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Stability Proof for a Well-Established Super-Twisting Parameter Setting

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Abstract

Several sufficient stability conditions exist for the super-twisting algorithm. For its tuning in practical applications, however, a popular parameter set is regularly used, which is provided in literature without any stability proof. This short note provides a novel, simple sufficient stability condition which includes this very popular parameter setting.

Key words: Sliding mode control; Super-twisting algorithm; Stability proof; Lyapunov function

1 Introduction

Sliding mode control provides several techniques for the design of controllers that are robust with respect to a wide range of unmodeled disturbances. A very popular control algorithm in this regard is the super-twisting algorithm [1]. It is given by the following differential equation, whose solutions are understood in the sense of Filippov [2],

$$\dot{x}_1 = -k_1 \sqrt{|x_1|} \operatorname{sign}(x_1) + x_2, \tag{1a}$$
$$\dot{x}_2 = -k_2 \operatorname{sign}(x_1) + \delta \tag{1b}$$

with an absolutely continuous perturbation $\delta(t)$ bounded by

$$|\delta| \le L. \tag{1c}$$

Challenges, that have been considered in literature, are both the tuning of the positive parameters k_1, k_2 and, to a much greater extent of course, the proof of this system's stability. For parameter tuning, the rules

$$k_1 = 1.5\sqrt{L}, \qquad k_2 = 1.1L. \tag{2}$$

were proposed in [3] as part of a parameter set for the robust exact differentiator, an extension to arbitrary system orders which reduces to (1) for order two. These rules have become very popular since, with them both being found in books and papers on the subject [4–7], as well as being used in applications [8–11]. Stability analyses of the algorithm have been presented for example in [12–15], with several of them constructing Lyapunov functions. A notable breakthrough was achieved in [14] with a family of Lyapunov functions yielding the sufficient condition

$$k_2 > L, \qquad k_1 > 2\sqrt{k_2 - \sqrt{k_2^2 - L^2}} \tag{3}$$

for finite time stability of the system (see also [16]). The popular parameter setting (2) *does not satisfy* these inequalities, however! Nor are any other stability proofs for this setting known to the authors. Simulations remedy this shortcoming only to a small extent, as finite time convergence for one particular perturbation $\delta(t)$ does not guarantee convergence for all bounded perturbations.

This note provides a strict Lyapunov function which extends the provably stable parameter range, for the first time *including* the popular parameter set (2).

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2 Main Result

Theorem 1. *System (1) is finite time stable if its parameters satisfy*

$$k_2 > L, \qquad k_1 > \sqrt{k_2 + L}. \tag{4}$$

Remark 2. As large values of the parameters k_1 and k_2 are associated with increased chattering (see for example [17]), small parameter values are often desirable in practice [1]. When this is the case, the proposed condition allows a substantial reduction of the parameter k_1 by almost 30 percent as k_2 approaches the lower bound L, from $k_1 > 2\sqrt{k_2}$ to $k_1 > \sqrt{2k_2}$.

Remark 3. Combining condition (4) with the existing criterion (3), one finds that finite time stability is guaranteed by $k_2 > L$ and

$$k_1 > \begin{cases} \sqrt{k_2 + L} & k_2 \leq \frac{8\sqrt{2}-3}{7} L \\ 2\sqrt{k_2 - \sqrt{k_2^2 - L^2}} & \text{otherwise.} \end{cases} \tag{5}$$

Remark 4. Note that the parameters (2) proposed in [3] satisfy condition (4), which proves the finite time stability of the super-twisting algorithm with this commonly used setting.

Proof. Select a positive parameter $\alpha < 1$ such that

$$k_1 > \frac{1}{\alpha}\sqrt{k_2 + L} \tag{6}$$

and define the region

$$\mathcal{M} := \left\{ [x_1 \quad x_2]^{\mathrm{T}} \mid x_1 \geq 0, x_2 \leq \alpha k_1 \sqrt{x_1} \right\}. \tag{7}$$

Introduce the state vector $\mathbf{x} := [x_1 \quad x_2]^{\mathrm{T}}$ and consider the Lyapunov candidate

$$V(\mathbf{x}) = \begin{cases} 2\sqrt{x_2^2 + 3\alpha^2 k_1^2 x_1} - x_2 & \mathbf{x} \in \mathcal{M} \\ 2\sqrt{x_2^2 - 3\alpha^2 k_1^2 x_1} + x_2 & -\mathbf{x} \in \mathcal{M} \\ 3\,|x_2| & \text{otherwise.} \end{cases} \tag{8}$$

One may easily check that this function is continuous, positive definite, piecewise differentiable and locally Lipschitz continuous everywhere except in the origin. For the parameter values $k_1 = 1.5$ and $\alpha = 0.97$ the function's level lines and values [2] are shown in Figure 1.

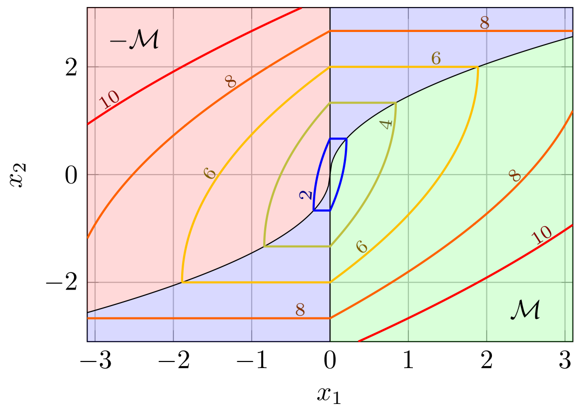

Fig. 1. Level lines and values of Lyapunov function (8) for $k_1 = 1.5$, $\alpha = 0.97$. The three regions distinguished in (8) are colored in green, red and blue, respectively.

For the third case in (8), i.e. $x_2^2 > \alpha^2 k_1^2 |x_1|$ and $x_1 x_2 > 0$, the time-derivative of V satisfies

$$\dot{V} = -3k_2 \operatorname{sign}(x_1 x_2) + 3 \operatorname{sign}(x_2)\,\delta \leq 3(L - k_2) < 0. \tag{9}$$

Consider now the case $\mathbf{x} \in \mathcal{M}$; computing the time-derivative of V yields

$$\dot{V} = (k_2 - \delta) + \frac{3\alpha^2 k_1^2 (x_2 - k_1\sqrt{x_1}) - 2x_2(k_2 - \delta)}{\sqrt{x_2^2 + 3\alpha^2 k_1^2 x_1}}. \tag{10}$$

This expression is a homogeneous function (see [18]) of degree zero with respect to $\sqrt{x_1}$ and x_2, i.e. the relation $\dot{V}(\beta^2 x_1, \beta x_2) = \dot{V}(x_1, x_2)$ holds for any positive β. It is thus sufficient to investigate the function's behavior for values of $\mathbf{x}$ satisfying $x_2^2 + 3\alpha^2 k_1^2 x_1 = 1$, i.e. for points lying on an ellipsoidal arc in the $\sqrt{x_1}$-x_2 half-plane. One consequently defines the function

$$\begin{aligned} g(x_2) &:= \dot{V}\Big|_{x_2^2 + 3\alpha^2 k_1^2 x_1 = 1} \\ &= (k_2 - \delta)(1 - 2x_2) + 3\alpha^2 k_1^2 x_2 - \alpha k_1^2 \sqrt{3 - 3x_2^2}. \end{aligned} \tag{11}$$

The restriction $\mathbf{x} \in \mathcal{M}$ is equivalent to the argument x_2 of this function lying in the interval $[-1, \frac{1}{2}] =: \mathcal{I}$. The second derivative of g is

$$\frac{d^2 g}{d x_2{}^2} = \frac{\sqrt{3}\alpha k_1^2}{(1 - x_2^2)^{\frac{3}{2}}} \geq 0 \qquad \forall x_2 \in \mathcal{I} \setminus \{-1\}, \tag{12}$$

implying that g is convex and its local maxima lie at the

[1] Strictly speaking, this is only the case when there is no reaching phase (e.g. when using integral sliding mode techniques) or when parameters are changed after the reaching phase is over. Otherwise, reaching times may be an issue, possibly making larger parameter values more desirable to obtain fast convergence.

[2] The Lyapunov function was constructed such that its level lines are composed of straight lines and parts of ellipsoids *in the* $\sqrt{x_1}$*-*x_2*-plane.*

border of the interval $\mathcal{I}$. A simple evaluation yields

$$g(-1) = 3(k_2 - \delta - \alpha^2 k_1^2) \leq 3(k_2 + L - \alpha^2 k_1^2), \quad (13a)$$

$$g\left(\frac{1}{2}\right) = \frac{3\alpha(\alpha-1)k_1^2}{2}. \quad (13b)$$

It follows from (6) and $0 < \alpha < 1$ that both of these values are strictly negative. Their maximum is an upper bound for g on the interval $\mathcal{I}$, and by homogeneity one concludes that

$$\dot{V} \leq 3\max\left(k_2 + L - \alpha^2 k_1^2, \frac{\alpha(\alpha-1)k_1^2}{2}\right) < 0 \quad (14)$$

holds in the entire region $\mathcal{M}$. By changing the sign of x_1 and x_2 in the above derivation[3] the same inequality is obtained for $-\mathbf{x} \in \mathcal{M}$.

Finally, one has to check for possible motions along the curves where V is not differentiable. It is well known that no sliding motion can occur along the algorithm's discontinuity set given by $x_1 = 0$ except in the origin, as $x_1(t) \equiv 0$ and (1a) imply $x_2(t) \equiv 0$. The set given by $x_2 = \alpha k_1 \sqrt{|x_1|}\,\mathrm{sign}\,(x_1)$, on the other hand, is an integral curve of system (1) if the following inclusion holds:

$$\begin{aligned} \dot{x}_2 &= \frac{1}{2}\alpha(\alpha-1)k_1^2\,\mathrm{sign}\,(x_2) \\ &= \frac{g(1/2)}{3}\,\mathrm{sign}\,(x_2) \in -k_2\,\mathrm{sign}\,(x_2) + [-L, L]. \quad (15) \end{aligned}$$

Despite V not being differentiable with respect to $\mathbf{x}$, its time-derivative along this curve may be computed as

$$\dot{V} = \frac{\mathrm{d}}{\mathrm{d}t} 3\,|x_2| = 3\dot{x}_2\,\mathrm{sign}\,(x_2) = g(1/2) < 0. \quad (16)$$

The time derivative of V along the trajectories of system (1) is thus bounded by a constant negative value for almost all time instants and the function V is a strict Lyapunov function certifying finite time stability of system (1). □

3 Extension to Unmatched Disturbances

The Lyapunov function may be used to prove finite time stability also in the presence of certain perturbations that are unmatched, i.e. acting on (1a). Assume that the system is changed from (1) to

$$\dot{x}_1 = -k_1\sqrt{|x_1|}\,\mathrm{sign}\,(x_1) + x_2 + \eta\sqrt{|x_1|}, \quad (17a)$$

$$\dot{x}_2 = -k_2\,\mathrm{sign}\,(x_1) + \delta \quad (17b)$$

[3] Note that both Lyapunov function (8) and system (1) are symmetric with respect to a sign change in both x_1 and x_2.

with the additional perturbation η bounded by a non-negative constant M, i.e.

$$|\eta| \leq M, \qquad |\delta| \leq L. \quad (17c)$$

In the proof, this changes only the case $\mathbf{x} \in \mathcal{M}$, where the second parenthesized expression in (10) has to be modified. This modification does not influence the convexity of g nor relation (13a); only relation (13b) changes to

$$g\left(\frac{1}{2}\right) = \frac{3\alpha k_1}{2}(\alpha k_1 - k_1 + M). \quad (18)$$

This expression is negative if $k_1 > M + \alpha k_1$ holds for some value of α constrained by (6).

System (17) is hence finite time stable if the conditions

$$k_2 > L, \qquad k_1 > M + \sqrt{k_2 + L} \quad (19)$$

are satisfied. For the commonly used parameter setting (2) the bound $M < 0.05\sqrt{L}$ is obtained.

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