrage (borrow money from the bank and store it under your pillow), however this is no problem with real interest rates. They are a "fictional" quantity and we are not able to trade them. We have to consider the possiblity of negative rates when we want to build an inflation model later.

1.3.2. Zero coupon inflation-indexed swaps

A second big market for inflation-linked derivatives is the inflation-indexed swap (IIS) market. There exist two different types of IIS, the zero coupon IIS (ZCIIS) and the year-on-year IIS (YYIIS).

Definition 1.3: *A ZCIIS with start date T_0 and maturity T is a contract where at time T the buyer receives the performance of the underlying CPI index*

$$\frac{I(T)}{I(T_0)} - 1$$

in exchange for paying a fixed amount C at time T.

The value of such a contract using arbitrage-free pricing theory is

$$\mathbb{E}^Q\left[\exp\left\{-\int_t^T r_n(s)\,\mathrm{d}s\right\}\left(\frac{I(T)}{I(T_0)}-1-C\right)\bigg|\mathcal{F}_t\right] = P_{ILB}(t,T_0,T)-(1+C)P_n(t,T),$$

where $P_n(t,T)$ denotes the time t price of a T-zero coupon bond (a bond paying 1 nominal unit at time T).

The market quotes not prices but the rates rendering a ZCIIS contracts with $T_0 = t$ and maturity T zero. Rates are usually available for several full year maturities from today ($T = T_0 + M, M \in \mathbb{N}$).

Definition 1.4: *The ZCIIS rate is the constant $K(t,T)$ solving the equation*

$$P_R(t,T) = P_{ILB}(t,t,T) = (1+K(t,T))^{T-t}P_n(t,T). \tag{1.3}$$

Note that this equation results from setting $C = (1+K(t,T))^{T-t}-1$ which lets us interprete $K(t,T)$ as an average annual inflation rate. The current values for US ZCIIS rates are plotted in figure 1.4.

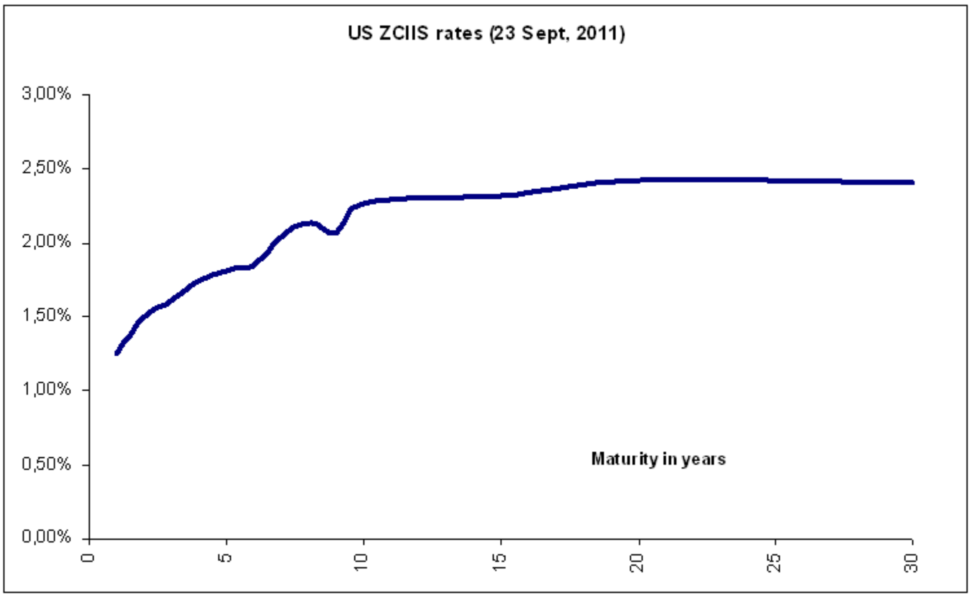


Figure 1.4.: US ZCIIS swap rates
Source: Bloomberg L.P. (2006), <USSWIT>, retrieved 23 Sep. 2011

1.3.3. Forward CPIs

Definition 1.5: *A forward CPI contract is the agreement to exchange at time T the (at time t unknown) CPI value $I(T)$ against a (at time t) fixed amount K.*

The value of such a contract is

$$\mathbb{E}^Q \left[\exp\left\{ -\int_t^T r_n(s)\,\mathrm{d}s \right\} (I(T)-K) \Big| \mathcal{F}_t \right] = I(t)P_{ILB}(t,T) - KP_n(t,T).$$

The rate K rendering such a contract zero is called the forward CPI.

Definition 1.6: *The time t forward CPI for maturity T is*

$$\mathcal{I}(t,T) := I(t)\frac{P_R(t,T)}{P_n(t,T)} = \frac{P_{ILB}(t,T)}{P_n(t,T)}. \tag{1.4}$$

Remark: Note that $\mathcal{I}(t,T)$ can be directly calculated out of the prices of nominal bonds and zero coupon ILBs. We will see that this is not possible for YYIIS rates. We also note the close correspondence between ZCIIS rates and forward CPIs. We have

$$\mathcal{I}(t,T) = I(t)(1+K(t,T))^{T-t}.$$

We now use the change of numeraire technique which is reviewed in section A.2. Under the measure Q_n^T induced by the numeraire $P_n(t,T)$ we have

$$P_{ILB}(t,T) = \mathbb{E}^Q \left[\exp\left\{ -\int_t^T r_n(s)\,\mathrm{d}s \right\} I(T) \Big| \mathcal{F}_t \right] = P_n(t,T)\mathbb{E}^{Q_n^T}\left[I(T) | \mathcal{F}_t \right].$$

This together with (1.4) proves

$$\mathcal{I}(t,T) = \mathbb{E}^{Q_n^T}\left[I(T) | \mathcal{F}_t \right]$$

and implies that $\mathcal{I}(t,T)$ is a Q_n^T-martingale. A very convenient approach to model inflation is to model CPI forwards under this T-forward measure Q_n^T. This leads to forward CPI market models.

1.3.4. Differences between ZCIIS implied and bond implied real rates

Looking at (1.3) one notices that if the nominal zero coupon prices are known we are able to calculate ZCIIS rates out of zero coupon ILB prices and vice versa. However in todays markets this is not so. The ZCIIS rates calculated out of (bootstrapped) prices of zero coupon ILBs don't necessarily coincide with market quoted ZCIIS rates. An example of the resulting differences can be seen in figure 1.5.

Considering only interest and inflation risk this is a clear arbitrage-opportunity. There exist several reasons causing those differences (as discussed in Fleckenstein et al. [12]). Some of them are

- Differences in liquidity for the underlying financial instruments,
- ILBs contain higher credit risk,

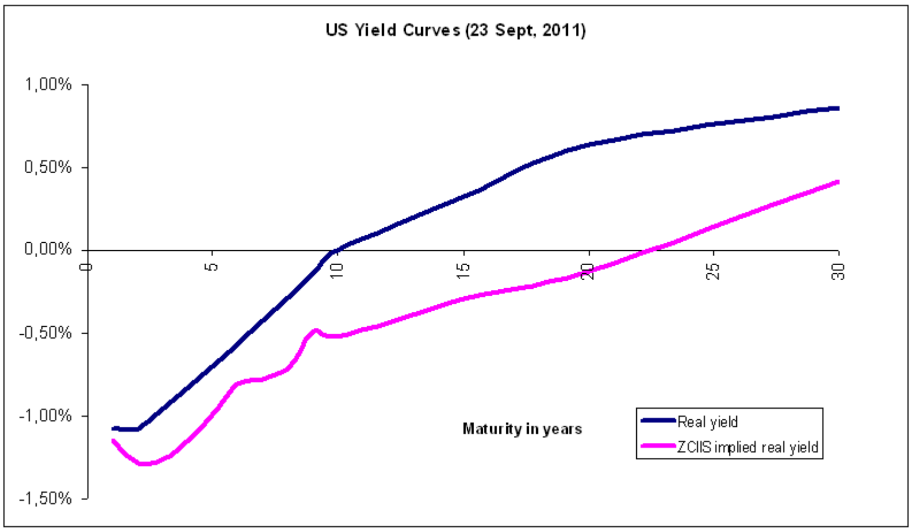


Figure 1.5.: Differences between real yield implied by TIPS and ZCIIS implied real yield

SOURCE: Bloomberg L.P. (2006), calculated from < USGG > and <USGGT>, <USSWIT> rates, retrieved 23 Sep. 2011

- Systemic risk,
- Asset purchases by central banks.

For calculation one has to choose which rates to use. In practice the market-quoted ZCIIS rates seem the better choice, since they possess less credit risk and prices don't include unaccounted options.

1.3.5. Inflation forwards

The annualized inflation forward rate $F_I(t, S, T)$ for a time interval $[S, T]$ is the rate C for which the time t value of a contract which exchanges the annualized CPI increase

$$\frac{1}{T-S}\left(\frac{I(T)}{I(S)}-1\right)$$

against C is zero. $F_I(t, S, T)$ is therefore determined by solving the following equation w.r.t. C:

$$\begin{aligned} 0 &= \mathbb{E}^Q\left[\exp\left\{-\int_t^T r_n(s)\,\mathrm{d}s\right\}\left(\frac{1}{T-S}\left(\frac{I(T)}{I(S)}-1\right)-C\right)\middle|\,\mathcal{F}_t\right], \\ &= P_n(t,T)\,\mathbb{E}^{Q_n^T}\left[\frac{1}{T-S}\left(\frac{I(T)}{I(S)}-1\right)\middle|\,\mathcal{F}_t\right]-CP_n(t,T). \end{aligned}$$

The second line is again the change of numeraire technique (see section A.2). We use this frequently throughout this work, most of the time not mentioning this explicitely.

Definition 1.7: *The annualized inflation forward rate for time* $[S,T]$ *is*

$$F_I(t,S,T) := \mathbb{E}^{Q_n^T}\left[\frac{1}{T-S}\left(\frac{I(T)}{I(S)}-1\right)\middle|\,\mathcal{F}_t\right] \quad 0 \le t \le T. \tag{1.5}$$

We cannot express $F_I(t,S,T)$ in terms of bond prices, therefore given only bonds and a riskless account as tradable assets one has to use a model to be able to calculate forward inflation rates.

Remark: After time S an inflation forward can actually be expressed by a CPI forward since then $I(S)$ is known and

$$\begin{aligned} F_I(t,S,T) &= \frac{1}{T-S}\left(\frac{1}{I(S)}\mathbb{E}^{Q_n^T}\left[\mathcal{I}(T)|\,\mathcal{F}_t\right]-1\right) \\ &= \frac{1}{T-S}\left(\frac{\mathcal{I}(t,T)}{I(S)}-1\right). \end{aligned}$$

Remark: Leung and Wu [29] propose a different definition for $F_I(\cdot,S,T)$, namely

$$\widetilde{F}_I(t,S,T) = \frac{1}{T-S}\left(\frac{\mathcal{I}(t,T)}{\mathcal{I}(t,S)}-1\right),$$

arguing that this rate is arbitrage-free for the annualized CPI increase. However their argument seems unfounded. To prove that this rate is arbitrage-free they use a non-adapted replication strategy. In the appendix they argue to buy (using the notation of this work) $\frac{I(S)}{I(t)}$ real forward contracts at time t which isn't possible since the amount $\frac{I(S)}{I(t)}$ isn't known at time t.

Although normally not directly traded in markets, inflation forwards are an important mathematical instrument. We will see that YYIIS are in fact a combination of several inflation forwards. The situation is similar to caps and caplets. While caps are traded in markets, caplets are the instruments one would like to consider in a mathematical context. We will also see that like with caps and caplets there is a simple bootstrapping procedure to calculate inflation forward rates out of YYIIS rates.

1.3.6. Year-on-year inflation-indexed swaps

The second type of IIS is the YYIIS. Contrary to ZCIIS the YYIIS is designed to exchange annual inflation. We consider a tenor structure $\mathbb{T} = \{T_0 < T_1 < \cdots < T_N\}$. At each payment date T_i the buyer receives

$$\frac{I(T_i)}{I(T_{i-1})} - 1$$

and pays the fixed amount $(T_i - T_{i-1})\,\mathcal{K}$. At time $t \le T_1$ the fair price of an individual payment of such a contract is

$$\begin{aligned}
&\mathbb{E}^Q\left[\exp\left\{-\int_t^{T_i} r(s)\,\mathrm{d}s\right\}\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - (T_i - T_{i-1})\,\mathcal{K}\right)\middle|\,\mathcal{F}_t\right] \\
&= P_n(t,T_i)\,\mathbb{E}^{Q_n^{T_i}}\left[\frac{I(T_i)}{I(T_{i-1})} - 1 - (T_i - T_{i-1})\,\mathcal{K}\,\middle|\,\mathcal{F}_t\right] \\
&= P_n(t,T_i)(T_i - T_{i-1})\,(F_I(t,T_{i-1},T_i) - \mathcal{K})\,,
\end{aligned}$$

and the fair price of the entire contract is

$$\sum_{i=1}^{N} P_n(t,T_i)(T_i - T_{i-1})(F_I(t,T_{i-1},T_i) - \mathcal{K}).$$

The YYIIS rate is defined as the rate $\mathcal{K}$ for which the above price is zero, i.e.

$$\mathcal{K} = \mathcal{K}(t,\mathbb{T}) = \frac{\sum_{i=1}^{N} P_n(t,T_i)(T_i - T_{i-1})F_I(t,T_{i-1},T_i)}{\sum_{i=1}^{N}(T_i - T_{i-1})P_n(t,T_i)}. \tag{1.6}$$

In YYIIS markets such rates are quoted. For a tenor structure $\mathbb{T} = \{T_0 < T_1 < \cdots < T_N\}$, where the T_i are annually spaced, we have rates $\mathcal{K}(t,T_i) := \mathcal{K}(t,\{T_0,\ldots,T_i\}), i \in \{1,\ldots,N\}$. As mentioned before one needs model assumptions to calculate $\mathcal{K}(t,T_i)$. We suppose most brokers use the later introduced Jarrow-Yildirim model. The situation changes if inflation caplets and floorlets are liquidly traded options. Then the call/put parity (see 1.3.8) can be used to calculate $\mathcal{K}(t,T_i)$ from market data. ZCIIS and YYIIS rates are normally quite similar. The shape of the current US YYIIS curve is therefore almost the same as the one in figure 1.4.

Bootstrapping

In interest rate markets one often wants to bootstrap caplet volatilities out of quoted cap implied volatilities because they include information over disjoint intervals. Similar to this we are interested in stripping annual inflation forward rates out of YYIIS rates. Let $F_I^i(t) := F_I(t,T_{i-1},T_i)$ be the inflation forward rate for the period $[T_{i-1},T_i]$. The procedure to bootstrap those rates out of annual YYIIS rates is straightforward. Consider two YYIIS rates with maturity T_i and T_{i-1}. Since the value of both contracts is zero by definition of the YYIIS rate we can subtract their contract values, so that all floating payments except the additional payment of the longer contract cancel out. The value of the last floating payment is $P_n(t,T_i)(F_I^i(t) - \mathcal{K}(t,T_i))$ and including the fixed payments we have

$$P_n(t,T_i)(F_I^i(t) - \mathcal{K}(t,T_i)) + \sum_{j=1}^{i-1} P_n(t,T_j)(\mathcal{K}(t,T_{i-1}) - \mathcal{K}(t,T_i)) = 0.$$

Hence

$$F_I^i(t) = \mathcal{K}(t, T_i) + \sum_{j=1}^{i-1} \frac{P_n(t, T_j)}{P_n(t, T_i)} \left(\mathcal{K}(t, T_i) - \mathcal{K}(t, T_{i-1})\right). \tag{1.7}$$

If nominal zero coupon bond prices are known we can bootstrap annual inflation forward rates out of market given YYIIS rates. The results of this procedure for euro YYIIS can be found in figure 1.6.

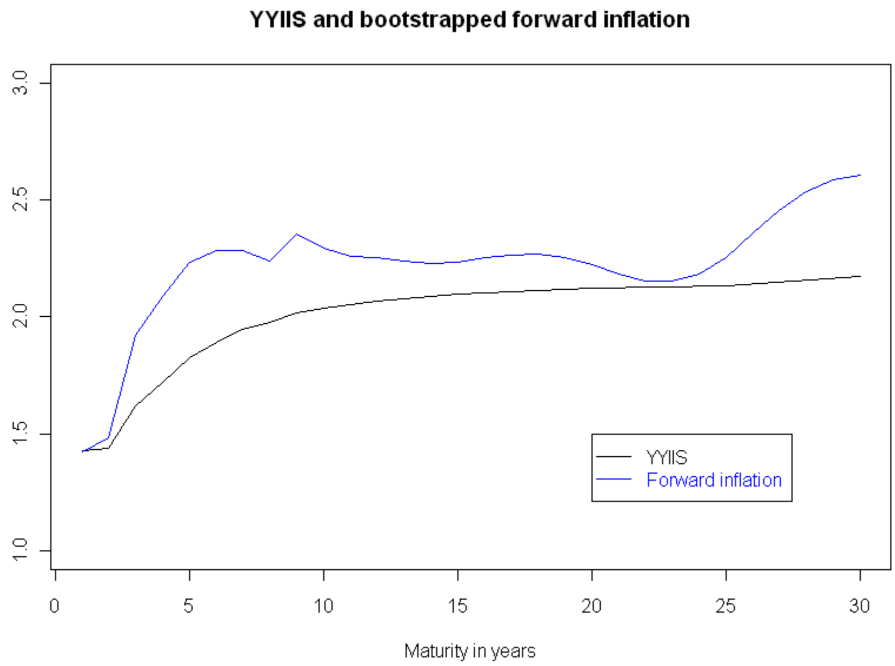


Figure 1.6.: Euro YYIIS rates and bootstrapped forward inflation rates
Source: Bloomberg L.P. (2006), <SWIL>, retrieved 18 Aug. 2011

We will get back to the valuation of YYIIS contracts later in chapter 2.

1.3.7. Inflation caps and floors

Inflation caps and floors are calls and puts on inflation rates. Similar to interest rate markets caps and floors consist of several caplets or floorlets. We denote inflation-linked caplets as IC and inflation-linked floorlets as IF. The payoff at time T (which is when the actual inflation rate $F_I(T, S, T) = \frac{1}{T-S}\left(\frac{I(T)}{I(S)} - 1\right)$ for the period is known) of a contract with strike κ is

$$(T - S)(F_I(T, S, T) - \kappa)_+$$

for a caplet and

$$(T - S)(\kappa - F_I(T, S, T))_+$$

for a floorlet. The fair price at time $t \leq T$ of a caplet $IC(t, S, T, \kappa)$

$$
\begin{aligned}
IC(t, S, T, \kappa) &= \mathbb{E}^{Q} \left[\exp\left\{ -\int_t^T r_n(s)\, \mathrm{d}s \right\} (T-S)(F_I(T,S,T) - \kappa)_+ \middle| \mathcal{F}_t \right] \\
&= (T-S) P_n(t,T)\, \mathbb{E}^{Q_n^T} \left[(F_I(T,S,T) - \kappa)_+ \middle| \mathcal{F}_t \right],
\end{aligned}
$$

and the fair price for a floorlet $IF(t, S, T, \kappa)$ is

$$
IF(t, S, T, \kappa) = (T-S) P_n(t,T)\, \mathbb{E}^{Q_n^T} \left[(\kappa - F_I(T,S,T))_+ \middle| \mathcal{F}_t \right].
$$

Prices of caps and floors then follow as sum of the current values of the individual caplets and floorlets. Cap and floor contracts almost always have a fixed tenor for the underlying inflation rates, quite often a year and consist of caplets and floorlets with the same strike κ.

1.3.8. The put/call parity in inflation markets

Consider the payoffs of an inflation caplet and floorlet for the time $[S, T]$ with the same strike κ. We have

$$
(T-S)(F_I(T,S,T) - \kappa)_+ - (T-S)(\kappa - F_I(T,S,T))_+ = (T-S)(F_I(T,S,T) - \kappa).
$$

Multiplying this by $P_n(t,T)$ and taking the expectation value under Q_n^T given $\mathcal{F}_t$ results in

$$
\begin{aligned}
IC(t, S, T, \kappa) - IF(t, S, T, \kappa) &= (T-S) P_n(t,T)\, \mathbb{E}^{Q_n^T} \left[(F_I(T,S,T) - \kappa) \right] \\
&= (T-S) P_n(t,T) \left(F_I(t,S,T) - \kappa \right).
\end{aligned} \tag{1.8}
$$

This put/call-parity for inflation markets is somewhat different from the classical put/call parity, since we don't really know the quantity $F_I(t, S, T)$. Given call and put prices for the same strikes we are able to calculate $F_I(t, S, T)$, which is in fact one way to determine $F_I(t, S, T)$ out of market data.

1.3.9. CPI caps and floors

CPI caps and floors are calls and puts on the CPI. A CPI cap/floor with strike K and maturity T guarantees a time T payoff of

$$
(\omega(I(T) - K))_+,
$$

where $\omega = 1$ for a cap and $\omega = -1$ for a floor. Therefore its value is

$$
P_n(t,T) \mathbb{E}^{Q_n^T} \left[(\omega(I(T) - K))_+ \middle| \mathcal{F}_t \right] = P_n(t,T) \mathbb{E}^{Q_n^T} \left[(\omega(\mathcal{I}(T,T) - K))_+ \middle| \mathcal{F}_t \right].
$$

If one assumes lognormal dynamics (see appendix B) for the forward CPI $\mathcal{I}(\cdot, T)$ we get a Black-like formula for the value of this cap/floor. This somewhat motivates the developement of a forward CPI market model (see chapter 2).

1.4. Break-even rates, expected inflation and inflation risk premia

Two very important concepts in inflation markets are the break-even rate and the inflation risk premia. The break-even rate is the difference between nominal and real yields. Quite often it is referred to as the expected inflation, motivated by the Fisher equation (Fisher [11]), which states inflation has to be the difference between nominal and real interest. This is not to be understood as a exact equation but as an economic principal and in reality this is a somewhat simplified description, since the break-even rate consists of more than just expected inflation. First of all the difference in bond prices is not just an inflation component. The difference between nominal and real yields is influenced by

- different liquidity premiums,
- deflation protection premiums for ILBs,
- different risk premiums for interest risk (since there are differences in duration for similar real and nominal bonds).

Removing all such effects we would observe a quantity which we call the inflation premia. Still this cannot be interpreted as expected inflation since it still includes an inflation risk premium an investor would expect for the risk of inflation. Thus we can summarize this in a diagram as seen in figure 1.7.

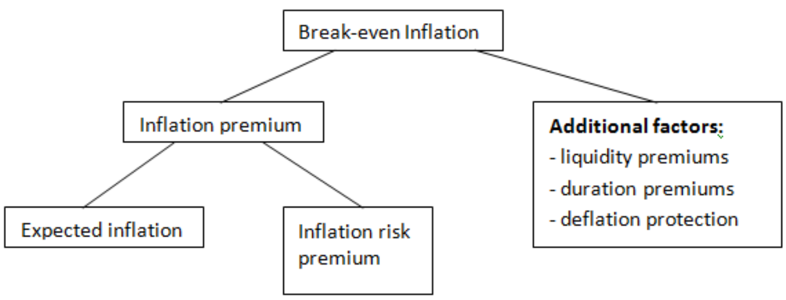


Figure 1.7.: Break-even rate, expected inflation and inflation risk premium

The question of how to estimate the above quantities has also been extensively treated in financial research, especially by central banks. Recently their have been attempts to use financial market data to estimate those quantities (see e.g. Hördahl and Tristani [23] or Garcia and Werner [13]).

2. Model building

2.1. The Heath-Jarrow-Morton model

A general framework for interest rate models using diffusion processes is the Heath-Jarrow-Morton (HJM) model as proposed in Heath et al. [20]. In this model the instantaneous forward rates $f_n(t,T)$ satisfy the stochastic differential equations (SDEs)

$$\mathrm{d}f_n(t,T) = \alpha_n(t,T)\,\mathrm{d}t + \sigma_n(t,T)^T\,\mathrm{d}B_n(t)\,, \quad 0 \le t \le T \le T^*, \tag{2.1}$$

where B_n is a possibly multidimensional Brownian motion. The index n stands for nominal and is introduced here to provide consistent notation throughout this work. The starting values $f_n(0,T)$ shall be given.

Remark: Given $f_n(t,T), t \le T \le T^*$ the zero coupon bond prices $P_n(t,T), t \le T \le T^*$ are

$$P_n(t,T) = \exp\left\{-\int_t^T f_n(t,u)\,\mathrm{d}u\right\}.$$

This implies

$$f_n(t,T) = -\frac{\frac{\partial}{\partial T}P_n(t,T)}{P_n(t,T)} = -\frac{\partial}{\partial T}\ln\left(P_n(t,T)\right).$$

Hence we can calculate $f_n(0,T)$ from given bond prices $P_n(0,T), 0 \le T \le T^*$ as

$$f_n(0,T) = -\frac{\partial}{\partial T}\ln\left(P_n(0,T)\right).$$

The dynamics of the forward rates define the evolving term structure of interests. For fixed T such a model is arbitrage-free (under mild technical assumptions). However, if T is allowed to vary the famous HJM drift condition has to be satisfied in order to produce consistent (arbitrage-free) prices for bonds with different maturities. Define

$$\alpha_n^*(t,T) := \int_t^T \alpha_n(t,u)\,\mathrm{d}u \qquad \text{and} \qquad \Sigma_n(t,T) := \int_t^T \sigma_n(t,u)\,\mathrm{d}u\,, \tag{2.2}$$

then the HJM drift condition is satisfied if the Girsanov kernel λ defining the equivalent martingale measure (EMM) change, given by

$$\alpha_n^*(t,T) = \frac{1}{2}\left\|\Sigma_n(t,T)\right\|^2 + \Sigma_n(t,T)^T\lambda(t), \tag{2.3}$$

is independent of T. Technical details are omitted here, since the derivation of the Jarrow Yildirim (JY) model in section 2.2 includes the results from the classical HJM model.

The dynamics of a nominal zero coupon bond $P_n(\cdot, T)$ in the HJM model is

$$\mathrm{d}P_n(t,T) = r_n(t)\,\mathrm{d}t + \Sigma_n(t,T)^T\,\mathrm{d}W_n(t)\,, \quad 0 \le t \le T,$$

where $r_n(t) = f_n(t,t)$ is the nominal short rate and W_n is a Brownian motion w.r.t. the EMM Q. The discrete forward rate $F_n(\cdot, S, T)$ for an interval $[S,T]$ defined by

$$F_n(t,S,T) = \frac{1}{T-S}\left(\frac{P_n(t,S)}{P_n(t,T)} - 1\right), \quad 0 \le t \le S$$

then follows the stochastic dynamics

$$\mathrm{d}F_n(t,S,T) = \frac{1}{T-S}\,(1 + (T-S)F_n(t,S,T))\,(\Sigma_n(t,T) - \Sigma_n(t,S))^T(\,\mathrm{d}W(t) + \Sigma_n(t,T)\,\mathrm{d}t\,).$$

By the change of numeraire technique (section A.2) and Girsanov's theorem we can rewrite this as

$$\mathrm{d}F_n(t,S,T) = \frac{1}{T-S}\,(1 + (T-S)F_n(t,S,T))\,(\Sigma_n(t,T) - \Sigma_n(t,S))^T\,\mathrm{d}W_n^T(t)\,, \tag{2.4}$$

where W_n^T is a Brownian motion under Q_n^T. Therefore $(F_n(t,S,T))_{0 \le t \le S}$ is a (local) martingale under Q_n^T. Modelling such forward rates as lognormally distributed random variables is the idea behind the LIBOR market model.

Remark: We define $F_n(t,S,T) = F_n(S,S,T)$ for $t \ge S$. We will encounter several processes X_t which are defined only up to some time S. If not stated otherwise we extend their definition by setting $X_t = X_S$ for $t > S$.

2.1.1. LIBOR market model (LMM)

In LMMs one chooses to model the discrete LIBOR forward rates $F_n^k(t) := F_n(t, T_{k-1}, T_k)$, $1 \le k \le N$ for a time grid $0 = T_0 < T_1, \cdots < T_N$, with the points usually 3, 6 or 12 month (referred to as the Tenor) apart. The idea is to model these forward rates as analytically easy tracable geometric Brownian motions, meaning that F_n^k satisfies the SDE

$$\mathrm{d}F_n^k(t) = F_n^k(t)\;\sigma_n^k(t)^T\,\mathrm{d}Z_{n,k}^k(t)\,, \quad 0 \le t \le T_{k-1},$$

where σ_n^k are positive deterministic functions and $Z_{n,k}^k$ are possibly correlated multidimensional Brownian motions under the T_k forward measures. The starting values (discrete forward rates) for these SDEs are obtained out of market data, e.g. swap rates and LIBOR rates (see e.g. Hull [25] p. 84).

Remark: The restriction that σ_n^k is positive is one of convenience. In fact if we don't assume σ_n^k positive, we can always set $\tilde{\sigma}_n^k(t) = |\sigma_n^k(t)|$ (this is to be understood as an equation in each component of the vector). If we further use that we can express the correlated Brownian motion $Z^k_{n,k} = A^k(t)\tilde{Z}^k_{n,k}(t)$ where $\tilde{Z}^k_{n,k}(t)$ is an uncorrelated Brownian motion, we have that $F^k_n(t)$ is given by

$$\mathrm{d}F^k_n(t) \,= F^k_n(t)\left(\tilde{\sigma}^k_n(t)\cdot \mathrm{sign}(\sigma^k_n(t))\right)^T A^k(t)\,\mathrm{d}\tilde{Z}^k_{n,k}(t)\,,$$

where the $\cdot$ denotes a pointwise product. We see that we can include the sign of the volatility function into the correlation structure of the underlying Brownian motion. Since we have assumed arbitrary correlations the class of possible models is still the same for σ^k_n being positive. The same holds for other market models introduced later and we always choose involved volatility functions of market models to be positive.

One can interprete a LMM as a special case of a HJM model by comparing the LMM forward rate dynamics with the general HJM setting, where F^k_n follows the SDE defined in (2.4). Setting $\delta^k = T_k - T_{k-1}$ and choosing $\Sigma_n(\cdot, T_k)$ such that

$$\frac{1}{\delta^k}\left(1+\delta^k F^k_n(t)\right)\left(\Sigma_n(t,T_k)-\Sigma_n(t,T_{k-1})\right)^T \mathrm{d}W^{T_k}_n(t) \,\stackrel{d}{=}\, F^k_n(t)\sigma^k_n(t)^T\,\mathrm{d}Z^k_{n,k}\,,$$

we can explicitely state such an HJM model.

Remark: Note that $W^{T_k}_n$ and $Z^k_{n,k}$ can be different Brownian motions with regards to dimension and correlation (although the dimension of $W^{T_k}_n$ in a comparable HJM model must be greater or equal than that of $Z^k_{n,k}$). Quite often one chooses $Z^k_{n,k}$ to be N-dimensional and σ^k_n to be 0 in all components other than k. This corresponds to choosing the Brownian motions $Z^k_{n,k}$ to be one-dimensional, which we will do in later parts of this work.

Dynamics under the forward spot measure

We have defined the dynamics of each forward rate using Brownian motions under the appropiate forward measure. However, for simulation purposes we need to have dynamics using Brownian motions under one common measure. One could fix a a forward measure and simulate under this measure, but it turns out that it is more convenient to use the so-called forward spot measure.

The forward spot measure is basically the discrete equivalent of the risk-neutral measure Q. Since the money account (the numeraire inducing Q) is continuous, while simulated rates are discrete, simulating under Q would produce a lot of difficulties. Therefore one chooses to simulate under the forward measure Q^d_n (the d stands for discrete) induced by the numeraire B^d_n defined by

$$B^d_n(t) = \prod_{j=1}^{\beta(t)-1}(1+\delta^j F^j_n(T_{j-1}))P_n(t,T_{\beta(t)-1}), \quad 0\le t\le T_N, \tag{2.5}$$

where $\beta(t) = \inf\{j \in \{1, \dots, N+1\} : t \leq T_{j-1}\}$ is the first forward rate that isn't fixed yet ($F_n^i(t)$ is fixed at time T_{i-1}). This means $\beta(0) = 1$, $\beta(t) = i+1$ for $T_{i-1} < t \leq T_i$. B_n^d is the price of a self-financing trading strategy where one invests 1 unit at time 0 with the fixed forward rate $F_n^1(0)$. At time $T_{i-1}, i = 2, \dots, N$ this is then reinvested using the now fixed forward rate $F_n^i(T_{i-1})$. Therefore for $T_{i-1} < t < T_i$ the money is invested until time T_i, when we will receive the amount $\prod_{j=1}^{i}(1 + \delta^j F_n^j(T_{j-1}))$. Discounting this with $P_n(t, T_i)$ we know its time t value and we see that the above equation is indeed the value of such a trading strategy. As will be shown later in the JY setting the dynamics of F_n^k with $Z_{n,k}^d$ a Q_n^d-Brownian motion are

$$\mathrm{d}F_n^k(t) = \sigma_n^k(t)^T F_n^k(t) \left(\sum_{j=\beta(t)}^{k} \frac{\delta^j \rho_n^{j,k}(t) \sigma_n^j(t) F_n^j(t)}{1 + \delta^j F_n^j(t)} \,\mathrm{d}t + \mathrm{d}Z_{n,k}^d(t) \right),$$

where $\rho_n^{j,k}$ denotes the instantaneous correlation between F_n^j and F_n^k.

Linking interest forward rates of longer tenor

We later see, that e.g. due to market instrument specifications it is necessary to link forward rates of different tenors. Consider two forward rates $F_n(t, T_{i-1}, T_i)$ and $F_n(t, T_i, T_{i+1})$ driven by one-dimensional Brownian motions $Z_{n,i}^i$ and $Z_{n,i+1}^{i+1}$. We know

$$1 + (T_{i+1} - T_{i-1}) F_n(t, T_{i-1}, T_{i+1}) = (1 + \delta^i F_n^i(t))(1 + \delta^{i+1} F_n^{i+1}(t))$$

and by Ito's lemma we get

$$\begin{aligned} \mathrm{d}F_n(t, T_{i-1}, T_{i+1}) = & \left(\frac{\delta^i F_n^i(t)}{T_{i+1} - T_{i-1}} + \frac{\delta^i \delta^{i+1} F_n^i(t) F_n^{i+1}(t)}{T_{i+1} - T_{i-1}} \right) \sigma_n^i(t) \,\mathrm{d}Z_{n,i}^i(t) \\ & + \left(\frac{\delta^{i+1} F_n^{i+1}(t)}{T_{i+1} - T_{i-1}} + \frac{\delta^i \delta^{i+1} F_n^i(t) F_n^{i+1}(t)}{T_{i+1} - T_{i-1}} \right) \sigma_n^{i+1}(t) \,\mathrm{d}Z_{n,i+1}^{i+1}(t) + \{\dots\} \,\mathrm{d}t\,. \end{aligned} \tag{2.6}$$

Denoting by $\sigma(t)$ the volatility of $F_n(t, T_{i-1}, T_{i+1})$, setting

$$\begin{aligned} u_1(t) &= \frac{1}{F_n(t, T_{i-1}, T_{i+1})} \left(\frac{\delta^i F_n^i(t)}{T_{i+1} - T_{i-1}} + \frac{\delta^i \delta^{i+1} F_n^i(t) F_n^{i+1}(t)}{T_{i+1} - T_{i-1}} \right), \\ u_2(t) &= \frac{1}{F_n(t, T_{i-1}, T_{i+1})} \left(\frac{\delta^{i+1} F_n^{i+1}(t)}{T_{i+1} - T_{i-1}} + \frac{\delta^i \delta^{i+1} F_n^i(t) F_n^{i+1}(t)}{T_{i+1} - T_{i-1}} \right), \end{aligned}$$

and taking quadratic variations in (2.6) we get

$$\sigma(t)^2 = u_1(t)^2 \sigma_n^i(t)^2 + u_2(t)^2 \sigma_n^{i+1}(t)^2 + 2\rho_n^{i,i+1}(t) u_1(t) \sigma_n^i(t) u_2(t) \sigma_n^{i+1}(t). \tag{2.7}$$

As opposed to σ_n^i, σ is usually not deterministic, since the weights u_1, u_2 are stochastic.

2.2. The Jarrow-Yildirim model (JY model)

The JY model was introduced in 1997 (Jarrow and Yildirim [26]). The general idea behind this model is based on a foreign-currency analogy, meaning that one models two "different" economies - the real interest and the nominal interest world - and links them using an exchange rate - in this case the CPI. This was motivated in section 1.3.1.

Jarrow and Yildirim propose two HJM models for real and nominal interest rates and assume that the CPI follows a geometrical Brownian motion. Hence the underlyings are assumed to have the following dynamics:

$$\mathrm{d}f_n(t,T) = \alpha_n(t,T)\,\mathrm{d}t + \sigma_n(t,T)^T\,\mathrm{d}B_n(t)\,, \quad 0 \le t \le T \le T^* \tag{2.8}$$

$$\mathrm{d}f_R(t,T) = \alpha_R(t,T)\,\mathrm{d}t + \sigma_R(t,T)^T\,\mathrm{d}B_R(t)\,, \quad 0 \le t \le T \le T^* \tag{2.9}$$

$$\mathrm{d}I(t) = I(t)\left(\mu_I(t)\,\mathrm{d}t + \sigma_I(t)^T\,\mathrm{d}B_I(t)\right), \quad 0 \le t \le T^* \tag{2.10}$$

where $f_n(t,T)$ and $f_R(t,T)$ denote the instantaneous nominal and real forward rates. While Jarrow and Yildirim use one-dimensional Brownian motions, we use general d-dimensional Brownian motions. So $B = (B_n, B_R, B_I)$ is a correlated Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$ generating the completed filtration $(\mathcal{F}_t)_{0\le t\le T^*}$ (which then satisfies the usual conditions allowing for the application of theorems out of stochastic analysis, see Protter [37]). B_n, B_R, B_I are Brownian motions of dimensions d_n, d_R, d_I with instantaneous correlations $\rho_{n,R} \in \mathbb{R}^{d_n\times d_R}$, $\rho_{n,I} \in \mathbb{R}^{d_n\times d_I}$, $\rho_{R,I} \in \mathbb{R}^{d_R\times d_I}$. The instantaneous correlation of the whole process B shall be denoted by ρ. Starting values are deterministic and given.

Assumption 1: *For the stochastic processes to be meaningfully defined we require for* $0 \le T \le T^*$*:*

- $\alpha_n(t,T), \alpha_R(t,T), \mu_I(t)$ *are* $\mathcal{F}_t$*-adapted one-dimensional measurable processes,*
- $\int_0^T |\alpha_n(t,T)| + |\alpha_R(t,T)| + |\mu_I(t)|\,\mathrm{d}t < \infty$ *P-a.s.,*
- $\sigma_n(t,T), \sigma_R(t,T), \sigma_I(t)$ *are* $\mathcal{F}_t$*-adapted measureable processes with dimensions* d_n*,* d_R*,* d_I*,*
- $\int_0^T \|\sigma_n(t,T)\|^2 + \|\sigma_R(t,T)\|^2 + \|\sigma_I(t)\|^2\,\mathrm{d}t < \infty$ *P-a.s..*

Let $k \in \{n, R\}$. The zero coupon bond prices are given by

$$P_k(t,T) = \exp\left\{-\int_t^T f_k(t,u)\,\mathrm{d}u\right\} \tag{2.11}$$

and the spotrates by $r_k(t) = f_k(t,t)$. The money accounts are defined by

$$S_k(t) = \exp\left\{\int_0^t r_k(s)\,\mathrm{d}s\right\}.$$

Assumption 2: *For the above instruments to be meaningfully defined (bounded and greater 0) we must have further regularity assumptions.*

- $\int_0^{T^*} |f_n(0,t)| + |f_R(0,t)| \,\mathrm{d}t < \infty$,
- $\int_0^{T^*} \int_0^t |\alpha_n(u,t)| + |\alpha_R(u,t)| \,\mathrm{d}u \,\mathrm{d}t < \infty \quad$ *P-a.s.*.

For $0 \le t \le T \le T^*$ *it has to hold:*

- $\int_0^t \| \int_t^T \sigma_n(u,v)\,\mathrm{d}v \|^2 + \| \int_t^T \sigma_R(u,v)\,\mathrm{d}v \|^2 \,\mathrm{d}u < \infty \quad$ *P-a.s.*,
- $\int_0^t \| \int_u^t \sigma_n(u,v)\,\mathrm{d}v \|^2 + \| \int_u^t \sigma_R(u,v)\,\mathrm{d}v \|^2 \,\mathrm{d}u < \infty \quad$ *P-a.s.*.

The first goal is to extend the no-arbitrage conditions of the HJM model to this extended framework. If we can find an EMM Q such that $\forall 0 \le T \le T^*$

$$\left(\frac{P_n(t,T)}{S_n(t)}\right)_{0\le t\le T}, \left(\frac{I(t)P_R(t,T)}{S_n(t)}\right)_{0\le t\le T}$$

are martingales, the model will be arbitrage-free.

Remark: The above assets and S_n are the tradable assets in our economy. Notice that we cannot trade $P_R(\cdot,T)$ but only $I(\cdot)P_R(\cdot,T)$ which is a nominal world price. Jarrow and Yildirim [26] additionally use a real short rate account, but as shown in Hinnerich [22], this is not necessary.

By Theorem A.5 for the here considered filtration every EMM can be described by an adapted process $\lambda(t) = (\lambda_n(t), \lambda_R(t), \lambda_I(t))$ for which the exponential process $(Z_t)_{0\le t\le T^*}$ with

$$Z_t = \exp\left\{ \int_0^t \lambda(s)^T \,\mathrm{d}B(s) - \frac{1}{2}\int_0^t \|\lambda(s)\|^2 \,\mathrm{d}s \right\}$$

is a martingale. Then $\frac{\mathrm{d}Q}{\mathrm{d}P} = Z_{T^*}$ and

$$W(t) = B(t) - \int_0^t \lambda(s)\,\mathrm{d}s \tag{2.12}$$

is a Brownian motion under Q with the same correlations as B.

Assumption 3: $(Z_t)_{0\le t\le T^*}$ *is a P-martingale. Sufficient for this is the Novikov condition*

$$\mathbb{E}^P\left[\exp\left\{\frac{1}{2}\int_0^{T^*} \|\lambda(s)\|^2 \,\mathrm{d}s\right\}\right].$$

Calculation of the differentials of the above quantities

Inserting the dynamics (2.8) and (2.9) of $f_k(t,T)$ into (2.11) we get

$$
\begin{aligned}
P_k(t,T) &= \exp\left\{-\int_t^T f_k(0,u)\,\mathrm{d}u - \int_t^T\int_0^t \alpha_k(s,u)\,\mathrm{d}s\;\mathrm{d}u - \int_t^T\int_0^t \sigma_k(s,u)^T\,\mathrm{d}B_k(s)\;\mathrm{d}u\right\}\\
&\overset{A.3}{=} \exp\left\{-\int_t^T f_k(0,u)\,\mathrm{d}u - \int_0^t\int_t^T \alpha_k(s,u)\,\mathrm{d}u\;\mathrm{d}s - \int_0^t\left(\int_t^T \sigma_k(s,u)\,\mathrm{d}u\right)^T\,\mathrm{d}B_k(s)\right\}.
\end{aligned}
$$

Furthermore since $r_n(t) = f_n(t,t)$ we can write

$$
\begin{aligned}
S_n(t) &= \exp\left\{\int_0^t f_n(u,u)\,\mathrm{d}u\right\}\\
&= \exp\left\{\int_0^t f_n(0,u)\,\mathrm{d}u + \int_0^t\int_0^u \alpha_n(s,u)\,\mathrm{d}s\;\mathrm{d}u + \int_0^t\int_0^u \sigma_n(s,u)^T\,\mathrm{d}B_n(s)\;\mathrm{d}u\right\}\\
&\overset{A.3}{=} \exp\left\{\int_0^t f_n(0,u)\,\mathrm{d}u + \int_0^t\int_s^t \alpha_n(s,u)\,\mathrm{d}u\;\mathrm{d}s + \int_0^t\left(\int_s^t \sigma_n(s,u)\,\mathrm{d}u\right)^T\,\mathrm{d}B_n(s)\right\}.
\end{aligned}
$$

Using the above relations and the representation (2.11) for $P_n(0,T)$ it follows that

$$
\begin{aligned}
\frac{P_n(t,T)}{S_n(t)} &= \exp\Bigg\{-\int_0^T f_n(0,u)\,\mathrm{d}u\\
&\qquad - \int_0^t\int_s^T \alpha_n(s,u)\,\mathrm{d}u\;\mathrm{d}s - \int_0^t\left(\int_s^T \sigma_n(s,u)\,\mathrm{d}u\right)^T\,\mathrm{d}B_n(s)\Bigg\}\\
&= \frac{P_n(0,T)}{S_n(0)}\exp\left\{-\int_0^t \alpha_n^*(s,T)\,\mathrm{d}s - \int_0^t \Sigma_n(s,T)^T\,\mathrm{d}B_n(s)\right\},
\end{aligned}
$$

where

$$
\alpha_k^*(t,T) := \int_t^T \alpha_k(t,u)\,\mathrm{d}u \qquad \text{and} \qquad \Sigma_k(t,T) := \int_t^T \sigma_k(t,u)\,\mathrm{d}u\,, \quad k\in\{n,R\}. \tag{2.13}
$$

By Ito's lemma we find

$$
\begin{aligned}
\mathrm{d}\frac{P_n(t,T)}{S_n(t)} &= \frac{P_n(t,T)}{S_n(t)}\left(-\alpha_n^*(t,T)\,\mathrm{d}t - \Sigma_n(t,T)^T\,\mathrm{d}B_n(t) + \frac{1}{2}\|\Sigma_n(t,T)\|^2\,\mathrm{d}t\right)\\
&\overset{(2.12)}{=} \frac{P_n(t,T)}{S_n(t)}\left(\left\{-\alpha_n^*(t,T) + \frac{1}{2}\|\Sigma_n(t,T)\|^2 - \Sigma_n(t,T)^T\lambda_n(t)\right\}\,\mathrm{d}t - \Sigma_n(t,T)^T\,\mathrm{d}W_n(t)\right)
\end{aligned}
$$

To be a (local) martingale the drift term of this SDE has to vanish. Hence

$$
\alpha_n^*(t,T) = \frac{1}{2}\|\Sigma_n(t,T)\|^2 - \Sigma_n(t,T)^T\lambda_n(t), \tag{2.14}
$$

which is the classical HJM drift condition. Differentiating (2.14) with respect to T yields the equivalent (since $\alpha^*(t,t)=0 \;\; \forall t$) formulation

$$\alpha_n(t,T) = \sigma_n(t,T)^T \Sigma_n(t,T) - \sigma_n(t,T)^T \lambda_n(t).$$

As in the case of the discounted nominal bond price we get

$$\mathrm{d}\frac{P_R(t,T)}{S_R(t)} = \frac{P_R(t,T)}{S_R(t)} \left(-\alpha_R^*(t,T)\,\mathrm{d}t - \Sigma_R(t,T)^T \,\mathrm{d}B_R(t) + \frac{1}{2}\|\Sigma_R(t,T)\|^2 \,\mathrm{d}t \right),$$

from which it follows by Ito's lemma that

$$\mathrm{d}P_R(t,T) = P_R(t,T) \left(r_R(t)\,\mathrm{d}t - \alpha_R^*(t,T)\,\mathrm{d}t - \Sigma_R(t,T)^T \,\mathrm{d}B_R(t) + \frac{1}{2}\|\Sigma_R(t,T)\|^2 \,\mathrm{d}t \right).$$

Again by Ito's lemma

$$\mathrm{d}\frac{P_R(t,T)}{S_n(t)} = \frac{P_R(t,T)}{S_n(t)} \left(\left\{ r_R(t) - r_n(t) - \alpha_R^*(t,T) + \frac{1}{2}\|\Sigma_R(t,T)\|^2 \right\} \mathrm{d}t - \Sigma_R(t,T)^T \,\mathrm{d}B_R(t) \right)$$

and

$$\begin{aligned}
\mathrm{d}\frac{I(t)P_R(t,T)}{S_n(t)} &= \frac{I(t)P_R(t,T)}{S_n(t)} \Bigg(\left\{ r_R(t) - r_n(t) - \alpha_R^*(t,T) + \frac{1}{2}\|\Sigma_R(t,T)\|^2 \right\} \mathrm{d}t \\
&\qquad - \Sigma_R(t,T)^T \,\mathrm{d}B_R(t) + \mu_I(t)\,\mathrm{d}t + \sigma_I(t)^T \,\mathrm{d}B_I(t) \\
&\qquad - \Sigma_R(t,T)^T \rho_{R,I}\; \sigma_I(t)\,\mathrm{d}t \Bigg) \\
&\overset{(2.12)}{=} \frac{I(t)P_R(t,T)}{S_n(t)} \Bigg(\{\mu_I(t) + r_R(t) - r_n(t) + \sigma_I(t)^T \lambda_I(t)\}\,\mathrm{d}t \\
&\qquad \Big\{ - \Sigma_R(t,T)^T \lambda_R(t) - \alpha_R^*(t,T) - \Sigma_R(t,T)^T \rho_{R,I}\; \sigma_I(t) \\
&\qquad + \frac{1}{2}\|\Sigma_R(t,T)\|^2 \Big\}\,\mathrm{d}t - \Sigma_R(t,T)^T \,\mathrm{d}W_R(t) + \sigma_I(t)^T \,\mathrm{d}W_I(t) \Bigg).
\end{aligned}$$

The drift term should vanish for all $T \geq t$. For $T = t$ we have $\Sigma_R(t,t) = \alpha^*(t,t) = 0$, therefore it must hold

$$\mu_I(t) = r_n(t) - r_R(t) - \sigma_I(t)^T \lambda_I(t).$$

For all $T > t$ we then must have

$$\alpha_R^*(t,T) = -\Sigma_R(t,T)^T \lambda_R(t) + \frac{1}{2}\|\Sigma_R(t,T)\|^2 - \Sigma_R(t,T)^T \rho_{R,I}\; \sigma_I(t). \tag{2.15}$$

Differentiating this with respect to T it follows

$$\begin{aligned}\alpha_R(t,T) &= -\sigma_R(t,T)^T\lambda_R(t) + \sigma_R(t,T)^T\Sigma_R(t,T) - \sigma_R(t,T)^T\rho_{R,I}\ \sigma_I(t)\\ &= \sigma_R(t,T)^T\left(\Sigma_R(t,T) - \lambda_R(t) - \rho_{R,I}\ \sigma_I(t)\right).\end{aligned}$$

Since $\alpha_R^*(t,t) = 0$ for all t this condition is equivalent to (2.15). Hence for the model to be arbitrage-free the following conditions have to be satisfied.

Assumption 4: Drift conditions in the JY model.
For all $0 \le t \le T \le T^*$ *it has to hold:*

$$\alpha_n(t,T) = \sigma_n(t,T)^T\Big(\Sigma_n(t,T) - \lambda_n(t)\Big), \tag{2.16}$$

$$\mu_I(t) = r_n(t) - r_R(t) - \sigma_I(t)^T\lambda_I(t), \tag{2.17}$$

$$\alpha_R(t,T) = \sigma_R(t,T)^T\Big(\Sigma_R(t,T) - \lambda_R(t) - \rho_{R,I}\ \sigma_I(t)\Big). \tag{2.18}$$

We have found necessary conditions for the model to be arbitrage-free. The assumptions so far guarantee that

$$\left(\frac{P_n(t,T)}{S_n(t)}\right)_{0\le t\le T}, \left(\frac{I(t)P_R(t,T)}{S_n(t)}\right)_{0\le t\le T}$$

are local martingales.

Assumption 5:

$$\left(\frac{P_n(t,T)}{S_n(t)}\right)_{0\le t\le T}, \left(\frac{I(t)P_R(t,T)}{S_n(t)}\right)_{0\le t\le T}$$

are Q-martingales $\forall\, 0 \le T \le T^*$. Sufficient for this are the Novikov conditions:

$$\mathbb{E}^Q\left[\exp\left\{\frac{1}{2}\int_0^T \|\Sigma_n(t,T)\|^2\,\mathrm{d}t\right\}\right] < \infty,$$

$$\mathbb{E}^Q\left[\exp\left\{\frac{1}{2}\int_0^T \|\sigma_I(t)\|^2 + \|\Sigma_R(t,T)\|^2 + 2\Sigma_R(t,T)^T\rho_{R,I}\sigma_I(t)\,\mathrm{d}t\right\}\right] < \infty.$$

We now summarize the results.

Theorem 2.1: *Under assumptions 1-5 the JY model is arbitrage-free. In terms of the Q-Brownian motions W_n, W_R, W_I we have the dynamics*

$$\mathrm{d}P_n(t,T) = P_n(t,T)\left(r_n(t)\,\mathrm{d}t - \Sigma_n(t,T)^T\,\mathrm{d}W_n(t)\right), \tag{2.19}$$

$$\mathrm{d}P_R(t,T) = P_R(t,T)\left([r_R(t) + \Sigma_R(t,T)^T\rho_{R,I}\ \sigma_I(t)]\,\mathrm{d}t - \Sigma_R(t,T)^T\,\mathrm{d}W_R(t)\right), \tag{2.20}$$

$$\mathrm{d}I(t) = I(t)\left([r_n(t) - r_R(t)]\,\mathrm{d}t + \sigma_I(t)^T\,\mathrm{d}W_I(t)\right), \tag{2.21}$$

$$
\begin{aligned}
\mathrm{d}f_n(t,T) &= \sigma_n(t,T)^T(\Sigma_n(t,T))\,\mathrm{d}t + \sigma_n(t,T)^T\,\mathrm{d}W_n(t)\,, && (2.22)\\
\mathrm{d}f_R(t,T) &= \sigma_R(t,T)^T(\Sigma_R(t,T) - \rho_{R,I}\;\sigma_I(t))\,\mathrm{d}t + \sigma_R(t,T)^T\,\mathrm{d}W_R(t)\,. && (2.23)
\end{aligned}
$$

2.2.1. Completeness of the JY model

A market with a finite number of assets is said to be complete if every (square) integrable claim is attainable, where we define a claim X to be a positive random variable which is measurable w.r.t. $\mathcal{F}_{T^*}$. Attainable means that there exist an admissible (self-financing and sufficiently regular) trading strategy replicating the payoff of X a.s.. For exact definitions we refer e.g. to Harrison and Pliska [19].

This is not directly translated in the JY or HJM setting where we consider an infinite amount of assets which are only tradable up to a certain time T instead of T^*. To adapt this concept for this type of model we pick K assets different from the bank account which are tradable at least until some time S. in $[0, S]$ we then consider the market consisting of those assets and the nominal bank account. For this market we can use the usual concept of a complete market. Note that the bonds up to time S not chosen as one of the K assets can be considered as attainable claims and their prices are uniquely defined. To rule out arbitrage their price dynamics have to coincide with the model dynamics in Theorem 2.1. This is the case if the JY drift condition (assumption 4) is satisfied.

Remark: For markets driven only by Brownian motions we usually have a relation between the number of assets (K) and the number of driving Brownian motions (N). In a complete market we are able to replicate every attainable claim. In order to hedge the randomness of one Brownian motion we usually need one asset, therefore in order to hedge claims depending on N Brownian motions we need $K = N$ assets.

It is shown in Harrison and Pliska [19] that the completeness of a market is equivalent to the uniqueness of the EMM Q. In the JY model Q is described by the process $\lambda = (\lambda_n, \lambda_R, \lambda_I)$ which has to satisfy assumption 4. If $\lambda(t)$ is uniquely defined on $[0, S]$ in terms of the original parameters $\alpha_n, \Sigma_n, \alpha_R, \Sigma_R, \mu_I, \sigma_I, \rho$ of the chosen K assets, we know that the EMM Q is unique on $\mathcal{F}_S$ and the considered market is complete.

Finally we say that the JY model is complete, if for every fixed S we can find K (K might depend on S) assets, such that the market consisting of those assets and the nominal bank account is complete.

So choose K assets and a fixed time horizon S. We then would like to know when λ is uniquely defined on $[0, S]$. Looking at assumption 4 we first observe that in general for $d_I > 1$ λ_I is not uniquely defined. Therefore in order for the JY model to be complete we require $d_I = 1$. This can be motivated by the following consideration. All the tradable asset prices depend directly on the value of the CPI. In case $d_I > 1$ we cannot hedge a asset depending only on W_I^i (one Brownian motion of the multidimensional Brownian motion W_I) and the model is not complete.

Hence we have to restrict a complete JY model to $d_I = 1$. Considering that we only want to price derivatives depending on the CPI index directly (and not on the underlying driving factors) this doesn't reduce the usefulness of this model. The effect of restricting to $d_I = 1$ is that we have fewer underlying Brownian motions and the generated Filtration is smaller. Therefore the amount of $\mathcal{F}_S$-measurable claims is smaller and we have a better chance that all are attainable.

Remark: We can always reduce a JY model with $d_I > 1$ to a JY model with $d_I = 1$ by setting $\tilde{\sigma}(t)^2 = \|\sigma(t)\|^2$ and substituting in (2.10)

$$\tilde{B}_I(t) = \frac{\sigma_I(t)^T}{\tilde{\sigma}(t)} B_I(t).$$

As long as all involved stochastic processes $(\alpha_k, \Sigma_k, \mu_I)$ are also measurable w.r.t. to the smaller filtration generated by $B_n, B_R, \tilde{B}_I$ the resulting models are equivalent for pricing derivatives depending only on $I(t)$ and interest rates.

Assuming now that $d_I = 1$ we consider the tradable assets availabe. Nominal zero coupon bonds depend only on W_n, while real zero coupon bonds (with price $I(\cdot)P_R(\cdot, T)$) depend on W_I, W_R. Motivated by the above remark we therefore might want to consider d_n nominal zero coupon bonds and $d_R + d_I = d_R + 1$ real zero coupon bonds in order to hedge arbitrary attainable claims. If for each $0 \leq S \leq T^*$ we can find bonds whose parameter functions uniquely define λ on $[0, S]$ by assumption 4 we know that the JY model is complete. A sufficient condition is that we can find bonds with parameter functions satisfying that the matrices

$$\left(\Sigma_n(t, T_n^1), \quad \dots \quad, \Sigma_n(t, T_n^{d_n})\right), \quad \left(\Sigma_R(t, T_R^1), \quad \dots \quad, \Sigma_R(t, T_R^{d_R+1})\right)$$

are Q-a.s. non-singular and for $0 \leq t \leq S$ $\sigma_I \neq 0$.

Assumption 6: *$d_I = 1$ and for each $S \leq T^*$ we can find nominal zero coupon bonds with maturities $S \leq T_n^1 < \cdots < T_n^{d_n} \leq T^*$ and real zero coupon bonds with maturities $S \leq T_R^1 < \cdots < T_R^{d_R+1} \leq T^*$ such that*

$$\left(\Sigma_n(t, T_n^1), \quad \dots \quad, \Sigma_n(t, T_n^{d_n})\right), \quad \left(\Sigma_R(t, T_R^1), \quad \dots \quad, \Sigma_R(t, T_R^{d_R+1})\right), \quad \sigma_I(t)$$

are Q-a.s. non-singular for all $t \in [0, S]$.

Theorem 2.2: *Under assumption 1-6 the JY model is complete.*

2.2.2. Changing the numeraire

Choosing the T-bond $P_n(\cdot, T)$ as numeraire, the corresponding EMM Q_n^T such that all price processes normalized by $P_n(\cdot, T)$ are martingales, is given by the Radon-Nikodym derivative (see section A.2)

$$\frac{\mathrm{d}Q_n^T}{\mathrm{d}Q} = \frac{P_n(T, T)}{S_n(T)} \frac{S_n(0)}{P_n(0, T)} = \frac{1}{S_n(T) P_n(0, T)} .$$

Therefore the density process $(Z_t)_{0\le t\le T}$ is given by

$$Z_t = \mathbb{E}^Q\left[\frac{\mathrm{d}Q_n^T}{\mathrm{d}Q}\,\bigg|\,\mathcal{F}_t\right] = \mathbb{E}^Q\left[\frac{1}{P_n(0,T)}\frac{P_n(T,T)}{S_n(T)}\,\bigg|\,\mathcal{F}_t\right] = \frac{P_n(t,T)}{S_n(t)P_n(0,T)}\ .$$

Since

$$\mathrm{d}\frac{P_n(t,T)}{S_n(t)} = -\frac{P_n(t,T)}{S_n(t)}\Sigma_n(t,T)^T\,\mathrm{d}W_n(t)$$

we have

$$Z_t = \exp\left\{\int_0^t -\Sigma_n(u,T)^T\,\mathrm{d}W_n(u) - \frac{1}{2}\int_0^t \|\Sigma_n(u,T)\|^2\,\mathrm{d}u\right\}.$$

By assumption 5 this is a Q-martingale.

We use Girsanov's theorem for correlated Brownian motions (Theorem A.4). Set $W = (W_n, W_R, W_I)$. In the notation of Theorem A.4 we have

$$\tilde{H}(u)^T\rho^{-1} = \left(-\Sigma_n(u,T)^T, \underline{0}_{d_R}^T, \underline{0}_{d_I}^T\right),$$

with $\underline{0}_d$ the column vector with d zeros. Hence

$$\tilde{H}(u) = -\begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\Sigma_n(u,T)$$

with $\rho_{R,n} = \rho_{n,R}^T$ and $\rho_{I,n} = \rho_{n,I}^T$ and by Theorem A.4

$$W^T(t) = W(t) + \int_0^t \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\Sigma_n(u,T)\,\mathrm{d}u\,,\quad 0\le t\le T \tag{2.24}$$

is a Brownian motion with correlation ρ under Q_n^T. Using (2.24) twice we get that the Brownian motion W^S (under Q_n^S) can be written as

$$\begin{aligned} W^S(t) &= W(t) + \int_0^t \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\Sigma_n(u,S)\,\mathrm{d}u \\ &= W^T(t) - \int_0^t \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\Sigma_n(u,T)\,\mathrm{d}u + \int_0^t \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\Sigma_n(u,S)\,\mathrm{d}u \\ &= W^T(t) - \int_0^t \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\left(\Sigma_n(u,T) - \Sigma_n(u,S)\right)\,\mathrm{d}u\,,\quad 0\le t\le \min\{S,T\}\ . \end{aligned} \tag{2.25}$$

2.2.3. The forward CPI

We want to calculate the dynamics of the forward CPI

$$\mathcal{I}(t,T) = \frac{I(t)P_R(t,T)}{P_n(t,T)}.$$

$(\mathcal{I}(t,T))_{0\le t\le T}$ is a martingale under Q_n^T, since it is a tradable asset divided by the numeraire $P_n(t,T)$. Its dynamics follow by repeatedly using Ito's lemma. Remembering

$$P_n(t,T) = P_n(0,T)\exp\left\{\int_0^t r_n(s)\,\mathrm{d}s \;-\int_0^t \Sigma_n(s,T)^T\,\mathrm{d}W_n(s)\right\},$$

it follows that

$$\mathrm{d}P_n(t,T)^{-1} \;= P_n(t,T)^{-1}\left(\left\{-r_n(t) + \|\Sigma_n(t,T)\|^2\right\}\,\mathrm{d}t \;+ \Sigma_n(t,T)^T\,\mathrm{d}W_n(t)\right), \tag{2.26}$$

Together with (2.20) this yields

$$\begin{aligned}
\mathrm{d}\frac{P_R(t,T)}{P_n(t,T)} \;&= \frac{P_R(t,T)}{P_n(t,T)}\Bigg(+ \left\{r_R(t) + \Sigma_R(t,T)^T\rho_{R,I}\;\sigma_I(t)\right\}\,\mathrm{d}t \;- \Sigma_R(t,T)^T\,\mathrm{d}W_R(t)\\
&\quad + \left\{-r_n(t) + \|\Sigma_n(t,T)\|^2\right\}\,\mathrm{d}t \;+ \Sigma_n(t,T)^T\,\mathrm{d}W_n(t)\\
&\quad + \Sigma_n(t,T)^T\rho_{n,R}\;\Sigma_R(t,T)\,\mathrm{d}t\Bigg)\;.
\end{aligned}$$

Combining this with (2.21) we get

$$\begin{aligned}
\mathrm{d}\frac{P_R(t,T)I(t)}{P_n(t,T)} \;&= \frac{P_R(t,T)I(t)}{P_n(t,T)}\Bigg(\left\{\|\Sigma_n(t,T)\|^2 + \Sigma_n(t,T)^T\rho_{n,R}\;\Sigma_R(t,T) + \Sigma_R(t,T)^T\rho_{R,I}\;\sigma_I(t)\right\}\,\mathrm{d}t\\
&\quad - \Sigma_R(t,T)^T\,\mathrm{d}W_R(t) \;+ \Sigma_n(t,T)^T\,\mathrm{d}W_n(t) \;+ \sigma_I(t)^T\,\mathrm{d}W_i(t)\\
&\quad + \Sigma_n(t,T)^T\rho_{n,I}\;\sigma_I(t)\,\mathrm{d}t \;- \Sigma_R(t,T)^T\rho_{R,I}\;\sigma_I(t)\,\mathrm{d}t\Bigg)\\
&= \frac{P_R(t,T)I(t)}{P_n(t,T)}\Bigg(\left\{\|\Sigma_n(t,T)\|^2 + \Sigma_n(t,T)^T\rho_{n,R}\;\Sigma_R(t,T) + \Sigma_n(t,T)^T\rho_{n,I}\;\sigma_I(t)\right\}\,\mathrm{d}t\\
&\quad - \Sigma_R(t,T)^T\,\mathrm{d}W_R(t) \;+ \Sigma_n(t,T)^T\,\mathrm{d}W_n(t) \;+ \sigma_I(t)^T\,\mathrm{d}W_i(t)\Bigg)\;.
\end{aligned}$$

Using (2.24) we finally obtain:

$$
\begin{aligned}
\mathrm{d}\mathcal{I}(t,T) = &\mathcal{I}(t,T)\Big(\Sigma_n(t,T)^T\,\mathrm{d}W_n(t) - \Sigma_R(t,T)^T\,\mathrm{d}W_R(t) + \sigma_I(t)^T\,\mathrm{d}W_I(t) \\
&\qquad + \|\Sigma_n(t,T)\|^2\,\mathrm{d}t - \Sigma_R(t,T)^T\rho_{R,n}\,\Sigma_n(t,T)\,\mathrm{d}t + \Sigma_n(t,T)^T\rho_{I,n}\,\sigma_I(t)\,\mathrm{d}t\Big) \\
= &\mathcal{I}(t,T)\Big(\Sigma_n(t,T)^T\,\mathrm{d}W_n^T(t) - \Sigma_R(t,T)^T\,\mathrm{d}W_R^T(t) + \sigma_I(t)^T\,\mathrm{d}W_I^T(t)\Big) \\
= &\mathcal{I}(t,T)\;\underline{\sigma}(t,T)^T\,\mathrm{d}W^T(t)\,,
\end{aligned}
\tag{2.27}
$$

where we write

$$
\underline{\sigma}(t,T) := \begin{pmatrix} \Sigma_n(t,T) \\ -\Sigma_R(t,T) \\ \sigma_I(t) \end{pmatrix} . \tag{2.28}
$$

Note that the forward CPIs, like the CPI I are therefore always strictly positive, as one would expect from a reasonable model.

2.2.4. Approaching forward inflation

We now take a look at inflation forward rates $F_I(\cdot,S,T)$. Since $\mathcal{I}(t,T) = \mathbb{E}^{Q_n^T}\left[I(T)|\,\mathcal{F}_t\right]$ we can rewrite (1.5) for $t \leq S$

$$
\begin{aligned}
F_I(t,S,T) &= \mathbb{E}^{Q_n^T}\left[\frac{1}{T-S}\left(\frac{I(T)}{I(S)}-1\right)\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^T}\left[\frac{1}{T-S}\left(\frac{\mathcal{I}(T,T)}{\mathcal{I}(S,S)}-1\right)\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^T}\left[\frac{1}{T-S}\left(\mathbb{E}^{Q_n^T}\left[\frac{\mathcal{I}(T,T)}{\mathcal{I}(S,S)}\middle|\,\mathcal{F}_S\right]-1\right)\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^T}\left[\frac{1}{T-S}\left(\frac{\mathcal{I}(S,T)}{\mathcal{I}(S,S)}-1\right)\middle|\,\mathcal{F}_t\right] .
\end{aligned}
\tag{2.29}
$$

By (2.25) and (2.27) we get

$$
\mathrm{d}\mathcal{I}(t,S) = \mathcal{I}(t,S)\underline{\sigma}(t,S)^T\left(\mathrm{d}W^T(t) - \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix}\left(\Sigma_n(t,T)-\Sigma_n(t,S)\right)\,\mathrm{d}t\right), \tag{2.30}
$$

and using Ito's lemma

$$\mathrm{d}\,\mathcal{I}(t,S)^{-1} = \mathcal{I}(t,S)^{-1}\Bigg(-\underline{\sigma}(t,S)^T \left(\mathrm{d}W^T(t) - \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} (\Sigma_n(t,T) - \Sigma_n(t,S))\ \mathrm{d}t \right) \\ + \underline{\sigma}(t,S)^T \rho\ \underline{\sigma}(t,T)\,\mathrm{d}t \Bigg)\ .$$

Combining this with (2.27) for $\mathcal{I}(\cdot,T)$ we arrive at

$$\begin{aligned} \mathrm{d}\frac{\mathcal{I}(t,T)}{\mathcal{I}(t,S)} &= \frac{\mathcal{I}(t,T)}{\mathcal{I}(t,S)}\Bigg(-\underline{\sigma}(t,S)^T \left(\mathrm{d}W^T(t) - \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} (\Sigma_n(t,T) - \Sigma_n(t,S))\ \mathrm{d}t \right) \\ &\qquad + \underline{\sigma}(t,S)^T \rho\ \underline{\sigma}(t,T)\,\mathrm{d}t + \underline{\sigma}(t,T)^T\,\mathrm{d}W^T(t) - \underline{\sigma}(t,S)^T \rho\ \underline{\sigma}(t,T)\,\mathrm{d}t \Bigg) \\ &= \frac{\mathcal{I}(t,T)}{\mathcal{I}(t,S)}\Bigg((\underline{\sigma}(t,T) - \underline{\sigma}(t,S))^T \left(\mathrm{d}W^T(t) - \rho^T \underline{\sigma}(t,S)\,\mathrm{d}t \right) \\ &\qquad + \underline{\sigma}(t,S) \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} (\Sigma_n(t,T) - \Sigma_n(t,S))\ \mathrm{d}t \Bigg). \end{aligned} \tag{2.31}$$

If $\Sigma_n(\cdot,T), \Sigma_R(\cdot,T), \sigma_I(\cdot)$ are deterministic, $\frac{\mathcal{I}(S,T)}{\mathcal{I}(S,S)} | \mathcal{F}_t$ is lognormally distributed (see section B.3) and we can find an explicit expression for $F_I(t,S,T)$. In general this is not the case. Since forward inflation rates are instruments we might want to calibrate models to, this is problematic. One way around this is to choose appropiate approximations of forward inflation rates as done in Mercurio [31] to get usable formulas. We will take a look at this in later sections.

2.2.5. Forward interest rates

We also want to calculate the dynamics of simple compounded forward interest rates

$$F_k(t,S,T) = \frac{1}{T-S}\left(\frac{P_k(t,S)}{P_k(t,T)} - 1 \right) \qquad k \in \{n,R\}, 0 \le t \le S. \tag{2.32}$$

They are the rates K zeroing the price of a forward rate contract at time t, which at time T exchanges the at time t unknown amount $F_n(S,S,T)$ against K.

A remark on the interpretation of real forward rates

In the nominal interest rate market the interpretation of the nominal forward rate is rather obvious. It's the rate one would be guaranteed now for a future investment. The forward contract difference at time T (ignoring the scaling with $(T-S)$) is

$$(K - F_n(S,S,T)) = K + 1 - \frac{1}{P_n(S,T)}.$$

Prooving that the definition of $F_n(t,S,T)$ is the only possible arbitrage-free one is then easily done.

Trade	t	S	T
sell a forward contract (at forward rate K)	0	0	$K+1-\frac{1}{P_n(S,T)}$
buy one unit of a S-bond	$P_n(t,S)$	1	0
at time S buy T-bonds for 1 unit	0	-1	$\frac{1}{P_n(S,T)}$
buy $\frac{P_n(t,S)}{P_n(t,T)}$ T-bonds	$-P_n(t,T)$	0	$\frac{P_n(t,S)}{P_n(t,T)}$
sum	0	0	$1+K-\frac{P_n(t,S)}{P_n(t,T)}$

Therefore the above definition is meaningful. By the foreign currency analogy this is assumed to hold for the real forward rate as well. But since we can only trade in the nominal world and not the real world, this needs some further attention. The economic meaning of the real forward rate is as follows. Assume we want to invest 1 unit of currency inflation-protected until time S, receiving the current real interest rate $F_R(t,t,S)$, and reinvest it at time S in the same manner at the real forward rate K. The alternative investment would be to directly invest it inflation protected until time T. Therefore if the market is arbitrage-free we must have the following relationship

$$(1 + F_R(t,t,S))\frac{I(S)}{I(t)}\;(1+K)\frac{I(T)}{I(S)} = (1 + F_R(t,t,T))\frac{I(T)}{I(t)}.$$

Hence a forward contract has to allow us to invest $I(S)$ at the rate K. This would then result in a payment of $(1+K)I(T)$ at time T. So a real forward contract is somewhat different from a nominal one since the nominal notional for this contract is only fixed in the future. The difference in the time T payment compared to investing this at the rate $F_R(S,S,T)$ is then

$$I(T)(K - F_R(S,S,T)) = (1+K)I(T) - \frac{I(T)}{P_R(S,T)}.$$

and a hedge can then be found by:

Trade	t	S	T
sell the forward contract	0	0	$(1+K)I(T) - \frac{I(T)}{P_R(S,T)}$
hedge $(1+K)$ T-CPI forwards	$P_n(t,T)(1+K)\,\mathcal{I}(t,T)$	0	$-(1+K)I(T)$
at S buy $\frac{1}{P_R(S,T)}$ real T-bonds	0	$-I(S)$	$\frac{I(T)}{P_R(S,T)}$
hedge 1 S-CPI forward	$-P_n(T,S)\,\mathcal{I}(t,S)$	I(S)	$\frac{P_n(t,S)}{P_n(t,T)}$
sum	$(1+K)I(t)P_R(t,T)$ $-I(t)P_R(t,S)$	0	0

Therefore we see that the definition in (2.32) is meaningful for real forward rates as well.

Calculation of the dynamics

We now want to calculate the dynamics of the forward rates starting with the nominal ones. Using Ito's lemma with (2.19) and (2.26) we have that

$$\begin{aligned}
\mathrm{d}\frac{P_n(t,S)}{P_n(t,T)} &= \frac{P_n(t,S)}{P_n(t,T)}\Big((\Sigma_n(t,T) - \Sigma_n(t,S))^T \,\mathrm{d}W_n(t) \,+ (\|\Sigma_n(t,T)\|^2 - \Sigma_n(t,S)^T\Sigma_n(t,T))\,\mathrm{d}t \Big) \\
&\overset{(2.24)}{=} \frac{P_n(t,S)}{P_n(t,T)}(\Sigma_n(t,T) - \Sigma_n(t,S))^T \,\mathrm{d}W_n^T(t) \;.
\end{aligned}$$

Then the dynamics of $F_n(t,S,T)$ is

$$\mathrm{d}F_n(t,S,T) \;= \frac{1}{T-S}\,(1 + (T-S)F_n(t,S,T))\,(\Sigma_n(t,T) - \Sigma_n(t,S))^T \,\mathrm{d}W_n^T(t)\,, \tag{2.33}$$

which of course coincides with the special case of the HJM model (2.4).

For the real forward rate note first that by (2.20)

$$\mathrm{d}P_R(t,T)^{-1} \;= \frac{1}{P_R(t,T)}\,\big(\big(\|\Sigma_R(t,T)\|^2 - r_R(t) - \Sigma_R(t,T)^T\rho_{R,I}\sigma_I(t)\big)\,\mathrm{d}t \,+ \Sigma_R(t,T)^T \,\mathrm{d}W_R(t) \big) \;.$$

Using (2.20) again we obtain

$$\begin{aligned}
\mathrm{d}\frac{P_R(t,S)}{P_R(t,T)} &= \frac{P_R(t,S)}{P_R(t,T)}\,\big(\|\Sigma_R(t,T)\|^2 - r_R(t) - \Sigma_R(t,T)^T\rho_{R,I}\sigma_I(t)\big)\,\mathrm{d}t \,+ \Sigma_R(t,T)^T \,\mathrm{d}W_R(t) \\
&\qquad + \big(r_R(t) + \Sigma_R(t,S)^T\rho_{R,I}\sigma_I(t)\big)\,\mathrm{d}t \,- \Sigma_R(t,S)^T \,\mathrm{d}W_R(t) \\
&\qquad - \Sigma_R(t,T)^T\Sigma_R(t,S)\,\mathrm{d}t \Big) \\
&= \frac{P_R(t,S)}{P_R(t,T)}\,(\Sigma_R(t,T) - \Sigma_R(t,S))^T \Big(\mathrm{d}W_R(t) \,+ (\Sigma_R(t,T) - \rho_{R,I}\sigma_I(t))\,\mathrm{d}t \Big)
\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(2.24)}{=}\frac{P_R(t,S)}{P_R(t,T)}\left(\Sigma_R(t,T)-\Sigma_R(t,S)\right)^T\Bigg(\mathrm{d}W_R^T(t)\\
&\qquad\qquad+\left(\Sigma_R(t,T)-\rho_{R,I}\sigma_I(t)-\rho_{R,n}\Sigma_n(t,T)\right)\mathrm{d}t\Bigg)\\
&=\frac{P_R(t,S)}{P_R(t,T)}\left(\Sigma_R(t,T)-\Sigma_R(t,S)\right)^T\left(\mathrm{d}W_R^T(t)\;-\left(\rho_{R,n},\quad I_{d_R},\quad \rho_{R,I}\right)\underline{\sigma}(t,T)\right),
\end{aligned}
$$

where we used the definition (2.28) for $\underline{\sigma}(\cdot,T)$. Hence

$$
\begin{aligned}
\mathrm{d}F_R(t,S,T)\;=&\frac{1}{T-S}\left(1+(T-S)F_R(t,S,T)\right)\left(\Sigma_R(t,T)-\Sigma_R(t,S)\right)^T\Bigg(\mathrm{d}W_R^T(t)\\
&\qquad+\left(\rho_{R,n},\quad I_{d_R},\quad \rho_{R,I}\right)\underline{\sigma}(t,T)\,\mathrm{d}t\Bigg).
\end{aligned}
\tag{2.34}
$$

Remark: W^T is a Brownian motion under Q_n^T. Hence $F_n(\cdot,S,T)$ is a (local) martingale, but $F_R(\cdot,S,T)$ is generally not a martingale under this measure. However, under the measure Q_R^T induced by the numeraire $I(\cdot)P_R(\cdot,T)$ it would be, since

$$
F_R(t,S,T)=\frac{1}{T-S}\left(\frac{I(t)P_R(t,S)-I(t)P_R(t,T)}{I(t)P_R(t,T)}\right)
$$

is then a sum of tradable assets divided by the numeraire. Therefore the term next to $\mathrm{d}W_R^T(t)$ in (2.34) must be the Girsanov kernel of the measure change from Q_n^T to Q_R^T.

Remark: Note that the dynamics of both forward rates are general enough to allow for positive and negative rates. While for nominal interest rates this is not really a desirable property, for real rates it is, since real rates can and have become negative.

2.2.6. The discrete bank account measure (forward spot measure)

We later introduce inflation market models, where one models discrete rates introduced earlier. Like with the classical LMM it is convenient to use a measure induced by a discrete bank account. We now show how to change to this measure by the change of numeraire technique (section A.2).
Consider some time structure $\mathbb{T}=\{0=T_0,T_1,\dots,T_N\}$ with $\delta^i=T_i-T_{i-1}$ and the in (2.5) defined discrete bank account

$$
B_n^d(t)=P_n(t,T_{\beta(t)-1})\prod_{j=0}^{\beta(t)-1}\left(1+\delta^i F_n(T_{j-1},T_{j-1},T_j)\right),\quad 0\le t\le T_N.
$$

Here $\beta(t) = \inf\{j \in \{1, \dots, N+1\} : t \le T_{j-1}\}$ for $0 \le t \le T_N$. The dynamics of this account depend on the dynamics of a single bond only

$$\mathrm{d}B_n^d(t) = \prod_{j=0}^{\beta(t)-1} (1 + \delta^i F_n(T_{j-1}, T_{j-1}, T_j))\, \mathrm{d}P_n(t, T_{\beta(t)-1}) \,.$$

Now consider for arbitrary $T \le T_N$ the forward measure Q_n^T (induced by $P_n(\cdot, T)$). The measure Q_n^d can then be described by

$$\frac{\mathrm{d}Q_n^d}{\mathrm{d}Q_n^T} = \frac{B_n^d(T)}{P_n(T,T)} \frac{P_n(0,T)}{B_n^d(0)} = B_n^d(T) P_n(0,T).$$

Since B_n^d is the value of a self-financing (admissible) trading strategy, $(\frac{B_n^d(t)}{P_n(t,T)})_{0 \le t \le T}$ is a martingale w.r.t. Q_n^T. The density process Z_t for this measure transformation is then given by

$$Z_t = \mathbb{E}^{Q_n^T} \left[P_n(0,T) \frac{B_n^d(T)}{P_n(T,T)} | \mathcal{F}_t \right] = P_n(0,T) \frac{B_n^d(t)}{P_n(t,T)}, \quad 0 \le t \le T.$$

By (2.19) and (2.26) we get the dynamics of Z_t as

$$\begin{aligned}
\mathrm{d}Z_t &= Z_t\big(r_n(t)\,\mathrm{d}t - \Sigma_n(t, T_{\beta(t)-1})^T\,\mathrm{d}W_n(t) - r_n(t)\,\mathrm{d}t - \|\Sigma_n(t,T)\|^2\,\mathrm{d}t \\
&\qquad + \Sigma_n(t,T)^T\,\mathrm{d}W_n(t) - \Sigma_n(t, T_{\beta(t)-1})^T \Sigma_n(t,T)\,\mathrm{d}t\big) \\
&= Z_t(\Sigma_n(t,T) - \Sigma_n(t, T_{\beta(t)-1}))^T \big(\mathrm{d}W_n(t) - \Sigma_n(t,T)\,\mathrm{d}t\big) \\
&\overset{(2.24)}{=} Z_t(\Sigma_n(t,T) - \Sigma_n(t, T_{\beta(t)-1}))^T\,\mathrm{d}W_n^T(t) \;.
\end{aligned}$$

Using the notation of Theorem A.4 we see

$$\tilde{H}(u) = \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} (\Sigma_n(u,T) - \Sigma_n(t, T_{\beta(t)-1})),$$

and hence

$$W^d(t) = W^T(t) - \int_0^t \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} (\Sigma_n(u,T) - \Sigma_n(t, T_{\beta(t)-1}))\,\mathrm{d}u\,, \quad 0 \le t \le T \qquad (2.35)$$

is a Brownian motion under Q_n^d with correlation ρ. In terms of W^d the dynamics of $F_n(t,S,T)$ reads

$$\begin{aligned}\mathrm{d}F_n(t,S,T) =& \frac{1+(T-S)F_n(t,S,T)}{T-S}(\Sigma_n(t,T)-\Sigma_n(t,S))\Big(\mathrm{d}W_n^d(t)\\ &+ (\Sigma_n(t,T)-\Sigma_n(t,T_{\beta(t)-1}))\,\mathrm{d}t\Big).\end{aligned} \tag{2.36}$$

Similary by (2.27)

$$\mathrm{d}\,\mathcal{I}(t,T) = \mathcal{I}(t,T)\underline{\sigma}(t,T)^T \left(\mathrm{d}W^d(t) + \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}(\Sigma_n(t,T)-\Sigma_n(t,T_{\beta(t)-1}))\,\mathrm{d}t\right). \tag{2.37}$$

2.3. Inflation models in practice

2.3.1. Short rate inflation model (Jarrow, Yildirim)

Although we concentrate on market models, we shortly report the model proposed in the original paper (Jarrow and Yildirim [26]). They assume extended Vasicek models for both nominal and real interest worlds. This means, that one models the short rate under the risk neutral measure according to the dynamics

$$\mathrm{d}r_k(t) = (\theta_k(t)-a_k r_k(t))\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_k(t)\,, \quad k\in\{n,R\},$$

where $\theta_k(t)$ is a deterministic functions and a_k,σ_k are constants. This process is mean reverting, a very desirable property, since interest rates tend to stay at a certain level. $\theta_k(t)$ is the mean level, a_k is the mean reverting speed and σ_k is the volatility. It can be shown that given instantaneous forward rates $(f_k(0,T))_{0\le T\le T^*}$ (one assumption of the JY model) the parameter $\theta_k(t)$ is uniquely defined by no-arbitrage conditions. Therefore the only free parameters in this model are a_k and σ_k. One can further show that such a model is equivalent to a JY model with the special choice

$$\sigma_k(t,T) = \sigma_k \mathrm{e}^{-a_k(T-t)}.$$

The model can be summarized as:

$$\begin{aligned}\mathrm{d}f_n(t,T) &= \sigma_n(t,T)^T(\Sigma_n(t,T)-\lambda_n(t))\,\mathrm{d}t + \sigma_n(t,T)^T\,\mathrm{d}W_n(t)\,,\\ \mathrm{d}f_R(t,T) &= \sigma_R(t,T)^T(\Sigma_R(t,T)-\lambda_R(t)-\rho_{R,I}\,\sigma_I(t))\,\mathrm{d}t + \sigma_R(t,T)^T\,\mathrm{d}W_R(t)\,,\\ \mathrm{d}I(t) &= I(t)\left([r_n(t)-r_R(t)]\,\mathrm{d}t + \sigma_I(t)^T\,\mathrm{d}W_I(t)\right).\end{aligned}$$

Further literature concerning how to price certain options in this model can be found in Henrard [21] or Huang and Yildirim [24].

2.3.2. Mercurio's first market model

Mercurio's first model (Mercurio [31]) is based on the idea to use LMM models for nominal and real forward rates. Consider a time structure $\mathbb{T} = \{0 = T_0, T_1, \dots T_N\}$ and define forward rates $F_k^i(t) := F_k(t, T_{i-1}, T_i), k \in \{n, R\}$. Set $\delta^i := T_i - T_{i-1}$. The dynamics of those rates in the JY model (see (2.33) and (2.34)) are given by

$$\begin{aligned} \mathrm{d}F_n^i(t) &= \frac{1}{\delta^i}\left(1 + \delta^i F_n^i(t)\right)\left(\Sigma_n(t, T_i) - \Sigma_n(t, T_{i-1})\right)^T \mathrm{d}W_n^{T_i}(t) \\ &= F_n^i(t)\frac{(1 + \delta^i F_n^i(t))}{\delta^i F_n^i(t)}(\Sigma_n(t, T_i) - \Sigma_n(t, T_{i-1}))^T \mathrm{d}W_n^{T_i}(t)\,, \qquad (2.38) \\ \mathrm{d}F_R^i(t) &= F_R^i(t)\frac{(1 + \delta^i F_R^i(t))}{\delta^i F_R^i(t)}\left(\Sigma_R(t, T_i) - \Sigma_R(t, T_{i-1})\right)\Bigg(\mathrm{d}W_R^{T_i}(t) \\ &\quad - \left(\rho_{R,n}, \quad I_{d_R}, \quad \rho_{R,I}\right)^T \underline{\sigma}(t, T_i)\,\mathrm{d}t\Bigg). \qquad (2.39) \end{aligned}$$

The idea behind LMMs is to choose the volatility $\left(\frac{\left(\mathrm{d}\ln\left(F_k^i(t)\right)\right)^2}{\mathrm{d}t}\right)^{\frac{1}{2}}$ of the chosen forward rates as deterministic. For nominal forward rates this can be done by choosing $\Sigma_n(\cdot, T_i)$ so that

$$\sigma_n^i(t) := \frac{(1 + \delta^i F_n^i(t))}{\delta^i F_n^i(t)}\|\Sigma_n(t, T_i) - \Sigma_n(t, T_{i-1})\|$$

is deterministic. Then define

$$Z_{n,i}^i(t) := \frac{(\Sigma_n(t, T_i) - \Sigma_n(t, T_{i-1}))^T}{\|\Sigma_n(t, T_i) - \Sigma_n(t, T_{i-1})\|}W_n^{T_i}(t).$$

$Z_{n,i}^i$ is a Brownian motion under $Q_n^{T_i}$ and we can write

$$\mathrm{d}F_n^i(t) = \sigma_n^i(t)F_n^i(t)\,\mathrm{d}Z_{n,i}^i(t)\,. \qquad (2.40)$$

Remark: $Z_{n,i}^i$ denotes a Brownian motion under the measure $Q_n^{T_i}$ (the superscript). The subscript i represents the underlying forward rate F_n^i modelled by this Brownian motion. We later also consider dynamics of rates using Brownian motions under a different measure, which is why we have to use two index. E.g. $Z_{n,i}^j$ would represent a Brownian motion under $Q_n^{T_j}$ and we could represent the dynamics of F_n^i as

$$\mathrm{d}F_n^i(t) = F_n^i(t)\left(\{\dots\}\,\mathrm{d}t + \sigma_n^i(t)\,\mathrm{d}Z_{n,i}^j(t)\right).$$

For real forward rates similary set

$$\sigma_R^i(t) := \frac{(1+\delta^i F_R^i(t))}{\delta^i F_R^i(t)} \|\Sigma_R(t,T_i) - \Sigma_R(t,T_{i-1})\|,$$
$$Z_{R,i}^i(t) := \frac{(\Sigma_R(t,T_i) - \Sigma_R(t,T_{i-1}))^T}{\|\Sigma_R(t,T_i) - \Sigma_R(t,T_{i-1})\|} W_R^{T_i}(t),$$

making $Z_{R,i}^i$ a $Q_n^{T_i}$-Brownian motion and choose $\Sigma_R(\cdot,T_i)$ so that σ_R^i is deterministic. Remember that in the JY model forward CPIs $\mathcal{I}^i(t) := \mathcal{I}(t,T_i)$ are given by (see (2.27)):

$$\mathrm{d}\mathcal{I}^i(t) = \mathcal{I}^i(t) \;\; \underline{\sigma}(t,T_i)^T \, \mathrm{d}W^{T_i}(t)\,.$$

The instantaneous correlation $\rho_{I,R}^{i,j}$ between $\mathcal{I}^i$ and F_R^j is

$$\begin{aligned}\rho_{I,R}^{i,j}(t) :=& \frac{\mathrm{d}I^i(t)\;\mathrm{d}F_R^j(t)}{\sqrt{\mathrm{d}I^i(t)^{\,2}\,\mathrm{d}F_R^j(t)^{\,2}}}\\ =& \frac{(\Sigma_R(t,T_j) - \Sigma_R(t,T_{j-1}))^T}{\|\Sigma_R(t,T_j) - \Sigma_R(t,T_{j-1})\|} \begin{pmatrix}\rho_{R,n}, & I_{d_R}, & \rho_{R,I}\end{pmatrix} \frac{\underline{\sigma}(t,T_i)}{\underline{\sigma}(t,T_i)^T \rho \underline{\sigma}(t,T_i)}.\end{aligned}$$

By setting

$$\sigma_I^i(t)^2 := \underline{\sigma}(t,T_i)^T \rho \underline{\sigma}(t,T_i)$$

we can write (2.39) as

$$\mathrm{d}F_R^i(t) = F_R^i(t)\left(-\rho_{I,R}^{i,i}\sigma_I^i(t)\sigma_R^i(t)\,\mathrm{d}t + \sigma_R^i(t)\,\mathrm{d}Z_{R,i}^i(t)\right). \tag{2.41}$$

Remark: Note that under the measure $Q_R^{T_i}$ induced by the numeraire $I(\cdot)P_R(\cdot,T_i)$ the real forward rate F_R^i is a martingale satisfying

$$\mathrm{d}F_R^i(t) = F_R^i(t)\sigma_R^i(t)\,\mathrm{d}\tilde{Z}_{R,i}^i(t),$$

where $\tilde{Z}_{R,i}^i$ is a $Q_R^{T_i}$-Brownian motion. Therefore by choosing σ_R^i deterministic we could value real interest rate caps and floors in real markets using Black's formula for the numeraire $I(\cdot)P_R(\cdot,T_i)$.

We choose $\rho_{I,R}^{i,j}, i,j = 1,\ldots,N$ in (2.41) as deterministic (which means choosing $\underline{\sigma}(t,T_i)$, $1 \le i \le N$ accordingly) and would like to choose $\sigma_I^i, i = 1,\ldots,N$ deterministic as well. However, it turns out that this is only possible for a single i, as first described in Schloegl [40]. The problem is that after choosing σ_I^i deterministic for one i, all the other σ_I^j are in fact stochastic. To motivate this choose an index i for which we set σ_I^i deterministic. Now take a look at

$$\frac{\mathcal{I}(t,T_i)}{\mathcal{I}(t,T_{i-1})} = \frac{P_R(t,T_i)}{P(t,T_{i-1})}\frac{P_n(t,T_{i-1})}{P_n(t,T_i)} = \frac{1+\delta^i F_n^i(t)}{1+\delta^i F_R^i(t)}, \quad t \le T_{i-1}. \tag{2.42}$$

Calculating the quadratic variation of the logarithm of both sides we find for the left side (using (2.31)):

$$(\underline{\sigma}(t,T_i) - \underline{\sigma}(t,T_{i-1}))^T \rho(\underline{\sigma}(t,T_i) - \underline{\sigma}(t,T_{i-1})) = \sigma_I^i(t)^2 + \sigma_I^{i-1}(t)^2 - 2\rho_I^{i,i-1}(t)\sigma_I^i(t)\sigma_I^{i-1}(t), \tag{2.43}$$

where $\rho_I^{i,i-1}$ is the deterministic chosen instantaneous correlation between $\mathcal{I}^i$ and $\mathcal{I}^{i-1}$. Since

$$\begin{aligned}
\mathrm{d}\ln(1+\delta^i F_n^i(t)) &= \{\dots\}\,\mathrm{d}t + \frac{\delta^i F_n^i(t)}{1+\delta^i F_n^i(t)}\sigma_n^i(t)\,\mathrm{d}Z_{n,i}^i(t)\,,\\
\mathrm{d}\ln(1+\delta^i F_R^i(t)) &= \{\dots\}\,\mathrm{d}t + \frac{\delta^i F_R^i(t)}{1+\delta^i F_R^i(t)}\sigma_R^i(t)\,\mathrm{d}Z_{R,i}^i(t)\,,
\end{aligned}$$

the quadratic variation for the right side is

$$\begin{aligned}
&\left(\left(\frac{\delta^i F_n^i(t)}{1+\delta^i F_n^i(t)}\right)^2 \sigma_n^i(t)^2 + \left(\frac{\delta^i F_R^i(t)}{1+\delta^i F_R^i(t)}\right)^2 \sigma_R^i(t)^2\right.\\
&\left.\quad + \frac{\delta^i F_R^i(t)}{1+\delta^i F_R^i(t)}\frac{\delta^i F_n^i(t)}{1+\delta^i F_n^i(t)}\sigma_n^i(t)\sigma_R^i(t)\rho_{n,R}^{i,i}(t)\right)\mathrm{d}t\,,
\end{aligned} \tag{2.44}$$

where $\rho_{n,R}^{i,i}$ is the deterministically chosen instantaneous correlation between F_n^i and F_R^i.

(2.44) is obviously stochastic while in (2.43) all factors except σ_I^{i-1} are chosen to be deterministic. Therefore σ_I^{i-1} has to be stochastic as well.

Mercurio suggests to approximate σ_I^{i-1} by freezing the forward rates at their current value. Using (2.43) and (2.44) after freezing the drifts we can solve this quadratic equation to approximate σ_I^{i-1} deterministically. However, in his second model Mercurio suggests a different framework not having this problem which is why we focus on this approach.

A big drawback of the first model is that by assuming a LMM for the real rates they have a lognormal distributions. Therefore real rates can never become negative. As mentioned earlier this is very undesirable and this is another reason why we focus on the second model.

Note that

$$Z_{I,i}^i(t) := \frac{\underline{\sigma}(t,T_i)}{\sigma_I^i(t)} W^{T_i}(t)$$

is $Q_n^{T_i}$-Brownian motion and that we can write the forward CPI dynamics (2.27) as

$$\mathrm{d}\mathcal{I}^i(t) = \mathcal{I}^i(t)\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^i(t)\,.$$

Remark: The problem of non-deterministic volatilities was first reviewed in Schloegl [40]. There Schloegl argues that in foreign exchange markets assuming lognormal dynamics for forward rates in both countries can only be combined with lognormal dynamics for one forward exchange rate (in this case forward CPIs). This is exactly the problem we encountered above. A second choice is to model forward rates and the forward exchange rates (here CPIs) to be lognormal. This will however result in nonlognormal distribution for the second market. This choice corresponds Mercurio's second model.

Summarizing the discussion above we get the following model, where we choose the index i of the lognormal forward CPI to be the index N.

Model: *Consider a time structure* $\mathbb{T} = \{0 = T_0, T_1, \dots T_N\}$ *and for* $i = 1, \dots, N$ *nominal and real forward rates* $F_n^i(t) = F_n(t, T_{i-1}, T_i)$, $F_R^i(t) = F_R(t, T_{i-1}, T_i)$ *as well as a single forward CPI* $\mathcal{I}^N(t) = \mathcal{I}(t, T_N)$ *given by*

$$\begin{aligned}
\mathrm{d}F_n^i(t) &= \sigma_n^i(t) F_n^i(t)\, \mathrm{d}Z_{n,i}^i(t)\,, \quad 0 \le t \le T_{i-1}, \\
\mathrm{d}F_R^i(t) &= F_R^i(t) \left(-\rho_{I,R}^{i,i}(t) \sigma_I^i(t) \sigma_R^i(t)\, \mathrm{d}t + \sigma_R^i(t)\, \mathrm{d}Z_{R,i}^i(t) \right), \quad 0 \le t \le T_{i-1}, \\
\mathrm{d}\,\mathcal{I}^N(t) &= \mathcal{I}^N(t) \sigma_I^N(t)\, \mathrm{d}Z_{I,N}^N(t)\,, \quad 0 \le t \le T_N,
\end{aligned}$$

where $Z_{k,i}^i$, $k \in \{n, R\}$, $Z_{I,N}^N$ *are* $Q_n^{T_i}$ ($Q_n^{T_N}$)*-Brownian motions with deterministic instantaneous correlations.* σ_n^i, σ_R^i, σ_I^N *are positive and deterministic and starting values of the SDEs are given.* $\rho_{I,R}^{i,i}$ *denotes the deterministic instantaneous correlation between* $\mathcal{I}^i = \mathcal{I}(t, T_i)$ *(defined in* (1.4)*) and* F_R^i*.*

Remark: The above dynamics are given using Brownian motions under different measures $Q_n^{T_i}$. However, the above model also specifies the dynamics of the rates with Brownian motions under a fixed measure $Q_n^{T_j}$ (j fixed). This follows from the change of numeraire technique (section A.2). It is easy to check that the measure change from $Q_n^{T_i}$ to $Q_n^{T_{i-1}}$ is given by the density process

$$\frac{P_n(t, T_{i-1})}{P_n(t, T_i)} = 1 + \delta^i F_n^i(t),$$

whose dynamics then follow from the model assumptions.

Remark: We choose all instantaneous correlations of this model as deterministic functions. In order to find an equivalent JY model we need to choose the dimensions of the JY-Brownian motions $W = (W_n, W_R, W_I)$ high enough, so that the parameter functions have enough degrees to allow instantaneous correlations between the Brownian motions $Z_{n,i}^i, Z_{R,i}^i, Z_{I,N}^N$ to be deterministic.

Note that the forward CPI volatilities for $i \neq N$ can then be calculated taking quadratic variations in (2.42).

Valuation of YYIIS

Given (approximately) deterministic forward CPI volatilities to price YYIIS in this model we have to calculate

$$F_I^i(t) = \mathbb{E}^{Q_n^{T_i}} \left[\frac{1}{\delta_i} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \middle| \mathcal{F}_t \right] = \frac{1}{\delta_i} \mathbb{E}^{Q_n^{T_i}} \left[\frac{I(T_i)}{I(T_{i-1})} \middle| \mathcal{F}_t \right] - 1,$$

where for $t \leq T_{i-1}$ by (1.4) and (2.32)

$$\begin{aligned}
\mathbb{E}^{Q_n^{T_i}}\left[\left.\frac{I(T_i)}{I(T_{i-1})}\right|\mathcal{F}_t\right] &= \mathbb{E}^{Q_n^{T_i}}\left[\left.\frac{1}{I(T_{i-1})}\mathbb{E}^{Q_n^{T_i}}\left[I(T_i)|F_{T_{i-1}}\right]\right|\mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^{T_i}}\left[\left.\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\right|\mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^T}\left[\left.\frac{P_R(T_{i-1},T_i)}{P_R(T_{i-1},T_i)}\frac{P_n(T_{i-1},T_{i-1})}{P_n(T_{i-1},T_i)}\right|\mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^T}\left[\left.\frac{1+\delta^i F_n^i(T_{i-1})}{1+\delta^i F_R^i(T_{i-1})}\right|\mathcal{F}_t\right].
\end{aligned}$$

All the coefficients in (2.40) and (2.41) are deterministic and

$$\left.\left(\ln\left(\frac{F_n^i(T_{i-1})}{F_n^i(t)}\right), \ln\left(\frac{F_R^i(T_{i-1})}{F_R^i(t)}\right)\right)\right|\mathcal{F}_t \sim N_2(\mu,\Sigma),$$

where the parameters of this two-dimensional normal distribution depend on

$$\sigma_I^i, \sigma_n^i, \sigma_R^i, \rho_{I,R}^i.$$

Therefore the expectation value can be calculated analytically (as done in Mercurio [31] for constant volatilities).

2.3.3. Mercurio's second market model (forward CPI market model)

Using the notation of the previous section we now want to model all forward CPIs $\mathcal{I}^i(t)$ and nominal forward rates $F_n^i(t)$ as geometric Brownian motions. From equation (2.42) we see that then also the processes F_R^i, $1 \leq i \leq N$ are fixed.

Model: *Consider a time structure* $\mathbb{T} = \{0 = T_0, T_1, \dots T_N\}$ *and for* $i = 1, \dots, N$ *nominal forward rates and forward CPIs given by*

$$\begin{aligned}
\mathrm{d}F_n^i(t) &= \sigma_n^i(t)F_n^i(t)\,\mathrm{d}Z_{n,i}^i(t)\,, \quad 0 \leq t \leq T_{i-1}, \\
\mathrm{d}\,\mathcal{I}^i(t) &= \mathcal{I}^i(t)\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^i(t)\,, \quad 0 \leq t \leq T_i,
\end{aligned}$$

Z_n^i *and* Z_I^i *are one-dimensional* $Q_n^{T_i}$*-Brownian motions with a deterministic instantaneous correlation structure given by*

$$\begin{pmatrix} (\rho_n^{i,j})_{i,j=1,\dots,N} & (\rho_{n,I}^{i,j})_{i,j=1,\dots,N} \\ (\rho_{n,I}^{j,i})_{i,j=1,\dots,N} & (\rho_I^{i,j})_{i,j=1,\dots,N} \end{pmatrix}.$$

The starting values and the parameter functions σ_n^i, σ_I^i *are positive and deterministic.*

This corresponds to a JY model by setting

$$
\begin{aligned}
Z^i_{n,i}(t) &:= \frac{(\Sigma_n(t,T_i) - \Sigma_n(t,T_{i-1}))^T}{\|\Sigma_n(t,T_i) - \Sigma_n(t,T_{i-1})\|} W^{T_i}_n(t), \\
Z^i_{I,i}(t) &:= \frac{\underline{\sigma}(t,T_i)^T}{\underline{\sigma}(t,T_i)^T \rho \underline{\sigma}(t,T_i)} W^{T_i}(t),
\end{aligned}
$$

and choosing $\underline{\sigma}(t,T_i), i = 1, \dots N$ so that

$$
\begin{aligned}
\sigma^i_n(t) &= \frac{(1+\delta^i F^i_n(t))}{\delta^i F^i_n(t)} \|\Sigma_n(t,T_i) - \Sigma_n(t,T_{i-1})\|, && (2.45) \\
\sigma^i_I(t)^2 &= \underline{\sigma}(t,T_i)^T \rho \; \underline{\sigma}(t,T_i), && (2.46) \\
\rho^{i,j}_n(t) &= \frac{\mathrm{d}Z^i_{n,i} \; \mathrm{d}Z^j_{n,j}}{\mathrm{d}t} = \frac{\mathrm{d}F^i_n(t) \; \mathrm{d}F^j_n(t)}{\sqrt{(\,\mathrm{d}F^i_n(t)\,)^2}\sqrt{\left(\,\mathrm{d}F^j_n(t)\,\right)^2}} \\
&= \frac{(\Sigma_n(t,T_i) - \Sigma_n(t,T_{i-1}))^T}{\|\Sigma_n(t,T_i) - \Sigma_n(t,T_{i-1})\|} \frac{(\Sigma_n(t,T_j) - \Sigma_n(t,T_{j-1}))}{\|\Sigma_n(t,T_j) - \Sigma_n(t,T_{j-1})\|}, && (2.47) \\
\rho^{i,j}_I(t) &= \frac{\mathrm{d}Z^j_{I,j} \; \mathrm{d}Z^i_{I,i}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{I}^j(t) \; \mathrm{d}\mathcal{I}^i(t)}{\sqrt{(\,\mathrm{d}\mathcal{I}^j(t)\,)^2}\sqrt{(\,\mathrm{d}\mathcal{I}^i(t)\,)^2}} \\
&= \frac{\underline{\sigma}(t,T_j)^T}{\underline{\sigma}(t,T_j)^T \rho \underline{\sigma}(t,T_j)} \rho \frac{\underline{\sigma}(t,T_i)^T}{\underline{\sigma}(t,T_i)^T \rho \underline{\sigma}(t,T_i)}, && (2.48) \\
\rho^{i,j}_{n,I}(t) &= \frac{\mathrm{d}Z^i_{n,i} \; \mathrm{d}Z^j_{I,j}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{I}^j(t) \; \mathrm{d}F^i_n(t)}{\sqrt{(\,\mathrm{d}\mathcal{I}^j(t)\,)^2}\sqrt{(\,\mathrm{d}F^i_n(t)\,)^2}} \\
&= \frac{\underline{\sigma}(t,T_j)^T}{\underline{\sigma}(t,T_j)^T \rho \underline{\sigma}(t,T_j)} \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} \frac{(\Sigma_n(u,T_i) - \Sigma_n(u,T_{i-1}))}{\|\Sigma_n(u,T_i) - \Sigma_n(u,T_{i-1})\|} && (2.49)
\end{aligned}
$$

are deterministic. Here we used the results and notation from (2.33) and (2.27).

Valuation of YYIIS

Using the introduced notation we can write (2.31) as

$$\begin{aligned}
\mathrm{d}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)} &= \frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\Big(\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^i(t) \;-\sigma_I^{i-1}(t)\,\mathrm{d}Z_{I,i-1}^{i-1}(t) \\
&\qquad + \left(\sigma_I^{i-1}(t)^2 - \sigma_I^{i-1}(t)\rho_I^{i,i-1}(t)\sigma_I^i(t)\right)\,\mathrm{d}t\Big) \\
&= \frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\Big(\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^i(t) \;-\sigma_I^{i-1}(t)\,\mathrm{d}Z_{I,i-1}^{i}(t) \qquad (2.50)\\
&\qquad + \left(\sigma_I^{i-1}(t)^2 - \sigma_I^{i-1}(t)\rho_I^{i,i-1}(t)\sigma_I^i(t) + \sigma_I^{i-1}(t)\frac{\sigma_n^i(t)\delta^iF_n^i(t)}{(1+\delta^iF_n^i(t))}\rho_{n,I}^{i,i-1}(t)\right)\,\mathrm{d}t\Big),
\end{aligned}$$

where using the above definitions of $\sigma_n^i, \sigma_I^i, \rho_{n,i}^{i,j}$ and the results from (2.25) we have that

$$\begin{aligned}
Z_{I,i-1}^i(t) &:= Z_{I,i-1}^{i-1}(t) + \int_0^t \frac{\sigma_n^i(s)\delta^iF_n^i(s)}{(1+\delta^iF_n^i(s))}\rho_{n,I}^{i,i-1}(s)\,\mathrm{d}s \\
&= \frac{\underline{\sigma}(t,T_{i-1})^T}{\sigma_I^{i-1}(t)}W^{T_{i-1}}(t) + \int_0^t \frac{\underline{\sigma}(s,T_{i-1})^T}{\sigma_I^{i-1}(s)}\begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}(\Sigma_n(s,T_i) - \Sigma_n(s,T_{i-1}))\,\mathrm{d}s \\
&= \frac{\underline{\sigma}(t,T_{i-1})^T}{\sigma_I^{i-1}(t)}W^{T_i}(t)
\end{aligned}$$

is a Brownian motion under $Q_n^{T_i}$. One would now like to use these dynamics to value forward inflation rates, which means calculating the expectation

$$\mathbb{E}^{Q_n^{T_i}}\left[\left.\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\right|\mathcal{F}_t\right].$$

The problem is that the drift term has stochastic components. However, freezing the value of $F_n^i(t)$ at its current value the term becomes deterministic and we can use the calculation of B.2. This approximation procedure yields the following result:

$$\delta^i(1+F_I^i(t)) = \mathbb{E}^{Q_n^{T_i}}\left[\left.\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\right|\mathcal{F}_t\right] \approx \mathrm{e}^{D^i(t)}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}, \tag{2.51}$$

where

$$D^i(t) = \int_t^{T_{i-1}} \sigma_I^{i-1}(u)^2 - \sigma_I^{i-1}(u)\rho_I^{i,i-1}(u)\sigma_I^i(u) + \sigma_I^{i-1}(u)\frac{\sigma_n^i(u)\delta^iF_n^i(t)}{(1+\delta^iF_n^i(t))}\rho_{n,I}^{i,i-1}(u)\,\mathrm{d}u\,.$$

Valuation of inflation caps & caplets

Consider a caplet for the inflation rate F_I^i with strike κ and price

$$IC(t,T_{i-1},T_i,\kappa) = \delta^i P_n(t,T_i)\,\mathbb{E}^{Q_n^{T_i}}\left[(F_I^i(T_i)-\kappa)_+ \middle|\, \mathcal{F}_t\right].$$

We can write the expectation as

$$\begin{aligned}
\delta^i\,\mathbb{E}^{Q_n^{T_i}}\left[(F_I^i(T_i)-\kappa)_+ \middle|\, \mathcal{F}_t\right] &= \mathbb{E}^{Q_n^{T_i}}\left[\left(\frac{I(T_i)}{I(T_{i-1})}-K\right)_+ \middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^{Q_n^{T_i}}\left[\frac{\mathbb{E}^{Q_n^{T_i}}[(I(T_i)-KI(T_{i-1}))_+ |\, \mathcal{F}_{T_{i-1}}]}{I(T_{i-1})}\middle|\, \mathcal{F}_t\right],
\end{aligned}$$

where $K = 1+\delta^i\kappa$. Then knowing that $I(T_i)|\,\mathcal{F}_{T_{i-1}}$ is distributed lognormally as a simple forward CPI we have by sections B.3 and B.4

$$\begin{aligned}
\mathbb{E}^{Q_n^{T_i}}[(I(T_i)-KI(T_{i-1}))_+ |\, \mathcal{F}_{T_{i-1}}] &= \mathcal{I}^i(T_{i-1})\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{KI(T_{i-1})}\right)+\frac{1}{2}\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}}\right) \\
&\quad - KI(T_{i-1})\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{KI(T_{i-1})}\right)-\frac{1}{2}\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}}\right).
\end{aligned}$$

Hence we have that

$$\begin{aligned}
\delta^i\,\mathbb{E}^{Q_n^{T_i}}\left[(F_I^i(T_i)-\kappa)_+ \middle|\, \mathcal{F}_t\right] = \mathbb{E}^{Q_n^{T_i}}\Bigg[&\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{KI(T_{i-1})}\right)+\frac{1}{2}\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}}\right) \\
&-K\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{KI(T_{i-1})}\right)-\frac{1}{2}\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}}\right)\Bigg|\,\mathcal{F}_t\Bigg].
\end{aligned} \tag{2.52}$$

Again fixing the drifts in (2.50) by section B.3 we know approximately that

$$\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\right)\Big|\,\mathcal{F}_t \sim N\left(\ln\left(\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\right)+D^i(t)-\frac{1}{2}V^i(t)^2, V^i(t)^2\right), \tag{2.53}$$

where

$$V^i(t)^2 = \int_t^{T_{i-1}} \sigma_I^i(u)^2 + \sigma_I^{i-1}(u)^2 - 2\rho_I^{i,i-1}(u)\sigma_I^{i-1}(u)\sigma_I^i(u)\,\mathrm{d}u\,.$$

Hence for the second part of (2.52) we can apply Lemma B.1 with

$$\begin{aligned}
\mu &= \ln\left(\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\right) + D^i(t) - \frac{1}{2}V^i(t)^2,\\
\sigma^2 &= V^i(t)^2,\\
a &= \frac{1}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}},\\
b &= \frac{-\ln(K) - \frac{1}{2}\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}},
\end{aligned}$$

so that

$$K\mathbb{E}^{Q_n^{T_i}}\left[\Phi\left(a\,\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\right)+b\right)\middle|\,\mathcal{F}_t\right] \approx K\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(t)}{K\,\mathcal{I}^{i-1}(t)}\right) + D_i(t) - \frac{1}{2}\mathcal{V}_i(t)^2}{\mathcal{V}_i(t)}\right),$$

where

$$\mathcal{V}^i(t)^2 = \int_t^{T_{i-1}}\sigma_I^i(u)^2\,\mathrm{d}u\, + V^i(t)^2.$$

For the first part we use Lemma B.2 with a, μ, σ as above and

$$b = \frac{-\ln(K) + \frac{1}{2}\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u}{\left(\int_{T_{i-1}}^{T_i}\sigma_I^i(u)^2\,\mathrm{d}u\right)^{\frac{1}{2}}}$$

to arrive at

$$\mathbb{E}^{Q_n^{T_i}}\left[\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\Phi\left(a\,\ln\left(\frac{\mathcal{I}^i(T_{i-1})}{\mathcal{I}^{i-1}(T_{i-1})}\right)+b\right)\middle|\,\mathcal{F}_t\right] \approx \frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\mathrm{e}^{D^i(t)}\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(t)}{K\,\mathcal{I}^{i-1}(t)}\right) + D^i(t) + \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right).$$

In the end we have to discount the expected cashflow to get

$$
\begin{aligned}
IC(t, T_{i-1}, T_i, \kappa) \approx P_n(t, T_i) & \left(\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)} \mathrm{e}^{D^i(t)} \Phi \left(\frac{\ln\left(\frac{\mathcal{I}^i(t)}{K\,\mathcal{I}^{i-1}(t)}\right) + D^i(t) + \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)} \right) \right. \\
& \left. - K\Phi \left(\frac{\ln\left(\frac{\mathcal{I}^i(t)}{K\,\mathcal{I}^{i-1}(t)}\right) + D^i(t) - \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)} \right) \right) \\
\overset{(2.51)}{\approx} P_n(t, T_i) & \left((1 + \delta^i F_I^i(t)) \Phi \left(\frac{\ln\left(\frac{(1+\delta^i F_I^i(t))}{1+\delta^i \kappa}\right) + \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)} \right) \right. \\
& \left. - (1 + \delta^i \kappa) \Phi \left(\frac{\ln\left(\frac{(1+\delta^i F_I^i(t))}{1+\delta^i \kappa}\right) - \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)} \right) \right),
\end{aligned}
$$

where for the second equation we used the approximation developed in (2.51). Note that this corresponds to a shifted lognormal valuation formula (see B.4.3). This is the reason we refer to this model as a market model, since typically such formulas are the first ones used by market practicioners.

Dynamics under the spot forward measure

We now calculate the dynamics of the modelled rates under a common measure, namely the earlier introduced forward spot measure induced by the discrete bank account. Remember the dynamics (2.36) and note that

$$
\Sigma_n(t, T_k) - \Sigma_n(t, T_{\beta(t)-1}) = \sum_{j=\beta(t)}^{k} (\Sigma_n(t, T_j) - \Sigma_n(t, T_{j-1})).
$$

Furthermore by (2.45) and (2.47)

$$
(\Sigma_n(t, T_i) - \Sigma_n(t, T_{i-1}))^T (\Sigma_n(t, T_j) - \Sigma_n(t, T_{j-1})) = \rho_n^{i,j}(t) \sigma_n^i(t) \frac{\delta^i F_n^i(t)}{1 + \delta^i F_n^i(t)} \sigma_n^j(t) \frac{\delta^j F_n^j(t)}{1 + \delta^j F_n^j(t)}.
$$

Therefore (2.36) reads

$$\begin{aligned}
\mathrm{d}F_n^i(t) &= \frac{1+\delta^i F_n^i(t)}{\delta^i}(\Sigma_n(t,T_i)-\Sigma_n(t,T_{i-1}))^T\,\mathrm{d}W_n^d(t) \\
&\quad + \frac{1+\delta^i F_n^i(t)}{\delta^i}\sum_{j=\beta(t)}^{i}(\Sigma_n(t,T_i)-\Sigma_n(t,T_{i-1}))^T(\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1}))\,\mathrm{d}t \\
&= F_n^i(t)\sigma_n^i(t)\left(\mathrm{d}Z_{n,i}^d(t) + \sum_{j=\beta(t)}^{i}\rho_n^{i,j}(t)\sigma_n^j(t)\frac{\delta^j F_n^j(t)}{1+\delta^j F_n^j(t)}\,\mathrm{d}t\right).
\end{aligned} \tag{2.54}$$

We now do the same thing for forward CPIs where by (2.49), (2.45) and (2.46) we have

$$\underline{\sigma}(t,T_i)\begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}(\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1})) = \rho_{n,I}^{j,i}(t)\sigma_I^i(t)\sigma_n^j(t)\frac{\delta^j F_n^j(t)}{1+\delta^j F_n^j(t)}.$$

Inserting this into (2.37) we get

$$\begin{aligned}
\mathrm{d}\,\mathcal{I}^i(t) &= \mathcal{I}^i(t)\left(\underline{\sigma}(t,T)^T\,\mathrm{d}W^d(t) + \sum_{j=\beta(t)}^{i}\underline{\sigma}(t,T)^T\begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}(\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1}))\,\mathrm{d}t\right) \\
&= \mathcal{I}^i(t)\sigma_I^i(t)\left(\mathrm{d}Z_{I,i}^d(t) + \sum_{j=\beta(t)}^{i}\rho_{n,I}^{j,i}(t)\sigma_n^j(t)\frac{\delta^j F_n^j(t)}{1+\delta^j F_n^j(t)}\,\mathrm{d}t\right).
\end{aligned} \tag{2.55}$$

Hence we have calculated everything we need for simulation.

2.3.4. Adjusting for different forward CPI tenors

The basic idea behind the LMM is to have a model resulting in Black formulas for interest cap markets. The underlyings of this market are mostly 6-month LIBOR rates, which is why we would like to model those rates. Contrary inflation markets (YYIIS, inflation caps) are based on annual inflation rates. Hence one would like to use a 1-year tenor for the modelled forward CPIs. We now adjust Mercurio's second model allowing for a natural calibration of both markets. We consider a nominal tenor structure

$$\mathbb{T}_n = \{T_0, T_1, \ldots, T_{2N}\},$$

where we want to have $\delta_n^i = T_i - T_{i-1} \approx 0.5, i \in \{1,\ldots,2N\}$ and an inflation tenor structure

$$\mathbb{T}_I = \{T_0, T_2, T_4, \ldots, T_{2N}\}$$

with $\delta_I^i = T_{2i} - T_{2(i-1)} \approx 1, i \in \{1, \dots, N\}$. Given this tenor structures we model the rates

$$F_n^i(t) = F_n(t, T_{i-1}, T_i) \qquad \text{and} \qquad \mathcal{I}^i(t) = \mathcal{I}(t, T_{2i})$$

according to the dynamics

$$\begin{aligned} \mathrm{d}F_n^i(t) &= F_n^i(t)\sigma_n^i(t)\,\mathrm{d}Z_{n,i}^i(t)\,, \quad i \in \{1, \dots, 2N\}, \\ \mathrm{d}\mathcal{I}^i(t) &= \mathcal{I}^i(t)\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^i(t)\,, \quad i \in \{1, \dots, N\}, \end{aligned}$$

where $Z_{n,i}^i$ are Brownian motions under $Q_n^{T_i}$ and $Z_{I,i}^i$ are Brownian motions under $Q_n^{T_{2i}}$, which have deterministic instantaneous correlations. Similar to the previous section this can again be interpreted as a JY model and we can also repeat the steps to derive dynamics under the measure Q_n^d induced by the discrete bank account (defined using the tenor structure $\mathbb{T}_n$). We can then summarize this model as follows

Model: *Consider a tenor structure $\{0 = T_0, T_1, \dots, T_{2N}\}$ with $\delta^i = T_i - T_{i-1}$ and forward rates $F_n^i, i = 1, \dots, 2N$ and forward CPIs $\mathcal{I}^i, i = 1, \dots, N$ following the SDEs*

$$\begin{aligned} \mathrm{d}F_n^i(t) &= F_n^i(t)\sigma_n^i(t)\left(\mathrm{d}Z_{n,i}^d(t) + \sum_{j=\beta(t)}^{i} \rho_n^{i,j}(t)\sigma_n^j(t)\frac{\delta^j F_n^j(t)}{1+\delta^j F_n^j(t)}\,\mathrm{d}t\right), \quad 0 \le t \le T_{i-1}, \\ \mathrm{d}\mathcal{I}^i(t) &= \mathcal{I}^i(t)\sigma_I^i(t)\left(\mathrm{d}Z_{I,i}^d(t) + \sum_{j=\beta(t)}^{2i} \rho_{n,I}^{j,i}(t)\sigma_n^j(t)\frac{\delta^j F_n^j(t)}{1+\delta^j F_n^j(t)}\,\mathrm{d}t\right), \quad 0 \le t \le T_{2i}, \end{aligned}$$

where $Z_{n,i}^d$ and $Z_{I,i}^d$ are Brownian motions under Q_n^d with deterministic instantaneous correlation structure

$$\begin{pmatrix} (\rho_n^{i,j})_{i,j=1,\dots,2N} & (\rho_{n,I}^{i,j})_{i=1,\dots,2N,j=1,\dots,N} \\ (\rho_{n,I}^{j,i})_{i=1,\dots,2N,j=1,\dots,N} & (\rho_I^{i,j})_{i,j=1,\dots,N} \end{pmatrix}.$$

and $\beta(t) = \inf\{j \in \{1, \dots, 2N+1\} : t \le T_{j-1}\}$. The parameter functions σ_n^i, σ_I^i and the starting values are positive and deterministic.

The formulas are very similar to (2.54) and (2.55). Only for $\mathcal{I}^i(t)$ the number of summands in the drift term has changed. Also prices for inflation forward rates and inflation caps can be derived in a similar fashion resulting in slightly adjusted formulas, which we state here. Set $\delta_I^i = T_{2i} - T_{2(i-1)}$, then

$$\delta_I^i(1 + F_I(t, T_{2(i-1)}, T_{2i})) = \mathbb{E}^{Q_n^{T_{2i}}}\left[\left.\frac{\mathcal{I}^i(T_{2(i-1)})}{\mathcal{I}^{i-1}(T_{2(i-1)})}\right|\mathcal{F}_t\right] \approx \exp\left\{D^i(t)\right\}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)},$$

where

$$D^i(t) = \int_t^{T_{2(i-1)}} \sigma_I^{i-1}(u)^2 - \sigma_I^{i-1}(u)\rho_I^{i,i-1}(u)\sigma_I^i(u)$$
$$+ \sigma_I^{i-1}(u)\left(\frac{\sigma_n^{2i}(u)\delta^{2i}F_n^{2i}(t)}{1+\delta^{2i}F_n^{2i}(t)}\rho_{n,I}^{2i,i-1}(u) + \frac{\sigma_n^{2i-1}(u)\delta^{2i-1}F_n^{2i-1}(t)}{1+\delta^{2i-1}F_n^{2i-1}(t)}\rho_{n,I}^{2i-1,i-1}(u)\right)\mathrm{d}u\,.$$

$$IC(t, T_{2(i-1)}, T_{2i}, \kappa) \approx P_n(t, T_{2i})\left(\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\mathrm{e}^{D^i(t)}\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(t)}{K\,\mathcal{I}^{i-1}(t)}\right) + D^i(t) + \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right)\right.$$
$$\left. - K\Phi\left(\frac{\ln\left(\frac{\mathcal{I}^i(t)}{K\,\mathcal{I}^{i-1}(t)}\right) + D^i(t) - \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right)\right),$$
$$\approx P_n(t, T_{2i})\left((1+\delta_I^i F_I^i(t))\Phi\left(\frac{\ln\left(\frac{(1+\delta_I^i F_I^i(t))}{1+\delta_I^i\kappa}\right) + \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right)\right.$$
$$\left. - (1+\delta_I^i\kappa)\,\Phi\left(\frac{\ln\left(\frac{(1+\delta_I^i F_I^i(t)}{1+\delta_I^i\kappa}\right) - \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right)\right), \tag{2.56}$$

where

$$\mathcal{V}^i(t)^2 = \int_t^{T_{2(i-1)}} \sigma_I^i(u)^2 + \sigma_I^{i-1}(u)^2 - 2\rho_I^{i,i-1}(u)\sigma_I^{i-1}(u)\sigma_I^i(u)\,\mathrm{d}u + \int_{T_{2(i-1)}}^{T_{2i}} \sigma_I^i(u)\,\mathrm{d}u\,.$$

Remark: Note that the formulas for YYIIS and inflation caplets only contain information about parts of the correlation structure, basically rates with similar times of expiry. If one wants to use this model for Monte Carlo simulation one needs the whole correlation matrix. Since there is no way of getting this information out of the reviewed market instruments, we have to resort to historical estimation to be able to determine these essential parameters (see section 3.1).

2.3.5. Forward CPI models - another approach

The idea of the so far presented forward CPI model is to model each forward CPI with a lognormal process driven by one Brownian motion. At all this results in N Brownian motions driving the forward CPIs. An alternative approach (as done in Mercurio and Moreni [34]) would be to use N Brownian motions and let the i-th forward CPI be driven by the first i Brownian motions.

So $\mathcal{I}^N(t)$ shall be driven by all N Brownian motions. We therefore choose the dynamcis

$$\mathrm{d}\mathcal{I}^N(t) = \mathcal{I}^N(t)\left(\sum_{j=\beta_I(t)}^{N}\sigma_I^j(t)\,\mathrm{d}Z_{I,j}^N(t)\right), \quad 0 \le t \le T_{2N}, \tag{2.57}$$

where $\beta_I(t) = 2\inf_j\{t \le T_{2j}\}$, $(Z_{I,j}^N)_{0\le t\le T_{2N}}$ are Brownian motions under $Q_n^{T_{2N}}$ with deterministic instantaneous correlation structure $(\rho_I^{k,l})_{k,l=1,\dots,N}$. Also we model the nominal interest market using a LMM as in section 2.3.4, where the Brownian motions $Z_{n,i}^i, i = 1,\dots,2N$ have a deterministic instantaneous correlation with $Z_{I,j}^N$ denoted again by $(\rho_{n,I}^{k,l})_{k=1,\dots,2N,l=1,\dots,N}$. We then define for $i = 1,\dots N-1$, $j = 1,\dots,i$

$$Z_{I,j}^i(t) := Z_{I,j}^N(t) - \int_0^t \sum_{k=2i}^{2N}\rho_{n,I}^{k,j}(s)\sigma_n^k(s)\frac{\delta^k F_n^k(s)}{1+\delta^k F_n^k(s)}\,\mathrm{d}s\,, \quad 0 \le t \le T_{2j} \tag{2.58}$$

and define the dynamics of the i-th forward CPI ($i = 1,\dots,N-1$) as

$$\mathrm{d}\mathcal{I}^i(t) = \mathcal{I}^i(t)\left(\sum_{j=\beta_I(t)}^{i}\sigma_I^j(t)\,\mathrm{d}Z_{I,j}^i(t)\right), \quad 0 \le t \le T_{2i}. \tag{2.59}$$

We now translate this in the JY setting and show that $Z_{I,j}^i$ is a Brownian motion under $Q_n^{T_{2i}}$. We have the economic interpretation that each Brownian motion models the inflation for one year. The i-th forward CPI depends on the inflation of all the years before its fixing time T_{2i} as represented in (2.59).

To interprete this in a JY setting we look at the dynamics of $\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}$, which using the definitions of (2.58) and (2.59) are

$$\begin{aligned}\mathrm{d}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)} =& \frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\left(\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^i(t) + \sum_{j=\beta_I(t)}^{i}\sigma_I^j(t)\rho_I^{i,j}(t)\sigma_I^i(t)\,\mathrm{d}t\right. \\ &\left. - \sum_{j=\beta_I(t)}^{i}\sigma_I^j(t)\left(\frac{\delta^{2i}\sigma_n^{2i}(t)F_n^{2i}(t)}{1+\delta^{2i}F_n^{2i}(t)}\rho_{n,I}^{2i,j}(t) + \frac{\delta^{2i-1}\sigma_n^{2i-1}(t)F_n^{2i-1}(t)}{1+\delta^{2i-1}F_n^{2i-1}(t)}\rho_{n,I}^{2i-1,j}(t)\right)\mathrm{d}t\right).\end{aligned} \tag{2.60}$$

Comparing this with (2.31) and taking quadratic variations of both we find

$$\sigma_I^i(t)^2 = (\underline{\sigma}(t,T_{2i}) - \underline{\sigma}(t,T_{2(i-1)}))^T\rho(\underline{\sigma}(t,T_{2i}) - \underline{\sigma}(t,T_{2(i-1)})),$$

and that we have the following relationship

$$Z_{I,j}^N(t) \stackrel{d}{=} \frac{(\underline{\sigma}(t,T_{2j}) - \underline{\sigma}(t,T_{2(j-1)}))^T}{\sigma_I^j(t)^2}W^{T_{2N}}(t), \quad j = 1,\dots,N. \tag{2.61}$$

We can express $\rho_I^{i,j}$ in an equivalent JY model as

$$\begin{aligned}\rho_I^{i,j}(t) &= \frac{\mathrm{d}Z_{I,i}^N(t)\ \mathrm{d}Z_{I,j}^N(t)}{\mathrm{d}t} = \frac{\mathrm{d}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\ \mathrm{d}\frac{\mathcal{I}^j(t)}{\mathcal{I}^{j-1}(t)}}{\sqrt{(\,\mathrm{d}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)})^2}\sqrt{(\,\mathrm{d}\frac{\mathcal{I}^j(t)}{\mathcal{I}^{j-1}(t)})^2}} \\ &\overset{(2.31)}{=} \frac{(\underline{\sigma}(t,T_{2i})-\underline{\sigma}(t,T_{2(i-1)}))^T}{\sigma_I^i(t)}\rho\frac{(\underline{\sigma}(t,T_{2j})-\underline{\sigma}(t,T_{2(j-1)}))}{\sigma_I^j(t)}\end{aligned}$$

and

$$\begin{aligned}\rho_{I,n}^{i,j}(t) &= \frac{\mathrm{d}Z_{I,i}^N(t)\ \mathrm{d}Z_{n,j}^j(t)}{\mathrm{d}t} = \frac{\mathrm{d}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}\ \mathrm{d}F_n^j(t)}{\sqrt{(\,\mathrm{d}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)})^2}\sqrt{(\,\mathrm{d}F_n^j(t)\,)^2}} \\ &= \frac{(\underline{\sigma}(t,T_{2i})-\underline{\sigma}(t,T_{2(i-1)}))^T}{\sigma_I^i(t)}\begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n}\end{pmatrix}\frac{\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1})}{\|\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1})\|}.\end{aligned}$$

Furthermore in a equivalent JY setting for the nominal LMM we again require (see 2.3.4)

$$\begin{aligned}\sigma_n^i(t) &= \frac{(1+\delta^i F_n^i(t))}{\delta^i F_n^i(t)}\|\Sigma_n(t,T_i)-\Sigma_n(t,T_{i-1})\|, \\ \rho_n^{i,j}(t) &= \frac{\mathrm{d}Z_{n,i}^i(t)\ \mathrm{d}Z_{n,j}^j(t)}{\mathrm{d}t} = \frac{\mathrm{d}F_n^i(t)\ \mathrm{d}F_n^j(t)}{\sqrt{(\,\mathrm{d}F_n^i(t)\,)^2}\sqrt{(\,\mathrm{d}F_n^j(t)\,)^2}} \\ &= \frac{(\Sigma_n(t,T_i)-\Sigma_n(t,T_{i-1}))^T}{\|\Sigma_n(t,T_i)-\Sigma_n(t,T_{i-1})\|}\frac{(\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1}))}{\|\Sigma_n(t,T_j)-\Sigma_n(t,T_{j-1})\|}.\end{aligned}$$

If we choose $\underline{\sigma}(t,T_i), i=1,\dots,2N$ so that all the above relations are fulfilled we can find such an equivalent JY model.

We can then use previous results to show that the stochastic processes defined in (2.58) are really Brownian motions under the appropiate forward measure. Since we know that $\underline{\sigma}(t,T_{\beta_I(t)-1})=0$ (the forward CPI is fixed at that time already) we have

$$\underline{\sigma}(t,T_{2i}) = \sum_{j=\beta_I(t)}^{i}(\underline{\sigma}(t,T_{2j})-\underline{\sigma}(t,T_{2(j-1)})).$$

Also

$$\Sigma_n(t,T_{2i})-\Sigma_n(t,T_{2(i-1)}) = (\Sigma_n(t,T_{2i})-\Sigma_n(t,T_{2i-1}))+(\Sigma_n(t,T_{2i-1})-\Sigma_n(t,T_{2(i-1)}))$$

so that we have

$$
\begin{aligned}
Z_{I,j}^{i}(t) =& Z_{I,j}^{N}(t) - \int_0^t \sum_{k=2i}^{2N} \rho_{n,I}^{k,j}(s)\sigma_n^k(s)\frac{\delta^k F_n^k(s)}{1+\delta^k F_n^k(s)}\,\mathrm{d}s \\
\overset{(2.61)}{=}& \frac{(\underline{\sigma}(t,T_{2j}) - \underline{\sigma}(t,T_{2(j-1)}))^T}{\sigma_I^j(t)} W^{T_{2N}}(t) \\
&- \int_0^t \sum_{k=2i}^{2N} \frac{(\underline{\sigma}(s,T_{2j}) - \underline{\sigma}(s,T_{2(j-1)}))^T}{\sigma_I^j(s)} \begin{pmatrix} I_{d_n} \\ \rho_{R,n} \\ \rho_{I,n} \end{pmatrix} (\Sigma_n(s,T_k) - \Sigma_n(s,T_{k-1}))\,\mathrm{d}s \\
\overset{(2.12)}{=}& \frac{(\underline{\sigma}(t,T_{2j}) - \underline{\sigma}(t,T_{2(j-1)}))^T}{\sigma_I^j(t)} W^{T_{2i}}(t)
\end{aligned}
$$

is indeed a Brownian motion under $Q_n^{T_{2i}}$. Similary using the result from (2.35) we have that

$$
Z_{I,i}^d(t) = Z_{I,i}^i(t) + \int_0^t \sum_{j=\beta(s)}^{2i} \frac{\delta^j \sigma_n^j(s) F_n^j(s)}{1+\delta^j F_n^j(s)} \rho_{n,I}^{j,i}(s)\,\mathrm{d}s
$$

is a Brownian motion under Q_n^d. We again want to use Q_n^d-Brownian motions for actual modelling. Comparing (2.59) and (2.60) we see that it is more convenient to model $\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}$ since under Q_n^d the dynamics of (2.59) would include a double sum. Hence we denote $Y^i(t) = \frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)}$ and model this quantity instead. Summarizing this we have the following model.

Model: *Consider a time structure* $\{0 = T_0, T_1, \ldots, T_{2N}\}$ *and define the following dynamics under* Q_n^d

$$
\begin{aligned}
\mathrm{d}F_n^i(t) &= F_n^i(t)\sigma_n^i(t)\left(\mathrm{d}Z_{n,i}^d(t) + \sum_{j=\beta(t)}^{i} \rho_n^{i,j}(t)\sigma_n^j(t)\frac{\delta^j F_n^j(t)}{1+\delta^j F_n^j(t)}\,\mathrm{d}t\right), \\
\mathrm{d}Y^i(t) &= Y^i(t)\Bigg(\sigma_I^i(t)\,\mathrm{d}Z_{I,i}^d(t) + \sigma_I^i(t)\sum_{j=\beta(t)}^{2i} \frac{\delta^j\sigma_n^j(t)F_n^j(t)}{1+\delta^j F_n^j(t)}\rho_{n,I}^{j,i}(t)\,\mathrm{d}t + \sum_{j=\beta_I(t)}^{i} \sigma_I^j(t)\rho_I^{i,j}(t)\sigma_I^i(t)\,\mathrm{d}t \\
&\quad - \sum_{j=\beta_I(t)}^{i} \sigma_I^j(t)\left(\frac{\delta^{2i}\sigma_n^{2i}(t)F_n^{2i}(t)}{1+\delta^{2i}F_n^{2i}(t)}\rho_{n,I}^{2i,j}(t) + \frac{\delta^{2i-1}\sigma_n^{2i-1}(t)F_n^{2i-1}(t)}{1+\delta^{2i-1}F_n^{2i-1}(t)}\rho_{n,I}^{2i-1,j}(t)\right)\mathrm{d}t\Bigg).
\end{aligned}
$$

$Z_{n,i}^d$ *and* $Z_{I,i}^d$ *are one-dimensional Brownian motions with a deterministic instantaneous correlation structure given by*

$$
\begin{pmatrix} (\rho_n^{i,j})_{i,j=1,\ldots,2N} & (\rho_{n,I}^{i,j})_{i=1,\ldots,2N,j=1,\ldots,N} \\ (\rho_{n,I}^{j,i})_{i=1,\ldots,2N,j=1,\ldots,N} & (\rho_I^{i,j})_{i,j=1,\ldots,N} \end{pmatrix}.
$$

The starting values of F_n^i *and* Y_i*, as well as the parameter functions* σ_n^i *and* σ_I^i *are positive* *and deterministic.*

Valuation of forward inflation rates and inflation caps

Valuation of YYIIS and inflation caplets can now be done using the same drift freezing approximations as in the previous approaches resulting in similar formulas:

$$F_I^i(t) \approx \frac{1}{\delta_I^i}\left(\mathrm{e}^{D^i(t)}\frac{\mathcal{I}^i(t)}{\mathcal{I}^{i-1}(t)} - 1\right),$$

$$IC(t, T_{2(i-1}, T_{2i}, \kappa) \approx P_n(t, T_{2i})\left((1+\delta_I^i F_I^i(t))\Phi\left(\frac{\ln\left(\frac{\delta_I^i(1+\delta_I^i F_I^i(t))}{1+\delta_I^i\kappa}\right) + \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right)\right.$$
$$\left. - (1+\delta_I^i\kappa)\,\Phi\left(\frac{\ln\left(\frac{\delta_I^i(1+\delta_I^i F_I^i(t)}{1+\delta_I^i\kappa}\right) - \frac{1}{2}\mathcal{V}^i(t)^2}{\mathcal{V}^i(t)}\right)\right),$$

where

$$D^i(t) = \int_t^{T_{2(i-1)}} \sum_{j=\beta_I(t)}^{i} \sigma_I^j(u)\rho_I^{i,j}(u)\sigma_I^i(u)$$
$$- \sum_{j=\beta_I(t)}^{i} \sigma_I^j(u)\left(\frac{\delta^{2i}\sigma_n^{2i}(u)F_n^{2i}(t)}{1+\delta^{2i}F_n^{2i}(t)}\rho_{n,I}^{2i,j}(u) + \frac{\delta^{2i-1}\sigma_n^{2i-1}(u)F_n^{2i-1}(t)}{1+\delta^{2i-1}F_n^{2i-1}(t)}\rho_{n,I}^{2i-1,j}(u)\right)\mathrm{d}u\,,$$
$$\mathcal{V}^i(t)^2 = \int_t^{T_{2i}} \sigma_I^i(u)^2\,\mathrm{d}u\,.$$

We see that there is one big difference. The convexity adjustment now depends not only on the subdiagonal of the correlation matrices but on the whole correlation matrix. This is an advantage, since it theoretically allows us to calibrate the whole correlation matrix to market data. In practice this is problematic since we will see that the order of magnitude of the convexity adjustment is in fact so small, that sometimes even bid/ask spreads are wide enough to include possible convexity adjustments. Therefore market data is not reliable enough to calibrate all the correlation parameters and we loose this advantage.

The main advantage of this model is that the k-th simulated Brownian motion can be interpreted as the CPI development for the k-th year, while in the earlier version the k-th Brownian motion could be interpreted as the CPI development until up to the k-th year. In this approach each Brownian motion represents disjoint development therefore allowing for a more intuitive interpretation of the parameters of the model.

The main drawback is that this results in more drift terms for the simulated rates, therefore making simulation and calibration computationally more costly.

2.3.6. Other models and extensions

Forward inflation market models

So far we have considered market models where we try to model the forward CPIs with starting values given by ZCIIS rates. To be able to price YYIIS we used approximations by fixing drifts to be deterministic. Following this approach we derived approximate dynamics of forward inflation rates, which are then of a shifted lognormal kind.

However, one could coil this up the other way around. We defined forward inflation rates to be martingales under the appropiate forward measure. One could choose to model those forward inflation rates as shifted lognormal martingales with some deterministic volatility. Then pricing inflation caps and floors would be straightforward resulting in Black-like formulas allowing for instant calibration to YYIIS rates (being starting values). The problem then is valuing ZCIIS, for which no closed form formulas have been found. One would have to use non-standard approximation techniques to get formulas usable for calibration. Since ZCIIS markets are far more liquid than YYIIS it seems the better choice to exactly (and instantly) fit ZCIIS rates (see Mercurio and Moreni [34]).

Further models

The here presented model can be extended to account for volatility smiles - either in nominal markets or in inflation markets. Mercurio and Moreni extend it first via Heston-like model (Mercurio and Moreni [33]) and later via SABR dynamics (Mercurio and Moreni [34]). Kenyon [27] proposes normal mixture models for inflation while Hinnerich [22] extends the JY framework allowing for jumps. All these models allow for smile modelling in inflation markets.

A different approach is chosen in Falbo et al. [10], where the authors model the nominal short-rate as a CIR (Cox, Ingersoll, Ross) process and the instantaneous inflation via a Vasicek model. Their approach has the advantage of additionally allowing to estimate the inflation risk premia.

3. Implemenation of a inflation market model

As we have seen, the JY model is an extension of the HJM model. In order to calibrate a JY model we have to take two steps. Before we can tackle the task of calibrating to inflation markets, we first have to calibrate an interest rate model (in our case a LMM). This is our first goal in this chapter. There are two approaches to calibrate a model:

- **Fitting market data:** For this approach one uses market prices of liquid instruments, where the prices depend on the parameters of the model one wants to calibrate. One then minimizes the difference between the theoretical model prices and the actual market prices, therefore generating a minimization problem to find the parameters.

- **Historical estimation:** Here one uses historical time series of market instruments and then tries to estimate the parameters used in the model.

Fitting is a forward-looking procedure (since market prices depend on future expectations) while historical estimation is backward looking. In general it is a good idea to first use historical data to get some idea about the shape of the parameters one wants to estimate and then fit parameters of a certain range using market data. Since we have no knowledge about the parameters used in our model yet, we first take a look at historical data.

3.1. Historical estimation of model parameters

In order to estimate the parametes of the reviewed models we want to observe nominal forward rates and forward CPIs (or fractions of those). The problem is that those quantities are not directly available, so one first has to calculate them out of available market data, in our case swap and ZCIIS rates.

We used daily data of the past 6 years, i.e. data from 1 Aug. 2005 to 29. Jul 2011. We normally split the data in 6 one-year periods allowing us to repeat the estimation procedure 6 times. This allows us to gain information about how parameters change over time.

3.1.1. Calculating the yield curve and forwards

The first step is to calculate the yield curve and thereof the nominal forward rates. We use euro swap rates with maturities of $1, 2, 3, 5, 7, 10, 20, 30$ years as input data (see table

	$S_1(t)$	$S_2(t)$	$S_3(t)$	$S_5(t)$	$S_7(t)$	$S_{10}(t)$	$S_{20}(t)$	$S_{30}(t)$
01.08.2005	2,252	2,418	2,572	2,849	3,098	3,393	3,830	3,927
02.08.2005	2,258	2,426	2,581	2,865	3,114	3,410	3,845	3,940
...								
29.07.2011	1,868	1,919	2,074	2,488	2,824	3,159	3,567	3,445

Table 3.1.: Swaprates (in percent) for maturities of $1, 2, 3, 5, 7, 10, 20, 30$ years from 1 Aug. 2005 to 29 Jul. 2011
SOURCE: Bloomberg L.P. (2006), <EUSA>, retreived 20 Aug. 2011

3.1). We are given data for 6 years (1 Aug. 2005 $\leq t \leq$ 29 Jul. 2011) and the swap rates $S_i(t) = S_{t,t+i}(t)$ (defined in section 3.2.2) at each time t are then given for maturities $t + i, i \in \{1, 2, 3, 5, 7, 10, 20, 30\}$.

Swap rates from bloomberg are quoted for terms of semiannual floating payments linked to the 6-month EURIBOR (ACT/360) and fixed payments of 1 year (30/360).

Step 1: Calculation of a yield curve:

We first need to interpolate the 8 given swap rates to get annual swap rates $S_k(t)$, $k = 1, \ldots, 30$. We decided to use B-splines (see Boor [4]) for this. Given annual swap rates we can calculate annual discount factors by

$$P_n(t, t+k) = \frac{1 - \sum_{i=1}^{k-1} S_k(t) P_n(t, t+i)}{1 + S_k(t)},$$

and then continuously compounded annual yield rates

$$y(t, t+k) = \frac{-\ln\left(P_n(t, t+k)\right)}{k}.$$

Interpolating these yields (again using B-splines) we are then given a yield curve $y(t, T), t \leq T \leq t + 30$ for each day of the six years.

Step 2: Calculation of the forward rates:

We now calculate semiannual forward rates for a fixed tenor structure $T_0 =$ 1. Aug. 2005, $T_1 =$ 1. Mar. 2006, ..., $T_{60} =$ 1. Aug. 2035. Given the above calculated yield curves we can do this by

$$F_n^i(t) := F_n(t, T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\mathrm{e}^{y(t,T_i)(T_i - t) - y(t,T_{i-1})(T_{i-1} - t)} - 1 \right), \quad \beta(t) \leq i \leq 60, \tag{3.1}$$

where $\beta(t) = \inf\{j \in \{1, \ldots, 61\} : t \leq T_{j-1}\}$ is the index of the first forward rate not fixed yet.

Finally we end up with forward rates $F_n^i(t)$ for $T_0 \leq t \leq \min\{T_{i-1}, 29 \text{ Jul. } 2011\}$ and we can use those for estimation.

3.1.2. Estimating volatility and correlation

We define volatility as the standard deviation of the annualized log returns of a financial asset and denote the annualized log return of an asset S over some period $[t_{i-1}, t_i]$ by

$$r_i = \ln\left(S_{t_i}\right) - \ln\left(S_{t_{i-1}}\right).$$

Remark: The term volatility is widely used in financial markets. Although originally motivated by the above definition it is nowadays used in a lot of contexts. A general exact definition is therefore not possible. Nevertheless one can always understand volatility as a concept of standard deviation of returns (a term also not uniquely defined).

If we assume that the price S of this asset follows a geometric Brownian motion with constant parameters μ, σ:

$$\mathrm{d}S(t) = S(t)\left(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\right),$$

then

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}.$$

Choosing n periods $(t_0 < t_1 < t_n)$ of equidistant spacing $\triangle$ we see that

$$r_i \overset{iid}{\sim} N\left(\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{\triangle}, \sigma^2\right)$$

and that a possible estimator for σ^2 is the classical variance estimator

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(r_i - \frac{1}{n}\sum_{i=1}^{n} r_i\right)^2.$$

Accordingly we have estimators for covariances and correlations.

Remark: Practicioners often define volatility as the annualized percentage returns of an asset. Given a geometric Brownian motion we have

$$\mathrm{d}\ln\left(S_t\right)\ \mathrm{d}\ln\left(S_t\right) = \frac{\mathrm{d}S_t}{S_t}\frac{\mathrm{d}S_t}{S_t} = \sigma^2\,\mathrm{d}t,$$

which is why infinitesimally log returns and percentage returns are the same. However, for a discrete equidistant grid the percentage returns are only assymptotically identically distributed. Nevertheless percentage returns are widely used for volatility estimation.

Remark: When using market data, one doesn't have equidistant spaced data, since there are holidays, weekends and sometimes data errors. Nevertheless for estimation we assume that each data point corresponds to one day of a year with about 250 days. Therefore in our case $\triangle \approx \frac{1}{250}$.

There are far more sophisticated procedures for estimating volatilities, however we stick to this simple standard approach. An overview of alternative estimators can be found in Brandt and Kinlay [5].

3.1.3. Estimating nominal forward volatilities and correlations

We go ahead and use the above estimator on the forward rates we calculated according to section 3.1.1. As mentioned earlier we partition the data in 6 intervals of one year. Using the notation of 3.1.1 this are the 6 intervals $I_i = [T_{2(i-1)}, T_{2i}), i = 1, \ldots, 6$. For each of the intervals we use the available calculated forward rates to estimate the volatility. Formally we assume the forward rates to follow a stochastic processes

$$\mathrm{d}F_n^i(t) = F_n^i(t)(\{\ldots\}\,\mathrm{d}t + \sigma_n^i(t)\,\mathrm{d}Z_{n,i}^P(t)),$$

where $Z_{n,i}^P$ are correlated P-Brownian motions and $\sigma_n^i(t)$ and the instantaneous correlations are constant on each interval $I_j, j = 1, \ldots, 6$. The drift term is in practice not constant, but since drift changes are of small magnitude we can ignore it anyway and use the classical variance and correlation estimates on each interval I_j.

Volatility

The resulting volatility curves (relating estimated volatility to the payment time of the underlying forward rate) are then presented in figure 3.1. The title of the individual graphs represents the data used for this estimation.

The results of this procedure are unexpected. We sometimes get strong rising volatility for longer maturities. This is definitely not a behaviour expected and described in literature (see e.g. Rebonato [38]). Taking a look at the volatilities of the used swap rates (figure 3.2) we find that this behaviour is not present at all (here we plotted the estimates of a constant volatility for $S_i(t)$ against its maturity i).

We think the effect for forward rate volatilities is due to numerical instabilities. If we consider equation (3.1), we see that one forward rate has to balance out the interest differences of possibly many years. While for short maturities this isn't too bad, for longer maturities this leads to oscillating behaviour.

One could argue that the observed behaviour should be accounted for in a model since the rates obviously show this feature. The problem is that only swap rates are traded and while their observed volatility should be realistic this doesn't have to be true for the non-traded forward rates. We think if forward rates were actually traded, their behaviour would

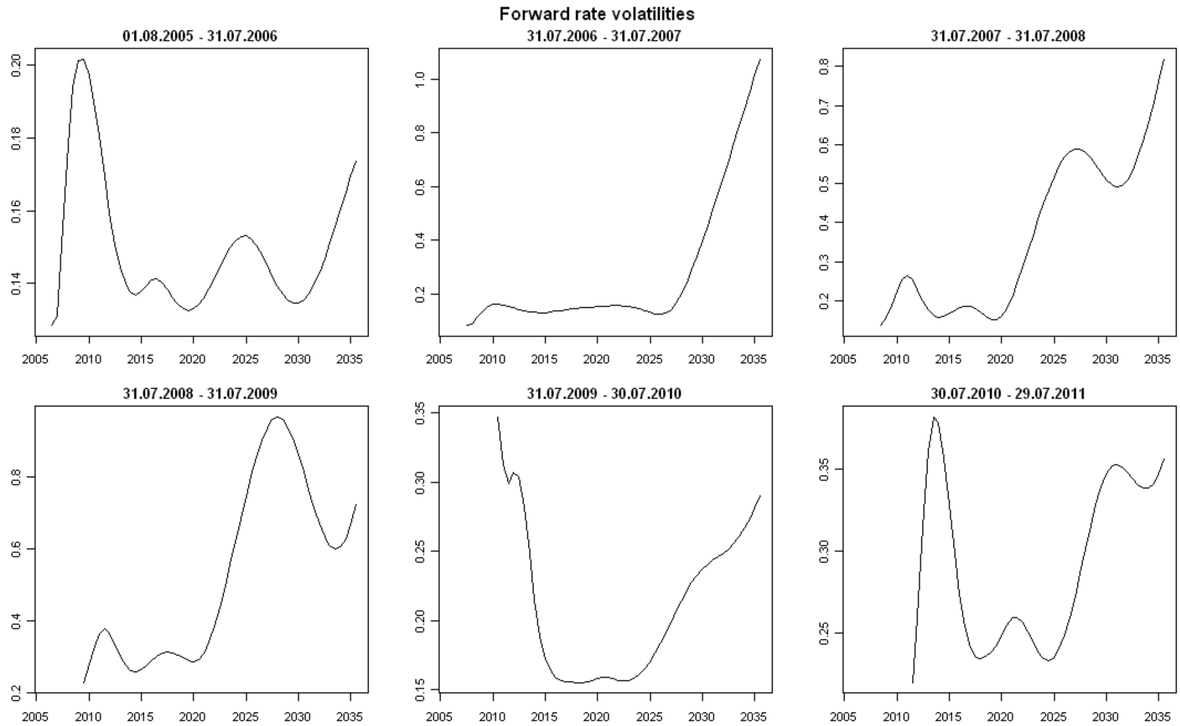


Figure 3.1.: Estimated volatilities of nominal forward rates with maturities up to 2035

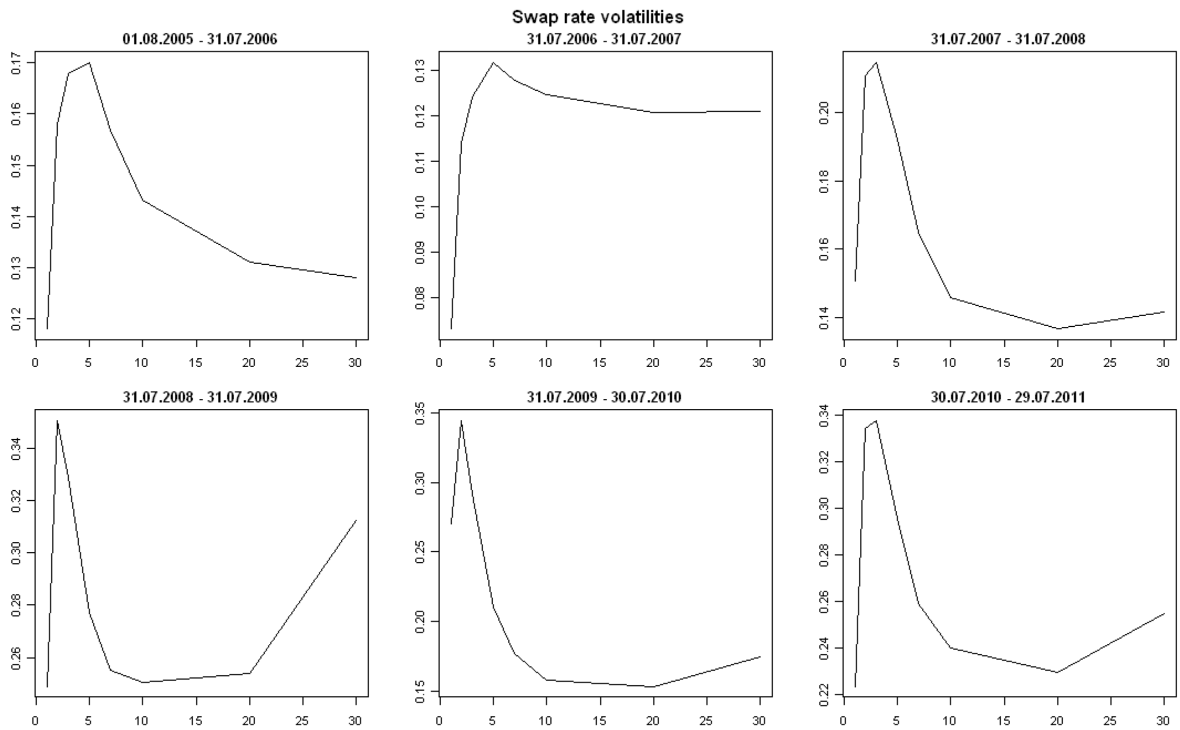


Figure 3.2.: Estimated volatilities of nominal swap rates

somewhat mirror that of swap rates. In fact if we look at forward rates with maturities up to 15 years, where the effect should not be as strong, we see (figure 3.3) that the behaviour is then similar to that of swap rates.

Then the volatility typically has a humped structure, meaning that volatilities are highest for forward rates with maturity (expiry) 2-3 years ahead and are decreasing for rates with farther ahead maturities. This is actually a behaviour stated in most literature and we include this behaviour in later model specification.

Correlation

The second nominal quantity of interest is the correlation matrix of forward rates. We analyze this by looking at heatmaps for the estimated correlations (figure 3.4).

Here we only used forward rates of maturities up to 2020 since we do not want to distort our estimations by the earlier described behaviour of forward rate volatilities. There is an obvious structure for forward rate correlations. Adjacent forward rates are highly correlated and the correlation decreases for farther apart rates. A second more subtle point is that adjacent forward rates with higher maturities are typically stronger correlated than adjacent ones with lower maturities. Correlation values are greater than zero. Parametric forms representing this behaviour will be discussed in section 3.2.4.

3.1.4. Estimation using forward CPIs

	$K_1(t)$	$K_2(t)$	$K_3(t)$	$K_5(t)$	$K_7(t)$	$K_{10}(t)$	$K_{20}(t)$	$K_{30}(t)$
01.08.2005	1,939	2,039	2,064	2,097	2,120	2,149	2,270	2,340
02.08.2005	1,942	2,042	2,067	2,100	2,130	2,160	2,273	2,350
...								
29.07.2011	1,868	1,888	1,941	2,033	2,139	2,240	2,367	2,453

Table 3.2.: ZCIIS rates (in percent) for maturities of $1, 2, 3, 5, 7, 10, 20, 30$ years from 1 Aug. 2005 to 29 Jul. 2011
SOURCE: Bloomberg L.P. (2006), <EUSWIT>, retreived 20 Aug. 2011

The second part of our historical analysis is about inflation. For this we consider ZCIIS rates (see table 3.2) based on the euro HICP index for the same maturities as the swap rates $(1, 2, 3, 5, 7, 10, 20, 30$ years). Hence we are given ZCIIS rates $K_i(t) := K(t, t + i)$ for 1 Aug. 2005 $\leq t \leq$ 29 Jul. 2011, where $K(t, T)$ was defined in (1.3).

We then want to calculate CPI forwards $\mathcal{I}^i(t) = \mathcal{I}(t, T_{2i})$ for the dates T_2 =1. Aug 2006, T_4 = 1. Aug. 2007, ..., T_{60} = 1. Aug. 2035. Here we use the relationship

$$\mathcal{I}(t, T) = I(t)(1 + K(t, T))^{T-t}.$$

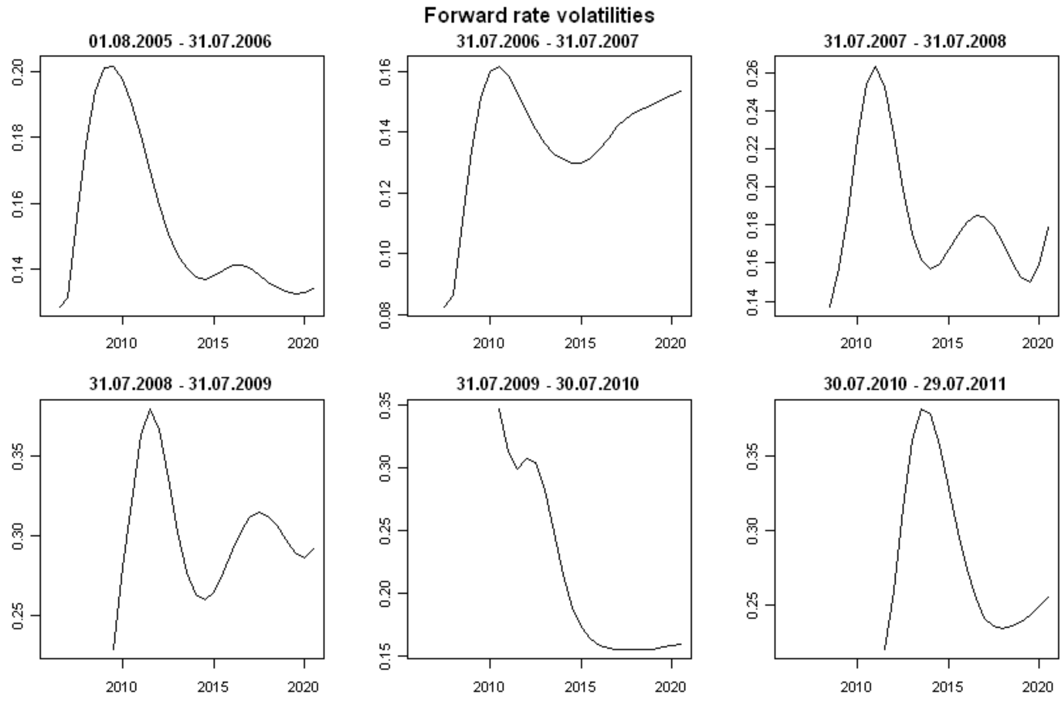


Figure 3.3.: Estimated volatilities for nominal forward rates with maturities up to 2020

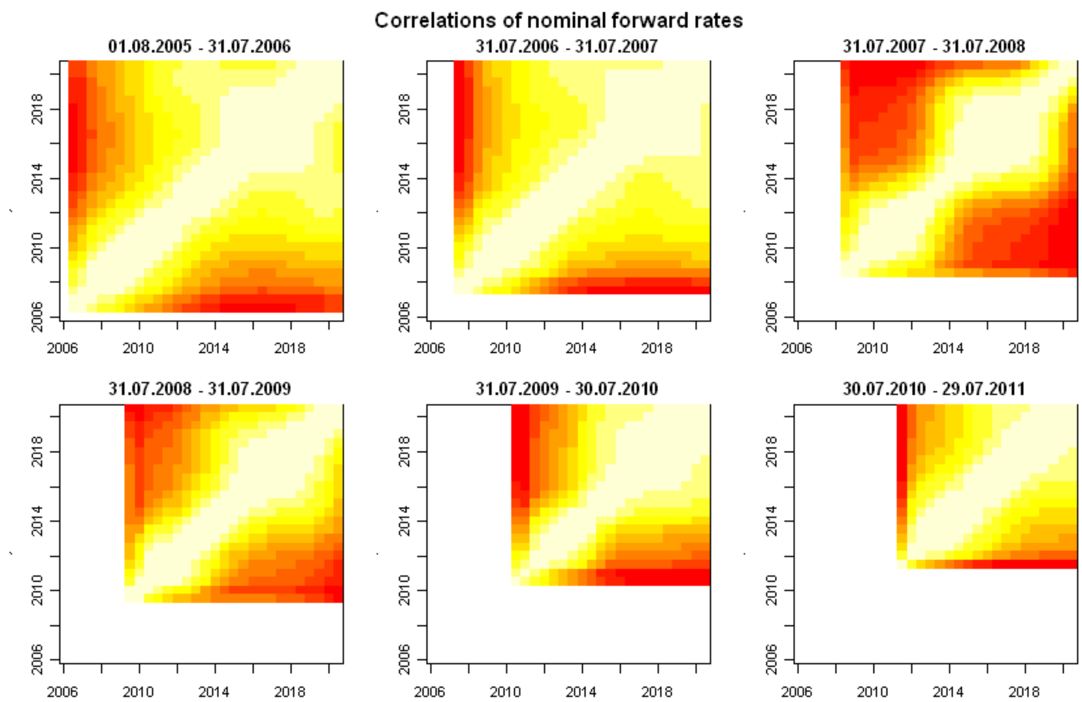


Figure 3.4.: Estimated correlations of nominal forward rates

We see that in order to calculate the CPI forwards we also need the historical values of the underlying index (see 1.2). The historical CPI values include seasonal effects while the rates $K(t,T)$ being only quoted for full year maturities don't. For the forward CPIs $\mathcal{I}^i$ to be without seasonal effects we therefore have to first seasonally adjust the historical CPI values. This is done by estimating monthly effects via linear regression and then adjusting the historical values of the CPI (see section 1.2.1). Given this seasonally detrended time series we then use linear interpolation to get daily historical CPI values.

Also note that in order to calculate $\mathcal{I}(t, T_{2i})$ we need ZCIIS rates for arbitrary T. Therefore we need to interpolate the rates $K_i(t)$. We again use smoothing B-splines for this. Combining all this we can calculate

$$\mathcal{I}^i(t) = I(t)(1 + K(t, T_{2i}))^{T_{2i}-t}, \quad \beta_I(t) \le i \le 30,$$

where $\beta_I(t) = 2\inf_j\{t \le T_{2j}\}$. Finally we end up with forward CPIs $\mathcal{I}^i(t)$ for $T_0 \le t \le \min\{T_{2i},\ 29 \text{ Jul. } 2011\}$.

Now we can calculate again log returns and use the standard variance estimation formula. Formally we have to assume that

$$\mathrm{d}\,\mathcal{I}^i(t) = \mathcal{I}^i(t)(\{\dots\}\,\mathrm{d}t + \sigma_I^i(t)\,\mathrm{d}Z_{I,i}^P,$$

where $Z_{I,i}^P$ are correlated P-Brownian motions and $\sigma_I^i(t)$ and instantaneous correlations (also those between $Z_{I,i}^P$ and $Z_{n,j}^P$) are constant on each one of the six intervals $I_j, j = 1,\dots,6$. Again the drift is not constant, but changes are of a small magitude.

Volatility

The estimated volatility of forward CPIs can be found in figure 3.5, where we again plot the estimated volatility against the maturity T_{2i} of the forward CPI. Again the title of each graph represents the interval of the data used for estimation.

We can see increasing volatilities with increasing maturity of the forward CPI. Remembering the instabilities for nominal forward rates we also take a look at the ZCIIS volatility. Since the ZCIIS rates can become negative , we can't assume lognormal dynamics. Therefore we first transform the ZCIIS rates to $1+K_i(t)$ and then use the same concept as before (this corresponds to a shifted lognormal model). The results can be found in figure 3.6. Here the estimated volatility is plotted against the maturity i of the ZCIIS rate.

We see that for ZCIIS rates the volatility is decreasing with increasing maturity. Comparing this to the behaviour of forward CPIs this is rather disturbing. Looking for an explanation we take a look at the definition of the forward CPI. Denoting by $\mathcal{I}(K_M)$ the forward CPI for M years depending on the ZCIIS rate ($\mathcal{I}(K_M) = (1+K_M)^M$) we see that

$$\frac{\frac{\partial(\mathcal{I}(K_M)}{\partial K_M}}{\mathcal{I}(K_M)} \Bigg/ \frac{\frac{\partial K_M}{\partial K_M}}{K_M} = \frac{(1+K_M)^{M-1} M K_M}{(1+K_M)^M} = \frac{M K_M}{1+K_M}. \tag{3.2}$$

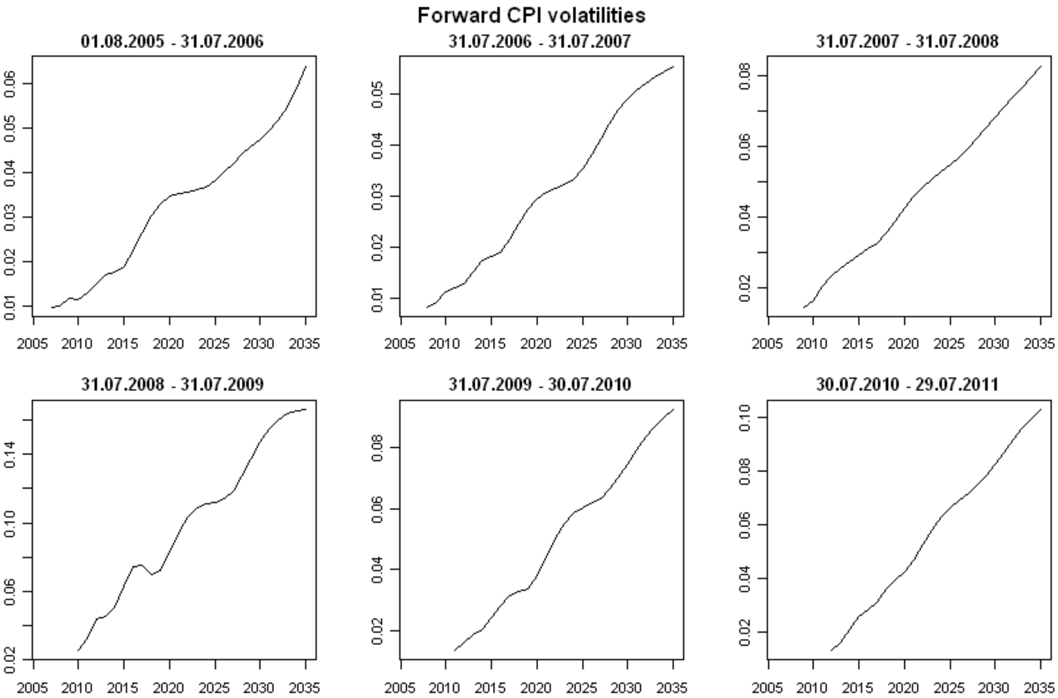


Figure 3.5.: Estimated forward CPI volatility

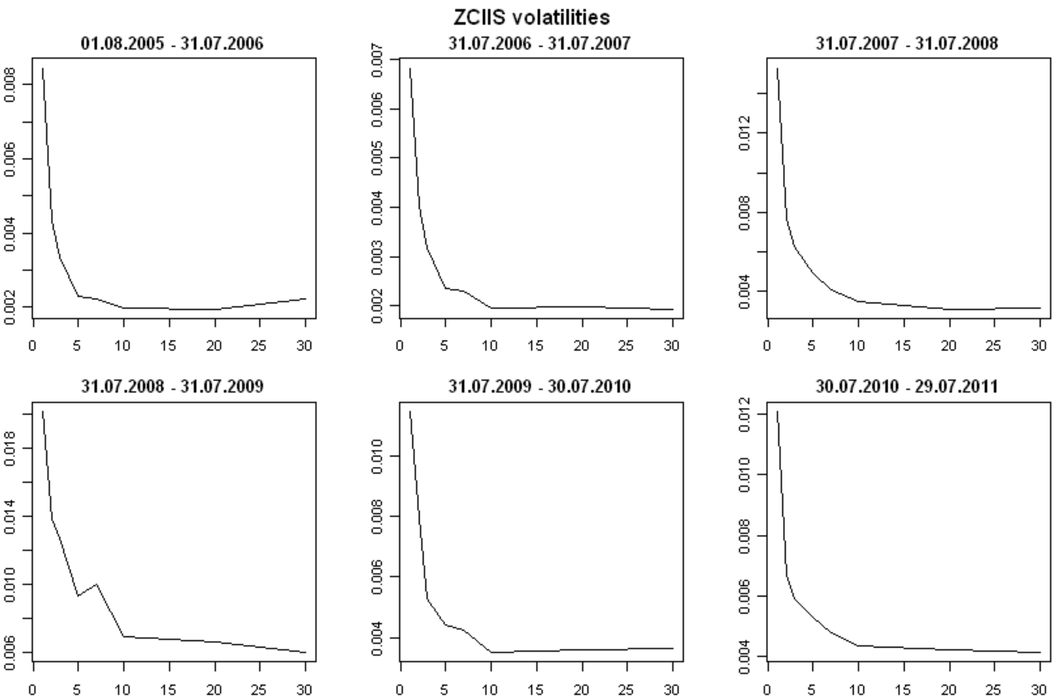


Figure 3.6.: Estimated ZCIIS volatility

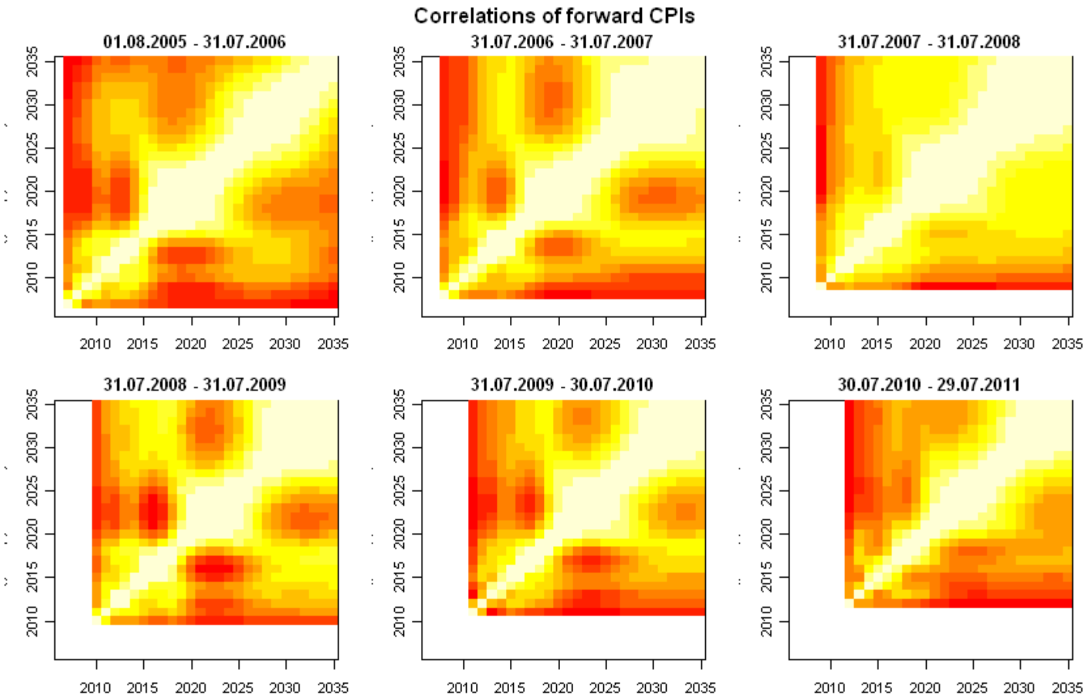


Figure 3.7.: Estimated forward CPI correlation

Since $|\frac{K_M}{1+K_M}|$ remains rather constant since ZCIIS rates typically around 2 %, we see that the percentage change of the forward CPI after a one percent change in the underlying ZCIIS rate is increasing linearly with increasing maturity.

This is an explanation for different forward CPI and ZCIIS volatilities, but we are still left with the question if we want to express the observed forward CPI volatility behaviour in a model. Again we think this to be rather dangerous since forward CPIs are not directly available. Hence we use this information carefully.

Correlations

We now take a look at the the forward CPI correlations (figure 3.7) using similar plots as for nominal forward rate correlation. We see a behaviour similar to nominal forward rate correlations. Forward CPIs with adjacent maturities are higly correlated and correlation is decreasing the further they are apart. Again correlation is rather increasing for farther apart maturities. Values are always positive and above .2. Again this is plausible and we will use this for modelling later.

The last step is the estimation of the correlations between nominal forward rates and forward CPIs. Resulting heat maps can be found in figure 3.8. There is no obvious behaviour visible. At best we can see that there is typically a peak between short-term interest rates and medium-term forward CPIs. This might be explained by the fact the central banks inflation targets are controlled by central banks interference on short term

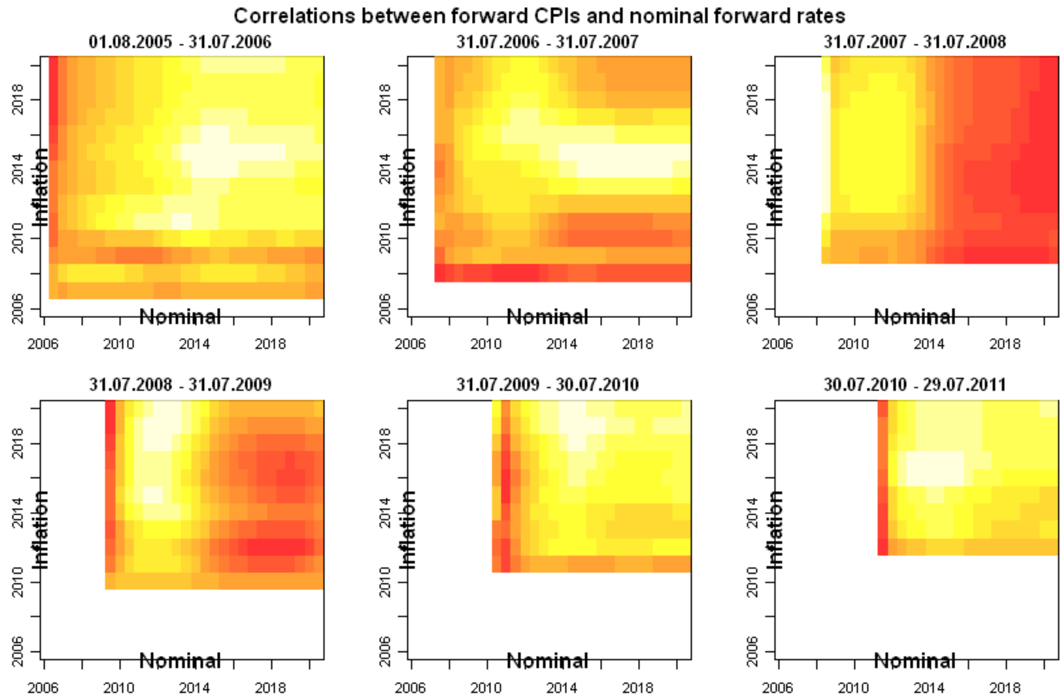


Figure 3.8.: Estimated correlation between nominal forward rates and forward CPIs

interest rate markets. However correlation values are mostly between .2 and .4. Looking at (assymptotic) confidence intervals, which are mostly bigger than .2 we conclude that correlation between different nominal forward rates and forward CPIs is rather similar. Therefore we propose to use constant correlation. The general interest rate level and the general inflation level are definitely correlated, but other than that not much can be said.

3.1.5. Estimation with forward CPI fractions

We have reviewed an alternative approach to model fractions of forward CPIs instead of forward CPIs in section 2.3.5. We now use those quantities (the returns of $Y_i(t) = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}$) instead of the returns of forward CPIs for estimation. Again formally assuming piecewise constant volatilities and instantaneous correlations (now for Y^i instead of $\mathcal{I}^i$) the results for volatility estimation can be found in figure 3.9.

First we notice the wild oscillating behaviour. This should be due to the interpolation of the ZCIIS. Notice that e.g. the lows for rates 15 or 25 years ahead from the considered data are where the biggest gaps in our available market data (1,2,3,5,7,10,20,30 year rates) are. Besides this oscilation we see that volatility is typically decreasing. One can even see a humped shape with humps at the 2 year rates. Therefore the behaviour is quite similar to that of the nominal forward rates and might be modelled accordingly.

The next step is to look at the correlation between indivdiual forward CPI fractions. The heat maps are plotted in figure 3.10. We see that the correlation is lowest for rates

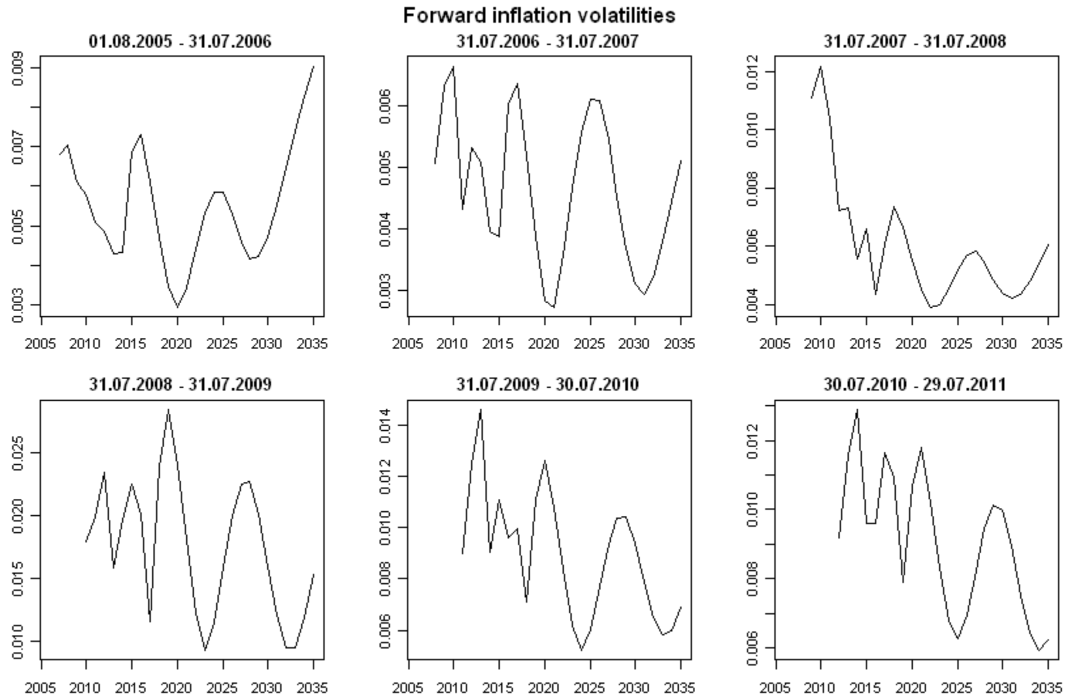


Figure 3.9.: Estimated forward CPI fraction volatility

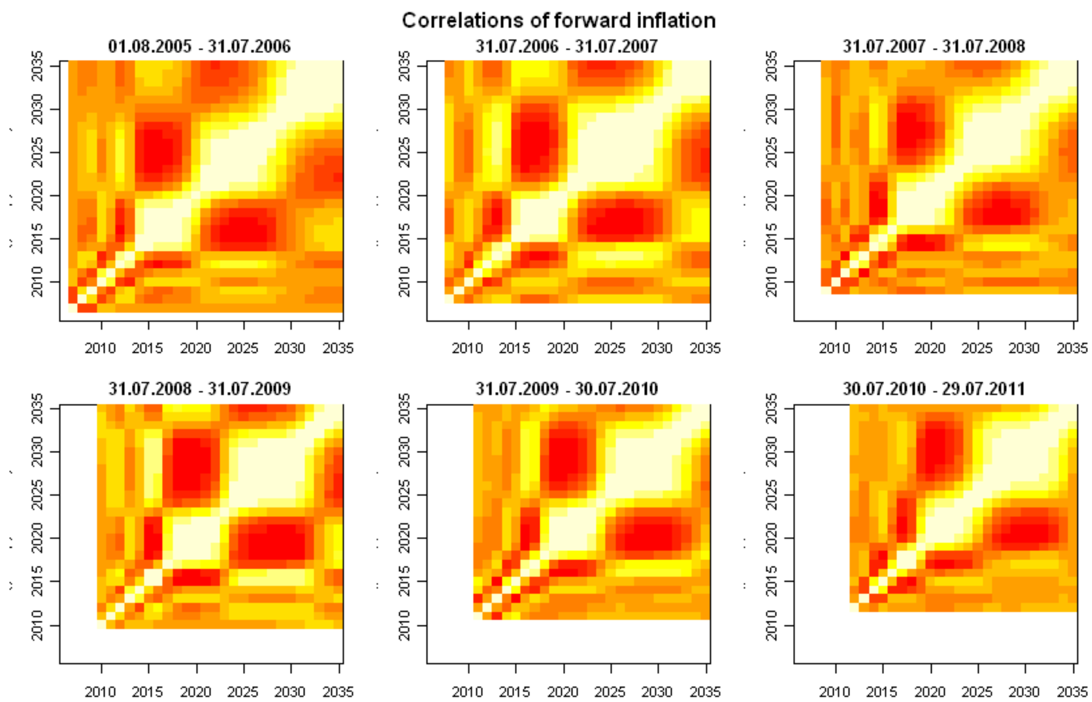


Figure 3.10.: Estimated forward CPI fraction correlation

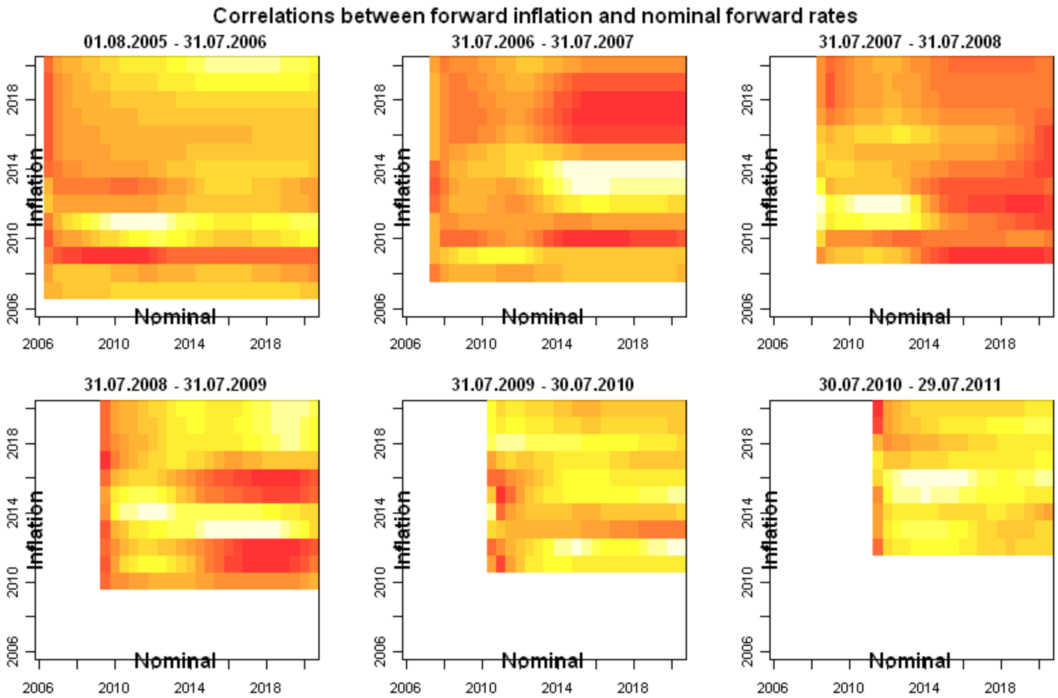


Figure 3.11.: Estimated correlation between nominal forward rates and forward CPI fractions

a few years apart (in fact clearly negative) and then increases for rates very close or far apart. Again rates with farther ahead maturities are longer correlated.

Finally we look again at the correlation between nominal forward rates and forward CPI fractions (figure 3.11). The results are similar to the forward CPI case. There is no obvious behaviour and correlation are at the same levels. Therefore this results in the same conclusion as for forward CPIs.

3.1.6. Comparison of the two approaches

One would expcect the CPI fraction approach to produce easier interpretable results. It turns out that this is not the case. We were able to guess the behaviour of volatilities, which are in fact somewhat similar to that of nominal forward rates, but especially for the correlation between nominal and inflation quantities we didn't get a clearer result. We would have expected that correlations are high for quantities with similar maturities considering that they contain information about the same time horizons (an idea in fact purchased in Mercurio and Moreni [34]). Surprisingly nothing like this was observed. Also the result on correlation of forward inflation rates was not necessarily expected and we have found no obvious explanation.

Because of this we do not want to draw any conclusion about which approach to favour. Considering that the first approach is numerically a bit faster, we stick to this approach in our implementation. Also as mentioned earlier because of the small order of magnitude of convexity adjustments, correlation calibration to market data is not meaningful for either of the two models and we have to make use of the results in this section to specify correlations. Therefore we especially focus on the results for correlation behaviour and here we favour the first approach. We take a closer look at this in section 3.3.

3.2. Calibration of a LIBOR market model

For calibrating LMMs one basically uses two types of instruments, interest rate caps and floors and interest rate swaptions. Both markets are very liquid and the motivation of the LMM was in fact to "intuitively" calibrate to caps and floors. Hence we first want to present the pricing formulas for interest rate caplets and floorlets and then discuss the matter of valuing swaptions in a LMM.

3.2.1. Caplets and floorlets

A caplet (Cpl) on the discrete forward rate $F_n(t, S, T)$ with strike K guarantees a time T payment of

$$(T - S)(F_n(S, S, T) - K)_+ .$$

Under a LMM the forward rate dynamics with Z a Q_n^T-Brownian motion are given by

$$\mathrm{d}F_n(t, S, T) = F_n(t, S, T)\sigma(t)\,\mathrm{d}Z(t) .$$

Hence by section B.4 the arbitrage free value of this contract is

$$\begin{aligned} Cpl(t, S, T, K) &= (T - S)P_n(t, T)\,\mathbb{E}^{Q_n^T}\left[(F_n(S, S, T) - K)_+ \,|\, \mathcal{F}_t\right] \\ &= (T - S)P_n(t, T)\left(F_n(t, S, T)\Phi(d_+) - K\Phi(d_-)\right), \end{aligned}$$

where

$$\begin{aligned} d_\pm &= \frac{1}{\overline{\sigma}\sqrt{S - t}}\left(\ln\left(\frac{F_n(t, S, T)}{K}\right) \pm \frac{1}{2}\overline{\sigma}^2(S - t)\right), \\ \overline{\sigma}^2 &= \frac{1}{S - t}\int_t^S \sigma(u)^2\,\mathrm{d}u \ . \end{aligned}$$

For a floorlet a similar result follows.

Remark: This formula is often referred to as Black's formula or Black76 formula and was motivated by the famous Black-Scholes formula for derivatives (Black and Fischer [3]). The rigorous mathermatical derivation by the change of numeraire technique, as in fact used above, was only introduced later. The market has been using this formula for a long time now and todays prices are quoted as the implied volatilities ($\overline{\sigma}$) calculated using this formula.

Note that the price of caplets only depends on the average (squared) volatility of the underlying forward rate. Caplets therefore contain no information about correlations between individual forward rates. We will get back to this in sections 3.2.3 and 3.2.4.

Caps and floors

Caps (or floors) are a combination of several caplets (or floorlets). Consider a time structure $T_0, T_1, \dots T_n$, then a cap with strike K consists of individual caplets with strike K on the forward rates $F_n^i(t)$, $i = 1, \dots, n$. Valuation is easily done by pricing the individual contracts and summing them up. The time differences normally are of a fixed tenor, mostly 3 or 6 month.

Market data & volatility smile

Caps	0,02	0,025	0,03	0,035	0,04	0,05	0,06	0,07	0,1
1	52,30%	50,70%	49,60%	49,00%	48,50%	47,90%	47,30%	47,10%	60,37%
2	60,10%	58,60%	57,30%	56,20%	55,30%	53,80%	52,10%	51,20%	49,40%
3	51,30%	48,50%	46,40%	45,00%	43,90%	42,60%	41,60%	41,20%	40,90%
5	47,10%	43,40%	40,40%	38,30%	36,80%	35,20%	34,30%	34,10%	35,99%
7	43,24%	39,80%	36,73%	34,60%	33,10%	31,40%	30,60%	30,50%	31,10%
10	39,29%	35,23%	32,75%	30,90%	29,20%	27,40%	26,60%	26,50%	27,00%
12	37,25%	34,00%	30,74%	28,90%	27,20%	25,40%	24,60%	24,40%	24,90%
15	35,27%	30,76%	28,83%	27,20%	25,50%	23,70%	23,00%	22,90%	23,50%
20	33,67%	29,48%	27,44%	26,30%	24,51%	23,20%	22,50%	22,40%	23,00%
30	32,40%	30,40%	26,58%	26,50%	23,92%	22,84%	22,48%	22,42%	22,79%

Table 3.3.: Implied volatilities for interest rate caps. Columns represent the strike level and rows the maturity of the caps. One and two year caps have a 3 month tenor, other maturities a 6 month tenor.
Source: Bloomberg L.P. (2006), <EUCV>, retreived 29. Sep. 2011

In markets one can usually not find caplet (implied) volatilities but implied volatilities for caps and floors. However one would like to have caplet volatilities since then each volatility contains information about a single forward rate and not about a combination of several. By no-arbitrage arguments these can be calculated with a bootstrapping method.

Caps and floors quoted in markets have a 3-month or 6-month tenor structure and are availabe for full-year maturities and for a certain range of strikes. Example data can be found in table 3.3. Note that the implied volatilities are different for different strike levels. This is called the volatility smile and shows that the classical LMM is not fully capable of representing market reality. There is a huge amount of literature how to extend the LMM to acchieve different implied volatilities for different strikes. Popular examples include the constant elasticity of variance (CEV) model (see Andersen and Andreasen [1]), the square root stochastic volatility model (see Wu and Zhang [42]) or the rather recent SABR model (Hagan and Lesniewski [16]).

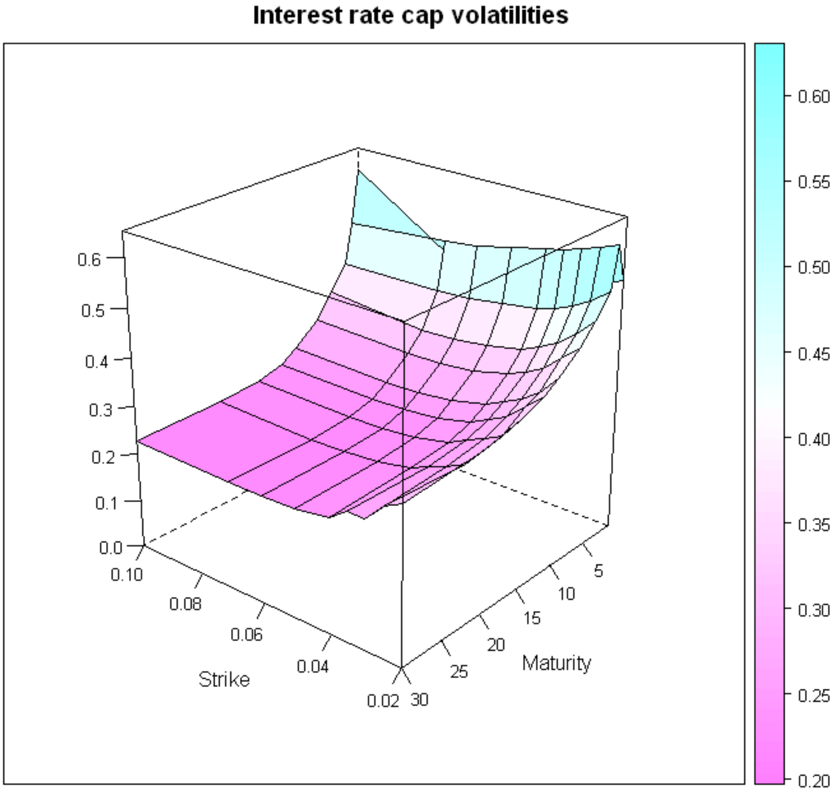


Figure 3.12.: Interest rates cap smile

Remark: This effect is often referred to volatility smile because of the original observed smile in market data for equities. Interest data is rather skewed (see 3.12), which makes term volatility skew also quite common. Today the term volatility smile is a synonym for arbitrary shapes.

Bootstrapping

The bootstrapping method can be described as follows. Consider given cap prices (e.g. given by implied volatility) for maturities with half-year differences (since most of the quoted contracts are based on 6-month LIBOR rates) and a common strike price K (this is essential for markets with volatility smiles). Then two adjacent caps only differ by one caplet. Therefore by substracting two prices of adjacent caps we get a fair price for this individual caplet. We can then calculate implied caplet volatilities out of those caplet prices. Although this procedure is straightforward in theory, implementing this faces several difficulties.

- As mentioned above market data is only available for full year maturities and even there not for every year. This means that we do not have cap prices with half year differences and one has to resort to some kind of interpolation. The easiest is to use linear interpolation on cap implied volatilites, but this can (and mostly will) result in implied caplet volatilities jumping up and down. To get rid of this undesired behaviour one can instead use smoothed B-splines (see Boor [4]). This is what we do in our implementation.

- Another problem is that some caps have 3-month LIBOR rates as underlyings. Although volatilities normally do not differ greatly for 3-month and 6-month caps, this isn't always true. Especially during the financial crisis starting in 2008 it was observed that those volatilities can be quite different (see Mercurio [32]).

  One can of course use the above bootstrapping method with a 3-month tenor for short-term maturities, but in the end one still has to convert those volatilities since for a LMM one normally wants to choose a fixed tenor, in our case a tenor of 6 month. A method to convert volatilities of different tenors is presented below.

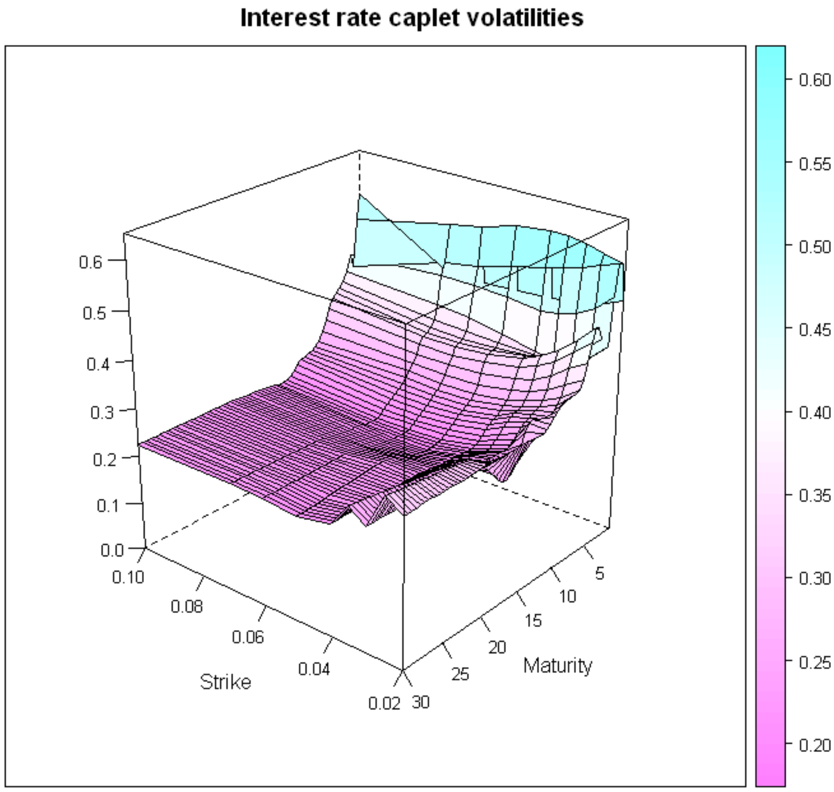


Figure 3.13.: Bootstrapped implied volatilities of interest rate caplets

Applying this procedure for different strike levels K one can calcualate implied caplet volatilities. The results are plotted in figure 3.13.

Converting volatilities of different tenors

As we have seen short-term caps have underlying LIBOR rates of 3 month, while most caps have underlying rates of 6 month. We now use (2.7) to convert the quoted 3-month volatilities to 6-month volatilities.

Consider a 3-month tenor structure $\mathbb{T}_{3M} = \{T_0, T_1, \dots, T_{2N}\}$ with a corresponding 6-month tenor structure $\mathbb{T}_{6M} = \{T_0, T_2, T_4, \dots, T_{2N}\}$. We are given implied volatilities $\overline{\sigma}^i$ for caplets with maturities in $\mathbb{T}_{3M}$ and want to calculate implied volatilities for caplets with maturities in $\mathbb{T}_{6M}$ (with according underlying rates). Assume a LMM for the 3-month forward rates

$$\mathrm{d}F_n^i(t) \; = F_n^i(t)\sigma_n^i(t)\,\mathrm{d}Z^i(t)\,.$$

Using (2.7) we can calculate the implied volatility $\overline{\sigma}(T_{2(i-1)}, T_{2i})$ of a 6-month caplet.

$$\begin{aligned}\overline{\sigma}(T_{2(i-1)}, T_{2i}) = \frac{1}{T_{2(i-1)}} &\int_0^{T_{2(i-1)}} u_1(t)^2 \sigma_n^{2i-1}(t)^2 + u_2(t)^2 \sigma_n^{2i}(t)^2 \,\mathrm{d}t \\ &+ 2\int_0^{T_{2(i-1)}} \rho_n^{2i,2i-1}(t) u_1(t) \sigma_n^{2i-1}(t) u_2(t) \sigma_n^{2i}(t)\,\mathrm{d}t\,,\end{aligned}$$

where

$$\begin{aligned}u_1(t) &= \frac{1}{F_n(t, T_{2(i-1)}, T_{2i})}\left(\frac{\delta^{2i-1}F_n^{2i-1}(t)}{T_{2i}-T_{2(i-1)}} + \frac{\delta^{2i-1}\delta^{2i}F_n^{2i-1}(t)F_n^{2i}(t)}{T_{2i}-T_{2(i-1)}}\right),\\ u_1(t) &= \frac{1}{F_n(t, T_{2(i-1)}, T_{2i})}\left(\frac{\delta^{2i}F_n^{2i}(t)}{T_{2i}-T_{2(i-1)}} + \frac{\delta^{2i-1}\delta^{2i}F_n^{2i-1}(t)F_n^{2i}(t)}{T_{2i}-T_{2(i-1)}}\right).\end{aligned}$$

$\overline{\sigma}(T_{2(i-1)}, T_{2i})$ is stochastic since the coefficients u_1, u_2 are stochastic. So first we freeze the forward rates at their current value allowing us the approximation

$$\begin{aligned}\overline{\sigma}(T_{2(i-1)}, T_{2i}) \approx u_1(0)^2 \frac{1}{T_{2(i-1)}}\int_0^{T_{2(i-1)}} \sigma_n^{2i-1}(t)^2\,\mathrm{d}t \; + u_2(0)^2 \frac{1}{T_{2(i-1)}}\int_0^{T_{2(i-1)}} \sigma_n^{2i}(t)^2\,\mathrm{d}t \\ + 2u_1(0)u_2(0)\frac{1}{T_{2(i-1)}}\int_0^{T_{2(i-1)}} \rho_n^{2i,2i-1}(t)\sigma_n^{2i-1}(t)\sigma_n^{2i}(t)\,\mathrm{d}t\,.\end{aligned}$$

Since we want to do those calculations before an actual calibration we do not know σ_n^i yet. Therefore we approximate them assuming the volatility functions to be constants equal to the given implied volatilities. Also we assume $\rho_n^{2i,2i-1}$ to be constant. This results in

$$\overline{\sigma}(T_{2(i-1)}, T_{2i}) \approx u_1(0)^2(\overline{\sigma}^{2i-1})^2 + u_2(0)^2(\overline{\sigma}^{2i})^2 + 2\rho_n^{2i,2i-1}u_1(0)u_2(0)\overline{\sigma}^{2i-1}\overline{\sigma}^{2i}. \tag{3.3}$$

Everything except $\rho_n^{2i,2i-1}$ in (3.3) is then given by market data. For $\rho_n^{2i,2i-1}$ we have to resort to historical estimates (see 3.1).

ATM caplets

As we mentioned above the LMM is not capable to capture prices across different strikes correctly. Since the topic of this work is inflation modelling, we want to concentrate on this and not on the topic of smile modelling. Therefore instead of using different strikes we use only one strike price for each maturity, namely the at-the-money (ATM) strike. For ATM caplets the current forward rate is their strike. In our notation for a caplet on the LIBOR rate $F_n^i(T_{i-1})$ this is its current value $F_n^i(0)$. To get the correct implied volatility we have to interpolate the bootstrapped volatility surface of figure 3.13. Certain smile models, e.g. the SABR model, have become so popular that they are sometimes used for interpolation (see Brigo and Mercurio [7]). For simplicity we resort to bilinear interpolation. One then calculates implied volatilities for a range of ATM caplets and uses them for calibration.

3.2.2. Swap rates and swaptions

Consider a time structure $\mathbb{T} = \{T_\alpha, T_{\alpha+1}, \ldots, T_\beta\}$. A swap with rate K is a contract guaranteeing the payments $\delta^i(F_n^i(T_{i-1}) - K)$ at each time $T_i, i \in \{\alpha+1, \ldots, \beta\}$. The time t value of such a contract is

$$\begin{aligned}\sum_{i=\alpha+1}^{\beta} P_n(t,T_i)\delta^i \mathbb{E}^{Q_n^{T_i}}\left[F_n^i(T_{i-1}) - K|\,\mathcal{F}_t\right] &= \sum_{i=\alpha+1}^{\beta} P_n(t,T_i)\delta^i(F_n^i(t) - K)\\ &= \sum_{i=\alpha+1}^{\beta} P_n(t,T_i)(\frac{P_n(t,T_i)}{P_n(t,T_{i-1})} - 1 - \delta^i K)\\ &= P_n(t,T_\alpha) - P_n(t,T_\beta) - \sum_{i=\alpha+1}^{\beta} P_n(t,T_i)\delta^i K.\end{aligned}$$

The rate K rendering the value of this contract 0 is called the swaprate $S_{\alpha,\beta}(t)$. It is

$$S_{\alpha,\beta}(t) = \frac{P_n(t,T_\alpha) - P_n(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}\delta^i P_n(t,T_i)} = \sum_{i=\alpha+1}^{\beta} w_i(t)F_n^i(t), \tag{3.4}$$

where

$$w_i(t) = \frac{\delta^i P_n(t,T_i)}{\sum_{k=\alpha+1}^{\beta}\delta^k P_n(t,T_k)}.$$

Remark: Notice that we could choose $\sum_{i=\alpha+1}^{\beta}\delta^i P_n(t,T_i)$ as a numeraire. This induces a measure $Q_n^{\alpha,\beta}$ under which the swaprate must be a martingale (two tradable assets divided by the numeraire). If one chooses to model swap rates under this measure, this leads to so-called swap market models. One can show that LMMs and swap market models are inconsistent, meaning that fixing lognormal LIBOR dynamics doesn't allow for lognormal swap rate dynamics and the other way around. Although swap market models allow for easy valuation of swaptions (but not of caps and floors), they have not become as popular as LMMs and nowadays mostly LMMs are used.

Swaptions

A swaption with strike K is a contract allowing one to enter a swap contract with rate K at time T_α. Hence its value at time T_α is

$$\left(P_n(T_\alpha,T_\alpha) - P_n(T_\alpha,T_\beta) - \sum_{i=\alpha+1}^{\beta} \delta^i K P_n(T_\alpha,T_i)\right)_+ = (S_{\alpha,\beta}(T_\alpha) - K)_+ \sum_{i=\alpha+1}^{\beta} \delta^i P_n(T_\alpha,T_i),$$

since we only exercise this option if the value at T_α is positive. Using the change of numeraire technique (see section A.2) we can price a swaption according to

$$\begin{aligned}\text{Swaption}(t,\alpha,\beta,K) &= \mathbb{E}^Q\left[\exp\left\{-\int_t^{T_\alpha} r_n(s)\,\mathrm{d}s\right\}(S_{\alpha,\beta}(T_\alpha)-K)_+\sum_{i=\alpha+1}^{\beta}\delta^i P_n(T_\alpha,T_i)\middle|\,\mathcal{F}_t\right]\\ &= \left(\sum_{i=\alpha+1}^{\beta}\delta^i P_n(t,T_i)\right)\mathbb{E}^{Q_n^{\alpha,\beta}}\left[(S_{\alpha,\beta}(T_\alpha)-K)_+\middle|\,\mathcal{F}_t\right].\end{aligned}$$

Assuming lognormal dynamics for the swap rate under $Q_n^{\alpha,\beta}$ the value of this swaption can be expressed with Black's formula. More precisely if

$$\mathrm{d}S_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)v^{\alpha,\beta}\,\mathrm{d}Z(t)\,,$$

where Z is a $Q_n^{\alpha,\beta}$-Brownian motion, we get

$$\text{Swaption}(t,\alpha,\beta,K) = \left(\sum_{i=\alpha+1}^{\beta}\delta^i P_n(t,T_i)\right)\left(S_{\alpha,\beta}(t)\Phi(d_+) - K\Phi(d_-)\right), \tag{3.5}$$

where

$$d_\pm = \frac{\ln\left(\frac{S_{\alpha,\beta}(t)}{K}\right) \pm \frac{1}{2}(v^{\alpha,\beta})^2(T_\alpha - t)}{v^{\alpha,\beta}\sqrt{T_\alpha - t}}.$$

The market again uses this and quotes implied volatilities according to this formula. However, since we focus on LMMs we want to value swaptions in a LMM setting, which doesn't allow for such an easy solution.

Valuation of swaptions in the LMM

There exist several ideas to price swaption in a LMM. A good overview can be found in Brigo and Mercurio [7]. We present two valuation formulas here, which are both based on the idea to approximate the LMM equivalent of the implied vola used in Black's formula

(3.5). Therefore we are interested in the quantity

$$(v_{LMM}^{\alpha,\beta})^2 T_\alpha = \int_0^{T_\alpha} (\,\mathrm{d}ln(S_{\alpha,\beta}(t)\,)(\,\mathrm{d}ln(S_{\alpha,\beta}(t)\,). \tag{3.6}$$

1. Rebonato's formula:

Remember that $S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_n^i(t)$ with

$$\begin{aligned} w_i(t) &= \frac{\delta^i P_n(t,T_i)}{\sum_{k=\alpha+1}^{\beta} \delta^k P_n(t,T_k)} = \frac{\delta^i \frac{P_n(t,T_i)}{P_n(t,T_\alpha)}}{\sum_{k=\alpha+1}^{\beta} \delta^k \frac{P_n(t,T_k)}{P_n(t,T_\alpha)}} \\ &= \frac{\delta^i \prod_{j=\alpha+1}^{i} (1+\delta^j F_n^j(t))^{-1}}{\sum_{k=\alpha+1}^{\beta} \delta^k \prod_{j=\alpha+1}^{k} (1+\delta^j F_n^j(t))^{-1}}. \end{aligned} \tag{3.7}$$

The $w_i(t)$ are stochastic. We freeze them at their current value and approximate

$$S_{\alpha,\beta}(t) \approx \sum_{i=\alpha+1}^{\beta} w_i(0) F_n^i(t).$$

Then

$$\mathrm{d}S_{\alpha,\beta}(t) \;\approx (\dots)\,\mathrm{d}t \;+ \sum_{i=\alpha+1}^{\beta} w_i(0)\,\mathrm{d}F_n^i(t)$$

and

$$\mathrm{d}\ln S_{\alpha,\beta}(t) \;= \frac{\mathrm{d}S_{\alpha,\beta}(t)}{S_{\alpha,\beta}(t)} \approx \{\dots\}\,\mathrm{d}t \;+ \frac{\sum_{i=\alpha+1}^{\beta} w_i(0)\,\mathrm{d}F_n^i(t)}{S_{\alpha,\beta}(t)}$$

resulting in

$$\begin{aligned} (v_{LMM}^{\alpha,\beta})^2 T_\alpha &\approx \int_0^{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) F_n^i(t) w_j(0) F_n^j(t) \rho_n^{i,j}(t) \sigma_n^i(t) \sigma_n^j(t)\,\mathrm{d}t}{S_{\alpha,\beta}(t)^2} \\ &\approx \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) F_n^i(0) w_j(0) F_n^j(0)}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \rho_n^{i,j}(t) \sigma_n^i(t) \sigma_n^j(t)\,\mathrm{d}t \end{aligned}$$

where we again froze the forward rates at their current value.

2. Hull & White's formula:

The difference is that for this formula drifts are frozen at a later point in the derivation, allowing for a slightly refined version of the above formula. First calculating the stochastic dynamics of $\mathrm{d}S_{\alpha,\beta}$ and only freezing the forward rates at their current value afterwards to calculate the quantity (3.6) leads to

$$(v_{LMM}^{\alpha,\beta})^2 T_\alpha \approx \sum_{i,j=\alpha+1}^{\beta} \frac{\overline{w}_i(0)F_n^i(0)\overline{w}_j(0)F_n^j(0)}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \rho_n^{i,j}(t)\sigma_n^i(t)\sigma_n^j(t)\,\mathrm{d}t\,,$$

where

$$\overline{w}_j(t) = w_j(t) + \sum_{i=\alpha+1}^{\beta} F_n^i(t)\frac{\delta^j w_i(t)}{1+\delta^j F_n^j(t)}\left(\frac{\sum_{k=j}^{\beta}\delta^k\prod_{l=\alpha+1}^{\beta}(1+\delta^l F_n^l(t))^{-1}}{\sum_{k=\alpha+1}^{\beta}\delta^k\prod_{l=\alpha+1}^{\beta}(1+\delta^l F_n^l(t))^{-1}} - \mathbb{I}\{i \geq j\}\right).$$

The adjustment of the weights is a correction for the first derivatives of w_i coming from only later freezing the forward rates. Brigo and Mercurio [7] argue that the difference between those two formulas is quite small. Therefore normally it is sufficient to use the first formula. This drift freezing procedure works quite well especially in times of low volatilities and therefore not too widely fluctuating forward rates. In turbulent times this is not always the case. Also for long-dated maturities these freezing procedures can be problematic. Therefore it might still be worth the extra effort to calculate those adjusted weights.

Market data

The market quotes swaption implied volas with maturities (T_α) from 1 to 30 years and underlying swap lifetimes ($T_\beta - T_\alpha$) of 1 to 30 years. An example matrix of such data can be found in table 3.4. The underlying swaps are for one year tenors and the strike rates are ATM - meaning they have a strike equal to the current forward swap rate.

	2	3	4	5	6	7	8	9	10	15	20	25
01	0,544	0,483	0,445	0,440	0,423	0,403	0,400	0,399	0,398	0,375	0,381	0,397
02	0,442	0,402	0,376	0,364	0,356	0,351	0,347	0,344	0,342	0,328	0,336	0,348
03	0,372	0,350	0,332	0,323	0,318	0,313	0,310	0,308	0,306	0,295	0,302	0,312
04	0,326	0,312	0,302	0,293	0,289	0,296	0,284	0,282	0,282	0,274	0,280	0,291
05	0,296	0,287	0,289	0,271	0,278	0,275	0,274	0,263	0,264	0,260	0,266	0,276
07	0,261	0,255	0,258	0,241	0,251	0,238	0,239	0,241	0,244	0,243	0,248	
10	0,217	0,216	0,216	0,217	0,219	0,221	0,225	0,229	0,233	0,232	0,234	
15	0,226	0,230	0,233	0,236	0,241	0,245	0,249	0,253	0,256	0,244		
20	0,260	0,261	0,263	0,264	0,267	0,269	0,308	0,310	0,310			
25	0,269	0,315	0,314	0,313								

Table 3.4.: Implied volatilites of swaption. Rows state the maturity and columns the underlying swap lifetime.
SOURCE: Bloomberg L.P. (2006), <EUSV>, retreived 29. Sep. 2011

Here we only have ATM swaptions, however like with caps, swaptions smile depending on the call level. One could use swaption values for different strikes (resulting in fact in a 3-dimensional grid: maturity, swap lifetime, strike) for calibration, but one would need to extend the LMM, e.g. in one of the ways proposed earlier, to be able to correctly fit data for different strikes. But again since the focus of this work is on pricing inflation instruments we do not focus on this.

Taking care of the different tenors

As mentioned earlier swaps underlying swaptions are of one-year tenor, while forward rates of caps are of 6-month tenor. Assuming one is simulating 6-month forwards one has to calculate the volatility of the one-year forward rates. This is very similar to the problem with short-term caplets and one could use the same approximations. However, for calibration purposes using the presented swaption formulas one may adept the derivation of the swaption formulas to get a more exact result.

Therefore consider the 6-month tenor structure $\mathbb{T}_n = \{0 = T_0, T_1, \ldots, \mathbb{T}_{2N}\}$ which underlying forward structure (denoted F_n^i) we want to model. Then we can write the annual swap rate (3.4) also as

$$S_{\alpha,\beta}(t) = \sum_{i=2\alpha+1}^{2\beta} w_i(t) F_n^i(t)$$

with

$$w_i(t) = \frac{\delta^i P_n(t, T_i)}{\sum_{k=\alpha+1}^{\beta}(T_{2k} - T_{2(k-1)}) P_n(t, T_{2k})} = \frac{\delta^i \prod_{j=2\alpha+1}^{i}(1 + \delta^j F_n^j(t))^{-1}}{\sum_{k=\alpha+1}^{\beta}(T_{2k} - T_{2(k-1)}) \prod_{j=2\alpha+1}^{2k}(1 + \delta^j F_n^j(t))^{-1}}$$

We can then apply the same drift freezing procedures as before and e.g. get in Rebonato's case

$$(v_{LMM}^{\alpha,\beta})^2 T_\alpha \approx \sum_{i,j=2\alpha+1}^{2\beta} \frac{w_i(0) F_n^i(0) w_j(0) F_n^j(0)}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \rho_n^{i,j}(t) \sigma_n^i(t) \sigma_n^j(t)\, \mathrm{d}t\,. \tag{3.8}$$

3.2.3. Instantaneous volatility

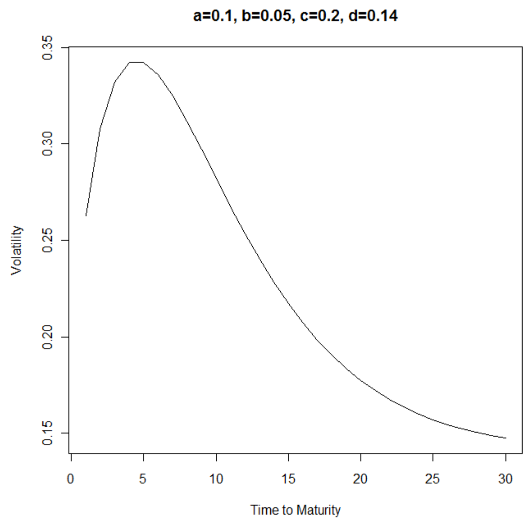
We have seen that e.g. caplet prices depend on the average volatility of forward rates. To specify a LMM we need to determine the volatility function, not just its average value. To do that we impose certain structural characteristics on the volatility functions. We can use the analysis of section 3.1 for this. One observable characteristic is time homogenity, meaning that the volatility only depends on the time to maturity (time till expiry) of a forward rate, mathematically

$$\sigma_n^k(t) = \sigma_n(T_k - t).$$

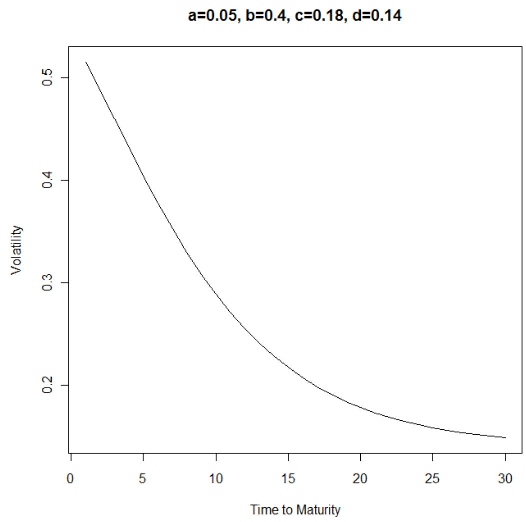
Although this already poses quite a restriction, there are still a lot of ways how to choose the volatility functions. Two approaches chosen commenly are to either use piecewise constant functions between the T_i or to impose a parametric shape on the volatility function. We resort to the second method. The volatility function we choose is

$$\sigma_n(\tau) = (a\tau + b)\mathrm{e}^{-c\tau} + d. \tag{3.9}$$

This volatility function can describe (next to other shapes) the two shapes most commonly found in interest rate caplet markets. Representative plots are found in figure 3.14 and we see that also estimated historical data (see 3.1) is of this type.



(a) Vola Shape in normal times



(b) Vola Shape in distress times

Figure 3.14.: Possible forward rate volatility shapes

An economical explanation for those two shapes is as follows. In normal times interest markets are greatly influenced by central bank decisions. Since central bank policies are aimed at a medium time horizon, which is a few years, this is the greatest source of uncertainty. Therefore the volatility is highest for these maturities. Contrary in times of distress markets become very volatile, which is why then the volatility is highest for short-dated forward rates, explaining the second type.

Remark: The proposed volatility function is able to capture both cases, but once chosen it is fixed. It doesn't allow for changes between the two shapes, therefore a chosen model does not allow e.g. later times of financial distress after currently quiet markets. Hence it is not capable of reproducing this aspect of reality, which is one reason why LMMs are not able to capture the volatility smile while certain extension (e.g. stochastic volatility extensions) are.

We now explain, how the individual parameters contribute to the shape of the yield curve.

d: This parameter determines the volatility level when the forward rate is still a long way from expiry (note that τ is big and the other terms are damped), b: This parameter explains the volatility level shortly before expiry (the level then is $b + d$),

a: This parameter influences the height and the direction of the hump in the volatility structure,

c: This represents the damping of the first term and therefore explains how long the volatility stays at its long-term level d before volatility typically increases at times closer to the forward expiry.

With this information we are able to find acceptable starting values for minimization procedures.

The proposed volatility function has some desirable features. However, having only four parameters it can be hard to get a reasonably good fit for a lot of market data. Since the LMM is actually designed to have the same valuation formulas for caps, one would even like to have an exact fit to this data. One solution to this is to add multiplicative factors ϕ_k to the general volatility function in order to acchieve a perfect fit. Given the parameters a, b, c, d one can easily calculate the necessary values of ϕ_k by solving

$$(\overline{\sigma}^k)^2 = \frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_n^k(t)^2 \, \mathrm{d}t \; = \phi_k^2 \frac{1}{T_{k-1}} \int_0^{T_{k-1}} ((a\tau + b)\mathrm{e}^{-c\tau} + d)^2 \, \mathrm{d}\tau \, .$$

Hence we have to set

$$\phi_k^2 = \frac{(\overline{\sigma}^k)^2}{\frac{1}{T_{k-1}} \int_0^{T_{k-1}} ((a\tau + b)\mathrm{e}^{-c\tau} + d)^2 \, \mathrm{d}\tau} .$$

If the resulting parameters ϕ_k are close to one, this doesn't really destroy the motivated structure of the instantaneous volatility, while still guaranteeing an exact fit. Doing this we will actually loose cap data as calibration instruments for the parameters a, b, c, d and we have to imply those parameters from other financial instruments, in our case swaptions.

3.2.4. Correlation structures

To specify an arbitrary $N \times N$-correlation matrix one has to determine $\frac{N(N-1)}{2}$ parameters (a correlation matrix is symmetric with ones in the diagonal). In most cases this number is too high for calibration purposes. Therefore one wants to reduce the number of parameters of such a correlation matrix.

Empirical analysis of section 3.1 shows that correlation structures of forward rates typically have the following two properties.

1. Correlation decreases for forward rates with farther apart maturities.

2. Correlation of forward rates with a fixed difference of maturities increases with larger maturities ($\rho_n^{i,j}$ with $j - i$ fixed is increasing in i).

For computational simplicity we decided to use constant correlation structures in our implementations. Nevertheless we also present some non-constant correlation parametrizations. Although computationally more efficient, constant correlations pose a problem if we want to include the second property above. It is no problem to find a constant correlation parametrizations having the second property. But assume that we use such a constant correlation structure. In a LMM we simulate forward rates with fixed maturities. Today the correlation structure is reasonable for these forward rates. But a few years later this is different. Now rates who had large maturities at the beginning are closer to maturity and should have a different correlation structure. But due to constant correlations they still have the same now too high correlations. Therefore satisfying the second condition might not be desirable when one wants to use constant correlation matrices.

We now present some possible parametrizations for correlation matrices:

1. Classical two-parameters parametrization:

$$\rho_n^{i,j} = \rho_\infty + (1-\rho_\infty)\exp\left\{-\beta|T_i - T_j|\right\}, \quad \beta \geq 0, |\rho_\infty| \leq 1 \tag{3.10}$$

This correlation structure satisfies only the first condition. The parameter β determines how fast correlation between rates with farther apart maturities decreases, while the parameter ρ_∞ determines the minimum correlation between interest rates. Although this allows for negative correlation ($\rho_\infty < 0$), in general nominal forward rates show positive correlation.

2. Two-parameters square-root parametrization:

$$\rho_n^{i,j}(t) = \rho_\infty + (1-\rho_\infty)\exp\left\{-\beta|\sqrt{T_i - t} - \sqrt{T_j - t}|\right\}, \quad \beta \geq 0, |\rho_\infty| \leq 1 \tag{3.11}$$

This form is proposed by Rebonato [38]. It satisfies both conditions. However the correlation is time-dependent, resulting in harder to compute integrals. One could substiute $T_i - t$ by T_i in the square roots therefore resulting in constants, however this would have the drawback mentioned above.

3. Three-parameters parametrization:

$$\rho_n^{i,j} = \rho_\infty + (1-\rho_\infty)\exp\left\{-\beta\exp\left\{-\alpha\min\{T_i, T_j\}\right\}|T_i - T_j|\right\}, \quad \alpha, \beta \geq 0, |\rho_\infty| \leq 1 \tag{3.12}$$

This form was also proposed by Rebonato and is another way to satisfy the second condition. The damping factor β is itself damped for larger maturities of the rates. This introduces a third parameter perhaps allowing for a better fit, but again has the drawbacks discussed above.

All the now proposed parametrization produce full rank matrices. The rank of the correlation matrix determines the number of driving Brownian motions of a LMM. Therefore all these parametrizations lead to full rank models. However, it might be desirable to reduce

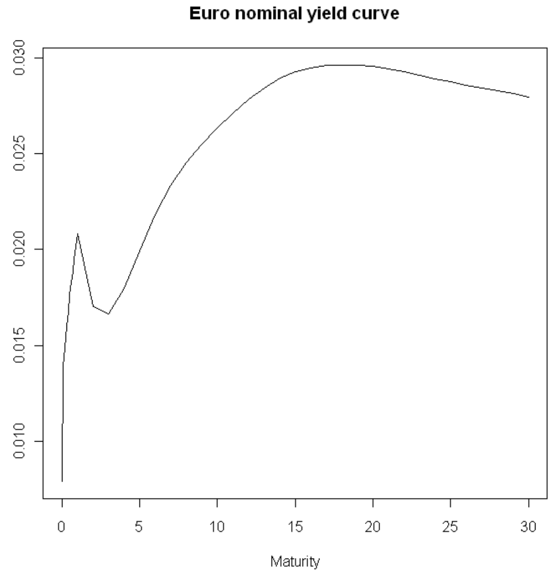


Figure 3.15.: Nominal euro yield curve

the number of driving Brownian motions and there exist several approaches how this can be acchieved. An overview can be found in Brigo and Mercurio [7] or Brigo [6].

3.2.5. Summary and results

We now give a short summary of the calibration of an LMM and also state our used specifications and data. All data is from Bloomberg L.P., retreived 29. Sep. 2011.

1. Preparing of market data
   - **Generate a yield curve out of market data (see section 3.1)**

     We use swap rates and additionally LIBOR rates for the short end of the curve. The calculated yield curve can be found in figure 3.15.
   - **Bootstrap implied 6-month ATM caplet volatilities out of cap data**

     This procedure was described in section 3.2.1. We used cap volatilities of different strikes and maturities (3.4) to bootstrap caplet data, afterwards tenor-adjusting volatilities for short-dated caplets. The bilinear interpolated ATM caplet volatilities can be found in figure 3.16.
2. Calibration
   - **Choose volatility function**

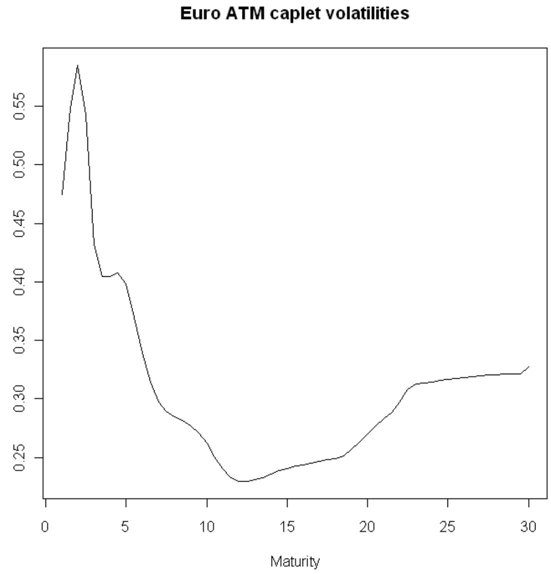


Figure 3.16.: ATM caplet volatilities

We choose our volatility functions to be of the following form:

$$\sigma_n^k(t) = \phi_k((a(T_{k-1} - t) + b)\mathrm{e}^{-c(T_{k-1}-t)} + d,$$
$$\phi_k^2 = \frac{(\overline{\sigma}^k)^2}{\frac{1}{T_{k-1}}\int_0^{T_{k-1}}((a\tau + b)\mathrm{e}^{-c\tau} + d)^2\,\mathrm{d}\tau},$$

guaranteeing an exact fit to ATM caplet data.

- **Specify correlation matrix**
  We decide to choose our correlation function as

$$\rho_n^{i,j} = \rho_\infty + (1 - \rho_\infty)\exp\{-\beta|T_i - T_j|\},$$

  since we want to use a constant correlation function.

- **Fitting procedure**
  We use a minimization procedure to fit the values of $a, b, c, d, \beta, \rho_\infty$ under reasonable parameter restrictions. We minimize the sum of quadratic differences between theoretical swaption prices (3.8) and market prices (table 3.4). Furthermore we introduce a penalty term for ϕ values which are not close to 1 and might destroy the desired volatility structure. For this we consider a treshold of .2 and then sum up the quadratic differences between ϕ_i and the area of $[0.8, 1.2]$ to include this as a penalty. The resulting parameters can be found in table 3.5. The resulting differences for swaption volatilities are plotted in figure 3.17.

a	b	c	d	β	ρ_∞
0.044	0.378	0.140	0.056	0.394	0.370

Table 3.5.: Fitted nominal parameters

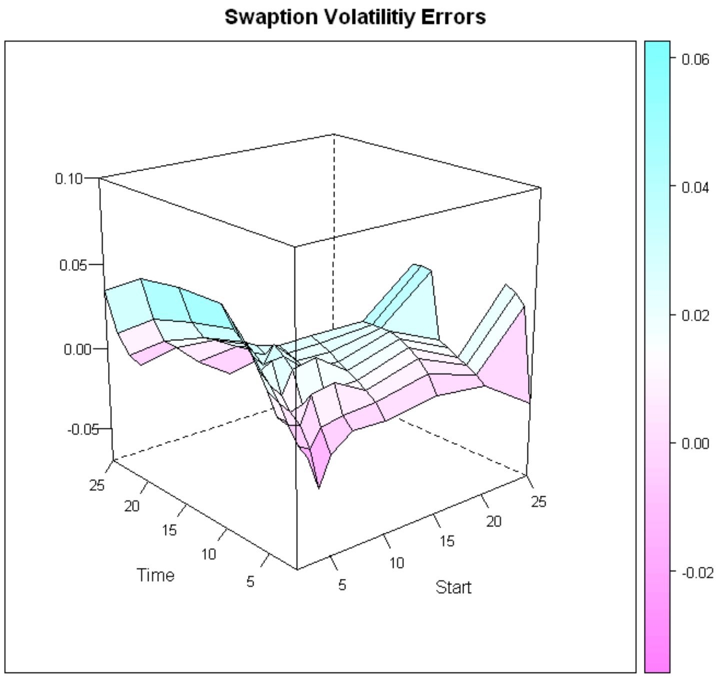


Figure 3.17.: Errors between calibrated model swaption prices and market swaption prices

3.3. Calibration of a inflation market model

Having calibrated a LMM to nominal data we go ahead to calibrate the inflation part. Note that out of the calibration of the LMM we are given the nominal volatilities σ_n^i and the correlation matrix ρ_n. So the parameters we still have to determine are

- the volatility functions of the forward CPIs - $\sigma_I^i(t)$,
- the correlation of the forward CPIs - $(\rho_I^{i,j})_{i,j=1,\dots,N}$,
- the correlation between forward CPIs and forward rates $(\rho_{n,I}^{i,j})_{i=1,\dots,2N,j=1,\dots,N}$.

We again choose correlations to be constant for computational simplicity.

3.3.1. Inflation swap rate data

Euro market data can be found in table 3.6. YYIIS rates are calculated by bloomberg with a JY kind of model.

We used the bootstrapping method developed in section 1.3.6 to calculate forward inflation

	ZCIIS	YYIIS
1 YR	1,396	1,395
2 YR	1,374	1,3826
3 YR	1,466	1,4747
4 YR	1,52	1,5265
5 YR	1,563	1,5686
6 YR	1,616	1,6194
7 YR	1,664	1,66275
8 YR	1,706	1,6985
9 YR	1,745	1,73565
10 YR	1,78	1,7673
12 YR	1,82	1,80365
15 YR	1,884	1,85865
20 YR	1,922	1,89445
25 YR	1,95	1,92025
30 YR	2,028	1,9766

Table 3.6.: Inflation-indexed swap data
SOURCE: Bloomberg L.P. <SWIL>, retreived 29. Sep. 2011

rates and compared them to the rates $Y_i(t) = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}$. As mentioned before the difference between those two rates is very small, in fact most of the time smaller than quoted bid/ask spreads for IIS rates.

Therefore this data shouldn't be used for calibration purposes. Remember that those rates are the only instruments we have reviewed (and which are available in the market), which would allow a fitting of correlations between nominal and inflation world. Hence we have to use historical estimates for correlations.

We also saw that the prices of inflation caplets/floorlets depend on this forward inflation rates. Considering that the differences between Y_i and the actual forward inflation are so small using the rates Y_i for the pricing of inflation caplets should not result in too much distortion. This is actually what practicioners do considering that reasonable YYIIS rates are hard to come by (note again that the data quoted in 3.6 is also only calculated via model assumptions and is no market-determined price).

3.3.2. Inflation caplets

Inflation cap data is available, e.g. on bloomberg using <SWIL>. Caps are quoted via several different methods. One is via implied volatilities, either for a shifted lognormal model or a normal model. Classical implied volatilities are not meaningful for inflation caplets, since inflation can be negative, which can't be accounted for with a lognormal model. The two formulas are presented in the appendix (section B.4). While the shifted lognormal formula is the obvious extension of the Black formula to allow for negative rates, the normal rate is used, since next to the here presented (shifted lognormal) market models, there have also been ideas to model inflation via normal dynamics (see e.g. Kenyon [27]).

Denote by $ICpl(t, S, T, \kappa)$ the price of an inflation caplet for the time $[S, T]$ with strike κ. Then under a normal model we have $F_I(T, S, T)|\,\mathcal{F}_t \sim N(F_I(t, S, T), \sigma^2(T-t))$ and by Lemma B.3 we have

$$\begin{aligned}\frac{ICpl(t,S,T,\kappa)}{(T-S)P_n(t,T)} &= \mathbb{E}^{Q_n^T}\left[(F_I(T,S,T)-\kappa)_+|\,\mathcal{F}_t\right] \\ &= (F_I(t,S,T)-\kappa)\Phi\left(\frac{F_I(t,S,T)-\kappa}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{\kappa - F_I(t,S,T)}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

For the shifted lognormal model, where we choose the shift parameter to be -1 (to allow for inflation as small as -100%), we assume the following forward inflation dynamics using the Q_n^T Brownian motion Z

$$\mathrm{d}(1+(T-S)F_I(t,S,T)) = (1+(T-S)F_I(t,S,T))\sigma\,\mathrm{d}Z(t)\,.$$

We then have that

$$\ln(1+(T-S)F_I(T,S,T))\,|\,\mathcal{F}_t \sim N(\ln(1+(T-S)F_I(t,S,T)) - \frac{1}{2}\sigma^2(T-t), \sigma^2(T-t))$$

and by B.4.3

$$\begin{aligned}\frac{ICpl(t,S,T,\kappa)}{P_n(t,T)} &= \mathbb{E}^{Q_n^T}\left[((T-S)F_I(t,S,T)-(T-S)\kappa)_+|\,\mathcal{F}_t\right] \\ &= (1+(T-S)F_I(t,S,T))\Phi\left(\frac{\ln\left(\frac{1+(T-S)F_I(t,S,T)}{1+(T-S)\kappa}\right) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}\right) \\ &\quad - (1+(T-S)\kappa)\Phi\left(\frac{\ln\left(\frac{1+(T-S)F_I(t,S,T)}{1+(KTS)\kappa}\right) - \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}\right)\end{aligned}$$

The formulas for floors follow directly. Solving those two equations for σ we get the implied volatilities $\overline{\sigma}_I^N, \overline{\sigma}_I^{SLN}$. Notice that in order to solve for this equations we already need to know the forward inflation rates (or equivalently YYIIS rates) or approximate them via $Y_i(t)$. This is one reason why most of the time data is also quoted in basispoints (10^{-5}), which then is the amount of money you have to pay for a caplet with nominal 1 and therefore a cashflow of $(T-S)(F_I(T,S,T)-K)_+$.

	Floor	Floor	Floor	Floor	Floor	Floor	Floor	Cap	Cap	Cap	Cap	Cap
	-0,03	-0,02	-0,015	-0,01	-0,005	0	0,01	0,02	0,03	0,04	0,05	0,06
1	1	2	2	3	5	8	10	20	6	2	1	1
2	8	10	13	16	21	30	67	56	23	12	7	5
3	21	29	35	45	57	71	128	128	68	44	34	28
4	42	54	63	79	97	117	193	206	118	81	65	55
5	69	108	118	131	148	172	261	285	166	120	96	81
6	95	115	132	160	191	221	331	383	239	175	147	129
7	121	198	213	231	255	288	408	472	297	224	187	164
8	149	178	202	242	284	324	463	572	370	279	239	212
9	178	211	239	285	333	377	530	666	437	333	288	257
10	209	324	345	370	403	447	603	748	484	374	318	284
15	340	516	543	555	619	656	858	1152	754	591	509	481
20	451	666	699	708	789	829	1067	1490	981	776	674	643
30	595	894	936	940	1052	1093	1389	2121	1381	1080	933	893

Table 3.7.: Inflation cap/floor data quoted in basis points. Rows represent the maturity, columns the strike level.

SOURCE: Bloomberg L.P. (2006), <EUISC> or <EUISF>, retreived 29. Sep. 2011

Market data and bootstrapping

Market data is available for several different strikes and maturities, example data can be found in table 3.7. Depending on the strike it is either a cap or a floor beeing quoted.

Similar to interest rate caps we have prices of caps, but for calibration purposes it would be more convenient to have caplet data. We can derive caplet prices out of cap prices by taking the differences for caps with maturities differing by one year. Since we do not want to interpolate prices we use the shifted lognormal formula, where we use the rates $Y_i(t)$ calculated out of ZCIIS rates, to calculate implied volatilities and interpolate those using B-splines. Using interpolated volatilities we can then calculate cap prices for arbitrary maturities, subtract them and then calculate shifted lognormal caplet volatilities. Since the formulas developed in chapter 2 are also of a shifted lognormal type we will later use those for calibration.

Like interest rate caplet inflation caplets/floorlets also display a volatility smile. Therefore we again have to apply the bootstrapping procedure for each strike individually and afterwards interpolate the surface to get ATM volatilities. Results are plotted in figure 3.18.

3.3.3. Forward CPI volatility

We now have to specify a certain structure for the volatilities of forward CPIs. Since we have not found a clear shape in the historical data we use the very basic assumption of constant volatilities for each forward CPI. This has the advantage that given correlations $\rho_I^{i,i-1}$ this allows for an exact fit to ATM inflation data by setting $\mathcal{V}^i/T_i$ in (2.56) equal to

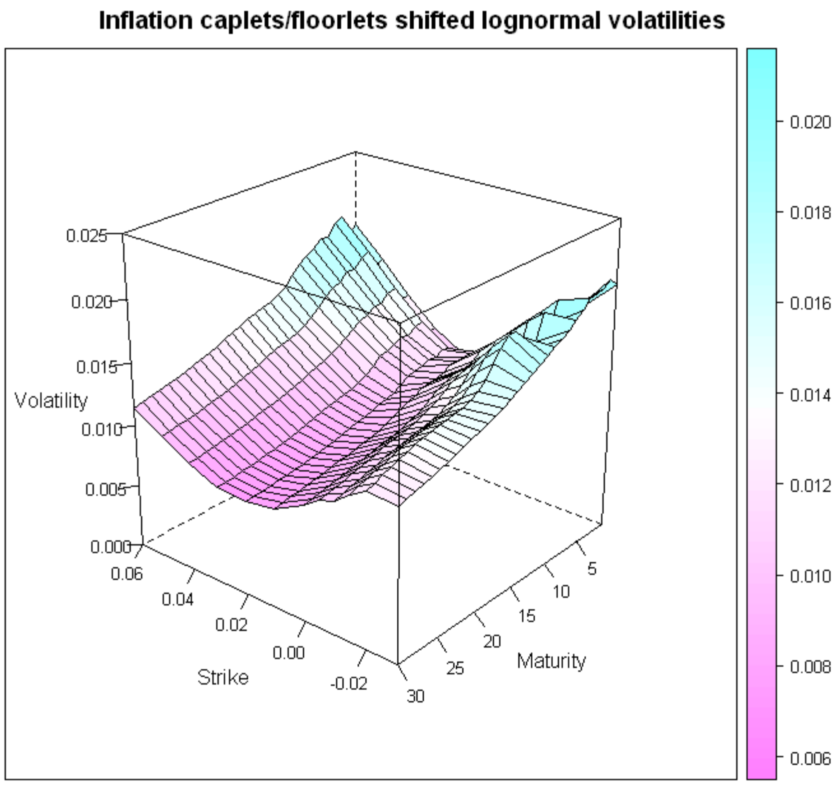


Figure 3.18.: Inflation caplet/floorlet smile - shifted lognormal volatilities

the market caplet volatility (here denoted by $\overline{\sigma}_I^i$). Then we have

$$(\sigma_I^i)_\pm = \rho_I^{i,i-1}\sigma_I^{i-1}\frac{T_{i-1}}{T_i} \pm \sqrt{\left(\frac{T_{i-1}}{T_i}\rho_I^{i,i-1}\sigma_I^{i-1}\right)^2 - \frac{T_{i-1}}{T_i}(\sigma_I^{i-1})^2 + (\overline{\sigma}_I^i)^2}. \tag{3.13}$$

Only the solution $(\sigma_I^i)_+$ makes sense, since otherwise we would have a monotonically decreasing behaviour.

3.3.4. Correlations

As mentioned earlier there are no market instruments available allowing for a market calibration of correlations and we have to resort to historical estimation. We use the data of section 3.1, i.e. the data of the last year (30 Jul. 2010 to 29 Jul. 2011). For the forward CPI correlation matrix we again fit a parametric correlation structure of the form

$$\rho_n^{i,j} = \rho_\infty + (1-\rho_\infty)\exp\left\{-\beta|T_i - T_j|\right\}.$$

Minimizing the sum of the quadratic differences we get the parameters of $\rho_\infty = 0.500, \beta = 0.107$. The estimated and fitted correlations are plotted in figure 3.19.

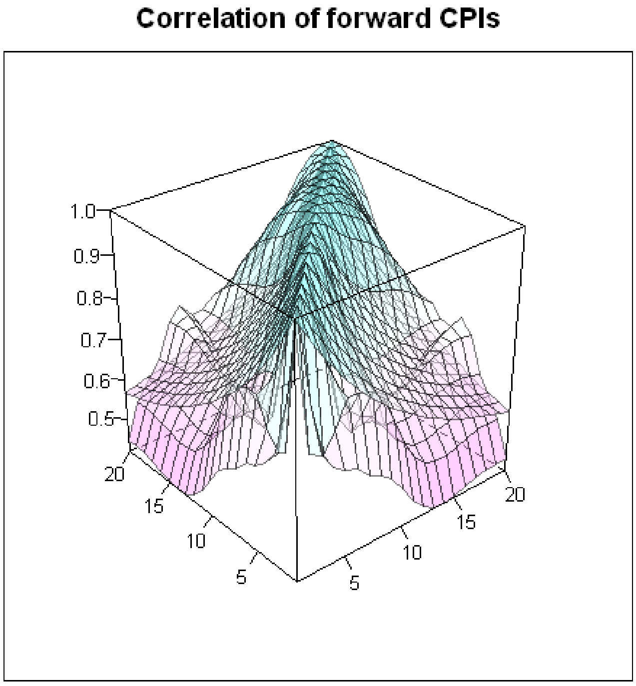


Figure 3.19.: Fitted forward CPI correlation

The second step is to estimate the correlations between the forward inflation rates and the forward CPI. As mentioned earlier we choose to represent this by a single constant correlation. Again fitting to the sum of the quadratic differences we get $\rho_{n,I} = 0.344$ (see figure 3.20).

Remark: We should make sure that the estimated correlation matrix is positive semidefinite. Since the correlation structures resulting from (3.10) are in fact positive definite as mentioned by Rebonato [38] and we just have an additional constant the resulting full correlation matrix is also positive definite.

3.3.5. Result of inflation calibration

Having specified the correlation strcuture we can now use (3.13) to exactly fit constant volatilities to ATM inflation caplet/floorlet data. The results can be found in figure 3.21. and we have then specified all parameters of the model and can go ahead to use the fitted model for Monte Carlo pricing.

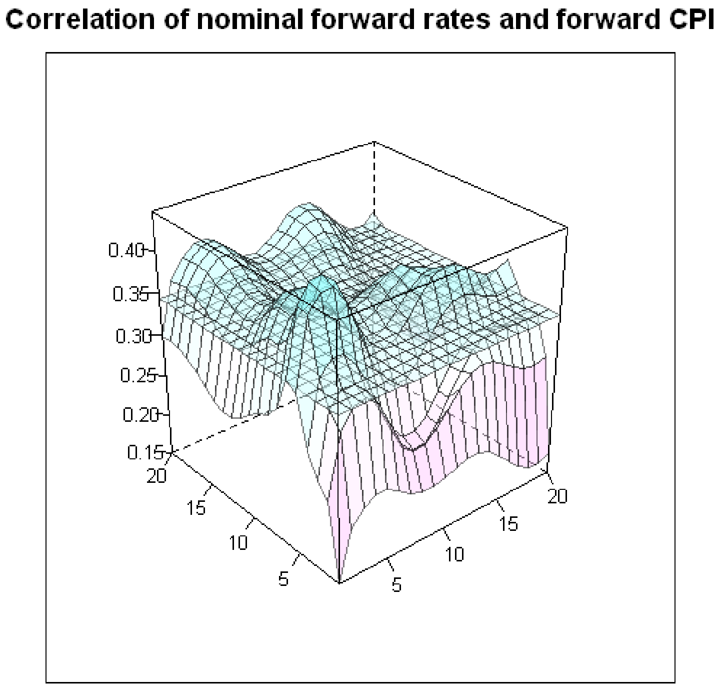


Figure 3.20.: Fitted correlation between forward CPIs and forward interest rates

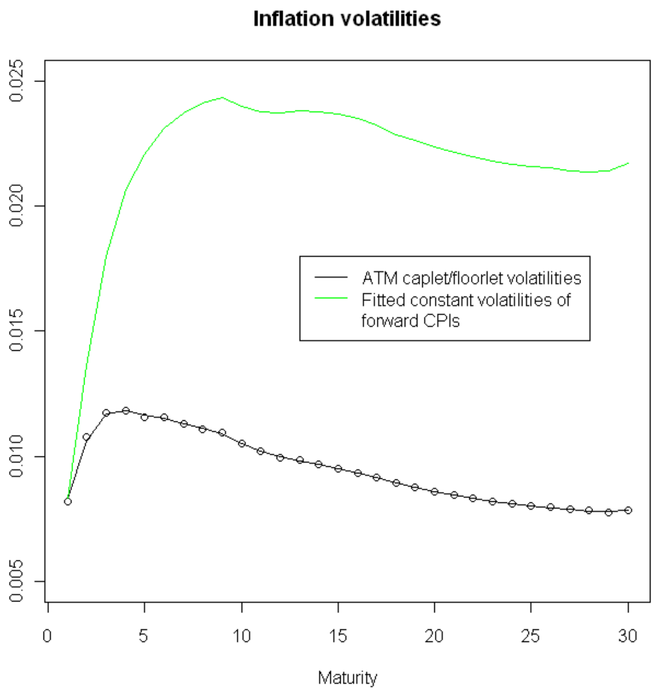


Figure 3.21.: Fitted forward CPI volatilities

3.4. Monte Carlo pricing

A general introdcution into Monte Carlo methods can be found in Glasserman [14] or Korn et al. [28]. In this work we only consider a few aspects and the interested reader is reffered to one of the two books.

3.4.1. Discretization of diffusion processes

The most famous discretization of a SDE is the Euler-Maruyama method. For this consider a SDE

$$\mathrm{d}X(t) \; = a(t, X(t))\,\mathrm{d}t \; + \sigma(t, X(t))^T\,\mathrm{d}W(t)\,, X(0) = x_0,$$

where $W(t)$ is a Brownian motion with correlation ρ.

Lets say we want to simulate $X(t)$ until time T. Then choose a time grid $0 = t_0, t_1, \dots,$ $t_n = T$, possibly with equidistant spacing, meaning $\triangle_i = t_{i+1} - t_i = \triangle$, and denote by $\hat{X}(t_k)$ the simulated value of the stochastic process. The Euler-Maruyama scheme then is

$$\hat{X}(t_{k+1}) = \hat{X}(t_k) + a(t_k, \hat{X}(t_k))\triangle_k + \sigma(t, \hat{X}(t_k))^T\sqrt{\triangle_k}\;\epsilon,$$

where $\epsilon \sim N(0, \rho)$ is a multivariate normal random variable. Setting $\hat{X}(t) = \hat{X}(t_k)$, where $k = \sup\{j : t_j \leq t\}$ one can show that under some regularity assumptions this discretization has a strong convergence of order $\frac{1}{2}$, which means

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t) - \hat{X}(t)|\right] < C\triangle^{\frac{1}{2}},$$

where C is a constant and $\triangle = \sup_k\{\triangle_k\}$. In fact under the additional restriction that $\sigma(t, X(t)) = \sigma(t)$, meaning the diffusion term does not depend on X, one can show that the Euler-Maruyama method coincides with the more sophisticated Milstein method and has in fact strong convergence of order 1. This will become important for us, since by using a log transformation of the forward rates the drift is in fact deterministic and we have convergence of order 1.

In the case of deterministic diffusion coefficients we have another benefit. Notice that

$$\int_{t_k}^{t_{k+1}} \sigma(t)^T\,\mathrm{d}W(t) \; \sim N\left(0, \triangle_k \int_{t_k}^{t_{k+1}} \sigma(t)^T\rho(t)\sigma(t)\,\mathrm{d}t\right).$$

For practical purposes we might acchieve higher accuracy by using

$$\hat{X}(t_{k+1}) = \hat{X}(t_k) + a(t_k, \hat{X}(t_k))\triangle_k + \epsilon,$$

with

$$\epsilon \sim N\left(0, \int_{t_k}^{t_{k+1}} \sigma(t)^T\rho(t)\sigma(t)\,\mathrm{d}t\right).$$

Application to inflation modelling

Remember the dynamics of forward interest rates and of forward CPIs developed in section 2.3.3 given by

$$\begin{aligned}\mathrm{d}F_n^i(t) &= F_n^i(t)\sigma_n^i(t)\left(\mathrm{d}Z_n^i(t) + \sum_{j=\beta(t)}^{i} \rho_n^{i,j}(t)\sigma_n^j(t)\frac{\delta F_n^j(t)}{1+\delta F_n^j(t)}\,\mathrm{d}t\right),\\ \mathrm{d}\,\mathcal{I}^i(t) &= \mathcal{I}^i(t)\sigma_I^i(t)\left(\mathrm{d}Z_I^i(t) + \sum_{j=\beta(t)}^{2i} \rho_{n,I}^{j,i}(t)\sigma_n^j(t)\frac{\delta F_n^j(t)}{1+\delta F_n^j(t)}\,\mathrm{d}t\right).\end{aligned}$$

The involved Brownian motions have a constant correlation structure denoted by ρ. The diffusion coefficient is stochastic, but taking the logarithm by Ito's lemma we have

$$\begin{aligned}\mathrm{d}\ln F_n^i(t) &= \sigma_n^i(t)\left(\mathrm{d}Z_n^i(t) + \sum_{j=\beta(t)}^{i} \rho_n^{i,j}(t)\sigma_n^j(t)\frac{\delta F_n^j(t)}{1+\delta F_n^j(t)}\,\mathrm{d}t - \frac{1}{2}\sigma_n^i(t)\,\mathrm{d}t\right),\\ \mathrm{d}\ln\mathcal{I}^i(t) &= \sigma_I^i(t)\left(\mathrm{d}Z_I^i(t) + \sum_{j=\beta(t)}^{2i} \rho_{n,I}^{j,i}(t)\sigma_n^j(t)\frac{\delta F_n^j(t)}{1+\delta F_n^j(t)}\,\mathrm{d}t - \frac{1}{2}\sigma_I^i(t)\,\mathrm{d}t\right),\end{aligned}$$

and simulating these stochastic processes applying the Euler-Maruyama scheme we have convergence of order 1.

Remark: This kind of simulation gives rise to another problem. The general idea in financial models is to generate arbitrage-free models for pricing. While theoretical models in this work are arbitrage-free, this is not necessarily true for the discretized versions. Glasserman and Zhao [15] research this problem and propose a solution by instead modelling transformations of the above quantities, which are martingales, therefore generating discretized models that are also arbitrage-free. However, this transformation results in stochastic diffusion coefficients and may reduce the quality of numerical simulation.

3.4.2. Monte Carlo standard error and variance reduction methods

We want to use Monte Carlo methods to calculate the risk-free value of stochastic cashflows. Denoting this cashflow by $\Pi(T)$ we therefore want to calculate

$$\mathbb{E}^Q\left[\exp\left\{-\int_0^T r(t)\,\mathrm{d}t\right\}\Pi(T)\right] = P_n(0,T)\mathbb{E}^{Q_n^T}\left[\Pi(T)\right] = \mathbb{E}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right].$$

We can estimate this expectation value by simulating S evolutions of interest rates and inflation rates to calculate

$$\frac{1}{S}\sum_{j=1}^{S}\frac{\Pi(T)_j}{B_n^d(T)_j}.$$

Notice that we have to simulate under the measure induced by the numeraire we use for discounting, in this case B_n^d. By the central limit theorem we then know that for $S \to \infty$

$$\frac{\sum_{j=1}^{S}\left(\frac{\Pi(T)_j}{B_n^d(T)_j} - \mathbb{E}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]\right)}{\sqrt{S\ \mathbb{V}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]}} \xrightarrow{d} N(0,1).$$

It follows that

$$\begin{aligned} Q_n^d\left(\left|\frac{1}{S}\sum_{j=1}^{S}\frac{\Pi(T)_j}{B_n^d(T)_j} - \mathbb{E}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]\right| < \epsilon\right) &\approx P\left(|N(0,1)| < \epsilon\sqrt{\frac{S}{\mathbb{V}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]}}\right) \\ &= 2\Phi\left(\epsilon\sqrt{\frac{S}{\mathbb{V}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]}}\right) - 1. \end{aligned}$$

Using

$$\widehat{\mathbb{V}} = \frac{1}{S}\sum_{j=1}^{S}\left(\frac{\Pi(T)_j}{B_n^d(T)_j} - \sum_{j=1}^{S}\frac{\Pi(T)_j}{B_n^d(T)_j}\right)^2$$

as an estimator for $\mathbb{V}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]$ we can then define approximative confidence intervals for the mean $\mathbb{E}^{Q_n^d}\left[\frac{\Pi(T)}{B_n^d(T)}\right]$. E.g. the $(1-\alpha)$ confidence interval then is

$$\left[\frac{1}{S}\sum_{j=1}^{S}\frac{\Pi(T)_j}{B_n^d(T)_j} - \Phi^{-1}(1-\alpha)\sqrt{\frac{\widehat{\mathbb{V}}}{S}},\quad \frac{1}{S}\sum_{j=1}^{S}\frac{\Pi(T)_j}{B_n^d(T)_j} - \Phi^{-1}(1-\alpha)\sqrt{\frac{\widehat{\mathbb{V}}}{S}}\right].$$

In order to generate small confidence intervals one has two options. One is to increase the number of simulations. However the central limit theorem only has a convergence of order $\frac{1}{2}$, therefore e.g. to improve the accuracy by a factor 10 we have to increase our simulations by a factor of 100. Since we normally want our valuation procedures to be fast this is only possible up to a certain point.

The second option is to somehow reduce the variance of the estimated quantity. There are a lot of suggestions how this can be done. Depending on the application some work well while other don't. In this case two ideas seem to work quite well. One is to use antithethic random variables, the other is to use control variate estimators. A good overview how to use those procedures can be found in Korn et al. [28].

3.4.3. Arbitragefree interpolation

A LMM and a market model for inflation only specifies a few discrete forward rates. A finite number of forward rates (CPIs) doesn't fully specify the yield curve or all forward CPI. If one needs rates not directly simulated, interpolation is necessary. Hence for a specification of a full model one has to specify an interpolation method. One can again resort to B-spline interpolation or even just use linear interpolation, but this interpolation methods have the drawback of possibly creating arbitrage possibilities. However, one can show that there are ways to interpolate simulated rates and still hold on to an arbitrage-free model.

In Werpachowski [41] the author presents an interpolation method that is arbitrage-free, consistent (calculated forward rates are fitted exactly) and guarantees positive rates as long as the simulated rates are positive. Additionally they provide an extension having the above properties and additionally providing a smooth volatility term structure meaning that interpolated rates have a volatility of the same order of magnitude as the simulated ones.

Using the standard notation (see section 2.3.4) the interpolation scheme is defined by

$$F_n(t,S,T_{\eta(S)}) = f_S F_n^{\eta(S)}(t), \quad \text{where } f_S := \frac{F_n(0,S,T_{\eta(S)})}{F_n^{\eta(S)}(0)}, \tag{3.14}$$

where $\eta(t) = \inf\{j \in 1\ldots,2N : t \leq T_j\}$ (not to confuse with $\beta(t) = \eta(t)+1$). As mentioned earlier we have also set $F_n^i(t) = F_n^i(T_{i-1})$ for $t \geq T_{i-1}$.

Using this together with the no-arbitrage relationship

$$(1+(T-S)F_n(t,S,T)) = (1+(T'-S)F_n(t,S,T'))(1+(T-T')F_n(t,T',T)), \tag{3.15}$$

we have

$$\begin{aligned} 1+(T-T_{\eta(T)-1})F_n(t,T_{\eta(T)-1},T) &= \frac{1+(T_{\eta(T)}-T_{\eta(T)-1})F_n^{\eta(T)}(t)}{1+(T_{\eta(T)}-T)F_n(t,T,T_{\eta(T)})} \\ &\overset{(3.14)}{=} \frac{1+(T_{\eta(T)}-T_{\eta(T)-1})F_n^{\eta(T)}(t)}{1+(T_{\eta(T)}-T)f_T F_n^{\eta(T)}(t)}. \end{aligned} \tag{3.16}$$

Combining (3.14), (3.15) and (3.16) we can then interpolate an arbitrary forward rate by

$$F_n(t,S,T) = \frac{1+(T_{\eta(S)}-S)f_S F_n^{\eta(S)}(t)}{1+(T_{\eta(T)}-T)f_T F_n^{\eta(T)}(t)} \prod_{i=\eta(S)+1}^{\eta(T)} (1+\delta^i F_n^i(t)), \tag{3.17}$$

where we use the convention that an empty product equals 1.

As discussed in Werpachowski [41] this interpolation of the forward rates is in fact equivalent to an discount factor interpolation. We define discount factors $\overline{P}(t, T_i)$ as

$$\overline{P}(t,T_i) = \begin{cases} (1 + (T_{\eta(t)} - t)F_n^{\eta(t)}(t))^{-1} \prod_{j=\eta(t)+1}^{i}(1 + \delta^j F_n^j(t))^{-1} & t \leq T_i \\ (1 + (T_{\eta(t)} - t)F_n^{\eta(t)}(t))^{-1} \prod_{j=i+1}^{\eta(t)}(1 + \delta^j F_n^j(t)) & T_i < t \end{cases}.$$

Then the discount factor interpolation reads as

$$P_n(t,T) = \frac{(T_{\eta(T)} - T)f_T}{T_{\eta(T)} - T_{\eta(T)-1}} \overline{P}_n(t, T_{\eta(T)-1}) + \left(1 - \frac{(T_{\eta(T)} - T)f_T}{T_{\eta(T)} - T_{\eta(T)-1}}\right) \overline{P}_n(t, T_{\eta(T)}), . \quad (3.18)$$

To extend this interpolation to provide a smooth volatility structure note that some of the forward rates used in (3.17) may have already been fixed at time t. Therefore they don't change anymore resulting in a lower volatility of interpolated rates. To fix this problem Werpachowski [41] simulate the used rates a little longer (F_n^i until T_i instead of T_{i-1}) and denote those zombie rates by $\tilde{F}_n^i(t)$. For those rates we then have

$$\tilde{F}_n^i(t) = \begin{cases} F_n^i(t) & t \leq T_{i-1} \\ \tilde{F}_n^i(t) \neq F_n^i(T_{i-1}) & T_{i-1} < t < T_i \end{cases}.$$

The interpolation scheme then reads as

$$F_n(t,S,T) = \frac{1 + (T_{\eta(S)} - S)f_S \tilde{F}_n^{\eta(S)}(t)}{1 + (T_{\eta(T)} - T)f_T \tilde{F}_n^{\eta(T)}(t)} \prod_{i=\eta(S)+1}^{\eta(T)} (1 + \delta^i \tilde{F}_n^i(t)). \quad (3.19)$$

The simulation of the zombie rates is done using the standard LMM drift and volatility terms. The only question remaining is how to choose the parameters which are not defined for $t \geq T_{i-1}$. Werpachowski [41] propose to have a smooth volatility structure it is sufficient to just flat extrapolate the parameter functions with their last defined value. We also stick to this approach.

Note that for the discount factor interpolation in (3.18) to use this approach one has to define the discount factors $\overline{P}_n(t, T_i)$ using the simulated zombie rates instead of the real ones.

Remark: For the forward CPI we choose to simply linearly interpolate the resulting forward CPI structure. We are well aware that the resulting forward CPIs need not be arbitrage-free, but considering we have already used procedures like seasonality adjustments this effect seems neglible.

3.4.4. Remarks on choosing the simulation grid and computational aspects

We have introduced discretization procedures in section 3.4.1. For practical purposes we have requirements for choosing the simulation grid. In addition to a small enough step size for numerical accuracy we want payment dates to be in the simulation grid, so that we are able to have simulated values at those dates. Additionally we want to have the fixing dates of the forward rates included in the simulation grid. Therefore it is not always possible to find a not too small equidistant spacing still including all those dates, which is why we have to use a non-equidistant time grid. One way to do this is to choose a stepsize and generate a grid according to this. Afterwards we simply add the dates mentioned above and choose this as our final simulation grid.

It is sometimes convenient to price several instruments at once to save time and resources. Since the biggest effort is the calculation of the simulated rates for a large amount of simulations we might save a lot of computing time by choosing a time grid with only one day difference, which would allow us to price arbitrary (concerning payment dates) instruments. However depending on the instrument there might be additional factors to consider and one maybe has to adjust simulation techniques or even model specifications itself.

Conclusion

We have calibrated a market model consistently to current market data and introdcued some aspects of how to use Monte Carlo simulation in this framework. There are still a lot of ways to extend and improve the involved models, be it with regards to smile modelling or with the application to special derivatives like multicallable bonds or similar instruments. However, using this framework we are able to price all except rather exotic financial derivatives linked to interest rates and inflation, which exist in todays markets.

A. A few theoretical results

A.1. Ito's lemma

Theorem A.1 (Ito's lemma)**:** *Let X be a d-dimensional stochastic process given by*

$$\mathrm{d}X_i(t) = a_i(t)\,\mathrm{d}t + \sigma_i(t)^T\,\mathrm{d}B(t)\,,$$

where B is a n-dimensional Brownian motion with correlation $\rho = (\rho^{i,j})_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ and a_i and σ_i are sufficiently regular adapted processes to allow for a solution to the SDE. Let F be a function in $C^2(\mathbb{R}^d,\mathbb{R})$. Then

$$\mathrm{d}F(X(t)) = \sum_{i=1}^{d}\frac{\partial F}{\partial x_i}(X(t))\,\mathrm{d}X_i(t) + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 F}{\partial x_i\partial x_j}(X(t))\sigma_i(X(t))^T\rho\sigma_j(X(t))\,\mathrm{d}t\,. \quad \text{(A.1)}$$

In the special case $d=2$, $\sigma_i = 0$ in all components except i, $\rho_{1,2} = \rho$ and $F(x,y) = xy$ this yields

$$\mathrm{d}(X_1(t)X_2(t)) = X_1(t)\,\mathrm{d}X_2(t) + X_2(t)\,\mathrm{d}X_1(t) + \rho\sigma_1(t)\sigma_2(t)\,\mathrm{d}t\ . \quad \text{(A.2)}$$

A.2. Arbitrage free pricing & numeraires

Consider a probability space $(\Omega,\mathcal{A},P)$ with a filtration $(\mathcal{F}_t)_{0\le t\le T^*}$ (all filtration throughout this work are assumed to be right continuous and P-complete, see e.g. Protter [37]) and $K+1$ traded assets whose price processes are modeled by a $K+1$-dimensional adapted positive semimartingale $(S_t)_{0\le t\le T^*}$. The asset S^0 represents a risk-free bank account and it's prices dynamics is

$$\mathrm{d}S_t^0 = r_t S_t^0\,\mathrm{d}t\,,$$

where r_t is the instantaneous shortrate. Let Q be an equivalent martingale measure (EMM), meaning that $Q\sim P$ and

$$\left(\frac{S_t^i}{S_t^0}\right)_{0\le t\le T^*} \text{ is a Q-martingale for } i=1,\ldots K.$$

Then Harrison and Pliska [18] show that the model is arbitrage-free and that the price π_t of any attainable (see Brigo and Mercurio [7]) payoff H at T is uniquely given by

$$\pi_t = \mathbb{E}^Q\left[\exp\left\{-\int_t^T r_s\,\mathrm{d}s\right\} H \,\middle|\, \mathcal{F}_t\right] = S_t^0\,\mathbb{E}^Q\left[\frac{H}{S_T^0}\,\middle|\,\mathcal{F}_t\right]. \tag{A.3}$$

In the definition of an EMM every price process is divided by S^0, which can be interpreted as expressing it in units of S^0. S^0 is called a numeraire. Sometimes it is more convenient to use a different numeraire than the shortrate account S^0.

Lemma A.2: *Let $(\frac{U_t}{S_t^0})_{0\le t\le T}$ be a strictly positive Q-martingale. Then there exists a probability measure $\tilde{Q} \sim Q$ such that for every Q-martingale $(\frac{S_t^1}{S_t^0})_{0\le t\le T}$ the process $(\frac{S_t^1}{U_t})_{0\le t\le T}$ is a $\tilde{Q}$-martingal and for every $\mathcal{F}_T$-measurable random variable C_T*

$$S_t^0\mathbb{E}^Q\left[\frac{C_T}{S_T^0}|\,\mathcal{F}_t\right] = U_t\mathbb{E}^{\tilde{Q}}\left[\frac{C_T}{U_T}|\,\mathcal{F}_t\right].$$

Furthermore, the Radon-Nikodym derivative defining $\tilde{Q}$ is given by

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q} = \frac{S_0^0}{U_0}\frac{U_T}{S_T^0}. \tag{A.4}$$

A.3. Fubini's theorem for stochastic integrals

Theorem A.3: *Let X be a semimartinale on $(\Omega, \mathcal{A}, P)$ and $(E, \mathcal{E}, \mu)$ a measurable space , where μ is a σ-finite measure. Consider a measurable stochastic process $H : \mathbb{R}_+ \times \Omega \times E \to \mathbb{R}$, such that*

$$\left(\int_E H(t,\omega,e)^2\mu(\,\mathrm{d}e\,)\right)^{\frac{1}{2}}$$

is integrable w.r.t. X and for each e $H^e(t) = H(t,\cdot,e)$ is a measurable process which is integrable w.r.t. X. Let $Z^e(t) = \int_0^t H^e(s)\,\mathrm{d}X(s)$ also be measurable and cadlag. Then we have

$$\int_E Z^e(t)\mu(\,\mathrm{d}e\,) = \int_E\int_0^t H^e(s)\,\mathrm{d}X(s)\,\mu(\,\mathrm{d}e\,)$$

exists and is a cadlag version of

$$\int_0^t\int_E H^e(t)\mu(\,\mathrm{d}e\,)\,\mathrm{d}X(s)\,.$$

Proof: see Protter [37] p. 207-208 □

Remark: Note that most assumptions in this theorem are about measurability and integrability. In our work, using only brownian motions this is quite easy to varify and similar to the classical Fubini theorem. The interesting assumption is the integrability of

$$\left(\int_E H(t,\omega,e)^2 \mu(\mathrm{d}e)\right)^{\frac{1}{2}}.$$

While for the classical Fubini theorem one just assumes integrability here we have to assume square-integrability. One can find counterexamples to show that this square integrability is really necessary.

A.4. Girsanov's theorem

Girsanov's theorem for correlated Brownian motion

Theorem A.4: *Let $(W(t))_{0\le t\le T}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$ with (positive definite) correlation ρ. Define*

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left\{\int_0^T \tilde{H}(u)^T \rho^{-1}\,\mathrm{d}W(u) - \frac{1}{2}\int_0^T \tilde{H}(u)^T\rho^{-1}\tilde{H}(u)\,\mathrm{d}u\right\}.$$

which defines an EMM Q if $\mathbb{E}^P\left[\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = 1$. Then $\tilde{W}(t) = W(t) - \int_0^t \tilde{H}(u)\,\mathrm{d}u$ is a Brownian motion with correlation ρ under Q.

Proof: Because ρ is positive definite we can write $\rho = LL^T$ and $B = L^{-1}W$ is a standard Brownian motion. By Girsanov's theorem for Q given by

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left\{\int_0^T H(u)^T\,\mathrm{d}B(u) - \frac{1}{2}\int_0^T \|H(u)\|^2\,\mathrm{d}u\right\}$$

$\tilde{B}(t) := B(t) - \int_0^t H(u)\,\mathrm{d}u$ is a Brownian motion under Q. Therefore

$$\tilde{W}(t) = L\tilde{B}(t) = LB(t) - \int_0^t LH(u)\,\mathrm{d}u =: W(t) - \int_0^t \tilde{H}(u)\,\mathrm{d}u$$

is a Brownian motion with correlation ρ under Q. The measure transformation $\frac{\mathrm{d}Q}{\mathrm{d}P}$ can be rewritten in terms of W as

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}P} &= \exp\left\{\int_0^T (L^{-1}LH(u))^T\,\mathrm{d}L^{-1}LB(u) - \frac{1}{2}\int_0^T \|L^{-1}LH(u)\|^2\,\mathrm{d}u\right\}\\ &= \exp\left\{\int_0^T \tilde{H}(u)^T\rho^{-1}\,\mathrm{d}W(u) - \frac{1}{2}\int_0^T \tilde{H}(u)^T\rho^{-1}\tilde{H}(u)\,\mathrm{d}u\right\}.\end{aligned}$$

□

Converse Girsanov theorem

Theorem A.5: *Let $(B_t)_{0\le t\le T}$ be a Brownian motion on the probability space $(\Omega,\mathcal{A},P)$ and be $(\mathcal{F}_t)_{0\le t\le T}$ the completed filtration generated by B. Assume there exists a probability measure $Q\sim P$ on $\mathcal{F}_T$. Let*

$$Z_t=\mathbb{E}[\frac{\mathrm{d}Q}{\mathrm{d}P}|\,\mathcal{F}_t]$$

be the density process of the measure transformation. Then there exists an adapted process $\lambda(t)$ such that

$$Z_t=\mathcal{E}\Bigg(\bigg(\int_0^t\lambda(s)^T\,\mathrm{d}B_s\bigg)_{0\le t\le T}\Bigg)_t,$$

where $\mathcal{E}(X)_t$ denotes the exponential process, that means the solution to the SDE

$$\mathrm{d}\mathcal{E}(X)_t=\mathcal{E}(X)_t\,\mathrm{d}X_t\,,\quad \mathcal{E}(X)_0=1.$$

Remarks:

- Since the density process Z is then an exponential process, the classical Girsanov theorem tell us that $B_t-\int_0^t\lambda(s)\,\mathrm{d}s$ is a Q-Brownian motion.
- This means that under the filtration generated by the Brownian motion every equivalent measure change is obtained by a Girsanov transformation (for arbitrary filtrations this is not the case).

Proof: Applying the martingale representation theorem to the integrable martingal Z_t (see Protter [37] p. 186) tells us that there exists a proces $\gamma(t)$ such that $\mathrm{d}Z_t=\gamma(t)^T\,\mathrm{d}B_t$. Setting

$$\lambda(t)=\frac{1}{Z_t}\gamma(t),$$

which is possible since $Z_t>0$ P-a.s. since $P\sim Q$. We arrive at

$$\mathrm{d}Z_t=Z_t\,(\lambda(t)^T\,\mathrm{d}B_t\,).$$

Therefore Z_t is given by an exponential process as statet in the theorem. □

Corollary A.6: *If B is a Brownian motion with positive definite correlation structure ρ the above result holds as well.*

Proof: Since ρ is positive definite we can find $L(t)$ so that $\rho=L(t)L(t)^T$. Then $W(t)=L(t)^{-1}B(t)$ is an uncorrelated brownian motion and the result holds. Then we have

$$Z_t=\mathcal{E}\Bigg(\bigg(\int_0^t\lambda(s)^T\,\mathrm{d}W_s\bigg)_{0\le t\le T}\Bigg)_t=\mathcal{E}\Bigg(\bigg(\int_0^T\lambda(s)^TL(t)^{-1}\,\mathrm{d}B_s\bigg)_{0\le t\le T}\Bigg)_t$$

and by choosing $\tilde{\lambda}(t)^T=\lambda(t)^TL^{-1}$ we have the result. □

B. Some useful calculations

B.1. $\mathbb{E}[\Phi(aX+b)]$

Lemma B.1: *Let $X \sim N(\mu, \sigma^2)$. Then we have that*

$$\mathbb{E}[\Phi(aX+b)] = \Phi\left(\frac{a\mu+b}{\sqrt{1+a^2\sigma^2}}\right).$$

Proof: We first prove this for $\tilde{X} \sim N(0,1)$.

$$\mathbb{E}\left[\Phi(a\tilde{X}+b)\right] = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\infty}^{ax+b} \mathrm{e}^{-\frac{x^2+y^2}{2}}\,\mathrm{d}y\;\mathrm{d}x$$

Now using the orthogonal transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{1+a^2}}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

we get

$$\mathbb{E}\left[\Phi(a\tilde{X}+b)\right] = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\infty}^{\frac{b}{\sqrt{1+a^2}}} \mathrm{e}^{-\frac{u^2+v^2}{2}}\,\mathrm{d}u\;\mathrm{d}v\; = \Phi\left(\frac{b}{\sqrt{1+a^2}}\right).$$

For arbitrary X we then have

$$\begin{aligned}\mathbb{E}[\Phi(aX+b)] &= \mathbb{E}\left[\Phi\left(a\left(\sigma\frac{X-\mu}{\sigma}+\mu\right)+b\right)\right] \\ &= \mathbb{E}\left[\Phi(a\sigma\tilde{X}+(a\mu+b))\right] = \Phi\left(\frac{a\mu+b}{\sqrt{1+a^2\sigma^2}}\right).\end{aligned}$$

□

Lemma B.2: *Let $X \sim N(\mu, \sigma^2)$. Then we have that*

$$\mathbb{E}\left[\mathrm{e}^X\Phi(aX+b)\right] = \mathrm{e}^{\mu+\frac{1}{2\sigma^2}}\,\Phi\left(\frac{a(\mu+\frac{1}{\sigma^2})+b}{\sqrt{1+a^2\sigma^2}}\right).$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^X \Phi(aX+b)\right] &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^x \int_{-\infty}^{ax+b} \mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \int_{-\infty}^{ax+b} \mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y\,\exp\left\{-\frac{(x-(\mu+\sigma^2)^2}{2\sigma^2}\right\}\mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\,\mathrm{d}x \\
&= \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\mathbb{E}\left[\Phi(a\tilde{X}+b)\right] \overset{\text{Lemma } B.1}{=} \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\Phi\left(\frac{a(\mu+\sigma^2)+b}{\sqrt{1+a^2\sigma^2}}\right),
\end{aligned}$$

where $\tilde{X} \sim N(\mu+\sigma^2, \sigma^2)$. □

B.2. Convexity adjustments

Consider an exponential process given by the SDE

$$\mathrm{d}X(t) = X(t)\left(D(t)\,\mathrm{d}t + \sigma(t)^T\,\mathrm{d}B(t)\right),$$

where $D(t)$ is deterministic. Define

$$Y(t) := \exp\left\{-\int_0^t D(u)\,\mathrm{d}u\right\} X(t).$$

Then

$$\mathrm{d}Y(t) = Y(t)\,\sigma(t)^T\,\mathrm{d}B(t),$$

and we see that $Y(t)$ is a local martingale. We suppose that Y_t is in fact a martingale, which is e.g. guaranteed if $\sigma(t)$ is deterministic or if the Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \sigma(t)^2\,\mathrm{d}t\right\}\right] < \infty$$

is satisfied. This then allows the following calculation

$$\begin{aligned}
\mathbb{E}[X(t)|\,\mathcal{F}_s] &= \mathbb{E}[\exp\left\{\int_0^t D(u)\,\mathrm{d}u\right\} Y(t)|\,\mathcal{F}_s] = \exp\left\{\int_0^t D(u)\,\mathrm{d}u\right\}\mathbb{E}[Y(t)|\,\mathcal{F}_s] \\
&= \exp\left\{\int_0^t D(u)\,\mathrm{d}u\right\} Y(s) = \exp\left\{\int_0^t D(u)\,\mathrm{d}u\right\}\exp\left\{-\int_0^s D(u)\,\mathrm{d}u\right\} X(s) \\
&= \exp\left\{\int_s^t D(u)\,\mathrm{d}u\right\} X(s).
\end{aligned}$$

The term $\exp\left\{\int_s^t D(u)\,\mathrm{d}u\right\}$ is often referred to as the convexity adjustment.

B.3. Lognormal distribution of SDEs

Consider a possibly multidimensional SDE of the following form

$$\mathrm{d}X(t) = X(t)\left(\mu(t)\,\mathrm{d}t + \sigma(t)^T\,\mathrm{d}B(t)\right)$$

where $B(t)$ is a Brownian motion with correlation structure ρ. Then

$$\mathrm{d}\ln(X(t)) = \mu(t)\,\mathrm{d}t + \sigma(t)^T\,\mathrm{d}B(t) - \frac{1}{2}\sigma(t)^T\rho\sigma(t)\,\mathrm{d}t\,,$$

and for deterministic parameter functions

$$\ln(X(t)) \sim N\left(\ln(X(0)) + D(t) - \frac{1}{2}V(t)^2, V(t)^2\right),$$

where

$$\begin{aligned} D(t) &= \int_0^t \mu(u)\,\mathrm{d}u\,, \\ V(t) &= \int_0^t \sigma(u)^T\rho\sigma(u)\,\mathrm{d}u\;. \end{aligned}$$

B.4. Valuation of calls/puts in simple models

In this section we value Calls/Put options. Set $\omega = 1$ for a call and $\omega = -1$ for a put.

B.4.1. Valuation in a normal model

Lemma B.3: *Consider a random variable $X \sim N(\mu, \sigma^2)$,*

$$\mathbb{E}\left[(\omega(X-K))_+\right] = \omega(\mu - K)\Phi\left(\omega\frac{K-\mu}{\sigma}\right) + \omega\sigma\phi\left(\omega\frac{K-\mu}{\sigma}\right)$$

Proof: Using $Z \sim N(0,1)$ we have

$$\begin{aligned} \mathbb{E}\left[(X-K)_+\right] = \mathbb{E}\left[(\mu+\sigma Z-K)_+\right] &= \int_{\frac{K-\mu}{\sigma}}^{\infty}(\mu+\sigma z-K)\phi(z)\,\mathrm{d}z \\ &= (\mu-K)\left(\Phi(\infty) - \Phi\left(\frac{K-\mu}{\sigma}\right)\right) + \sigma\int_{\frac{K-\mu}{\sigma}}^{\infty} z\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z \\ &= (\mu-K)\Phi\left(\frac{\mu-K}{\sigma}\right) + \sigma\phi\left(\frac{K-\mu}{\sigma}\right). \end{aligned}$$

For a put this follows in the same way. □

B.4.2. Valuation in a lognormal model

Lemma B.4: *Consider a lognormally distributed random variable X such that* $\ln(X) \sim N(\mu, \sigma^2)$*. Then*

$$\mathbb{E}\left[(\omega(X-K))_+\right] = \omega \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \Phi\left(\omega \frac{\mu - \ln(K) + \sigma^2}{\sigma}\right) - \omega K \Phi\left(\omega \frac{\mu - \ln(K)}{\sigma}\right).$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[(\omega(X-K))_+\right] &= \mathbb{E}\left[(\omega(\exp\left\{\sigma \frac{\ln(X)-\mu}{\sigma} + \mu\right\} - K))_+\right] \\
&= \int_{-\infty}^{\infty} (\omega(\exp\{\sigma z + \mu\} - K))_+ \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z \\
&= \int_{\frac{\ln(K)-\mu}{\sigma}}^{\omega\infty} (\omega(\exp\{\sigma z + \mu\} - K)) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \int_{\frac{\ln(K)-\mu}{\sigma}}^{\omega\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{(z-\sigma)^2}{2}}\,\mathrm{d}z - K \int_{\frac{\ln(K)-\mu}{\sigma}}^{\omega\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{z^2}{2}}\,\mathrm{d}z \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \int_{\frac{\ln(K)-\mu}{\sigma}+\sigma}^{\omega\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u - K\left(\Phi(\omega\infty) - \Phi\left(\frac{\ln(K)-\mu}{\sigma}\right)\right) \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \omega \Phi\left(-\omega \frac{\ln(K) - \mu + \sigma^2}{\sigma}\right) - K \Phi\left(-\omega \frac{\ln(K)-\mu}{\sigma}\right).
\end{aligned}$$

□

B.4.3. Valuation in a shifted lognormal model

Lemma B.5: *Consider a random variable $Y = X + c$ where* $\ln(X) \sim N(\mu, \sigma^2)$*. Then*

$$\begin{aligned}
\mathbb{E}\left[\omega(Y-K)_+\right] &= \mathbb{E}\left[\omega(X-(K-c))_+\right] \\
&\overset{B.4}{=} \omega \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \Phi\left(\omega \frac{\mu - \ln(K-c) + \sigma^2}{\sigma}\right) - \omega(K-c) \Phi\left(\omega \frac{\mu - \ln(K-c)}{\sigma}\right),
\end{aligned}$$

Remark: The parameter c is referred to as the shift parameter. Quite often one uses $c = -1$, since the random variable then assume values in $(-1, \infty)$. In financial markets one often observes returns (or in our case inflation rates) and by modelling these with a shifted lognormal random variable the possible values range from total loss (-100 %) to infinite earings (∞).

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