

THE POISSON BOUNDARY OF LAMPLIGHTER RANDOM WALKS ON TREES

ANDERS KARLSSON AND WOLFGANG WOESS

ABSTRACT. Let \mathbb{T}_q be the homogeneous tree with degree $q + 1 \geq 3$ and \mathcal{G} a finitely generated group whose Cayley graph is \mathbb{T}_q . The associated lamplighter group is the wreath product $\mathcal{Z}_r \wr \mathcal{G}$, where \mathcal{Z}_r is the cyclic group of order r . For a large class of random walks on this group, we prove almost sure convergence to a natural geometric boundary. If the probability law governing the random walk has finite first moment, then the probability space formed by this geometric boundary together with the limit distribution of the random walk is proved to be maximal, that is, the Poisson boundary. We also prove that the Dirichlet problem at infinity is solvable for continuous functions on the active part of the boundary, if the lamplighter “operates at bounded range”.

1. INTRODUCTION

Let $\mathbb{T} = \mathbb{T}_q$ be the homogeneous tree with degree $q + 1 \geq 3$. Assume that at each vertex $x \in \mathbb{T}$ there is a lamp which may be switched off or on in r different states of “intensity”, encoded by the set $\{0, \dots, r-1\}$, where the state 0 represents “off”. We think of $\{0, \dots, r-1\}$ as the vertex set of the complete graph \mathbb{K}_r (all pairs of distinct elements are adjacent). As an introductory example, consider the following random process: at the beginning, all lamps are switched off. A “lamplighter” starts a random walk at a vertex of \mathbb{T} . With probability θ , he chooses to move, that is, he makes a step to a randomly selected neighbouring vertex (without changing the lamps). With probability $1 - \theta$, he chooses to “switch”, that is, he randomly modifies the state of the lamp where he stands (and does not move). At each step we observe the actual position in the tree and the configuration of the lamps that are switched on. Thus, our process evolves on the state space $\mathbb{K}_r \wr \mathbb{T}$ consisting of pairs (η, x) , where $x \in \mathbb{T}$ and the configuration $\eta : \mathbb{T} \rightarrow \mathbb{K}_r$ is a function with finite support. The set of all configurations is denoted \mathcal{C} . It can be equipped with the structure of an abelian group with pointwise addition modulo r , and writing $\eta - \eta'$ below refers to this operation. In the sequel, we shall often write ηx instead of (η, x) .

We equip $\mathbb{K}_r \wr \mathbb{T}$ with a graph structure, where the neighbourhood relation is given by

$$(1.1) \quad (\eta, x) \sim (\eta', x') : \iff \begin{cases} x \sim x' \text{ in } \mathbb{T} \text{ and } \eta = \eta', & \text{or} \\ x = x' \text{ in } \mathbb{T} \text{ and } \text{supp}(\eta - \eta') = \{x\}. \end{cases}$$

We view \mathbb{T} as a Cayley graph of a finitely generated group \mathcal{G} , a free product of two-element and infinite cyclic groups. Thus, vertices of \mathbb{T} are (in one-to-one correspondence with) elements of \mathcal{G} , and x^{-1} and xx' refer to the operations in this group. By a slight deviation from the usual notation, we write o for the group identity and think of it as a root (origin) of the tree. Then our *lamplighter graph* $\mathbb{K}_r \wr \mathbb{T}$ is a Cayley graph of the *wreath product*

Date: June 1, 2006.

2000 Mathematics Subject Classification. 60J50; 05C05, 20E08, 31C20.

Key words and phrases. Random walk, wreath product, tree, Poisson boundary, Dirichlet problem.

Supported by ESF program RDSES and by Austrian Science Fund (FWF) P15577.

$\Gamma = \mathcal{Z}_r \wr \mathcal{G}$, where $\mathcal{Z}_r = \mathbb{Z}/(r\mathbb{Z})$. More precisely, every $x \in \mathcal{G}$ acts on \mathcal{C} by the translation T_x , where $T_x\eta(y) = \eta(x^{-1}y)$. The resulting semidirect product is

$$(1.2) \quad \mathcal{Z}_r \wr \mathcal{G} = \mathcal{C} \rtimes \mathcal{G}, \quad \text{with} \quad (\eta, x)(\eta', x') = (\eta + T_x\eta', xx'),$$

Our random process is a Markov chain $Z_n = (Y_n, X_n)$ on $\mathbb{K}_r \wr \mathbb{T}$, where X_n is the position and Y_n the configuration at time n . Its one-step transition probabilities $p(\eta x, \eta' x') = \Pr[Z_{n+1} = \eta' x' | Z_n = \eta x]$ are given by

$$(1.3) \quad p(\eta x, \eta' x') = \begin{cases} \theta/(q+1), & \text{if } x \sim x' \text{ and } \eta = \eta', \\ (1-\theta)/(r-1), & \text{if } x = x' \text{ and } \text{supp}(\eta - \eta') = \{x\} \\ 0 & \text{in all other cases.} \end{cases}$$

Thus, $p(\eta x, \eta' x') = \mu((\eta x)^{-1}(\eta' x'))$, where μ is a probability measure on the group $\mathcal{Z}_r \wr \mathcal{G}$. In particular, we can view Z_n as a random walk on that group. The random walk is transient, that is, with probability 1, it visits every finite subset of $\mathbb{K}_r \wr \mathbb{T}$ only finitely many times. Thus, Z_n tends “to infinity”, and the purpose of this note is to relate this property in a more detailed way with the underlying structure. Below we shall also consider more general random walks on $\mathcal{Z}_r \wr \mathcal{G}$.

We shall determine the *Poisson boundary* of a general class of lamplighter random walks on $\mathbb{K}_r \wr \mathbb{T}$, resp. $\mathcal{Z}_r \wr \mathcal{G}$ that includes the basic example (1.3). This boundary can be defined in several equivalent ways, see KAIMANOVICH AND VERSHIK [20] and KAIMANOVICH [18]. The Poisson boundary of a random walk on a group is a measure space, determined uniquely up to measure-theoretic isomorphism. One quick definition is to say that it is the space of ergodic components in the trajectory space of the random walk. Another approach is via bounded harmonic functions, see below. Here, we take a more topological viewpoint. We attach to $\mathbb{K}_r \wr \mathbb{T}$ a natural boundary Π at infinity, defined purely in geometric terms, such that $(\mathbb{K}_r \wr \mathbb{T}) \cup \Pi$ is a metrizable space (not necessarily compact or complete) on which the group $\Gamma = \mathcal{Z}_r \wr \mathcal{G}$ acts by homeomorphisms, and every point in Π is the accumulation point of a sequence in $\mathbb{K}_r \wr \mathbb{T}$. We then show that in that topology, (Z_n) converges almost surely to a Π -valued random variable Z_∞ . Let ν be the distribution of Z_∞ , given that the initial position and configuration of the random walk at time $n = 0$ are o and the zero configuration 0 . The measure ν is often called the *harmonic measure* or *limit distribution*. The pair (Π, ν) provides a model for the behaviour at infinity (in time and space) of the random walk. We give a quite simple proof that this is indeed the Poisson boundary under rather general assumptions. This tells us that up to sets with measure 0, we have found the finest model for distinguishing limit points at infinity of the random walk. The (geometric) tool that we shall use for proving that (Π, ν) is the Poisson boundary is the *strip criterion* of KAIMANOVICH [18, §6], [21, Thm. 5.19].

In section 4, we prove that the Dirichlet problem is solvable with respect to this natural geometric boundary. Again, in the spirit of this article, the focus is not on proving the most general result possible.

Let us conclude the introduction with a brief and incomplete overview of recent work on lamplighter random walks and identifications of the Poisson boundary.

The abovementioned paper of KAIMANOVICH AND VERSHIK [20] may serve as a major source for the previous literature, different equivalent definitions of the Poisson boundary, and a wealth of results and methods. There one also finds the first interesting results on the Poisson boundary of lamplighter random walks, namely on $\mathcal{Z}_r \wr \mathbb{Z}^d$. The techniques were refined in the subsequent body of work of KAIMANOVICH, see e.g. [17], [18], [21] and the references therein. Within the study of random walks on groups, wreath products (lamplighter walks) have been the object of intensive studies in the last decade. Wreath products exhibit interesting types of asymptotic behaviour of n -step return probabilities, see SALOFF-COSTE AND PITTET [26], [27], REVELLE [29]. The rate of escape has been studied by LYONS, PEMANTLE AND PERES [24], ERSCHLER [11], [12], REVELLE [28] and, for simple lamplighter walks on trees, by GILCH [15]. For the spectrum of transition operators, see GRIGORCHUK AND ŽUK [16], DICKS AND SCHICK [10] and BARTHOLDI AND WOESS [1]. The positive harmonic functions for a class of random walks on $\mathcal{Z}_r \wr \mathbb{Z}$ have been determined by WOESS [31] and BROFFERIO AND WOESS [4], who have also determined the full Martin compactification in a “nearest neighbour” case [3]. For the Poisson boundary and bounded harmonic functions for various types of lamplighter random walks, besides [20] and [17], see once more the impressive work of ERSCHLER [13].

Concerning the Dirichlet problem in the discrete setting we refer to WOESS [30, Chapter IV] for more information.

2. CONVERGENCE TO THE GEOMETRIC BOUNDARY

The lamplighter graph $\mathbb{K}_r \wr \mathbb{T}$ with neighbourhood defined (1.1) is far from being tree-like (it has one end). It is easy (and well known) to describe the graph metric. A shortest path from ηx to $\eta' x'$ in the lamplighter graph must be such that the lamplighter starts at x , walks along the tree and visits every $y \in \text{supp}(\eta' - \eta)$, where he has to switch the lamp from state $\eta(y)$ to state $\eta'(y)$, and at the end, he must reach x' . Thus, $d(\eta x, \eta' x') = \ell + |\text{supp}(\eta' - \eta)|$, where ℓ is the smallest length of a “travelling salesman” tour (walk) from x to x' that visits each element of $\text{supp}(\eta' - \eta)$. This description of the metric does not require that the base graph is a tree, however note that in a tree, the “travelling salesman” algorithm required for finding such a tour is easy to implement, see e.g. PARRY [25] and CECCHERINI-SILBERSTEIN AND WOESS [7, Example 2].

Since the group $\Gamma = \mathcal{Z}_r \wr \mathcal{G}$ is non-amenable, the random walks $Z_n = (Y_n, X_n)$ that we consider here are all *transient*, that is, $d(Z_n, Z_0) \rightarrow \infty$ almost surely, see [30, Thm. 3.24] for this result going back to KESTEN [22], [23]. The main question considered here is whether we can describe in a more detailed, geometric way how (Z_n) behaves at infinity.

For this purpose, we first briefly recall the *end compactification* of \mathbb{T} , whose graph metric we denote also by $d(\cdot, \cdot)$. A *geodesic path*, resp. *geodesic ray*, resp. *infinite geodesic* in \mathbb{T} is a finite, resp. one-sided infinite, resp. doubly infinite sequence (x_n) of vertices of \mathbb{T} such that $d(x_i, x_j) = |i - j|$ for all i, j . Two rays are *equivalent* if, as sets, their symmetric difference is finite. An *end* of \mathbb{T} is an equivalence class of rays. The space of ends is denoted $\partial\mathbb{T}$, and we write $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. For all $w, z \in \widehat{\mathbb{T}}$ there is a unique geodesic \overline{wz} that connects the two. In particular, if $x \in \mathbb{T}$ and $\mathbf{u} \in \partial\mathbb{T}$ then $\overline{x\mathbf{u}}$ is the ray that starts at x and represents \mathbf{u} . Furthermore, if $\mathbf{u}, \mathbf{v} \in \partial\mathbb{T}$ ($\mathbf{u} \neq \mathbf{v}$) then $\overline{\mathbf{u}\mathbf{v}}$ is the infinite geodesic whose

two halves (split at any vertex) are rays that represent \mathbf{u} and \mathbf{v} , respectively. If $w, z \in \widehat{\mathbb{T}}$, then their *confluent* $c = w \wedge z$ with respect to the root vertex o (=the identity element of \mathcal{G}) is defined by $\overline{ow} \cap \overline{oz} = \overline{oc}$. Let $(w|z) = d(o, c)$, which is finite unless $w = z \in \partial\mathbb{T}$. We can define a metric ρ on $\widehat{\mathbb{T}}$ by

$$(2.1) \quad \rho(w, z) = \begin{cases} q^{-(w|z)}, & \text{if } z \neq w, \\ 0, & \text{if } z = w. \end{cases}$$

This makes $\widehat{\mathbb{T}}$ a compact ultrametric space with \mathbb{T} as a dense, discrete subset. Each isometry $g \in \text{Aut}(\mathbb{T})$ extends to a homeomorphism of $\widehat{\mathbb{T}}$.

The natural compactification of \mathcal{C} in the topology of pointwise convergence is the set $\widehat{\mathcal{C}} = \mathbb{K}_r^{\mathbb{T}} = \{\zeta : \mathbb{T} \rightarrow \mathbb{K}_r\}$ of *all*, finitely or infinitely supported, configurations. Since the vertex set of the lamplighter graph is $\mathcal{C} \times \mathbb{T}$, the space $\widehat{\mathbb{K}_r \wr \mathbb{T}} = \widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ is a natural compactification, and $\partial(\mathbb{K}_r \wr \mathbb{T}) = (\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}) \setminus (\mathcal{C} \times \mathbb{T})$ is a natural “geometric” boundary at infinity of the lamplighter graph. We shall see that this boundary contains many points where the random walk (Z_n) does not accumulate. We define

$$(2.2) \quad \begin{aligned} \Pi &= \bigcup_{\mathbf{u} \in \partial\mathbb{T}} \mathcal{C}_{\mathbf{u}} \times \{\mathbf{u}\}, \quad \text{where} \\ \mathcal{C}_{\mathbf{u}} &= \{\zeta \in \widehat{\mathcal{C}} : \text{supp}(\zeta) \text{ is finite or accumulates just at } \mathbf{u}\}. \end{aligned}$$

Since $\mathcal{C}_{\mathbf{u}}$ is dense in $\widehat{\mathcal{C}}$, the closure of Π is the part $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ of $\partial(\mathbb{K}_r \wr \mathbb{T})$. The action of the group $\mathcal{Z}_r \wr \mathcal{G}$ on $\mathbb{K}_r \wr \mathbb{T}$ by multiplication from the left extends by homeomorphisms to $\widehat{\mathbb{K}_r \wr \mathbb{T}}$ and leaves the Borel subset Π invariant. Indeed, if $g = (\eta, x) \in \Gamma = \mathcal{Z}_r \wr \mathcal{G}$ and $\beta = (\zeta, \mathbf{u}) \in \widehat{\mathcal{C}} \times \widehat{\mathbb{T}}$ then, precisely as in (1.2),

$$(2.3) \quad g\beta = (\eta, x)(\zeta, \mathbf{u}) = (\eta + T_x\zeta, x\mathbf{u}).$$

(Addition of configurations is pointwise modulo r .) If in addition $\zeta \in \mathcal{C}_{\mathbf{u}}$, where $\mathbf{u} \in \partial\mathbb{T}$, then $\eta + T_x\zeta \in \mathcal{C}_{x\mathbf{u}}$, since adding η modifies $T_x\zeta$ only in finitely many points.

For the basic example (1.3), it is quite clear that Z_n converges almost surely to a random element $Z_{\infty} = (Y_{\infty}, X_{\infty}) \in \Pi$. Indeed, the \mathbb{T} -coordinate X_n of Z_n is the random walk on \mathbb{T} with transition probabilities $\tilde{p}(x, x') = \theta/(q+1)$, if $x' \sim x$, and $\tilde{p}(x, x) = 1 - \theta$ (and $\tilde{p}(x, y) = 0$ if $d(y, x) \geq 2$). An elementary argument using only transience yields that X_n converges to a random element $X_{\infty} \in \partial\mathbb{T}$. Also, only the states of those lamps can be modified which are visited by (X_n) , and by transience, each vertex is visited finitely often: after the last visit at a given vertex, the state of lamp sitting there remains unchanged. Therefore Y_n must converge pointwise to a random configuration Y_{∞} which can accumulate at no point besides X_{∞} .

We shall prove an analogous result for a much larger class of random walks on $\Gamma = \mathcal{Z}_r \wr \mathcal{G}$, or equivalently, on $\mathbb{K}_r \wr \mathbb{T}$. They are specified by a probability measure μ on the group Γ , and we suppose that $\text{supp}(\mu)$ generates Γ . We can model the random walk on the probability space (Ω, Pr) , where $\Omega = \Gamma^{\mathbb{N}}$ and $\text{Pr} = \mu^{\mathbb{N}}$, with $\mathbb{N} = \{1, 2, 3, \dots\}$. The n -th projections $\mathbf{g}_n = (\eta_n, \mathbf{x}_n) : \Omega \rightarrow \mathbb{Z}_r \wr \mathcal{G}$ are a sequence of independent, Γ -valued random variables with common distribution μ . If $g_0 = (\eta_0, x_0) \in \Gamma$ then the sequence of random

variables

$$Z_n = (Y_n, X_n) = g_0 \mathbf{g}_1 \cdots \mathbf{g}_n, \quad n \geq 0$$

is the *right random walk* on Γ with law μ and starting point g_0 . Its one-step transition probabilities are given by $p(g, g') = \mu(g^{-1}g')$, where $g = \eta x$ and $g' = \eta' x' \in \Gamma$. Note that

$$(2.4) \quad X_n = x_0 \mathbf{x}_1 \cdots \mathbf{x}_n \quad \text{and} \quad Y_n = \eta_0 + \sum_{k=1}^n T_{X_{k-1}} \boldsymbol{\eta}_k,$$

with pointwise addition modulo r . In particular, (X_n) is the right random walk on the “base” group \mathcal{G} with starting point x_0 whose law is the projection $\tilde{\mu}$ of μ ,

$$(2.5) \quad \tilde{\mu}(x) = \sum_{\eta} \mu(\eta x).$$

Before stating a result on convergence of (Z_n) to Π , we have to specify additional properties. We say that the lamplighter *operates at bounded range*, if

$$(2.6) \quad R(\mu) = \max\{\min\{d(y, o), d(y, x)\} : \mu(\eta x) > 0, y \in \text{supp}(\eta)\} < \infty.$$

This means that when the lamplighter steps from x to x' in \mathbb{T} then while doing so, he can modify the current configuration of lamps only at points which are at bounded distance from x or x' . (Note that here we do not require that μ itself has finite support.)

The law μ of the random walk is said to have *finite first moment*, if

$$(2.7) \quad \sum_{\eta x \in \mathcal{G}} d(o, \eta x) \mu(\eta x) < \infty.$$

In this case, it is a well known consequence of Kingman’s subadditive ergodic theorem (see e.g. DERRIENNIC [9]) that there are finite constants $\ell(P)$ and $m(P)$ such that

$$\frac{d(Z_n, Z_0)}{n} \rightarrow \ell(P) \quad \text{and} \quad \frac{d(X_n, X_0)}{n} \rightarrow m(P) \quad \text{almost surely.}$$

It is clear that $\ell(P) \geq m(P)$. Recent [15] and ongoing work by GILCH suggests that $\ell(P) > m(P)$ strictly.

(2.8) Lemma. *If P has finite first moment then $m(P) > 0$.*

Proof. This follows from the fact that \mathcal{G} is non-amenable, see [30, Thm. 8.14]. Again, for random walks with finite range on a group, this goes back to KESTEN [22], [23]. \square

(2.9) Theorem. *Let $Z_n = (Y_n, X_n)$ be a random walk with law μ on $\Gamma = \mathcal{Z}_r \wr \mathcal{G}$, such that $\text{supp}(\mu)$ generates Γ .*

If the lamplighter operates at finite range, or if μ has finite first moment, then there is a Π -valued random variable $Z_\infty = (Y_\infty, X_\infty)$ such that $Z_n \rightarrow Z_\infty$ in the topology of $\widehat{\mathbb{K}_r} \wr \mathbb{T}$, almost surely for every starting point $g_0 = \eta_0 x_0$.

Furthermore, the distribution of X_∞ is a continuous measure on $\partial\mathbb{T}$ (it carries no point mass), and consequently the same is true for the distribution of Z_∞ on Π .

Proof. We may suppose without loss of generality that $g_0 = id$, where $id = (0, o)$ is the identity of $\mathcal{Z}_r \wr \mathcal{G}$.

The law $\tilde{\mu}$ of the projected random walk (X_n) on \mathcal{G} is such that its support generates \mathcal{G} . CARTWRIGHT AND SOARDI [6] have shown that without any moment assumption, such a random walk on $\mathbb{T} \equiv \mathcal{G}$ converges almost surely to a random end, that is, a $\partial\mathbb{T}$ -valued random variable X_∞ .

Now suppose first that the lamplighter operates at bounded range. Let (y_n) be an unbounded sequence in \mathbb{T} with $y_n \in \text{supp}(Y_n)$, i.e., y_n is a vertex where the lamp is “on” at time n . Then we see from (2.4) that y_n must be at bounded distance from the initial trajectory $\{X_0, X_1, \dots, X_n\}$. Therefore, we must have $y_n \rightarrow X_\infty$. Indeed, one sees immediately from the definition (2.1) of the metric ρ that the end compactification $\hat{\mathbb{T}}$ has the following property: if $(x_n), (y_n)$ are two sequences in \mathbb{T} such that $x_n \rightarrow \mathbf{u} \in \partial\mathbb{T}$ and $\sup_n d(y_n, x_n) < \infty$, then also $y_n \rightarrow \mathbf{u}$.

Next, assume that the random walk has finite first moment. Define the integer-valued random variables $M_n = \max\{d(y, o) : y \in \text{supp}(\boldsymbol{\eta}_n)\}$. They are independent and identically distributed and also have finite first moment. Therefore $M_n/n \rightarrow 0$ almost surely. This implies that the following holds with probability 1.

(2.10) If (y_n) is a sequence in \mathbb{T} such that $y_n \in \text{supp}(T_{X_{n-1}}\boldsymbol{\eta}_n)$ for each n , then

$$d(y_n, X_{n-1})/n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Recall that almost surely $X_n \rightarrow X_\infty \in \partial\mathbb{T}$ and $d(X_n, o)/n \rightarrow m(P)$, with $m(P) > 0$ by Lemma 2.8. Thus, we get for the confluent $c_n = y_n \wedge X_{n-1}$ that $d(c_n, X_{n-1})/n \leq d(y_n, X_{n-1})/n \rightarrow 0$, whence $d(c_n, o)/n \rightarrow m(P)$. Therefore, for the metric (2.1) of $\hat{\mathbb{T}}$, we obtain $\rho(y_n, X_{n-1}) \rightarrow 0$. Consequently $y_n \rightarrow X_\infty$.

Now observe that by (2.4),

$$\text{supp}(Y_n) \subset \bigcup_{k=1}^n \text{supp}(T_{X_{k-1}}\boldsymbol{\eta}_k),$$

which is a union of finite sets. Thus, almost surely by the above, if (y_n) is an unbounded sequence in \mathbb{T} with $y_n \in \text{supp}(Y_n)$, it must converge to X_∞ .

Finally, from [6] it is also known (without moment hypothesis) that the distribution of X_∞ is a continuous measure on $\partial\mathbb{T}$. \square

3. THE POISSON BOUNDARY

Under the assumptions of Theorem 2.9, we can define the limit distribution ν of the random walk. This is the probability measure defined for Borel sets $U \subset \Pi$ by

$$(3.1) \quad \nu(U) = \Pr[Z_\infty \in U \mid Z_0 = id]$$

Recall the natural action (2.3) of Γ on Π . If $g_0 \in \Gamma$, then

$$\Pr[Z_\infty \in U \mid Z_0 = g] = \nu(g^{-1}U).$$

This implies that ν satisfies the convolution equation $\mu * \nu = \nu$. In particular, the measure space (Π, ν) is a *boundary* of the random walk on Γ in the sense of FURSTENBERG [14].

A general boundary is a suitable probability space (\mathbf{B}, λ) such that Γ acts on \mathbf{B} by measurable bijections and $\mu * \lambda = \lambda$. As is explained in [14], one can then typically (i.e., when \mathbf{B} carries a topology and Γ acts continuously) construct a topology on $\Gamma \cup \mathbf{B}$ such

that (Z_n) converges almost surely to a \mathbf{B} -valued random variable whose distribution is λ , given that $Z_0 = id$. This means that (\mathbf{B}, λ) is a model for describing in detail how (Z_n) tends to infinity, that is, for distinguishing limit points of (Z_n) as $n \rightarrow \infty$. When comparing two boundaries, this has of course to be done modulo sets with measure 0. For example, in the case of the boundary (Π, ν) it may appear more natural to consider ν as a measure on the closure $\bar{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$ of Π in $\widehat{\mathbb{K}_r} \wr \mathbb{T}$, with the same definition as in (3.1) but Borel sets $U \subset \widehat{\mathcal{C}} \times \partial\mathbb{T}$. Since ν charges only the (dense) subset Π , the measure spaces (Π, ν) and $(\widehat{\mathcal{C}} \times \partial\mathbb{T}, \nu)$ are isomorphic.

In this spirit, our question is whether the boundary (Π, ν) is *maximal*. This means that for every other boundary (\mathbf{B}, λ) , up to sets with measure 0, there is a measure-preserving surjection of (Π, ν) onto (\mathbf{B}, λ) . A more heuristic interpretation of maximality is that (Π, ν) is the finest model for distinguishing limit points at infinity of the random walk. Existence and uniqueness of the maximal boundary is a general fact [20], and it is called the *Poisson boundary*.

KAIMANOVICH [18], [21] has provided a useful geometric tool for checking maximality. Consider the “reflected” right random walk on Γ

$$\check{Z}_n = g_0 \mathbf{g}_1^{-1} \cdots \mathbf{g}_n^{-1}$$

starting at g_0 (we shall only consider $g_0 = id$). Its law is the probability measure $\check{\mu}$ on Γ , where $\check{\mu}(g) = \mu(g^{-1})$.

(3.2) Proposition. [Kaimanovich]. *Let μ be a probability measure on Γ with a finite first moment, and let (\mathbf{B}, λ) and $(\check{\mathbf{B}}, \check{\lambda})$ be a μ - and a $\check{\mu}$ -boundary, respectively. Suppose that there is a measurable G -equivariant map S assigning to $(\lambda \times \check{\lambda})$ -almost every pair of points $(\beta, \check{\beta}) \in \mathbf{B} \times \check{\mathbf{B}}$ a non-empty “strip” $S(\beta, \check{\beta}) \subset \Gamma$ such that for the ball $B(id, n)$ of radius n in the metric of Γ ,*

$$\frac{1}{n} \log |S(\beta, \check{\beta}) \cap B(id, n)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for $(\lambda \times \check{\lambda})$ -almost every $(\beta, \check{\beta}) \in \Pi \times \check{\Pi}$, then (\mathbf{B}, λ) and $(\check{\mathbf{B}}, \check{\lambda})$ are the Poisson boundaries of the random walks with law μ and $\check{\mu}$, respectively.

(3.3) Theorem. *Let $Z_n = (Y_n, X_n)$ be a random walk with law μ on $\Gamma = \mathcal{Z}_r \wr \mathcal{G}$, such that μ has finite first moment and $\text{supp}(\mu)$ generates Γ . If Π is defined as in (2.2) and ν is the limit distribution on Π of (Z_n) starting at id , then (Π, ν) is the Poisson boundary of the random walk.*

Proof. By Theorem 2.9, each of the random walks (Z_n) and (\check{Z}_n) starting at id converges almost surely to a Π -valued random variable. Let ν and $\check{\nu}$ be their respective limit distributions. Then the spaces (Π, ν) and $(\Pi, \check{\nu})$ are boundaries of the respective random walks. If $\beta = (\zeta, \mathbf{u}) \in \Pi$ then let $\mathbf{u}(\beta) = \mathbf{u}$. Also, if $y \in \mathbb{T}$ and $\mathbf{u} \in \partial\mathbb{T}$ then let $\mathbb{T}_y(\mathbf{u})$ be the subtree of \mathbb{T} which is the component of \mathbf{u} in $\mathbb{T} \setminus \{y\}$ (that is, every ray that represents \mathbf{u} has all but finitely many vertices in this component of $\mathbb{T} \setminus \{y\}$).

By continuity of ν and $\check{\nu}$ (Theorem 2.9), we have

$$\nu \times \check{\nu}(\{(\beta, \check{\beta}) \in \Pi \times \Pi : \mathbf{u}(\beta) = \mathbf{u}(\check{\beta})\}) = 0.$$

Therefore, we only need to construct the strips $S(\beta, \check{\beta})$ when $\mathbf{u}(\beta) \neq \mathbf{u}(\check{\beta})$. Thus, let $\beta = (\zeta, \mathbf{u}), \check{\beta} = (\check{\zeta}, \mathbf{v}) \in \Pi$ and $\mathbf{u} \neq \mathbf{v}$. For any vertex y on the (two-sided infinite) geodesic $\overline{\mathbf{u}\mathbf{v}}$, let $\eta_y(\beta, \check{\beta})$ be the configuration which coincides with ζ on $\mathbb{T}_y(\mathbf{v})$ and with $\check{\zeta}$ on $\mathbb{T} \setminus \mathbb{T}_y(\mathbf{v})$. This configuration has finite support, since $\text{supp}(\zeta)$ can only accumulate at \mathbf{u} and $\text{supp}(\check{\zeta})$ can only accumulate at \mathbf{v} , while $\mathbb{T}_y(\mathbf{v})$ does not accumulate at \mathbf{u} and $\mathbb{T} \setminus \mathbb{T}_y(\mathbf{v})$ does not accumulate at \mathbf{v} . Then define

$$S(\beta, \check{\beta}) = \{(\eta_y(\beta, \check{\beta}), y) : y \in \overline{\mathbf{u}\mathbf{v}}\}.$$

This is a subset of Γ . We check that the map $(\beta, \check{\beta}) \mapsto S(\beta, \check{\beta})$ is Γ -equivariant: let $g = (\eta, x) \in \Gamma$. We have to show that

$$(3.4) \quad g S(\beta, \check{\beta}) = S(g\beta, g\check{\beta}).$$

Now, if $y \in \overline{\mathbf{u}\mathbf{v}}$ then $xy \in \overline{(x\mathbf{u})(x\mathbf{v})}$. Also, $T_x \eta_y(\beta, \check{\beta}) = \eta_{xy}(\beta', \check{\beta}')$, where $\beta' = (T_x \zeta, x\mathbf{u})$ and $\check{\beta}' = (T_x \check{\zeta}, x\mathbf{v})$. Therefore

$$\begin{aligned} \eta + \eta_{xy}(\beta', \check{\beta}') &= \eta_{xy}(\beta'', \check{\beta}''), \quad \text{where} \\ \beta'' &= (\eta + T_x \zeta, x\mathbf{u}) = g\beta \quad \text{and} \quad \check{\beta}'' = (\eta + T_x \check{\zeta}, x\mathbf{v}) = g\check{\beta}. \end{aligned}$$

Thus, for $y \in \overline{\mathbf{u}\mathbf{v}}$,

$$g(\eta_y(\beta, \check{\beta}), y) = (\eta + T_x \eta_y(\beta, \check{\beta}), xy) = (\eta_{xy}(g\beta, g\check{\beta}), xy)$$

This proves (3.4).

Finally, if $\eta y \in S(\beta, \check{\beta})$ and $d(o, \eta y) \leq n$ then $d(o, y) \leq n$. Since

$$|\{y \in \overline{\mathbf{u}\mathbf{v}} : d(o, y) \leq n\}| \leq 2n,$$

we see that all conditions of Proposition 3.2 are satisfied. \square

4. THE DIRICHLET PROBLEM AT INFINITY

In this section we shall assume in addition that our random walk on Γ is *irreducible* in the sense that its law μ is such that $\text{supp}(\mu)$ generates Γ as *semigroup*. Equivalently, this means that for every pair of elements $g = \eta x, g' = \eta' x' \in \Gamma$ the probability that the random walk starting at g ever visits g' is > 0 .

Also, it will be convenient to consider the limit distribution ν of (3.1) as a Borel measure on the compact set $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$. The irreducibility hypothesis implies that

$$(4.1) \quad \text{supp}(\nu) = \overline{\Pi},$$

that is, the whole of $\overline{\Pi}$ is active (as we shall see below).

With respect to our random walk with transition probabilities $p(g, g') = \mu(g^{-1}g')$ on Γ , a function $h : \Gamma \rightarrow \mathbb{R}$ is called *harmonic*, if it satisfies the weighted mean value property

$$h(g) = \sum_{g'} p(g, g') h(g') \quad \text{for all } g \in \Gamma.$$

For $g \in \Gamma$, we define ν_g by $\nu_g(U) = \nu(g^{-1}U)$, where U runs through Borel subsets of $\overline{\Pi}$. It is a basic feature of the Poisson boundary that every bounded harmonic function h on

Γ has a unique integral representation

$$(4.2) \quad h(g) = \int_{\widehat{\mathcal{C}} \times \partial\mathbb{T}} f d\nu_g,$$

where $f \in L^\infty(\nu)$, see e.g. [20].

Conversely, once we have convergence of the random walk to the boundary, any function $f \in L^\infty(\nu)$ gives rise to a harmonic function via the integral formula (4.2). In fact, for this we do not need to know that $\overline{\Pi}$ is the Poisson boundary; the only point is that otherwise we will not get *all* bounded harmonic functions via (4.2).

Related to existence of the limit measure on the boundary, there is the question whether the *Dirichlet problem at infinity* is solvable. In our case it reads as follows:

Does every continuous function on $\overline{\Pi} = \widehat{\mathcal{C}} \times \partial\mathbb{T}$ extend continuously to a function on $\Gamma \cup \overline{\Pi}$ which is harmonic on Γ ?

If the answer is positive, then that harmonic function must be given by (4.2), and we would like that whenever f is continuous it should hold that $\lim_{g \rightarrow \beta} h(g) = f(\beta)$ for every $\beta \in \overline{\Pi}$. We then say that the *Dirichlet problem at infinity is solvable* for continuous functions on $\overline{\Pi}$.

We note that usually, the Dirichlet problem at infinity refers to the harmonic extension of any continuous function that is given on the *whole* boundary in a (suitable) compactification of the state space, compare with [30, §20]. In our case, the whole boundary is the set $(\widehat{\mathcal{C}} \times \widehat{\mathbb{T}}) \setminus (\mathcal{C} \times \mathbb{T})$ which contains $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ as a proper, compact subset. However, the complement of $\widehat{\mathcal{C}} \times \partial\mathbb{T}$ is not charged by ν , so that boundary data given on that complement have no effect on the harmonic function h of (4.2), and we cannot expect continuity at those points. Therefore we have to restrict to continuous functions on $\widehat{\mathcal{C}} \times \partial\mathbb{T}$.

The *Green kernel* of the projected random walk (X_n) on $\mathbb{T} \equiv \mathcal{G}$ with law $\tilde{\mu}$ as in (2.5) is

$$\tilde{G}(x, y) = \sum_{n=0}^{\infty} \tilde{p}^{(n)}(x, y) = \sum_{n=0}^{\infty} \tilde{\mu}^{(n)}(x^{-1}y), \quad x, y \in \mathbb{T},$$

where $\tilde{p}^{(n)}(\cdot, \cdot)$ denotes n -step transition probabilities and $\tilde{\mu}^{(n)}$ is the n -th convolution power of $\tilde{\mu}$. By irreducibility and transience, $0 < \tilde{G}(x, y) < \infty$. This is the expected number of times that (X_n) visits y , given that $X_0 = 0$.

(4.3) Lemma. *If $\tilde{\mu}$ is irreducible, then Green kernel vanishes at infinity, that is,*

$$\lim_{d(x,y) \rightarrow \infty} \tilde{G}(x, y) = 0.$$

Proof. The Group \mathcal{G} is non-amenable. Let $S = \text{supp}(\tilde{\mu})$. Since S generates \mathcal{G} as a semigroup, it is a well-known exercise to show that the subgroup of \mathcal{G} generated by $S \cdot S^{-1}$ is a finite-index normal subgroup, whence also non-amenable. Therefore one can apply Théorème 2 of DERRIENNIC AND GUIVARC'H [8] and/or Théorème 2 of BERG AND CHRISTENSEN [8] to obtain that the measure $\sum_{n=0}^{\infty} \tilde{\mu}^{(n)}$ defines a bounded convolution operator on $\ell^2(\mathcal{G})$. It follows that the Green kernel vanishes at infinity. This may also be deduced by applying the main theorem of [23] to $\tilde{\mu} * \tilde{\mu}$. \square

Below, we shall need the quantity

$$\tilde{F}(x, y) = \tilde{G}(x, y) / \tilde{G}(y, y) = \Pr[\exists n \geq 0 : X_n = y \mid X_0 = 0],$$

which also vanishes at infinity.

(4.4) Theorem. *If μ is irreducible and the lamplighter operates at finite range, then the Dirichlet problem at infinity for continuous functions on $\bar{\Pi} = \hat{\mathcal{C}} \times \partial\mathbb{T}$ is solvable.*

Proof. The typical ‘‘probabilistic’’ method for proving this (see e.g. [30, §20]) is to show the following: (i) the random walk Z_n converges to the boundary, and (ii) for the corresponding harmonic measure class $\{\nu_g : g = \eta x \in \Gamma\}$, one has

$$(4.5) \quad \lim_{g \rightarrow \beta} \nu_g = \delta_\beta \quad \text{weakly for every } \beta = (\zeta, \mathbf{u}) \in \hat{\mathcal{C}} \times \partial\mathbb{T}.$$

Point (i) is affirmed by Theorem 2.9. For proving (ii), it will be convenient to consider ν_g as a measure on $\hat{\mathcal{C}} \times \hat{\mathbb{T}}$ which charges only the set $\bar{\Pi} = \hat{\mathcal{C}} \times \partial\mathbb{T}$. Thus, we show that for any neighbourhood U in $\hat{\mathcal{C}} \times \hat{\mathbb{T}}$ of $\beta = (\zeta, \mathbf{u}) \in \bar{\Pi}$, we have

$$\lim_{g \rightarrow \beta} \nu_g(U^c) = 0,$$

where $U^c = (\hat{\mathcal{C}} \times \hat{\mathbb{T}}) \setminus U$. Here, it is sufficient to take U in a suitable neighbourhood basis of β . Such a basis is obtained as follows: take a vertex $y \in \mathbb{T}$ and let $\hat{\mathbb{T}}_y(\mathbf{u})$ be the closure in $\hat{\mathbb{T}}$ of the subtree $\mathbb{T}_y(\mathbf{u})$ (the component of \mathbf{u} in $\mathbb{T} \setminus \{y\}$, see the proof of Theorem 3.3). The family of all $\hat{\mathbb{T}}_y(\mathbf{u})$, $y \in \mathbb{T}$, is a neighbourhood basis of \mathbf{u} in $\hat{\mathbb{T}}$. Also, the family of all sets $\hat{\mathcal{C}}_A(\zeta) = \{\zeta' \in \hat{\mathcal{C}} : \zeta' = \zeta \text{ on } A\}$, where $A \subset \mathbb{T}$ is finite, is a neighbourhood basis of ζ in $\hat{\mathcal{C}}$ for the topology of pointwise convergence of configurations. Thus,

$$\{U_{y,A}(\beta) = \hat{\mathbb{T}}_y(\mathbf{u}) \times \hat{\mathcal{C}}_A(\zeta) : y \in \overline{\sigma\mathbf{u}}, A \subset \mathbb{T} \text{ finite}\}$$

is a neighbourhood basis of $\beta = (\zeta, \mathbf{u})$. Consider $U = U_{x,A}(\beta)$. Then

$$U^c \subset V_1 \cup V_2, \quad \text{where } V_1 = (\hat{\mathbb{T}} \setminus \hat{\mathbb{T}}_y(\mathbf{u})) \times \hat{\mathcal{C}} \quad \text{and} \quad V_2 = \hat{\mathbb{T}}_y(\mathbf{u}) \times (\hat{\mathcal{C}} \setminus \hat{\mathcal{C}}_A(\zeta)).$$

We now prove that $\nu_g(V_i) \rightarrow 0$ for $i = 1, 2$ as $g = \eta x \rightarrow \beta$.

Regarding V_1 , we first remark that the Dirichlet problem at infinity for continuous functions on $\partial\mathbb{T}$ is solvable for the random walk (X_n) with law $\tilde{\mu}$ on $\mathcal{G} \equiv \mathbb{T}$, see [6] and [30, Cor. 21.12]. Therefore

$$\lim_{g \rightarrow \beta} \nu_g(V_1) = \lim_{x \rightarrow \mathbf{u}} \Pr_x[X_\infty \in \partial\mathbb{T} \setminus \hat{\mathbb{T}}_y(\mathbf{u})] = 0.$$

Regarding V_2 , let $R = R(\mu)$ be the bound on the range of (2.6). Set $A^R = \{v \in \mathbb{T} : d(v, A) \leq R\}$. Suppose $\eta = \eta(g)$ is sufficiently close to the limit ζ so that $\eta(v) = \zeta(v)$ for every $v \in A$. If the random walk $Z_n = (Y_n, X_n)$ starting at g converges to a limit point in V_2 then (X_n) must visit A^R in order to modify the states of the lamps at the points in A . Therefore

$$\nu_g(V_2) \leq \sum_{v \in A^R} \tilde{F}(x, v),$$

which tends to zero when $x \rightarrow \mathbf{u}$ by Lemma 4.3. This concludes the proof. \square

5. FINAL REMARKS

Theorem 3.3 is yet one more application of the very useful strip criterion of V. A. KAIMANOVICH who, in private communication, has informed us that in the unpublished paper [19] he uses a method in a somewhat similar spirit to the proof of Theorem 3.3 to describe the Poisson boundary for random walks on $\mathcal{Z}_r \wr \mathbb{Z}^d$, where the projection of the random walk onto the integer grid \mathbb{Z}^d has non-zero drift.

Regarding the Dirichlet problem at infinity, we repeat here that this type of question can be asked whenever one has a compactification of the state space in whose topology the random walk converges almost surely to the boundary at infinity. Solvability of the Dirichlet problem for continuous functions on the boundary (or rather its active part, i.e., the support of the limit measure) is by no means the same as having determined the Poisson (or even Martin) boundary, as one finds erroneously stated every now and then in published work. For example, in our case, we know that $(\overline{\Pi}, \nu)$ is the Poisson boundary when μ has finite first moment, while we proved that the corresponding Dirichlet problem is solvable when the lamplighter operates at finite range, but μ need not have finite first moment for that. Thus, we have given positive answers to both questions simultaneously only when μ has finite first moment *and* the lamplighter operates at finite range.

The methods that we have used can be extended to space-homogenous (in the sense of [21]) lamplighter random walks over hyperbolic graphs, graphs with infinitely many ends, and other classes of transitive base graphs that can be used in the construction of lamplighter graphs according to (1.1). Also, the fact that the states of the lamps are encoded by the complete graph \mathbb{K}_r , resp. the cyclic group \mathcal{Z}_r , does not play an essential role. Finally, with additional effort, the result regarding the Dirichlet problem can apparently be extended. The detailed elaboration of these general facts remains reserved to future work, while the present note has aimed at giving a short and hopefully readable explanation of the basic aspects.

REFERENCES

- [1] Bartholdi, L., and Woess, W.: *Spectral computations on lamplighter groups and Diestel-Leader graphs*, J. Fourier Anal. Appl. **11** (2005) 175 - 202.
- [2] Berg, C., and Christensen, J. P. R.: *Sur la norme des opérateurs de convolution*, Invent. Math. **23** (1974) 173–178.
- [3] Brofferio, S., and Woess, W.: *Green kernel estimates and the full Martin boundary for random walks on lamplighter groups and Diestel-Leader graphs*, Annales Inst. H. Poincaré (Prob. & Stat.) **41** (2005) 1101–1123.
- [4] Brofferio, S., and Woess, W.: *Positive harmonic functions for semi-isotropic random walks on trees, lamplighter groups, and DL-graphs*, Potential Analysis **24** (2006) 245–265.
- [5] Cartwright, D. I., Kaimanovich, V. A., and Woess, W.: *Random walks on the affine group of local fields and of homogeneous trees*, Ann. Inst Fourier (Grenoble) **44** (1994) 1243–1288.
- [6] Cartwright, D. I., and Soardi, P. M.: *Convergence to ends for random walks on the automorphism group of a tree*, Proc. Amer. Math. Soc. **107** (1989) 817–823
- [7] Ceccherini-Silberstein, T., and Woess, W.: *Growth and ergodicity of context-free languages*, Trans. Amer. Math. Soc. **354** (2002) 4597–4625.
- [8] Derriennic, Y., and Guivarc’h, Y.: *Théorème de renouvellement pour les groupes non moyennables*, C. R. Acad. Sci. Paris, Série A **277** (1973) 613–615.

- [9] Derriennic, Y.: *Quelques applications du théorème ergodique sous-additif*, Astérisque **74** (1980) 183–201.
- [10] Dicks, W., and Schick, Th.: *The spectral measure of certain elements of the complex group ring of a wreath product*, Geom. Dedicata **93** (2002) 121–137.
- [11] Erschler, A. G.: *On the asymptotics of the rate of departure to infinity* (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **283** (2001) 251–257, 263.
- [12] Erschler, A. G.: *On drift and entropy growth for random walks on groups*, Ann. Probab. **31** (2003) 1193–1204.
- [13] Erschler, A. G.: *Liouville property for groups and manifolds*, Invent. Math. **155** (2004) 55–80.
- [14] Furstenberg, H.: *Random walks and discrete subgroups of Lie groups*. In *Advances in Probability and Related Topics*, **1** (P. Ney, ed.), pp. 1–63, M. Dekker, New York, 1971.
- [15] Gilch, L. A.: *Rate of escape on the lamplighter tree*, Preprint, TU Graz (2006).
- [16] Grigorchuk, R. I., and Żuk, A.: *The lamplighter group as a group generated by a 2-state automaton, and its spectrum*, Geom. Dedicata **87** (2001) 209–244.
- [17] Kaimanovich, V. A.: *Poisson boundaries of random walks on discrete solvable groups*, in: *Probability Measures on Groups X* (ed. H. Heyer), pp. 205–238, Plenum, New York, 1991.
- [18] Kaimanovich, V. A.: *The Poisson formula for groups with hyperbolic properties*, Annals of Math. **152** (2000) 659–692.
- [19] Kaimanovich, V. A.: *Poisson boundary of discrete groups, a survey*, unpublished manuscript, <http://name.math.univ-rennes1.fr/vadim.kaimanovich/list.htm>
- [20] Kaimanovich, V. A., and Vershik, A. M.: *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983) 457–490.
- [21] Kaimanovich, V. A., and Woess, W.: *Boundary and entropy of space homogeneous Markov Chains*, Ann. Probab. **30** (2002) 323–363.
- [22] Kesten, H.: *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959) 336–354.
- [23] Kesten, H.: *Full Banach mean values on countable groups*, Math. Scand. **7** (1959) 146–156.
- [24] Lyons, R., Pemantle, R., and Peres, Y.: *Random walks on the lamplighter group*, Ann. Probab. **24** (1996) 1993–2006.
- [25] Parry, W.: *Growth series of some wreath products*, Trans. Amer. Math. Soc. **331** (1992) 751–759.
- [26] Pittet, C., and Saloff-Coste, L.: *Amenable groups, isoperimetric profiles and random walks*, in: *Geometric Group Theory Down Under* (Canberra, 1996), pp. 293–316, de Gruyter, Berlin, 1999.
- [27] Pittet, C., and Saloff-Coste, L.: *On random walks on wreath products*, Ann. Probab. **30** (2002) 948–977.
- [28] Revelle, D.: *Rate of escape of random walks on wreath products*, Ann. Probab. **31** (2003) 1917–1934.
- [29] Revelle, D.: *Heat kernel asymptotics on the lamplighter group*, Electron. Comm. Probab. **8** (2003), 142–154.
- [30] Woess, W.: *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics **138**, Cambridge University Press, Cambridge, 2000.
- [31] Woess, W.: *Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions*, Combinatorics, Probability & Computing **14** (2005) 415–433.
- [32] Woess, W.: *A note on the norms of transition operators on lamplighter graphs and groups*, Internat. J. Algebra Comput. **15** (2005) 1261–1272.

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY
100 44 STOCKHOLM, SWEDEN

E-mail address: akarl@math.kth.se

INSTITUT FÜR MATHEMATIK C, TECHNISCHE UNIVERSITÄT GRAZ,
STEYRERGASSE 30, A-8010 GRAZ, AUSTRIA

E-mail address: woess@TUGraz.at