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Tandem Queues for Inventory Management under Random Perturbations

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Using the theory of M/M/1 queues at stationarity, we provide criteria of stability (recurrence) for a stochastic inventory model with an observed selling rate and an optimally chosen buying rate. Optimality is based on the maximum gain under stability, where buying and selling prices, as well as shop- and stock-keeping costs are incorporated into the model. An important aspect is to achieve robustness of the stocking process by minimizing the fluctuation of the predicted gain. This robustness can be achieved by controlling intermediate transfer rates of the assumed stochastic tandem network. Stochastic simulations demonstrate the applicability of the stability criteria under several scenarios of differing intensities of perturbation. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: Queueing theory; tandem networks; inventory management

1. Introduction

1.1. Motivation

In inventory science, the prediction of reordering points and optimal stock levels given an observed selling rate has always been of central interest[1]. If prices concerning production, selling, and stock-keeping costs are constant and demand is uncertain, then determining the optimal stock level is known as the *Newsboy problem*. The question of the influence on these predictions concerning optimality if more than one decision-making party is involved has been modeled with so-called 'echelon-systems'[2]. For example, a serial two-echelon system may be given by a simple supply chain consisting of the manufacturer and the retailer, who represent two independent parties with possibly different views of the market[3]. Optimal policies have been given for models in which several retailers (representing different 'parallel' parties) are ordering at different frequencies independently[4, 5].

In queueing theory, the situation of several processes being active at the same time, independently, can be captured in the case of purely Markovian queues if the service process is divided into independent Poisson processes. Queues have been successfully applied to inventory models[6]. In particular, the theory of M/M/1-queues involving the stationary measure of the underlying Markov chain could be used effectively[7, 8]. Our approach is a direct extension of this model wherein a tandem-series of two M/M/1-queues is considered.

Recently, Huang *et al.*[9] investigated the influence of the limited capacity of transport as an influencing factor for optimal inventory policies. Similarly, we study changes of the system under unfavorable *perturbations* of the initial system parameters, namely:

1. Changes of the probability to handle incoming goods as opposed to serve demanding customers (These changes will correspond to changes of the transition probabilities of a Markov chain on the integers: $\langle p,q\rangle \mapsto \langle p',q'\rangle$ where $p+q=p'+q'=1$).
2. Changes of the customer's disposition to buy (involving only changes of buying *rates* $\mu \mapsto \mu'$, with μ, $\mu'>0$).

Moreover, we use a tandem queueing network to differentiate between stock and shop. In this way, the influence of a cheaper stock-keeping cost can be investigated.

Tandem queues have successfully been used for situations in which a succession of two processes has to be performed, such that 'in the long run' the work gets done without an arrival pileup. Here, the combined process of buying, stocking, and selling goods in a retailer's and logistics environment is modeled by a tandem queueing network under the assumption of independence between the events of single transactions, i.e. a series of two M/M/1-queues[10]. Selling goods, happening between exponentially

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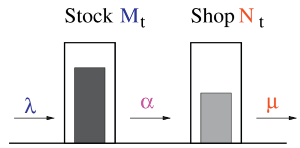

Figure 1. Tandem queue with stock and shop level, showing buying rate λ, selling rate μ and transfer rate α between stock and shop. μ is subject to a perturbation: sometimes, it is replaced by $\mu' < \mu$

distributed periods of time, represents the *service* process, buying goods the *arrival* process in the queue (with exponentially distributed interarrival times), whereas queue length processes characterize the amount of available goods in the storage room and the shelf of the store. Markovian rates for buying, transferring (from stock to shop), and selling determine the simple tandem queueing network, as any decrement of the stock due to good transfer is accompanied by an increment of the shop.

Figure 1 exhibits the applied stochastic network consisting of the stock and the shop level. Incoming goods arriving at an *arrival rate* λ, are transferred to the shop floor with *transfer rate* α and are sold to the customer at *selling rate* μ. A typical application in a retailer environment may be: goods are ordered by a distribution center and arrive at a specific buying rate λ. Upon request and depending on the availability in the shop those goods are transferred from the stock at an intermediate transfer rate α. The consumer enters the shop and demands goods at a specific rate μ.

We use A_t for the number of arrivals of goods up to time t, and D_t for the number of demands in the interval $[0, t]$. These are Poisson processes with rates λ and μ, respectively. We presume the validity of a *gain process* g_t depending on the size M_t and the increase A_t of the stock inventory, as well as the size N_t and decrease D_t of the shop inventory. It determines the amount earned by sales at price P, and losses due to buying-cost C, as well as the cost B_1 for stock keeping, and maintaining goods within the shop (B_2). It is given by

$$g_t(\omega) = P \cdot S_t(\omega) - B_1 \int_0^t M_s(\omega) \mathrm{d}s - B_2 \int_0^t N_s(\omega) \mathrm{d}s - C \cdot A_t(\omega), \tag{1}$$

where S_t is the number of demands during availability of the goods in the shop.

Using this model, when given an observed density of demand μ, an optimal buying (λ) and stocking (α) strategy is sought maximizing the asymptotic gain rate $G = G(\lambda, \alpha)$, which is defined by

$$G = \lim_{t \to \infty} \frac{1}{t} g_t(\omega).$$

This strategy mainly consists of two parts:

(A) The first goal is to find the optimal buying rate λ, and rate of good-transfer α between stock and shop, under the dependence of the observed selling rate μ.

(B) The second goal is to give stability criteria to the network, to make the above gain-optimization also realizable under time-dependent perturbations of the parameter.

We define *stability* of the retail process by *recurrence* of the combined process of stock- and shop size (the two queue length processes of the tandem network). While the question of recurrence of this Markov chain has been answered for fixed rates, we will present the stability criteria for the situation, where one of the rates, namely the selling rate μ will be perturbed. This perturbation is time-inhomogeneous in nature, reflecting the real-life situation of fluctuations in the model-defining parameters. We arrive at a combination of two conditions guaranteeing optimality and stability (recurrence).

Our model is a tandem queueing network, with arrival rate λ, transfer rate α, and service rate μ. It is well known that this queueing system is positive recurrent if $\lambda < \alpha$, and $\lambda < \mu$. Note that it is *not* essential that, $\alpha < \mu$, a condition which ensures stability of the second queue by itself[11]. Owing to Burke's theorem[12], the output process of the first queue in equilibrium (presuming its existence) is a Poisson process of intensity λ, the arrival rate, representing a discrete form of the continuity equation. Again, in equilibrium, this rate is the arrival rate for the second queue, and therefore the condition for equilibrium is $\lambda < \mu$. This also follows from the work of Fayolle, Malyshev, and Menshikov on random walks on the quarter plane with a specific (unsymmetric) set of generators. Although the model is non-reversible, the authors give geometric criteria[13] of ergodicity (implying positive recurrence) depending on the mean one-step displacement. These are equivalent to the conditions $\lambda < \alpha$ and $\lambda < \mu$.

1.2. *What is new in our model?*

The proposed model inherits three novelties not covered in Reference[13]:

- The problem of a random walk on the quarter plane with random conductances is currently unsolved[13]. We go one step towards a solution by allowing *one* of the parameters (the selling rate μ) to be different for each step of the random walk in $\mathbb{Z}_+^2$.

- We consider a system in which goods are transferred independently to the stock, from the stock to the shop, and from the shop to the consumer. Owing to this independence, the system may be considered a two-echelon system, in which independent parties agitate each with its own rate in time.
- Since we also incorporate the effect of *perturbations* of the selling rate μ by switching to μ', the system may also be considered a parallel two-layer system, due to two independent types of consumers acting simultaneously.

Summarizing, we have a composition of a two-echelon system in series (stock and shop) with a two-layer system in parallel (two different consumer types).

We organize the remainder of this paper in the following way: Section 2 contains a detailed definition of the model and proofs of recurrence of an extension of the tandem queue. Section 3 offers a simulation analysis of a real-life application, assuming random perturbations of the selling process.

As for notation, let $\mathbb{N}_0$ denote the non-negative integers, and $\mathbb{R}^0_+ := [0, \infty)$. When we speak of tandem queues, we mean two queues in series: We call the first queue the one *in front*, and the second one the *rear queue*, along the direction of the bypassing goods (from left to right).

2. Criteria of stability

2.1. The unperturbed two-echelon

First, consider two $M/M/1$-queues in tandem, the arrival process of the first queue, with length process M_t is Poisson with parameter λ and has exponentially distributed service with rate α; the second has a length process denoted by N_t and has an exponentially distributed service process with rate μ.

Each of these standard queues is a continuous-time Markov chain (birth–death process) with constant birth rate λ, and constant death rate μ (independent of the state of the queue length). The tandem process is defined on the set of right continuous integer-valued functions with the probability measure of a continuous time Markov process determined by an embedded discrete-time process, with transition probabilities $(P_{kl}), k, l \in \mathbb{Z}^2_+$. This is done in the following way[14]: After picking an initial state according to some initial distribution, let $j=1$ and proceed in the following way:

1. Wait a time T_j, which is an independent exponentially distributed variable with parameter equal to one.
2. Make a jump from the current state to one of its neighbors. Choose among the neighbors with probability distribution P_{kl}, where $k \in \mathbb{Z}^2_+$ is the current state and $l \in \mathbb{Z}^2_+$ is among the accessible neighbors of k.
3. Increase j and go to Step 1.

This is repeated, and since all the waiting times are independent, the process is Markovian. Being at state $k = \langle k_1, k_2 \rangle$ means that the stock inventory fulfills $M_t = k_1$ and $N_t = k_2$. Possible neighbors of $k = \langle M_t, N_t \rangle$ include those that allow a transition to either increase M_t, decrease N_t, or simultaneously decrease M_t and increase N_t. To these three transitions, we assign the transition probabilities $\lambda/(\lambda+\mu+\alpha)$, $\mu/(\lambda+\mu+\alpha)$, and $\alpha/(\lambda+\mu+\alpha)$, respectively (see Figure 2).

Remark

Usually the process is defined with waiting times which obey an exponential distribution with parameter equal to $\lambda+\mu+\alpha$ (instead of one), corresponding to the *first of the three* exponentially distributed waiting times belonging to the three possible transitions. However, since the sum of these rates is equal at every state, we can scale the time by a factor $\lambda+\mu+\alpha$ to obtain exactly the process defined above.

2.2. Two different types of perturbations

As an alternative to the 'unperturbed' standard tandem queueing network defined above, consider now a modification of the second (rear) M/M/1-queue. Note first that N_t can be 'lifted' to a random walk X_t on the fence-graph with vertices $S = \mathbb{N}_0 \times \{0, 1\}$. For any given step of a path of N_t, the Markov chain X_t just performs—with probability $\frac{1}{2}$—an additional switch between the upper and lower rails of the fence. As the fence-graph is a covering of the non-negative integers with nearest neighbors, the projection of X_t onto its first coordinate is just N_t.

Consider now the following modification Y_t of the continuous-time process. Replace Step 2 of the above protocol by

2.′ Make a jump from the current state to one of its neighbors. With probability $\frac{1}{2}$, use the transition probabilities as already defined. Otherwise, use the probabilities $\lambda/(\lambda+\mu'+\alpha)$, $\mu'/(\lambda+\mu'+\alpha)$, and $\alpha/(\lambda+\mu'+\alpha)$, i.e. the same probabilities as above, only with μ' instead of μ.

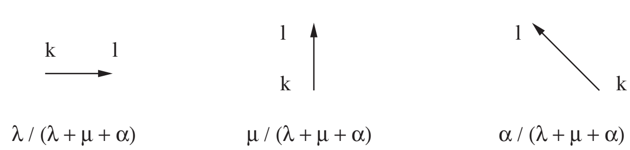

Figure 2. The three possible transitions on the tandem queue and their probabilities P_{kl}

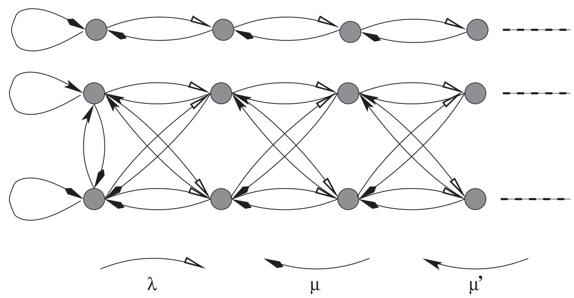

Figure 3. (Upper) The M/M/1 Queue N_t. (Lower) The state space of the lifted random walk and its perturbation in the lazy customers model with different values for μ and μ'. (For the shy staff model, the same graph results as the state space of the extended process, but the switching of the layers is performed with probability $\frac{1}{2}$.) The projection of this process onto $\mathbb{Z}_+$ is the *perturbed rear queue* $\tilde{N}_t$

As another possibility, consider a modification of the tandem process, in which the alternate set of transition probabilities is not chosen with equal probabilities. Instead, the range of possible transitions is doubled (from three to six), and the corresponding transition probabilities are chosen proportional to the corresponding rate parameters. To this end, define the process to run on the state space $\mathbb{Z}_+^2 \times \{0,1\}$. Thus, for every state $k \in \mathbb{Z}_+^2$, there is an additional state variable $k_3 \in \{0,1\}$:

2.″ Make a jump from the current state to one of its neighbors, i.e. perform one of the transitions by adding to the state variable one of $(1,0,0)$, $(0,-1,0)$, $(-1,1,0)$, $(1,0,1)$, $(0,-1,1)$, and $(-1,1,1)$. With $R=(\lambda+\lambda+\alpha+\alpha+\mu+\mu')$, choose these transitions with probabilities λ/R, μ/R, α/R, λ, μ'/R, and α/R, respectively.

Thus, either modification is an extension ($=:Y_t$) of the birth–death process belonging to the tandem queue as a process with state space $\mathbb{Z}_+^2 \times \{0,1\}$: An additional layer is introduced. One belongs to set λ, α, μ, and the other to λ, α, μ'. In perturbation 2.′, the layer is switched with probability equal to $\frac{1}{2}$ in each step. In 2.″, the switching is performed with a probability which is determined by the quotient of μ and μ'.

Looking at the rear queueing process N_t, separately, the modification may be described as in Figure 3. Instead of performing steps between nearest neighbors on the positive integers, an additional layer switching is allowed. The only new parameter is μ', the modified selling rate. If $\mu'<\mu$, the interpretation of model 2.′ is an inventory model, in which *the tendency to receive goods from the stock as opposed to serve the demanding customer is increased*. Model 2.", on the other hand, represents *an exchange of half of the customers with ones that arrive at the shop with reduced rate* μ'.

→ We call 2.′ the *shy staff* model and 2.″ the *lazy customers* model.

We now define the perturbation Z_t of the initial tandem process on $\mathbb{Z}_+$ to be the projection of the extended process Y_t onto the positive integers, i.e. Z_t is Y_t with the third (*layer*-) coordinate suppressed. The perturbed rear queue, considered separately as in Figure 3 will be denoted by $\tilde{N}_t$.

2.3. *Stability of the time-inhomogeneous birth–death process*

We now present the essential theorem belonging to part (B) of our agenda: We give the stability criteria of the inventory model where stability is understood as recurrence of the Markov chain belonging to the tandem queue. First, we give the recurrence criteria for the queues, separately. Then we use Burke's theorem of the asymptotic independence of the queues involved in a tandem to formulate the condition for recurrence of the whole tandem network.

Theorem 1

The following are sufficient criteria for the inventory defined by the perturbed tandem queue to be stable.

1. For the 'shy staff' model,

$$\lambda^2+\lambda\alpha\leq\mu\mu'+\alpha(\mu+\mu')/2$$

2. In the 'lazy customers' model

$$\lambda\leq\frac{1}{2}(\mu+\mu').$$

Remark

The smaller the α, the more the geometrical mean is the relevant measure for comparison of λ with μ and μ'. Since the geometrical mean is always smaller or equal to the arithmetic mean, the theorem states that it is more likely that 'shy customers' destabilize the inventory than 'lazy customers'.

First, we show that our definition of the perturbation Z_t of the tandem queue is in fact a Markov process.

Lemma 2
The perturbed process Z_t is Markovian for the shy staff model and the lazy customers model.

Proof
Shy staff model: Note, that during one step of the discrete-time chain, the probability to change the layer is always equal to $\frac{1}{2}$. Therefore, all possible sequences of level switchings or maintaining the level have the same probability. Hence, there is no possibility to extract information from the past on top of the information given by the initial Markov chain on $\mathbb{Z}_+^2$, while the latter, however, is Markovian. Hence, since the Markov chain Y_t defined on $\mathbb{Z}_+^2 \times \{0,1\}$ is Markovian, there is no loss of information when the projection onto $\mathbb{Z}_+^2$ is performed.

Lazy customers model: At equilibrium, there is independence of the front queue from the rear queue. By Burke's theorem, the output process of the front queue is Poisson-distributed with parameter λ. Therefore, the arrival process of the rear queue also has rate λ. Now, the perturbed rear queue is a birth-death process with a Poisson-distributed arrival process having $(\lambda+\lambda)$ as its rate, and a Poisson service process with rate $\mu+\mu'$, scaled in time by the factor $1/(\lambda+\lambda+\mu+\mu')$. Therefore, it is a time-scaled $M/M/1$-queueing process with parameters 2λ and $\mu+\mu'$. □

Lemma 3
At stationarity for the shy staff model, the perturbed rear queue is recurrent if $\lambda^2+\lambda\alpha \leq \mu\mu'+\alpha(\mu+\mu')/2$, whereas for the lazy customers model, it is recurrent if $2\lambda \leq \mu+\mu'$.

Proof
Shy staff model: As noted in the previous proof, at stationarity, the front and rear queue are independent. $\tilde{N}_t$, the rear process projected onto $\mathbb{Z}_+$ is Markovian and has transition rates probabilities, which are mixtures of the corresponding parameters involving $\langle\lambda,\mu\rangle$ and $\langle\lambda,\mu'\rangle$. Thus, it is a birth–death chain, which is recurrent if

$$\frac{1}{2}\frac{\lambda}{\lambda+\alpha+\mu}+\frac{1}{2}\frac{\lambda}{\lambda+\alpha+\mu'} \leq \frac{1}{2}\frac{\mu}{\lambda+\alpha+\mu}+\frac{1}{2}\frac{\mu'}{\lambda+\alpha+\mu'}$$

This is equivalent with the inequality stated.

Lazy customers model: At stationarity, due to Lemma 2, the rear process projected onto $\mathbb{Z}_+$ of the lazy customers model is a birth–death chain with transition probabilities $(\lambda+\lambda)/(\lambda+\lambda+\mu+\mu')$ and $(\mu+\mu')/(\lambda+\lambda+\mu+\mu')$. The process is recurrent if the forward conductances are smaller than the backward conductances[15], which is equivalent to

$$(\lambda+\lambda)/(\lambda+\lambda+\mu+\mu') \leq (\mu+\mu')/(\lambda+\lambda+\mu+\mu').$$

□

Theorem 4
The perturbed tandem queue Z_t is positive recurrent if $\lambda<\alpha$, and

- *for the shy staff model*: $\lambda(\lambda+\alpha)<\sqrt{\mu\mu'}+\alpha(\mu+\mu')/2$ and
- *for the lazy customers*: $\lambda<\frac{1}{2}(\mu+\mu')$.

Proof
By Burke's theorem, at equilibrium, the first queue has an output process which is the same as the arrival process, namely a Poisson process with parameter λ. If $\lambda<\alpha$, the positive recurrent queue is exponentially ergodic. Therefore, the image of the initial distribution under the semi-group of the first queue is the stationary distribution plus an exponentially fast decaying measure. The arrival process of the rear part of the tandem network is therefore the sum of the Poisson distribution with parameter λ and an exponentially fast decaying measure. Therefore, by the assumption and Lemma 3, the rear perturbed queue $\tilde{N}_t$ with arrival rate λ is positively recurrent. Since under stationarity, the front queue and the rear queue are independent, the limit measure is the convex combination of the product measure of the two invariant measures belonging to the two queues, and an exponentially decaying finite measure. Hence, the perturbed tandem network Z_t is positive recurrent. □

2.4. The optimal gain

In this section we present results concerning part (A) of our intended goals.

Owing to ergodicity of the Markov chain with stationary measure $\pi=\pi^M \otimes \pi^N$ (product measure of stationary measures of stock and shop), the asymptotic gain per unit time $G:=\lim_{t\to\infty}\frac{1}{t}\int_0^t g_s(\omega)ds$ with (1) is given by

$$G=\mathbb{E}_\pi[g_t]=P\cdot\lambda-B_1\cdot\frac{\lambda}{\alpha+\lambda}-B_2\cdot\frac{\lambda}{\mu+\lambda}-C\cdot\lambda. \tag{2}$$

The fact that $\mathbb{E}_\pi[S_t]=\lambda$ can be seen from Burke's theorem, which says that the output of an M/M/1-queue with arrival rate λ at stationarity is a Poisson process with parameter λ. (Note that B_1, B_2 are cost-*rates*, whereas P, C are price and cost of goods' sale and purchase.)

Burke's theorem also implies that at stationarity, the two queues of the tandem are independent. From this it follows directly that the optimal value of λ given μ and α is the solution of the quartic equation

$$(P-C)\cdot(\alpha-\lambda)^2\cdot(\mu-\lambda)^2 = B_1\alpha(\mu-\lambda)^2 + B_2\mu(\alpha-\lambda)^2. \tag{3}$$

Little's theorem, which says that the expected queue length ℓ is the expected total waiting time w (time it takes to pass through whole queue) times the arrival rate ($\ell=\lambda w$). The above, therefore, yields formulas for gain and optimal prices in terms of the mean total waiting times for stock (w_M) and shop (w_N), and expected sizes of stock ($\ell_M = \mathbb{E}_\pi[M_t]$) and shop ($\ell_N = \mathbb{E}_\pi[N_t]$):

$$G = \lambda\cdot(P-C-B_1 w_M - B_2 w_N), \tag{4}$$

and at optimality (highest expected gain G^*): $P-C = B_1\ell_M w_M + B_2\ell_N w_N$, hence

$$G^* = \lambda^*\cdot((\ell_M-1)B_1\omega_M + (\ell_N-1)B_2\omega_N), \tag{5}$$

where λ^* is the solution of (3). (Note, $w_M = 1/(\alpha-\lambda)$ and $w_N = 1/(\mu-\lambda)$.)

Owing to the asymptotic independence of the stock- and shop size (queue-length processes) and the arrival and demand processes, the variance $\mathrm{Var}_\pi(g_t) = \mathbb{E}_\pi[g_t^2] - \mathbb{E}_\pi[g_t]^2$ becomes

$$\mathrm{Var}_\pi(g_t) = \lambda(P^2+C^2) + B_1^2\frac{\lambda\alpha}{(\alpha-\lambda)^2} + B_2^2\frac{\lambda\mu}{(\mu-\lambda)^2}. \tag{6}$$

We discuss these results by studying Tables I and II, and the following examples:

Example 1

If $\alpha=\mu$, the optimal solution is given by

$$\lambda^* = \mu - \sqrt{\mu}\sqrt{\frac{B_1+B_2}{P-C}}. \tag{7}$$

This gives a criterion for the value of the difference $P-C$ of price and buying cost has to be, to make positive gain. From $\lambda*>0$ it follows that

$$P-C > \frac{(B_1+B_2)}{\mu}.$$

This is the optimal solution in case of only one single queue. Thus $\alpha=\mu$ ignores the possibility to optimize further by making use of the tandem queue. From the simulations below, it becomes clear the optimal value of α is smaller than that of μ, in case $B_1 > B_2$.

Example 2

Suppose that it is intended to make a gain of at least G_o, and to do this at a variance of at most V_o, then formulas (4)–(6) allow to give the following necessary condition for this goal to be met:

$$\frac{V_0}{G_0} \geq \frac{P^2+C^2+\alpha B_1^2 w_M^2 + \mu B_2^2 w_N^2}{B_1(\ell_M-1)w_M + B_2(\ell_N-1)w_N}.$$

The characteristic parameters w_M, w_N, ℓ_M, ℓ_M are easily measurable in real situations. Thus, this criterion provides a practical method to check for the feasability of a specific inventory realization.

Table I. Gain under varying λ: The selling rate is $\mu=5$, the transfer rate $\alpha=5$, the selling price is $P=20$, the buying cost is $C=15$, $B_1=1$ per day, and $B_2=1$ per day

.	$\lambda=2.5$	$\lambda=3.0$	$\lambda^*=3.27$	$\lambda=3.5$	$\lambda=4.0$	$\lambda=4.25$	$\lambda=4.5$
Comp.(1)	9.5	10.5	10.67	10.5	8.0	4.25	−4.5
Simul. m/std	10.3 ± 2.8	12.7 ± 1.5	12.8 ± 1.9	11.5 ± 2.5	9.15 ± 5.1	5.1 ± 7.5	1.3 ± 10.6

Table II. Gain under varying stock-keeping cost-parameter B_1: The buying rate is $\lambda=3$, the transfer rate $\alpha=5$, the selling rate is $\mu=5$, the selling-price is $P=20$, the buying-cost is $C=15$, $B_2=2$ per day

	$B_1=0.01$	$B_1=0.1$	$B_1=0.5$	$B_1=1.0$	$B_1=1.5$	$B_1=2.0$	$B_1=2.5$
Comp.(1)	11.985	11.85	11.25	10.5	9.75	9.0	8.25
Simul.m/std	14.2 ± 1.2	13.8 ± 1.4	12.8 ± 1.9	12.8 ± 1.6	11.8 ± 1.9	11.2 ± 1.6	10.3 ± 2.0

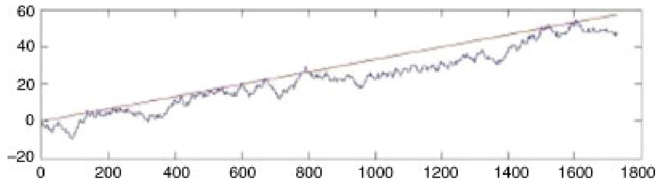

Figure 4. Lazy customers: Simulation of gain in Euro over a time span of 3 months (75 days) without perturbation for selling rate $\mu = 1.0$, transfer rate $\alpha = 1.0$, and optimal (Example 1) buying rate $\lambda = 0.97$, when $P = €20$, $C = €15$, and $B_1 = B_2 = €1/\text{month}$

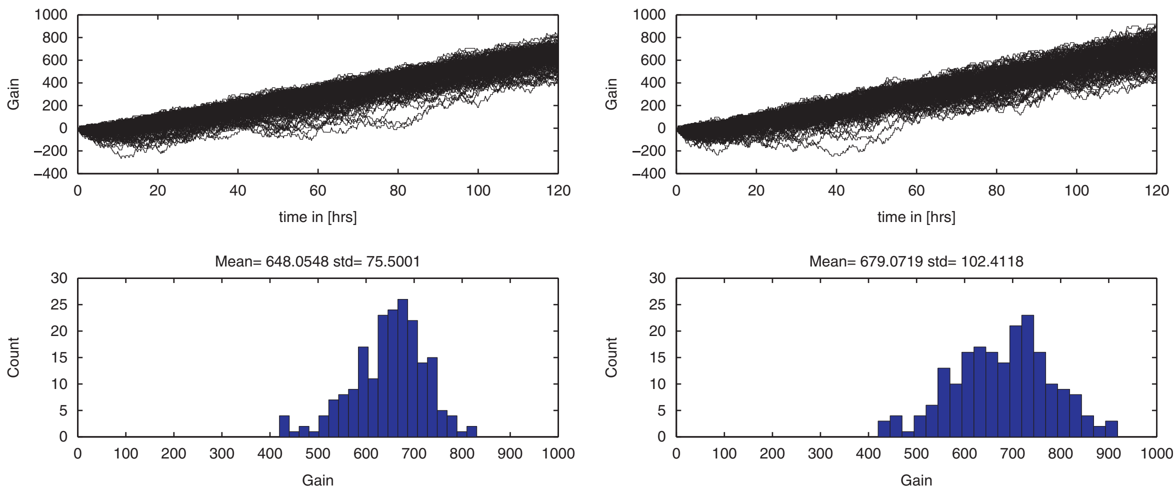

Figure 5. (Left) Family of trajectories for gain (in Euro) under choice of rate parameters $\lambda = 3.0$, $\alpha = 4.0$, and $\mu = 5.0$ without perturbation ($n = 200$ runs). (Right) Family of trajectories for gain under choice of rate parameters $\lambda = 3.0$, $\alpha = 4.0$, and $\mu = 5.0$ *with* perturbation of amplitude $\xi = 1.0$ ($n = 200$ runs). As ξ lies well within the allowed range ($\xi < 2(\mu - \lambda)$) for stability, the perturbation causes no significant decrease in the gain

3. Applications

We present some simulations involving the perturbed inventory process for the *lazy customers* model (see Section 2.2). We cover the complete range of perturbation amplitudes and exhibit the behavior of mean and variance in each regime.

3.1. *Constant selling rate without perturbation*

The first application is performed to predict the gain over a period of 75 days (1800 h). Under the assumption of a tandem network with the parameters: empiric selling rate $\mu = 1.0$ items/hour and transfer rate $\alpha = 1.0$ items/h, the optimum gain rate under stability is calculated as $\lambda = 0.79$ items/h. We assume a selling price of €20 per item, a stock-keeping cost of €1 per month as well as a shop-keeping cost of €1 per month.

The development of the gain as a function of time is shown in Figure 4 (straight line). The analytical solution is compared with a stochastic simulation of the underlying buying/stocking/selling process. The figure shows one specific simulated trajectory of the stochastic process.

3.2. *Constant selling rates with perturbation*

The simulation of the lazy customers model is shown in Figure 5 (Left). Assuming the parameters of the tandem network parameters $\mu = 5.0$, transfer rate $\alpha = 4.0$ and buying rate $\lambda = 3.0$ we show the development of the gain during 120 days for 200 trajectories, as well as the histogram at the end of the period. First, no random perturbation of the selling process is assumed. The histogram shows an empirical mean value $m =$ €648 and a standard deviation $s =$ €75 at the end of the observation period.

We call ξ the amplitude of the perturbation, i.e. in the simulation, we modify the selling rate μ' by the following rule: $\mu' = \mu - \xi$. In Table III we have $\lambda = 3$, and $\mu = 5$.

Assuming the same values for the selling rate, intermediate transfer rate, and mean buying rate, Figure 5 (Right) shows the distribution of the gain assuming a moderate perturbation ξ of the buying rate μ. The histogram of the gain at the end of the observation period shows a mean value $m =$ €679 and an increased standard deviation $s =$ €102.

The criterion of stability (Theorem 1) is supplemented by experimental results by showing that approximately near the critical value of $\xi_c = 4 = 2|\lambda - \mu|$, the observed standard deviation for the first time exceeds half of the empirical mean. This observation is confirmed in the final pair of histograms with values of ξ close to ξ_c (see Figure 6).

Table III. Gain after 5 days: mean±std (**Lazy Customer Model**)

ξ	$\alpha=3.1$	$\alpha=3.5$	$\alpha=4$	$\alpha=5$	$\alpha=6$	$\alpha=8$
0.0	30.0±21.9	47.5±12.8	45.4±9.2	64.6±6.0	68.1±5.7	74.8±6.8
1.0	27.2±22.4	42.2±11.4	57.2±7.2	63.0±7.3	66.2±6.2	72.1±6.7
2.0	24.8±21.2	45.8±14.7	54.7±11.2	60.1±8.8	64.8±10.0	71.7±8.2
3.0	18.5±23.4	37.9±17.3	47.8±14.1	57.4±11.6	64.3±12.1	70.1±8.7
3.5	9.8±27.9	30.4±21.7	45.4±17.2	55.5±12.9	61.6±15.2	68.2±10.3
4.0	1.3±32.1	12.1±27.0	37.6±24.6	53.1±15.5	62.5±14.5	66.5±11.3

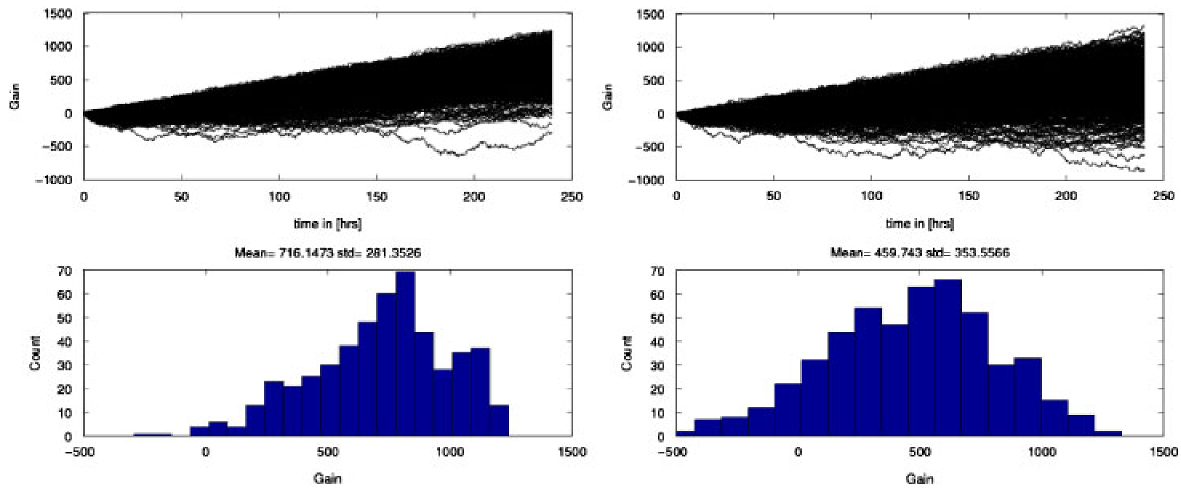

Figure 6. (Left) Simulation with $\xi=3.8$ (critical value $\xi_c=4.0$); $\lambda=3$, $\alpha=8$, $\mu=5$ over a period of 10 days and $B_1=10$, $B_2=20$ Euros/month. (Right) Same as Left, except for perturbation amplitude, which is now $\xi=4.2$. The catastrophe becomes visible by positive losses occurring with 'high probability'

4. Conclusion

We developed two stochastic models for buying/stocking/selling processes assuming perturbations of the instantaneous step rates. We compare the two models by giving the criteria for stability for each. The underlying process consists of a tandem network imitating the stock and the shelf in a retailer's environment. We derived analytical solutions for the stability, the optimal gain, and its variance. Further investigations have been performed studying the robustness of the queues assuming random perturbations. In this case, we focused on the Lazy Customers model due to the stability threshold, which is independent of the transfer rate. These applications show the suitability of the derived models for various inventory environments. The analysis guarantees the stability (recurrence) of the tandem queue. The criteria given allow for optimizing the buying rate and transfer rate between stock and shop, having knowledge of the selling rate.

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