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WAVE PROPAGATION IN INCOMPRESSIBLE MODELLED POROUS CONTINUA WITH BOUNDARY ELEMENTS

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Abstract

The Boundary Element Method (BEM) is preferred to solve wave propagation problems in semi-infinite continua numerically. One crucial condition to establish a BE formulation is the knowledge of a fundamental solution. For poroelastic constitutive equations, up to now, such a solution has only been available for the case of compressible constituents in Laplace/Fourier domain.

Here, the Laplace domain fundamental solutions for incompressible constituents are derived using Hörmanders method for two different sets of unknowns, for the solid displacements and fluid pressure and for the solid and fluid displacements. The solutions are used in a time-dependent BE formulation based on the Convolution Quadrature Method, which only requires the Laplace domain fundamental solutions.

There are three wave types in poroelastic continua, the fast compressional wave with solid and fluid moving in-phase, the shear wave, and the second (slow) compressional wave, which has no equivalent in elastic materials, with solid and fluid moving in opposite directions. With incompressible constituents, the propagation speed of the fast compressional wave becomes infinite. Some studies concerning the influence of this infinite wave speed are shown as well as results of BEM calculations. The numerical examples are calculated for the unknowns solid displacements and fluid pressure.

Keywords: Poroelasticity, Biot's theory, Incompressible, Boundary Elements, Fundamental Solutions.

INTRODUCTION

The efficiency of the Boundary Element Method (BEM) in dealing with semi-infinite domain problems, e.g., soil-structure interaction, have long been recognized by researchers and engineers. For soil, a fluid saturated material, a poroelastic constitutive model should be used in connection with a time-dependent BE formulation to model wave propagation problems correctly. Dynamic poroelastic BE formulations are published in frequency domain, e.g., (Cheng et al. 1991), in Laplace domain, e.g., (Chen and Dargush 1995), and in time domain (Chen and Dargush 1995; Schanz 2001a). In all of these formulations, Biot's theory is used assuming compressible constituents. Beside the compressibility of the constituents also a structural compressibility exists and is modelled in Biot's theory. For some materials, e.g., soil, the compression modulus of the constituents itself is much larger than the compressibility of the structure. In these cases, it is sufficient to approximate both the fluid and solid constituents as incompressible, i.e., only the structural compressibility remains.

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Here, Biot's model for this special case is discussed for two different sets of independent variables, the solid displacements and pore pressure, and solid and fluid displacements. Subsequently the novel fundamental solutions for the incompressible case are derived using the method of Hörmander (1963). Aiming on wave propagation problems, the time-dependent BE formulation based on the Convolution Quadrature Method as proposed by Schanz (2001b) is used here. A BEM formulation is established employing the fundamental solutions using solid displacements and pore pressure as independent variables, which are a sufficient set according to Bonnet (1987). As the unknowns are the same as in the compressible model and, further, the principal structure of the set of governing equations is similar, the same procedure as in the above mentioned BE formulations can be followed.

To demonstrate the limits of the incompressible approximation for a poroelastic medium, two different cases of materials, a rock and a soil, are used. The results of the incompressible modelling are compared with the compressible modelling at the example of a half space.

Throughout this paper, the summation convention is applied over repeated indices and Latin indices receive the values 1,2, and 1,2,3 in two-dimensions (2-d) and three-dimensions (3-d), respectively. Commas $()_{,i}$ denote spatial derivatives and dots $\dot{()}$ denote the time derivative. As usual, the Kronecker delta is denoted by δ_{ij}.

BIOT'S THEORY – GOVERNING EQUATIONS

Following Biot's approach to model the behaviour of porous media, an elastic skeleton with a statistical distribution of interconnected pores is considered (Biot 1955). This porosity is denoted by

$$\phi = \frac{V^f}{V} , \tag{1}$$

where V^f is the volume of the interconnected pores contained in a sample of bulk volume V. Contrary to these pores the sealed pores will be considered as part of the solid. Full saturation is assumed leading to $V = V^f + V^s$ with V^s the volume of the solid, i.e., a two-phase material is given.

Constitutive assumptions

If the constitutive equations are formulated for the elastic solid and the viscous interstitial fluid, a partial stress formulation is obtained (Biot 1955)

$$\sigma_{ij}^s = 2G\varepsilon_{ij}^s + \left(K - \frac{2}{3}G + \frac{Q^2}{R}\right)\varepsilon_{kk}^s\delta_{ij} + Q\varepsilon_{kk}^f\delta_{ij} \tag{2a}$$

$$\sigma^f = -\phi p = Q\varepsilon_{kk}^s + R\varepsilon_{kk}^f , \tag{2b}$$

with $()^s$ and $()^f$ indicating either solid or fluid, respectively. The respective stress tensor is denoted by σ_{ij}^s and σ^f and the corresponding strain tensor by ε_{ij}^s and ε_{kk}^f. The elastic skeleton is assumed to be isotropic and homogeneous where the two elastic material constants compression modulus K and shear modulus G refer to the bulk material. The coupling between the solid and the fluid is characterised by the two parameters Q and R. In the above, the sign conventions for stress and strain follow that of elasticity, namely, tensile stress and strain is denoted positive. Therefore, in equation (2b) the pore pressure p is the negative hydrostatic stress in the fluid σ^f.

An alternative representation of the constitutive equation (2) is used in Biot's earlier work (1941). There, the total stress $\sigma_{ij} = \sigma_{ij}^s + \sigma^f\delta_{ij}$ is introduced and with Biot's effective stress coefficient $\alpha = \phi(1 + Q/R)$ the constitutive equation with the solid strain ε^s_{ij} and the pore pressure p

$$\sigma_{ij} = 2G\varepsilon^s_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon^s_{kk}\delta_{ij} - \alpha\delta_{ij}p \tag{3a}$$

is obtained. Additionally to the total stress σ_{ij} as a second constitutive equation the variation of fluid volume per unit reference volume ζ is introduced

$$\zeta = \alpha\varepsilon^s_{kk} + \frac{\phi^2}{R}p\;. \tag{3b}$$

This variation of fluid ζ is defined by the mass balance over a reference volume, i.e., by the continuity equation

$$\dot{\zeta} + q_{i,i} = a \tag{4}$$

with the specific flux $q_i = \phi\left(\dot{u}^f_i - \dot{u}_i\right)$ and a source term $a(t)$. Equation (4) identify ζ as a kind of strain describing the displacements of the fluid u^f_i relative to the solid displacements u_i which takes a source in the fluid into account. This source term is not motivated by any physical reason[1] but it is later needed for the derivation of the fundamental solutions.

In a two-phase material not only each constituent, the solid and the fluid, may be compressible on a microscopic level but also the skeleton itself possesses a structural compressibility. If the compression modulus of one constituent is much larger on the microscale than the compression modulus of the bulk material, this constituent is assumed to be materially incompressible. A common example for a materially incompressible solid constituent is soil. In this case, the individual grains are much stiffer than the skeleton itself. The respective conditions for such incompressibilities are (Detournay and Cheng 1993)

$$\frac{K}{K^s} \ll 1 \quad \text{incompressible solid,} \qquad \frac{K}{K^f} \ll 1 \quad \text{incompressible fluid}\;, \tag{5}$$

where K^s denotes the compression modulus of the solid grains and K^f the compression modulus of the fluid.

To find the respective constitutive equations for incompressible constituents the material parameters α and R have to be rewritten in a different way. Considerations of constitutive relations at micro mechanical level as given in Detournay and Cheng (1993) lead to a more rational model for this purpose

$$\alpha = 1 - \frac{K}{K^s} \approx 1 \qquad \text{and} \tag{6a}$$

$$R = \frac{\phi^2 K^f {K^s}^2}{K^f\left(K^s - K\right) + \phi K^s\left(K^s - K^f\right)} \to \infty\;. \tag{6b}$$

The relation $R \to \infty$ expresses that the value of R becomes large, however, due to physical reasons it is in any case limited. But, the condition that R becomes large is used to neglect in (3b) the influence of the pore pressure. This condition and (6a) results in the incompressible constitutive assumptions

$$\sigma_{ij} = 2G\varepsilon^s_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon^s_{kk}\delta_{ij} - \delta_{ij}p \tag{7a}$$

$$\zeta = \varepsilon^s_{kk} \tag{7b}$$

[1] e.g., chemical reactions

for the total stress formulation.

For the partial stress formulation (2), a different point of view may be considered. Biot (1955) gives as condition for incompressible constituents

$$(1-\phi)\,\varepsilon^s_{kk} + \phi\varepsilon^f_{kk} = 0\,, \tag{8}$$

i.e., it is assumed that the dilatation of the bulk material vanishes. Realising the relation

$$\frac{Q}{R} = \frac{1-\phi}{\phi} \qquad \Rightarrow \qquad \frac{Q}{R}\varepsilon^s_{kk} + \varepsilon^f_{kk} = 0 \tag{9}$$

also in the partial stress formulation the case of incompressible constituents can be included resulting in the constitutive assumptions

$$\sigma^s_{ij} = 2G\varepsilon^s_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon^s_{kk}\delta_{ij} \tag{10a}$$

$$\sigma^f = -\phi p = R\left(\frac{Q}{R}\varepsilon^s_{kk} + \varepsilon^f_{kk}\right) \stackrel{!}{=} 0\,. \tag{10b}$$

To achieve the zero value in equation (10b), the condition that the value R becomes large but is limited must be used.

Contrary to the incompressible model formulated for the total stress formulation (7), in the partial stress formulation the assumption of incompressibility (8) results in an uncoupling of the solid and the fluid in the constitutive assumptions. Therefore, the two incompressible models (7) and (10) are different whereas the underlying compressible models (3) and (2), respectively, are identical. This is not really a contradiction. Keeping in mind that an incompressible model is always an approximation for the more realistic compressible case, it is clear that different approximations can exist. However, the question which approximation is best can only be answered by the respective application.

Aiming at the equation of motion to model wave propagation phenomena, it is sufficient to formulate a linear kinematic equation. Hence, in the following, the relation of the solid/fluid strain to the solid/fluid displacement is chosen linear, respectively

$$\varepsilon^s_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \varepsilon^f_{kk} = u^f_{k,k} \tag{11}$$

assuming small deformation gradients.

Governing equations: Compressible model

In the preceding section, the constitutive equations and the kinematics have been given. The next step is to state the balances of momentum. In any two-phase material there are three possibilities to formulate the balances of momentum. First, the balance of momentum for the solid, second, for the fluid or, third, for the bulk material, have to be fulfilled.

The first two balances are used by Biot (1956) with the solid displacements and the fluid displacements as unknowns

$$\sigma^s_{ij,j} + (1-\phi)\,f^s_i = (1-\phi)\,\varrho_s\ddot{u}_i + \varrho_a\left(\ddot{u}_i - \ddot{u}^f_i\right) + \frac{\phi^2}{\kappa}\left(\dot{u}_i - \dot{u}^f_i\right) \tag{12a}$$

$$\sigma^f_{,i} + \phi f^f_i = \qquad \phi\varrho_f\ddot{u}^f_i - \varrho_a\left(\ddot{u}_i - \ddot{u}^f_i\right) - \frac{\phi^2}{\kappa}\left(\dot{u}_i - \dot{u}^f_i\right)\,. \tag{12b}$$

The first balance equation (12a) is that for the solid skeleton and the second (12b) is that for the interstitial fluid. In equation (12), the body forces in the solid skeleton f_i^s and in the fluid f_i^f are introduced. Further, the respective densities are denoted by ϱ_s and ϱ_f. To describe the dynamic interaction between fluid and skeleton an additional density, the apparent mass density ϱ_a, has been introduced by Biot (1956). It can be written as $\varrho_a = C\phi\varrho_f$ where C is a factor depending on the geometry of the pores and the frequency of excitation.

The third above mentioned balance of momentum for the mixture is formulated in Biot's earlier work (1941) for quasistatics and in Biot (1956) for dynamics. This dynamic equilibrium is given by

$$\sigma_{ij,j} + F_i = \varrho_s (1-\phi) \ddot{u}_i + \phi \varrho_f \ddot{u}_i^f \; , \tag{13}$$

with the bulk body force per unit volume $F_i = (1-\phi) f_i^s + \phi f_i^f$. It is obvious that adding the two partial balances (12a) and (12b) results in the balance of the mixture (13).

In most papers using the total stress formulation, now, the constitutive assumption for the fluid transport in the interstitial space is given by Darcy's law. Here, it is also used, however, with the balance of momentum in the fluid (12b) Darcy's law is already given. Rearranging (12b) and taking the definition of the flux $q_i = \phi \left(\dot{u}_i^f - \dot{u}_i \right)$ as well as $\sigma^f = -\phi p$ into account the dynamic version of Darcy's law

$$q_i = -\kappa \left(p_{,i} + \frac{\varrho_a}{\phi} \left(\ddot{u}_i^f - \ddot{u}_i \right) + \varrho_f \ddot{u}_i^f - f_i^f \right) \tag{14}$$

is achieved.

Aiming at the equation of motion, the constitutive equations have to be combined with the corresponding balances of momentum and the kinematic conditions. To do this, first, the degrees of freedom must be determined. There are several possibilities: i) to use the solid displacements u_i and the fluid displacement u_i^f (two vectors, i.e., six unknowns in 3-d) or ii) a combination of the pore pressure p and the solid displacements u_i (one vector and one scalar, i.e., four unknowns in 3-d). As shown in Bonnet (1987), it is sufficient to use the latter choice. Here, for completeness, both choices will be presented.

First, the equations of motion for a poroelastic body are presented for the unknowns solid displacement u_i and fluid displacement u_i^f. Inserting in (12) the constitutive equations (2) written for the partial stress tensors and the linear strain displacement relations (11) yields a set of equations of motion in time domain

$$G u_{i,jj} + \left(K + \frac{1}{3} G \right) u_{j,ij} + Q \left(\frac{Q}{R} u_{j,ji} + u_{j,ji}^f \right) + (1-\phi) f_i^s = \tag{15a}$$

$$(1-\phi) \varrho_s \ddot{u}_i + \varrho_a \left(\ddot{u}_i - \ddot{u}_i^f \right) + \frac{\phi^2}{\kappa} \left(\dot{u}_i - \dot{u}_i^f \right)$$

$$R \left(\frac{Q}{R} u_{j,ji} + u_{j,ji}^f \right) + \phi f_i^f = \phi \varrho_f \ddot{u}_i^f - \varrho_a \left(\ddot{u}_i - \ddot{u}_i^f \right) - \frac{\phi^2}{\kappa} \left(\dot{u}_i - \dot{u}_i^f \right) . \tag{15b}$$

Second, the respective equations of motion are presented for the pore pressure p and the solid displacements u_i as unknowns. To achieve this formulation the fluid displacement u_i^f has to be eliminated from equations (13), (14), (3), and (4). In order to do this, Darcy's law (14) is rearranged to obtain $u_i^f - u_i$. Since this relative displacement is given as second time derivative in (14) and the flux is related to its first oder time derivative by $q_i = \phi \left(\dot{u}_i^f - \dot{u}_i \right)$, this is only possible in Laplace domain. After transformation to Laplace domain, the relative fluid to solid displacement is

$$\hat{u}_i^f - \hat{u}_i = -\underbrace{\frac{\kappa \varrho_f \phi^2 s^2}{\phi^2 s + s^2 \kappa \left(\varrho_a + \phi \varrho_f\right)}}_{\beta} \frac{1}{s^2 \phi \varrho_f} \left(\hat{p}_{,i} + s^2 \varrho_f \hat{u}_i - \hat{f}_i^f\right) . \tag{16}$$

In equation (16), the abbreviation β is defined for further usage and $\mathscr{L}\left\{f\left(t\right)\right\} = \hat{f}\left(s\right)$ denotes the Laplace transform with the complex variable s. Moreover, vanishing initial conditions for u_i and u_i^f are assumed here and in the following. Now, the final set of differential equations for the displacement $\hat{u}_i$ and the pore pressure $\hat{p}$ is obtained by inserting the constitutive equations (3) into the Laplace transformed dynamic equilibrium (13) and continuity equation (4) with $\hat{u}_i^f - \hat{u}_i$ from equation (16). This leads to the final set of differential equations for the displacement $\hat{u}_i$ and the pore pressure $\hat{p}$

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - \left(\alpha - \beta\right)\hat{p}_{,i} - s^2\left(\varrho - \beta\varrho_f\right)\hat{u}_i = \beta\hat{f}_i^f - \hat{F}_i \tag{17a}$$

$$\frac{\beta}{s\varrho_f}\hat{p}_{,ii} - \frac{\phi^2 s}{R}\hat{p} - \left(\alpha - \beta\right)s\hat{u}_{i,i} = -\hat{a} + \frac{\beta}{s\varrho_f}\hat{f}_{i,i}^f . \tag{17b}$$

In the above equation (17), the bulk density $\varrho = \varrho_s\left(1-\phi\right) + \phi\varrho_f$ is used. This set of equations describes the behaviour of a poroelastic continuum completely as well as the formulation (15). Contrary to the formulation using the solid and fluid displacement (15) an analytical representation in time domain is only possible for $\kappa \to \infty$. This case would represent a negligible friction between the solid and the interstitial fluid.

Governing equations: Incompressible model

As mentioned above, often the approximation of incompressible constituents can be used. Inserting in (15) the incompressibility condition (9), the governing equations are given by

$$Gu_{i,jj} + \left(K + \frac{1}{3}G\right)u_{j,ij} + \left(1-\phi\right)f_i^s = \left(1-\phi\right)\varrho_s\ddot{u}_i + \varrho_a\left(\ddot{u}_i - \ddot{u}_i^f\right) + \frac{\phi^2}{\kappa}\left(\dot{u}_i - \dot{u}_i^f\right) \tag{18a}$$

$$\phi f_i^f = \phi\varrho_f\ddot{u}_i^f - \varrho_a\left(\ddot{u}_i - \ddot{u}_i^f\right) - \frac{\phi^2}{\kappa}\left(\dot{u}_i - \dot{u}_i^f\right) \tag{18b}$$

using the solid displacements and fluid displacements as unknowns. In this incompressible version of the equations of motion, the uncoupling of the fluid and solid in the constitutive assumptions is clearly observed as commented in the last section. So, in equations (18) only the coupling by the acceleration and damping terms remains. Further, the second equation (18b) is no longer independent. It can not be used to eliminate the fluid displacement u_i^f in (18a). As additional equation the incompressibility condition (8) has to be used.

On the contrary, if the solid displacement and the pore pressure are used as unknowns, a sufficient set of differential equations is obtained. Simply inserting the conditions (6) in (17), i.e., setting $\alpha = 1$ and taking the limit $R \to \infty$, the equations of motion under the assumption of incompressible constituents are achieved resulting in

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - (1-\beta)\,\hat{p}_{,i} - s^2\,(\varrho - \beta\varrho_f)\,\hat{u}_i = \beta\hat{f}_i^f - \hat{F}_i \tag{19a}$$

$$\frac{\beta}{s\varrho_f}\hat{p}_{,ii} - (1-\beta)\,s\hat{u}_{i,i} = -\hat{a} + \frac{\beta}{s\varrho_f}\hat{f}_{i,i}^f\,. \tag{19b}$$

The equation for the pore pressure (19b) shows that this variable is no longer a degree of freedom. Integrating of (19b) would yield the gradient of the pore pressure which could then be eliminated in (19a). Physically interpreted, in this case the pore pressure is only determined by the deformation of the solid skeleton and no longer by any deformation of the fluid.

FUNDAMENTAL SOLUTIONS

Fundamental solutions for the above given systems of differential equations are known in closed form only in Fourier or Laplace domain. But, even in the transformed domain only the general case of compressible constituents is published. The fundamental solutions for the Laplace transformed system of (15) is given in Manolis and Beskos (1989) and for the Laplace transformed system of (17) in Chen (1994b) and Chen (1994a).

Here, the fundamental solutions for the incompressible case are presented. Not only for completeness, the fundamental solutions for the compressible case are recalled, also to show how the physical approximation of incompressibility is represented in the mathematic formulas. In order to deduce these solutions, an operator notation is useful. So, for the representation with solid displacements and pore pressure as unknowns the governing equations of the compressible case (17) as well as the incompressible case (19) are reformulated as

$$\mathbf{B4}\begin{bmatrix}\hat{u}_i\\ \hat{p}\end{bmatrix} + \begin{bmatrix}\hat{F}_i\\ \hat{a}\end{bmatrix} = \mathbf{0} \tag{20}$$

with the differential operators

$$\mathbf{B4}^{comp} = \begin{bmatrix}\left(G\nabla^2 - s^2\,(\varrho - \beta\varrho_f)\right)\delta_{ij} + \left(K + \frac{1}{3}G\right)\partial_i\partial_j & -(\alpha-\beta)\,\partial_i\\ -s\,(\alpha-\beta)\,\partial_j & \dfrac{\beta}{s\varrho_f}\nabla^2 - \dfrac{\phi^2 s}{R}\end{bmatrix} \tag{21a}$$

$$\mathbf{B4}^{incomp} = \begin{bmatrix}\left(G\nabla^2 - s^2\,(\varrho - \beta\varrho_f)\right)\delta_{ij} + \left(K + \frac{1}{3}G\right)\partial_i\partial_j & -(1-\beta)\,\partial_i\\ -s\,(1-\beta)\,\partial_j & \dfrac{\beta}{s\varrho_f}\nabla^2\end{bmatrix}\,. \tag{21b}$$

In equations (20) and (21), the operator is denoted by $\mathbf{B4}$ independently whether it is in 2-d ($i,j = 1,2$, i.e., 3 unknowns) or 3-d ($i,j = 1,2,3$, i.e., 4 unknowns). The corresponding representation of a poroelastic continuum using the solid displacements and the fluid displacements as unknowns is

$$\mathbf{B6}\begin{bmatrix}\hat{u}_i\\ \hat{u}_i^f\end{bmatrix} + \begin{bmatrix}(1-\phi)\,\hat{f}_i^s\\ \phi\hat{f}_i^f\end{bmatrix} = \mathbf{0} \tag{22}$$

with the differential operators

$$\mathbf{B6}^{comp} = \begin{bmatrix} \left(G\nabla^2 - s^2\left((1-\phi)\varrho_s + \varrho_a\right) - s\frac{\phi^2}{\kappa}\right)\delta_{ij} + \left(K + \frac{1}{3}G + \frac{Q^2}{R}\right)\partial_i\partial_j & Q\partial_i\partial_j + \left(s^2\varrho_a + s\frac{\phi^2}{\kappa}\right)\delta_{ij} \\ Q\partial_i\partial_j + \left(s^2\varrho_a + s\frac{\phi^2}{\kappa}\right)\delta_{ij} & R\partial_i\partial_j - \left(s^2\left(\phi\varrho_f + \varrho_a\right) - s\frac{\phi^2}{\kappa}\right)\delta_{ij} \end{bmatrix} \tag{23a}$$

$$\mathbf{B6}^{incomp} = \begin{bmatrix} \left(G\nabla^2 - s^2\left((1-\phi)\varrho_s + \varrho_a\right) - s\frac{\phi^2}{\kappa}\right)\delta_{ij} + \left(K + \frac{1}{3}G\right)\partial_i\partial_j & \left(s^2\varrho_a + s\frac{\phi^2}{\kappa}\right)\delta_{ij} \\ \left(s^2\varrho_a + s\frac{\phi^2}{\kappa}\right)\delta_{ij} & \left(-s^2\left(\phi\varrho_f + \varrho_a\right) - s\frac{\phi^2}{\kappa}\right)\delta_{ij} \end{bmatrix} . \tag{23b}$$

As before in (21), the operator name $\mathbf{B6}$ is the same whether it is in 2-d (4 unknowns) or 3-d (6 unknowns). In the following, the same material parameter in both representations (23) and (21) will be used, so Q is replaced by $Q = R(\alpha/\phi - 1)$ to have comparable representations.

In equations (21) and (23), the partial derivative $()_{,i}$ is denoted by ∂_i and $\nabla^2 = \partial_{ii}$ is the Laplacian operator. Note, all the operators (21) and (23) are elliptic but the operators $\mathbf{B6}$ in (23) are self adjoint whereas the operators $\mathbf{B4}$ in (21) are not self adjoint. Therefore, in the latter case for the deduction of fundamental solutions the adjoint operator to $\mathbf{B4}$ has to be used which in the following will not be indicated separately.

A fundamental solution is mathematically spoken a solution of the equation $\mathbf{BG} + \mathbf{I}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ where the matrix of fundamental solutions is denoted by $\mathbf{G}$, the identity matrix by $\mathbf{I}$, and the Dirac distribution by $\delta(\mathbf{x} - \mathbf{y})$. Physically interpreted the solution at point $\mathbf{x}$ due to a single force at point $\mathbf{y}$ is looked for. Concerning the interpretation of the 'single force' the difference in the fundamental solutions for both representations of poroelastic governing equations (20) and (22) becomes obvious. In the system (22), the right hand side consists of forces acting in the solid part $(1-\phi)\hat{f}_i^s$ and in the fluid part $\phi\hat{f}_i^f$ of the porous media, respectively. Contrary, in the system (20), the right hand side consists of a bulk body force $\hat{F}_i = (1-\phi)\hat{f}_i^s$ and a source term $\hat{a}$, i.e., no forces in the fluid $\hat{f}_i^f$ are present. Due to this, it can not be expected that the fundamental solutions of both systems coincide. Only the displacement solution due to a single force in the solid has the same physical meaning.

To find these solutions, the same method can be chosen for both representations. In all cases, for compressible as well as incompressible constituents and for both representations, respectively, the method of Hörmander (1963) is used. The idea of this method is to reduce the highly complicated operators (21) and (23) to simple well known operators. For this purpose the definition of the inverse matrix operator $\mathbf{B}^{-1} = \mathbf{B}^{co}/\det(\mathbf{B})$ with the matrix of cofactors $\mathbf{B}^{co}$ is used. The ansatz $\mathbf{G} = \mathbf{B}^{co}\varphi$ for the matrix of fundamental solutions with an unknown scalar function φ inserted in the operator equation $\mathbf{BG} + \mathbf{I}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ yields a more convenient representation of equations (20) and (22)

$$\begin{aligned} \mathbf{BB}^{co}\varphi + \mathbf{I}\delta(\mathbf{x} - \mathbf{y}) = \det(\mathbf{B})\,\mathbf{I}\varphi + \mathbf{I}\delta(\mathbf{x} - \mathbf{y}) = \mathbf{0} \\ \rightsquigarrow \quad \det(\mathbf{B})\,\varphi + \delta(\mathbf{x} - \mathbf{y}) = 0 \, . \end{aligned} \tag{24}$$

With this reformulation, the search for a fundamental solution is reduced to solve the simpler scalar equation (24). An overview of this method is found in the original work by Hörmander (1963) and more exemplary in Schanz (2001b) and Rashed (2002).

First, this method is applied to the compressible operators in (21).

Compressible model Following Hörmander's idea, first, the determinants of the operators $\mathbf{B4}^{comp}$ and $\mathbf{B6}^{comp}$ are calculated, preferably with the aid of computer algebra. In the following only 3-d will be considered, 2-d being handled in the same way (Schanz and Pryl 2004). This yields the results

$$\det\left(\mathbf{B4}^{comp}\right) = \frac{G^2\beta}{s\varrho_f}\left(K+\frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right) \tag{25}$$

$$\det\left(\mathbf{B6}^{comp}\right) = \left(\frac{s^2G\phi^2\varrho_f}{\beta}\right)^2\left(K+\frac{4}{3}G\right)R\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right) \tag{26}$$

with the roots $\lambda_i,\ i = 1, 2, 3$

$$\begin{aligned}\lambda_{1,2}^2 =& \frac{1}{2}\Bigg[\frac{\phi^2\varrho_f}{\beta R} + \frac{\varrho-\beta\varrho_f}{K+\frac{4}{3}G} + \frac{\varrho_f\left(\alpha-\beta\right)^2}{\beta\left(K+\frac{4}{3}G\right)} \\ &\pm\sqrt{\left(\frac{\phi^2\varrho_f}{\beta R} + \frac{\varrho-\beta\varrho_f}{K+\frac{4}{3}G} + \frac{\varrho_f\left(\alpha-\beta\right)^2}{\beta\left(K+\frac{4}{3}G\right)}\right)^2 - 4\frac{\phi^2\varrho_f\left(\varrho-\beta\varrho_f\right)}{\beta R\left(K+\frac{4}{3}G\right)}}\Bigg] \\ \lambda_3^2 =& \frac{\varrho-\beta\varrho_f}{G}\ .\end{aligned} \tag{27}$$

Expressing the determinant using this roots the scalar equation corresponding to (24) is given by

$$\left(\nabla^2 - s^2\lambda_3^2\right)\left(\nabla^2 - s^2\lambda_1^2\right)\left(\nabla^2 - s^2\lambda_2^2\right)\psi + \delta\left(\mathbf{x}-\mathbf{y}\right) = 0 \tag{28}$$

using an appropriate abbreviation ψ for every operator, i.e.,

$$\begin{aligned}\mathbf{B4}^{comp}: \quad \psi &= G^2\frac{\beta}{s\varrho_f}\left(K+\frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)\varphi \\ \mathbf{B6}^{comp}: \quad \psi &= G^2\left(\frac{s^2\phi^2\varrho_f}{\beta}\right)^2\left(K+\frac{4}{3}G\right)R\left(\nabla^2 - s^2\lambda_3^2\right)\varphi\ .\end{aligned} \tag{29}$$

The solution of the modified higher order Helmholtz equation (28) is

$$\psi = \frac{1}{4\pi r s^4}\left[\frac{e^{-\lambda_1 sr}}{\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_1^2-\lambda_3^2\right)} + \frac{e^{-\lambda_2 sr}}{\left(\lambda_2^2-\lambda_1^2\right)\left(\lambda_2^2-\lambda_3^2\right)} + \frac{e^{-\lambda_3 sr}}{\left(\lambda_3^2-\lambda_2^2\right)\left(\lambda_3^2-\lambda_1^2\right)}\right] \tag{30}$$

with the distance between the two points $\mathbf{x}$ and $\mathbf{y}$ is denoted by $r = |\mathbf{x}-\mathbf{y}|$.

Having in mind that the Laplace transformation of the function describing a travelling wave front with constant speed c is $e^{-rs/c} = \mathscr{L}\left\{H\left(t - r/c\right)\right\}$ (in 3-d), it is obvious that the above solution (30) represents three waves. However, as the roots λ_i are functions of s, here, the wave speeds are time dependent representing the attenuation in a poroelastic continuum. This is in accordance with the well known three wave types of a poroelastic continuum (Biot 1956). The roots λ_1, λ_2, and λ_3 correspond to the wave velocities of the slow and fast compressional wave and to the shear wave, respectively. It should be remarked that the root λ_3 representing the shear wave is in 3-d a double root whereas it is in 2-d only a single root, which, as in elasticity, corresponds to the number of polarisation planes (Royer and Dieulesaint 2000).

From a pure mathematical point of view, the determinant of the operator $\mathbf{B6}^{comp}$ can have six roots in 3-d. However, in (26) only four roots are found. As above discussed, each root represents a different wave type whereas the shear wave is in 3-d a double root. So, from physics it is obvious that the operator $\mathbf{B6}^{comp}$ can not have more roots as given in (26). As a consequence, it can be concluded that the representation of a poroelastic continuum with solid displacements and fluid displacements is overdetermined, i.e., the representation with pore pressure and solid displacements is sufficient. This confirms the considerations of Bonnet (1987).

The next steps are to insert the solution ψ back in the definition $\mathbf{G} = \mathbf{B}^{co}\varphi$ taking into account the proper relation (29) between φ and ψ. After calculating the respective matrix of cofactors $\mathbf{B}^{co}$ the fundamental solutions are found for the solid displacement/pore pressure representation

$$\mathbf{G4}^{comp} = \begin{bmatrix} \hat{U}^s_{ij} & \hat{U}^f_i \\ \hat{P}^s_j & \hat{P}^f \end{bmatrix} = \tag{31}$$
$$\frac{s\varrho_f}{G\beta\left(K+\frac{4}{3}G\right)} \begin{bmatrix} \left(F\nabla^2 + AD\right)\delta_{ij} - F\partial_{ij} & -A\left(\alpha-\beta\right)s\partial_i \\ -A\left(\alpha-\beta\right)\partial_i & A\left(\left(K+\frac{4}{3}G\right)\nabla^2 + A\right) \end{bmatrix} \psi$$

with the abbreviations $A = G\nabla^2 - s^2\left(\varrho - \beta\varrho_f\right)$, $D = \beta/\left(s\varrho_f\right)\nabla^2 - \phi^2 s/R$, $F = \left(K + 4/3G\right)D - \left(\alpha-\beta\right)s$, and for the solid/fluid displacement representation

$$\mathbf{G6}^{comp} = \begin{bmatrix} \hat{U}^{ss}_{ij} & \hat{U}^{sf}_{ij} \\ \hat{U}^{fs}_{ij} & \hat{U}^{ff}_{ij} \end{bmatrix} = \tag{32}$$
$$\frac{-\beta}{Gs^2\phi^2\varrho_f\left(K+\frac{4}{3}G\right)} \begin{bmatrix} M_3\partial_{ij} + \left(M_5 - M_3\nabla^2\right)\delta_{ij} & M_1\partial_{ij} + \left(M_4 - M_1\nabla^2\right)\delta_{ij} \\ M_1\partial_{ij} + \left(M_4 - M_1\nabla^2\right)\delta_{ij} & M_2\partial_{ij} + \left(M_6 - M_2\nabla^2\right)\delta_{ij} \end{bmatrix} \psi$$

with the abbreviations

$$\begin{aligned}
M_1 &= CE\left[\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right] - C^2\left(\frac{\alpha}{\phi}-1\right) + C\nabla^2\left(K+\frac{1}{3}G\right) \\
&\quad + B\left[C - E\left(\frac{\alpha}{\phi}-1\right)\right] \\
M_2 &= 2BC\left(\frac{\alpha}{\phi}-1\right) - B^2 - B\nabla^2\left(K+\frac{1}{3}G\right) - C^2\left[\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right] \\
M_3 &= E\nabla^2\left(K+\frac{1}{3}G\right) + 2EC\left(\frac{\alpha}{\phi}-1\right) - C^2 - E^2\left[\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right] \\
M_4 &= \frac{s^2\varrho_f\phi^2 G}{\beta}\left(\nabla^2 - s^2\lambda_3^2\right)\left[\nabla^2\left(\frac{\alpha}{\phi}-1\right) + \frac{C}{R}\right] \\
M_5 &= \frac{s^2\varrho_f\phi^2 G}{\beta}\left(s^2\lambda_3^2 - \nabla^2\right)\left[\nabla^2 + \frac{E}{R}\right] \\
M_6 &= \frac{s^2\varrho_f\phi^2 G}{\beta}\left(s^2\lambda_3^2 - \nabla^2\right)\left[\nabla^2\left(\frac{K+\frac{1}{3}G}{R} + \left(\frac{\alpha}{\phi}-1\right)^2\right) + \frac{B}{R}\right] \\
B &= G\nabla^2 - s^2\left(1-\phi\right)\varrho_s - C \qquad C = s\frac{\phi^2}{\kappa} + s^2\varrho_a \qquad E = -s^2\phi\varrho_f - C\,.
\end{aligned}$$

The difference of the 2-d solution and the 3-d solution lies only in different functions ψ. The explicit expressions for the fundamental solutions for the unknowns solid displacements u_i and pore pressure p can be found in the appendix I and their 2-d counterparts in Schanz and Pryl (2004).

Incompressible model In the incompressible approximation the procedure is the same as before. First, the determinants with their respective roots are calculated. However, here, both representation have different roots indicating that two different incompressible models are considered as discussed before. For each representation the determinants and roots are either

$$\det\left(\mathbf{B4}^{incomp}\right) = \frac{G^2\beta}{s\varrho_f}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(\nabla^2 - s^2\lambda_1^2\right)\nabla^2 \tag{33}$$

with the roots

$$\lambda_1^2 = \frac{\varrho + \varrho_f\left(\frac{1}{\beta} - 2\right)}{K + \frac{4}{3}G} \qquad \lambda_3^2 = \frac{\varrho - \beta\varrho_f}{G}\ , \tag{34}$$

and

$$\det\left(\mathbf{B6}^{incomp}\right) = \frac{s^6 G^2\phi^6\varrho_f^3}{\beta^3}\left(K + \frac{4}{3}G\right)\left(\nabla^2 - s^2\lambda_3^2\right)^2\left(s^2\lambda_1^2 - \nabla^2\right) \tag{35}$$

with the roots

$$\lambda_1^2 = \frac{\varrho - \beta\varrho_f}{K + \frac{4}{3}G} \qquad \lambda_3^2 = \frac{\varrho - \beta\varrho_f}{G}\ . \tag{36}$$

The solutions of the equation (28) corresponding to the determinant (33) is

$$\psi = \frac{1}{4\pi r s^4}\left[\frac{e^{-\lambda_1 s r}}{\left(\lambda_1^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)} + \frac{1}{\lambda_1^2\lambda_3^2} + \frac{e^{-\lambda_3 s r}}{\lambda_3^2\left(\lambda_3^2 - \lambda_1^2\right)}\right] \tag{37}$$

whereas the the solutions corresponding to the determinant (35) is

$$\psi = \frac{1}{4\pi r s^2}\frac{1}{\lambda_3^2 - \lambda_1^2}\left[e^{-\lambda_3 s r} - e^{-\lambda_1 s r}\right]\ . \tag{38}$$

The incompressible fundamental solutions are the limit values of the compressible results for $\lambda_2 \to 0$. In both representations, the third root λ_3 corresponding to the shear wave velocity is not changed because incompressibility can only affect volumetric changes. Contrary, the compressional waves have to change as observed by the vanishing root λ_2 and the different root λ_1. Here, also the difference between both formulations is obvious. In the pore pressure/solid displacement representation the smaller value λ_2, corresponding to the faster compression wave, goes to zero. The larger value λ_1, corresponding to the slower compressional wave, survive. Reflecting the physics behind these two compressional waves this behaviour is explainable. In case of the fast compressional wave, the solid and the fluid moves in phase. If the solid is assumed to be incompressible it have no longer any deformation and, subsequent, the wave speed tends to infinity respective the corresponding λ_2 to zero. In case of the slow compressional wave, the solid and fluid moves in opposite phase. This relative movement is still possible if the solid and fluid are incompressible.

These physical considerations are well represented in the pore pressure/solid displacement formulation. Contrary, in the solid/fluid displacement formulation, no root λ_2 exists, i.e., the determinant (35) is only of third order in ∇^2. This represents the fact that this incompressible model is not achieved by a limit as in the pore pressure/solid displacement formulation. Only from physics it can be concluded that the fast compressional wave vanishes, however, the surviving wave have a different wave velocity compared to the other formulation.

The incompressible fundamental solutions are found for the solid displacement/pore pressure representation

$$\mathbf{G4}^{incomp} = \begin{bmatrix} \hat{U}^s_{ij} & \hat{U}^f_i \\ \hat{P}^s_j & \hat{P}^f \end{bmatrix} = \tag{39}$$

$$\frac{s\varrho_f}{G\beta\left(K+\frac{4}{3}G\right)} \begin{bmatrix} \left(F\nabla^2+AD\right)\delta_{ij} - F\partial_{ij} & -A\left(1-\beta\right)s\partial_i \\ -A\left(1-\beta\right)\partial_i & A\left(\left(K+\frac{4}{3}G\right)\nabla^2+A\right) \end{bmatrix}\psi$$

with the abbreviations $A = G\nabla^2 - s^2\left(\varrho - \beta\varrho_f\right),\ D = \beta/\left(s\varrho_f\right)\nabla^2,\ F = \left(K+4/3G\right)D - \left(1-\beta\right)s$, where D and F differ from the compressible case, and ψ is defined in (37). For the solid/fluid displacement representation the matrix of fundamental solutions is

$$\mathbf{G6}^{incomp} = \begin{bmatrix} \hat{U}^{ss}_{ij} & \hat{U}^{sf}_{ij} \\ \hat{U}^{fs}_{ij} & \hat{U}^{ff}_{ij} \end{bmatrix} = \tag{40}$$

$$\frac{\beta^2}{Gs^4\phi^4\varrho_f^2\left(K+\frac{4}{3}G\right)} \begin{bmatrix} (M_0E+M_1\nabla^2)\delta_{ij} - M_1\partial_{ij} & (-M_0C+M_2\nabla^2)\delta_{ij} - M_2\partial_{ij} \\ (-M_0C+M_2\nabla^2)\delta_{ij} - M_2\partial_{ij} & A(M_0+\nabla^2M_3)\delta_{ij} - M_5\partial_{ij} \end{bmatrix}\psi$$

with the abbreviations $M_0 = E\,A - C^2,\ M_1 = B\,E^2,\ M_2 = -B\,C\,E,\ M_3 = B\,E,\ M_5 = B\,C^2,\ A = G\nabla^2 - s^2((1-\phi)\rho_s - \phi\rho_f + \rho_f\frac{\phi^2}{\beta}),\ B = K + \frac{1}{3}G,\ C = s^2\rho_f(\frac{\phi^2}{\beta} - \phi),\ E = -s^2\rho_f\frac{\phi^2}{\beta}$. In equation (40), the function ψ has to be taken from (38).

As for the compressible case, the explicit expressions for the incompressible fundamental solutions for the unknowns solid displacements u_i and pore pressure p can be found in the appendix I and their 2-d counterparts in Schanz and Pryl (2004).

BOUNDARY ELEMENT METHOD

The boundary integral equation for dynamic poroelasticity in Laplace domain can be obtained using either the corresponding reciprocal work theorem (Cheng et al. 1991) or the weighted residuals formulation (Domínguez 1992). Here, the approach with weighted residuals is used. Only the solid displacements u_i and pore pressure p, i.e., one vector and one scalar, are used for the set of independent variables. This set is sufficient (Bonnet 1987) and has fewer degrees of freedom than with the solid and relative fluid to solid displacements u_i, u_i^f, i.e., two vectors of independent variables.

Following the procedure as described in Schanz (2001a) gives the time dependent integral equation for poroelasticity on a domain with boundary Γ

$$\int_0^t \int_\Gamma \begin{bmatrix} U^s_{ij}(t-\tau,\mathbf{y},\mathbf{x}) & -P^s_j(t-\tau,\mathbf{y},\mathbf{x}) \\ U^f_i(t-\tau,\mathbf{y},\mathbf{x}) & -P^f(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} \begin{bmatrix} t_i(\tau,\mathbf{x}) \\ q(\tau,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau =$$
$$\int_0^t \oint_\Gamma \begin{bmatrix} T^s_{ij}(t-\tau,\mathbf{y},\mathbf{x}) & Q^s_j(t-\tau,\mathbf{y},\mathbf{x}) \\ T^f_i(t-\tau,\mathbf{y},\mathbf{x}) & Q^f(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} \begin{bmatrix} u_i(\tau,\mathbf{x}) \\ p(\tau,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau + \begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(t,\mathbf{y}) \\ p(t,\mathbf{y}) \end{bmatrix} \tag{41}$$

with the integral free terms c_{ij} and c known from elastostatics and acoustics, respectively, and with the Cauchy principal value integral $\oint$. T^s_{ij}, T^f_i, Q^s_j, and Q^f denote the fundamental solutions for tractions and flux as defined in Schanz (2001a), which can be computed from U^s_{ij}, U^f_i, P^s_j, and P^f.

A boundary element formulation is achieved following the usual procedure. First, the boundary surface Γ is discretized by E elements Γ_e where F polynomial shape functions $N^f_e(\mathbf{x})$ are defined. Hence, the following ansatz functions are used with the time dependent nodal values $u^{ef}_i(t), t^{ef}_i(t), p^{ef}(t)$, and $q^{ef}(t)$

$$\begin{aligned} u_i(\mathbf{x},t) &= \sum_{e=1}^{E}\sum_{f=1}^{F} N^f_e(\mathbf{x})\,u^{ef}_i(t) \quad & t_i(\mathbf{x},t) &= \sum_{e=1}^{E}\sum_{f=1}^{F} N^f_e(\mathbf{x})\,t^{ef}_i(t) \\ p(\mathbf{x},t) &= \sum_{e=1}^{E}\sum_{f=1}^{F} N^f_e(\mathbf{x})\,p^{ef}(t) \quad & q(\mathbf{x},t) &= \sum_{e=1}^{E}\sum_{f=1}^{F} N^f_e(\mathbf{x})\,q^{ef}(t)\ . \end{aligned} \tag{42}$$

In equations (42), the shape functions of all four variables are denoted by the same function $N^f_e(\mathbf{x})$ indicating the same approximation level of all variables. This is not mandatory but usual, non-isoparametric element types employing different ansatz functions for displacements and pressure, as common in finite elements (Lewis and Schrefler 1998), can also be used (Pryl and Schanz 2004). Inserting these ansatz functions (42) in the time dependent integral equation (41) yields

$$\begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(\mathbf{y},t) \\ p(\mathbf{y},t) \end{bmatrix} =$$
$$\sum_{e=1}^{E}\sum_{f=1}^{F}\Bigg\{ \int_0^t \int_\Gamma \begin{bmatrix} U^s_{ij}(t-\tau,\mathbf{y},\mathbf{x}) & -P^s_j(t-\tau,\mathbf{y},\mathbf{x}) \\ U^f_i(t-\tau,\mathbf{y},\mathbf{x}) & -P^f(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} N^f_e(\mathbf{x}) \begin{bmatrix} t^{ef}_i(\tau) \\ q^{ef}(\tau) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau$$
$$- \int_0^t \oint_\Gamma \begin{bmatrix} T^s_{ij}(t-\tau,\mathbf{y},\mathbf{x}) & Q^s_j(t-\tau,\mathbf{y},\mathbf{x}) \\ T^f_i(t-\tau,\mathbf{y},\mathbf{x}) & Q^f(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} N^f_e(\mathbf{x}) \begin{bmatrix} u^{ef}_i(\tau) \\ p^{ef}(\tau) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau \Bigg\}\ . \tag{43}$$

Next, a time discretization has to be introduced. Since no time dependent fundamental solutions are known, the convolution quadrature method (briefly summarised in appendix II) is the most effective method. Another possibility is an inverse Laplace transform of the Laplace domain fundamental solutions at every collocation point in every time step using a series expansion (Chen and Dargush 1995).

Hence, after dividing the time period t in N intervals of equal duration Δt, i.e., $t = N\Delta t$, the convolution integrals between the fundamental solutions and the nodal values in (43) are approximated by the convolution quadrature method, i.e., the quadrature formula (55) is applied to the integral equation (43). This results in the following boundary element time stepping formulation for $n = 0, 1, \ldots, N$

$$\begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(\mathbf{y}, n\Delta t) \\ p(\mathbf{y}, n\Delta t) \end{bmatrix} = \sum_{e=1}^{E} \sum_{f=1}^{F} \sum_{k=0}^{n} \Bigg\{ \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{U}_{ij}^{s}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}_{j}^{s}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{U}_{i}^{f}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}^{f}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} t_i^{ef}(k\Delta t) \\ q^{ef}(k\Delta t) \end{bmatrix} - \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{T}_{ij}^{s}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}_{j}^{s}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{T}_{i}^{f}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}^{f}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} u_i^{ef}(k\Delta t) \\ p^{ef}(k\Delta t) \end{bmatrix} \Bigg\} \tag{44}$$

with the weights corresponding to (57), e.g.,

$$\omega_{n-k}^{ef}\Big(\hat{U}_{ij}^{s}, \mathbf{y}, \Delta t\Big) = \frac{\mathscr{R}^{-(n-k)}}{L} \sum_{\ell=0}^{L-1} \int_{\Gamma} \hat{U}_{ij}^{s}\left(\frac{\gamma\left(e^{i\ell\frac{2\pi}{L}}\mathscr{R}\right)}{\Delta t}, \mathbf{y}, \mathbf{x}\right) N_e^f(\mathbf{x})\, \mathrm{d}\Gamma\, e^{-i(n-k)\ell\frac{2\pi}{L}} . \tag{45}$$

Note, the calculation of the integration weights is only based on the Laplace transformed fundamental solutions which are available. Therefore, with the time stepping procedure (44) a boundary element formulation for poroelastodynamics is given without time dependent fundamental solutions.

To calculate the integration weights ω_{n-k}^{ef} in (44), spatial integration over the boundary Γ has to be performed. Because the essential constituents of the Laplace transformed fundamental solutions are exponential functions in 3-d, i.e., the integrand is smooth, the regular integrals are evaluated by standard Gaussian quadrature rule, while the weakly singular parts of the integrals in (44) are regularized by polar coordinate transformation. The strongly singular integrals in (44) are equal to those of elastostatics or acoustics, respectively, and, hence, the regularization methods known from these theories can be applied, e.g., the method suggested by Guiggiani and Gigante (1990). Moreover, to obtain for equation (44) a system of algebraic equations, collocation is used at every node of the shape functions $N_e^f(\mathbf{x})$.

According to $t - \tau = (n - k)\Delta t$, the integration weights ω_{n-k}^{ef} are only dependent on the difference $n - k$. This property is analogous to elastodynamic time domain BE formulations (see, e.g., Domínguez (1993)) and can be used to establish a recursion formula ($m = n - k$)

$$\omega_0(\mathbf{C})\,\mathbf{d}^n = \omega_0(\mathbf{D})\,\bar{\mathbf{d}}^n + \sum_{m=1}^{n} \left(\omega_m(\mathbf{U})\,\mathbf{t}^{n-m} - \omega_m(\mathbf{T})\,\mathbf{u}^{n-m}\right) \qquad n = 1, 2, \ldots, N \tag{46}$$

with the time dependent integration weights ω_m containing the Laplace transformed fundamental solutions $\mathbf{U}$ and $\mathbf{T}$, respectively (see, equation (45)). Similarly, $\omega_0(\mathbf{C})$ and $\omega_0(\mathbf{D})$ are the corresponding integration weights of the first time step related to the unknown and known boundary data in time step n $\mathbf{d}^n$ and $\bar{\mathbf{d}}^n$, respectively. Finally, a direct equation solver is applied.

Table 1. Material data of Berea sandstone (rock) and sand (soil)

	K $\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right)$	G $\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right)$	$\varrho\left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)$	ϕ	K$_s$ $\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right)$	$\varrho_f\left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)$	K$_f$ $\left(\frac{\mathrm{N}}{\mathrm{m}^2}\right)$	$\kappa\left(\frac{\mathrm{m}^4}{\mathrm{Ns}}\right)$
rock	$8 \cdot 10^9$	$6 \cdot 10^9$	2458	0.19	$3.6 \cdot 10^{10}$	1000	$3.3 \cdot 10^9$	$1.9 \cdot 10^{-10}$
soil	$2.1 \cdot 10^8$	$9.8 \cdot 10^7$	1884	0.48	$1.1\ 10^{10}$	1000	$3.3 \cdot 10^9$	$3.55\ 10^{-9}$

EXAMPLE: WAVE PROPAGATION IN POROELASTIC HALF SPACE

In order to demonstrate the effect of modelling the constituents incompressible, wave propagation phenomenon in a poroelastic half space is considered. Results obtained by the incompressible model are compared to those of the compressible model. For the comparison, a long strip ($6\,\mathrm{m} \times 33\,\mathrm{m}$) is discretized with 396 triangular linear elements on 242 nodes (see Fig. 1). The time step size used is $\Delta t = 0.00008\,\mathrm{s}$ in case of rock and $\Delta t = 0.00032\,\mathrm{s}$ for

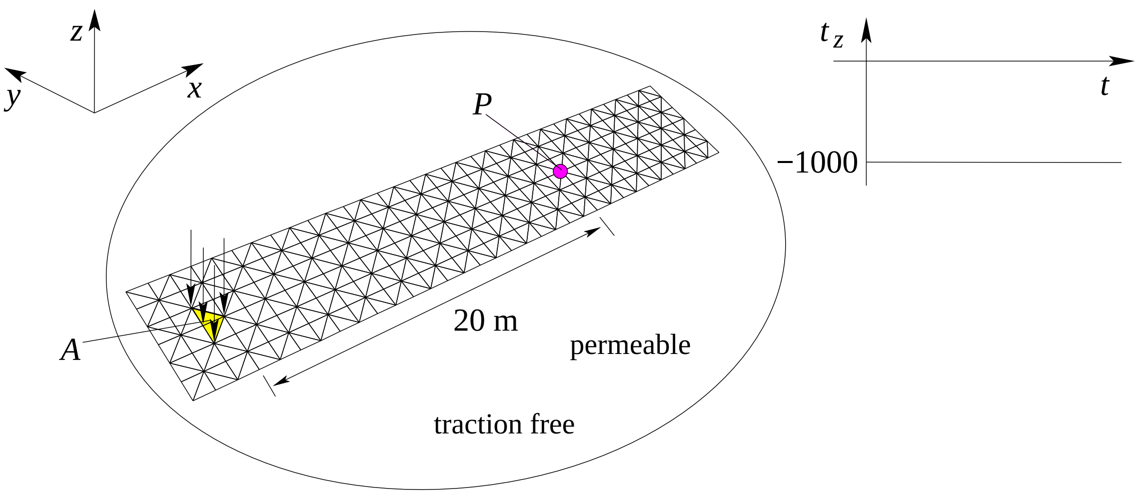
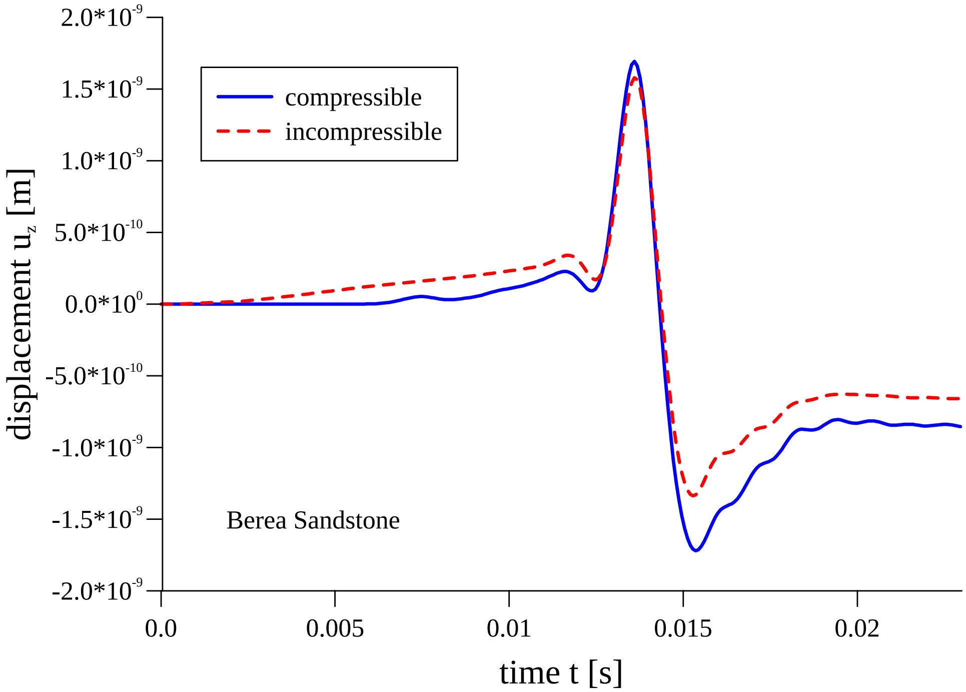


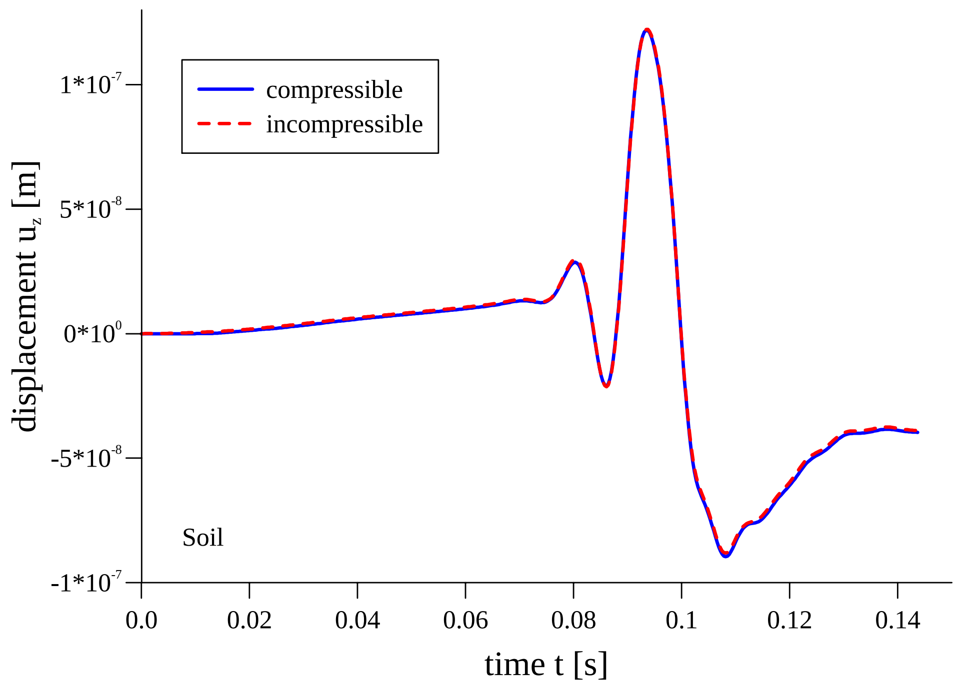
Figure 1. Half space under vertical load: Discretization and load history

soil. The modelled half space is loaded on area $A\ (1\,\mathrm{m}^2)$ by a vertical total stress vector $t_z = -1000\,\mathrm{N/m}^2\,H(t)$ and the remaining surface is traction free. The pore pressure is assumed to be zero all over the surface, i.e., the surface is permeable. The material properties are those corresponding to a rock (Berea sandstone) and a soil (coarse sand) given in Tab. 1. The interstitial fluid is water.

Before looking at the results, it may be convenient to look at the ratios of the compression moduli. For rock there is: $K/K_s = 0.22$, $K/K_f = 2.42$ and for soil $K/K_s = 0.019$, $K/K_f = 0.064$. Hence, it can be expected that the incompressible modelling for the rock fails and give good results for the soil. Exactly this is confirmed by the results given in Fig. 2a for the rock and Fig. 2b for the soil. There, the horizontal displacement at the marked point on the mesh, $20\,\mathrm{m}$ from the margin of the loaded area, for an incompressible as well as compressible model is plotted versus time t. Clearly, for the rock there are large differences whereas for the soil both results are almost indistinguishable. Also, from the incompressible rock results it can be observed that the arrival of the fast compressional wave, the first deviation from zero, tends to zero. The Rayleigh wave, e.g., the large amplitude at $t \approx 0.015\,\mathrm{s}$ for Berea sandstone, is not affected by the different modellings. This is in accordance with the theory where one wave



(a) Berea sandstone (rock)



(b) Soil

Figure 2. Vertical displacement versus time

vanishes, i.e., one of the roots (27), (34) is zero, and the third root, corresponding to the shear wave, is not influenced.

In Fig. 3, the pore pressure p at a point 3m under the loaded area is plotted versus time t for both the incompressible and compressible model. Again, there are large differences for the rock whereas for soil the results only show minor differences. In the incompressible results, the instant arrival of the fast compressional wave can be observed better than in the displacement plots. The results with increased permeability κ show that the slow compressional wave speed is also changed, which corresponds to the theory. Clearly, for the soil also this wave type does not exhibit any significant differences between the two models. Finally, it should be remarked that the usage of the incompressible fundamental solutions needs about 20 % less CPU time.

CONCLUSIONS

To summarise, an overview of the differences between the compressible and incompressible models is given. Assuming both constituents incompressible, the propagation speed of the fast compressional wave becomes infinite, the slow compressional wave survives with changed speed, and the shear wave remains unchanged. The incompressible modelling also changes the steady state (consolidation) displacement.

Because there are no noticeable differences for some materials (e.g., loose grain with fluid), the use of the incompressible model can be recommended for them to achieve better performance. The speedup compared to the compressible computation is about 20 %.

Appendix I. EXPLICIT EXPRESSIONS FOR THE FUNDAMENTAL SOLUTIONS

The explicit expressions of the poroelastodynamic fundamental solutions for the unknowns solid displacements u_i and pore pressure p and for solid displacements and fluid displacements u_i and u_i^f are given in the following for a 3-d continuum, for compressible as well as incompressible constituents.

Solid displacements u_i and pore pressure p

Compressible The elements of the matrix $\mathbf{G4}$ (31) are the displacements caused by a Dirac force in the solid:

$$\hat{U}_{ij}^s = \frac{1}{4\pi r\left(\varrho - \beta\varrho_f\right)s^2}\left[R_1\frac{\lambda_4^2-\lambda_2^2}{\lambda_1^2-\lambda_2^2}e^{-\lambda_1 sr} - R_2\frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2-\lambda_2^2}e^{-\lambda_2 sr} + \left(\delta_{ij}\lambda_3^2 s^2 - R_3\right)e^{-\lambda_3 sr}\right] \tag{47a}$$

with $R_k = \frac{3r_{,i}r_{,j}-\delta_{ij}}{r^2} + \lambda_k s\frac{3r_{,i}r_{,j}-\delta_{ij}}{r} + \lambda_k^2 s^2 r_{,i}r_{,j}$ and $\lambda_4^2 = \frac{(\varrho-\beta\varrho_f)}{K+\frac{4}{3}G}$. The pressure caused by the same load is

$$\hat{P}_j^s = \frac{(\alpha-\beta)\,\varrho_f r_{,j}}{4\pi\beta s\left(K+\frac{4}{3}G\right)r\left(\lambda_1^2-\lambda_2^2\right)}\left[\left(\lambda_1 s + \frac{1}{r}\right)e^{-\lambda_1 sr} - \left(\lambda_2 s + \frac{1}{r}\right)e^{-\lambda_2 sr}\right]\ . \tag{47b}$$

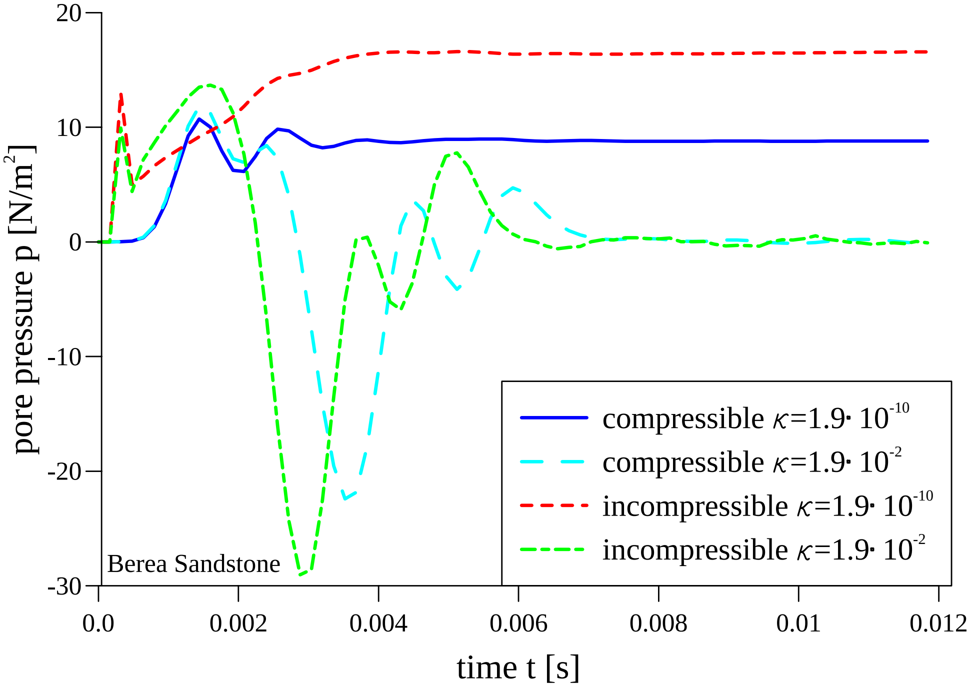
For a Dirac source in the fluid the respective displacement solution is

$$\hat{U}_i^f = s\hat{P}_i^s \tag{47c}$$

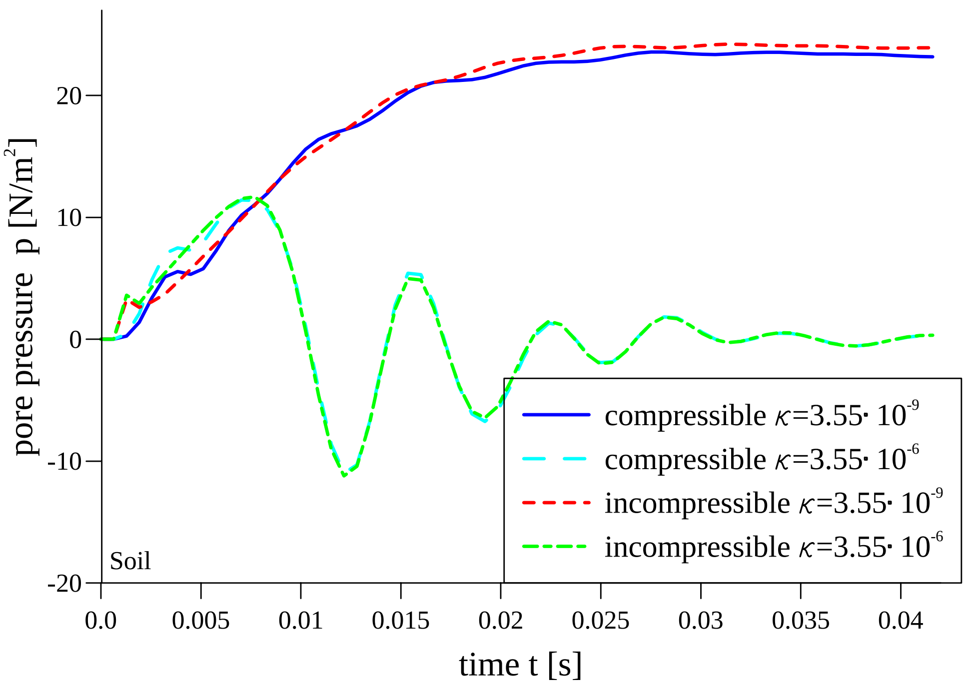
and the pressure

$$\hat{P}^f = \frac{s\varrho_f}{4\pi r\beta\left(\lambda_1^2-\lambda_2^2\right)}\left[\left(\lambda_1^2-\lambda_4^2\right)e^{-\lambda_1 sr} - \left(\lambda_2^2-\lambda_4^2\right)e^{-\lambda_2 sr}\right]\ . \tag{47d}$$

In the above given solutions the roots $\lambda_i,\ i=1,2,3$ from (27) are used.



(a) Berea sandstone (rock)



(b) Soil

Figure 3. Pore pressure versus time

Incompressible For the case of incompressible constituents, the displacements caused by a Dirac force in the solid are

$$\hat{U}_{ij}^{s} = \frac{1}{4\pi r\left(\rho-\beta\rho_f\right)s^2}\left[R_1\frac{\lambda_4^2}{\lambda_1^2}e^{-\lambda_1 sr} - R_2\frac{\lambda_4^2-\lambda_1^2}{\lambda_1^2} + \left(\delta_{ij}\lambda_3^2 s^2 - R_3\right)e^{-\lambda_3 sr}\right] \tag{48a}$$

with $R_k = \left(3r_{,i}r_{,j}-\delta_{ij}\right)/r^2 + \lambda_k s\left(3r_{,i}r_{,j}-\delta_{ij}\right)/r + \lambda_k^2 s^2 r_{,i}r_{,j}$, $\lambda_{1,3}$ from (34), and $\lambda_4^2 = \frac{(\rho-\beta\rho_f)}{K+\frac{4}{3}G}$. The pressure caused by the same load is

$$\hat{P}_j^s = \frac{(\alpha-\beta)\,\rho_f r_{,j}}{4\pi s\beta\left(K+\frac{4}{3}G\right)r\lambda_1^2}\left[\left(\lambda_1 s+\frac{1}{r}\right)e^{-\lambda_1 sr}-\frac{1}{r}\right]. \tag{48b}$$

For a Dirac source in the fluid the respective displacement solution is

$$\hat{U}_i^f = s\hat{P}_i^s \tag{48c}$$

and the pressure

$$\hat{P}^f = \frac{s\rho_f}{4\pi r\beta\lambda_1^2}\left[\left(\lambda_1^2-\lambda_4^2\right)e^{-\lambda_1 sr}+\lambda_4^2\right]. \tag{48d}$$

Solid and fluid displacements u_i, u_i^f

Compressible The explicit expressions of the poroelastodynamic fundamental solutions are given in the following. The four elements of the matrix **G6** (32) are the displacements caused by a Dirac force in the solid:

$$\hat{U}_{ij}^{ss} = \frac{\rho_f}{4\pi r}\Bigg\{ \tag{49}$$

$$\frac{e^{-\lambda_1 sr}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\Bigg[R_1\left(\frac{\phi^2}{\beta}\left(K+\frac{1}{3}G\right)\left(\lambda_1^2R-\rho_f\frac{\phi^2}{\beta}\right)-\rho_f\frac{\left((\frac{\phi^2}{\beta}-\phi)R+Q\frac{\phi^2}{\beta}\right)^2}{R}\right)$$

$$-\delta_{ij}\,s^2\,\frac{\phi^2}{\beta}\left(\frac{\phi^2}{\beta}\rho_f\rho_M+\left(-R(1-\phi)\rho_s-\frac{Q^2}{R}\rho_f\phi-\rho_f\frac{(R+Q)^2}{R}(\frac{\phi^2}{\beta}-\phi)\right)\lambda_1^2\right.$$

$$\left.\left.+\lambda_1^2\left(K+\frac{4}{3}G\right)\left(\lambda_1^2R-\rho_f\frac{\phi^2}{\beta}\right)\right)\right]$$

$$+\frac{e^{-\lambda_2 sr}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)}\Bigg[R_2\left(\frac{\phi^2}{\beta}\left(K+\frac{1}{3}G\right)\left(\lambda_2^2R-\rho_f\frac{\phi^2}{\beta}\right)-\rho_f\frac{\left((\frac{\phi^2}{\beta}-\phi)R+Q\frac{\phi^2}{\beta}\right)^2}{R}\right)$$

$$-\delta_{ij}\,s^2\,\frac{\phi^2}{\beta}\left(\frac{\phi^2}{\beta}\rho_f\rho_M+\left(-R(1-\phi)\rho_s-\frac{Q^2}{R}\rho_f\phi-\rho_f\frac{(R+Q)^2}{R}(\frac{\phi^2}{\beta}-\phi)\right)\lambda_2^2\right.$$

$$\left.\left.+\lambda_2^2\left(K+\frac{4}{3}G\right)\left(\lambda_2^2R-\rho_f\frac{\phi^2}{\beta}\right)\right)\right]$$

$$+\frac{e^{-\lambda_3 sr}}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_1^2)}\Bigg[R_3\left(\frac{\phi^2}{\beta}\left(K+\frac{1}{3}G\right)\left(\lambda_3^2R-\rho_f\frac{\phi^2}{\beta}\right)-\rho_f\frac{\left((\frac{\phi^2}{\beta}-\phi)R+Q\frac{\phi^2}{\beta}\right)^2}{R}\right)$$

$$- \delta_{ij}\, s^2\, \frac{\phi^2}{\beta}\left(\frac{\phi^2}{\beta}\rho_f\rho_M + \left(-R(1-\phi)\rho_s - \frac{Q^2}{R}\rho_f\phi - \rho_f\frac{(R+Q)^2}{R}(\frac{\phi^2}{\beta}-\phi)\right)\lambda_3^2\right.$$

$$\left.\left. + \lambda_3^2\left(K+\frac{4}{3}G\right)\left(\lambda_3^2 R - \rho_f\frac{\phi^2}{\beta}\right)\right)\right]\Bigg\}$$

with the roots $\lambda_i,\ i = 1,2,3$ from (27) and $\rho_M = \rho_s(1-\phi) + \rho_f(\phi-\beta)$. The relative fluid displacements caused by the same load are identical to the solid the solid displacements caused by a force in the fluid

$$\hat{U}_{ij}^{sf} = \hat{U}_{ij}^{fs} = \frac{\rho_f}{4\pi r}\Bigg\{ \tag{50}$$

$$\frac{e^{-\lambda_1 sr}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\left[R_1\left(\left(\left(\frac{\phi^2}{\beta}-\phi\right)(K+\frac{4}{3}G)R + G\frac{\phi^2}{\beta}Q\right)\lambda_1^2 - \rho_f\left(\frac{\phi^2}{\beta}-\phi\right)^2 Q\right.\right.$$

$$\left. - \frac{\phi^2}{\beta}Q\rho_{\beta M} - \rho_f\left(\frac{\phi^2}{\beta}-\phi\right)(K+\frac{1}{3}G+\frac{Q^2}{R})\frac{\phi^2}{\beta} - R\left(\frac{\phi^2}{\beta}-\phi\right)\rho_{\beta M}\right)$$

$$- \delta_{ij}\, s^2\left(\frac{\phi^2}{\beta}-\phi\right)\left(\left(\rho_f\frac{\phi^2}{\beta}-\lambda_1^2 R\right)\left((1-\phi)\rho_s - (K+\frac{4}{3}G)\lambda_1^2\right)\right.$$

$$\left.\left. - \left(\left(\frac{\phi^2}{\beta}-\phi\right)\frac{(R+Q)^2}{R} + \phi\frac{Q^2}{R}\right)\lambda_1^2\rho_f + \rho_f^2\phi\left(\frac{\phi^2}{\beta}-\phi\right)\right)\right]$$

$$+ \frac{e^{-\lambda_2 sr}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)}\left[R_2\left(\left(\left(\frac{\phi^2}{\beta}-\phi\right)(K+\frac{4}{3}G)R + G\frac{\phi^2}{\beta}Q\right)\lambda_2^2 - \rho_f\left(\frac{\phi^2}{\beta}-\phi\right)^2 Q\right.\right.$$

$$\left. - \frac{\phi^2}{\beta}Q\rho_{\beta M} - \rho_f\left(\frac{\phi^2}{\beta}-\phi\right)(K+\frac{1}{3}G+\frac{Q^2}{R})\frac{\phi^2}{\beta} - R\left(\frac{\phi^2}{\beta}-\phi\right)\rho_{\beta M}\right)$$

$$- \delta_{ij}\, s^2\left(\frac{\phi^2}{\beta}-\phi\right)\left(\left(\rho_f\frac{\phi^2}{\beta}-\lambda_2^2 R\right)\left((1-\phi)\rho_s - (K+\frac{4}{3}G)\lambda_2^2\right)\right.$$

$$\left.\left. - \left(\left(\frac{\phi^2}{\beta}-\phi\right)\frac{(R+Q)^2}{R} + \phi\frac{Q^2}{R}\right)\lambda_2^2\rho_f + \rho_f^2\phi\left(\frac{\phi^2}{\beta}-\phi\right)\right)\right]$$

$$+ \frac{e^{-\lambda_3 sr}}{(\lambda_2^2-\lambda_3^2)(\lambda_1^2-\lambda_3^2)}\left[R_3\left(\left(\left(\frac{\phi^2}{\beta}-\phi\right)(K+\frac{4}{3}G)R + G\frac{\phi^2}{\beta}Q\right)\lambda_3^2 - \rho_f\left(\frac{\phi^2}{\beta}-\phi\right)^2 Q\right.\right.$$

$$\left. - \frac{\phi^2}{\beta}Q\rho_{\beta M} - \rho_f\left(\frac{\phi^2}{\beta}-\phi\right)(K+\frac{1}{3}G+\frac{Q^2}{R})\frac{\phi^2}{\beta} - R\left(\frac{\phi^2}{\beta}-\phi\right)\rho_{\beta M}\right)$$

$$- \delta_{ij}\, s^2\left(\frac{\phi^2}{\beta}-\phi\right)\left(\left(\rho_f\frac{\phi^2}{\beta}-\lambda_3^2 R\right)\left((1-\phi)\rho_s - (K+\frac{4}{3}G)\lambda_3^2\right)\right.$$

$$\left.\left. - \left(\left(\frac{\phi^2}{\beta}-\phi\right)\frac{(R+Q)^2}{R} + \phi\frac{Q^2}{R}\right)\lambda_3^2\rho_f + \rho_f^2\phi\left(\frac{\phi^2}{\beta}-\phi\right)\right)\right]\Bigg\}$$

with $\rho_{\beta M} = \left(\frac{\phi^2}{\beta} - \phi\right)\rho_f + (1-\phi)\rho_s$. For a Dirac force in the fluid the respective relative fluid displacement solution is

$$\hat{U}_{ij}^{ff} = \frac{1}{4\pi r}\Bigg\{ \tag{51}$$

$$\frac{e^{-\lambda_1 sr}}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\Bigg[R_1\Bigg(-G(K+\frac{4}{3}G)R\lambda_1^4 + \left((K+\frac{7}{3}G)R\rho_{\beta M} + 2GQ\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)\right)\lambda_1^2$$

$$-\frac{\left(\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)Q + \rho_{\beta M}R\right)^2}{R} - (K+\frac{1}{3}G)\rho_f^2\left(\frac{\phi^2}{\beta}-\phi\right)^2\Bigg)$$

$$-\delta_{ij}\, s^2\Bigg(\rho_f\frac{\phi^2}{\beta}\rho_M\rho_{\beta M} - (K+\frac{4}{3}G)GR\lambda_1^6$$

$$-\Bigg(\rho_{\beta M}^2 R + \rho_f\rho_{\beta M}\left(2\left(\frac{\phi^2}{\beta}-\phi\right)Q + \frac{\phi^2}{\beta}(K+\frac{1}{3}G+\frac{Q^2}{R})\right)$$

$$+\rho_f\frac{\phi^2}{\beta}\left(2(1-\phi)\rho_s + \rho_f(\frac{\phi^2}{\beta}-\beta)\right)G\Bigg)\lambda_1^2$$

$$+\left((K+\frac{7}{3}G)\rho_{\beta M}R + G\rho_f\left(2\left(\frac{\phi^2}{\beta}-\phi\right)Q + \frac{\phi^2}{\beta}(K+\frac{4}{3}G+\frac{Q^2}{R})\right)\right)\lambda_1^4\Bigg)\Bigg]$$

$$+\frac{e^{-\lambda_2 sr}}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)}\Bigg[R_2\Bigg(-G(K+\frac{4}{3}G)R\lambda_2^4 + \left((K+\frac{7}{3}G)R\rho_{\beta M} + 2GQ\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)\right)\lambda_2^2$$

$$-\frac{\left(\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)Q + \rho_{\beta M}R\right)^2}{R} - (K+\frac{1}{3}G)\rho_f^2\left(\frac{\phi^2}{\beta}-\phi\right)^2\Bigg)$$

$$-\delta_{ij}\, s^2\Bigg(\rho_f\frac{\phi^2}{\beta}\rho_M\rho_{\beta M} - (K+\frac{4}{3}G)GR\lambda_2^6$$

$$-\Bigg(\rho_{\beta M}^2 R + \rho_f\rho_{\beta M}\left(2\left(\frac{\phi^2}{\beta}-\phi\right)Q + \frac{\phi^2}{\beta}(K+\frac{1}{3}G+\frac{Q^2}{R})\right)$$

$$+\rho_f\frac{\phi^2}{\beta}\left(2(1-\phi)\rho_s + \rho_f(\frac{\phi^2}{\beta}-\beta)\right)G\Bigg)\lambda_2^2$$

$$+\left((K+\frac{7}{3}G)\rho_{\beta M}R + G\rho_f\left(2\left(\frac{\phi^2}{\beta}-\phi\right)Q + \frac{\phi^2}{\beta}(K+\frac{4}{3}G+\frac{Q^2}{R})\right)\right)\lambda_2^4\Bigg)\Bigg]$$

$$+\frac{e^{-\lambda_3 sr}}{(\lambda_2^2-\lambda_3^2)(\lambda_1^2-\lambda_3^2)}\Bigg[R_3\Bigg(-G(K+\frac{4}{3}G)R\lambda_3^4 + \left((K+\frac{7}{3}G)R\rho_{\beta M} + 2GQ\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)\right)\lambda_3^2$$

$$-\frac{\left(\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)Q + \rho_{\beta M}R\right)^2}{R} - (K+\frac{1}{3}G)\rho_f^2\left(\frac{\phi^2}{\beta}-\phi\right)^2\Bigg)$$

$$-\delta_{ij}\, s^2\Bigg(\rho_f\frac{\phi^2}{\beta}\rho_M\rho_{\beta M} - (K+\frac{4}{3}G)GR\lambda_3^6$$

$$- \left(\rho_{\beta M}^2 R + \rho_f \rho_{\beta M} \left(2 \left(\frac{\phi^2}{\beta} - \phi \right) Q + \frac{\phi^2}{\beta} (K + \frac{1}{3} G + \frac{Q^2}{R}) \right) \right.$$
$$\left. + \rho_f \frac{\phi^2}{\beta} \left(2(1-\phi)\rho_s + \rho_f (\frac{\phi^2}{\beta} - \beta) \right) G \right) \lambda_3^2$$
$$+ \left((K + \frac{7}{3} G) \rho_{\beta M} R + G \rho_f \left(2 \left(\frac{\phi^2}{\beta} - \phi \right) Q + \frac{\phi^2}{\beta} (K + \frac{4}{3} G + \frac{Q^2}{R}) \right) \right) \lambda_3^4 \Bigg) \Bigg] \Bigg\}$$

Incompressible The explicit expressions of the poroelastodynamic fundamental solutions are given in the following. The four elements of the matrix **G6** (40) are the displacements caused by a Dirac force in the solid:

$$\hat{U}_{ij}^{ss} = \frac{s^4 \rho_f^3}{4\pi r} \frac{\phi^6}{\beta^3} \left(\frac{e^{-\lambda_1 s r}}{(\lambda_1^2 - \lambda_3^2)\lambda_1^2} \left[R_1 (K + \frac{1}{3} G) \left(G \lambda_1^2 - \rho_M \right) \right. \right. \tag{52}$$
$$\left. - \delta_{ij}\, s^2 \left(G (K + \frac{4}{3} G) \lambda_1^4 - \lambda_1^2 \rho_M (K + \frac{7}{3} G) + \rho_M^2 \right) \right]$$
$$- \frac{e^{-\lambda_3 s r}}{(\lambda_1^2 - \lambda_3^2)\lambda_3^2} \left[R_3 (K + \frac{1}{3} G) \left(G \lambda_3^2 - \rho_M \right) \right.$$
$$\left. - \delta_{ij}\, s^2 \left(G (K + \frac{4}{3} G) \lambda_3^4 - \lambda_3^2 \rho_M (K + \frac{7}{3} G) + \rho_M^2 \right) \right]$$
$$\left. - \frac{\rho_M}{\lambda_1^2 \lambda_3^2} \left(\frac{3 r_{,i} r_{,j} - \delta_{ij}}{r^2} (K + \frac{1}{3} G) + \delta_{ij} s^2 \rho_M \right) \right).$$

with λ_1 and λ_3 from Eq.(36). The relative fluid displacements caused by the same load are identical to the solid the solid displacements caused by a force in the fluid

$$\hat{U}_{ij}^{sf} = \hat{U}_{ij}^{fs} = \left(1 - \frac{\beta}{\phi} \right) \hat{U}_{ij}^{ss} \tag{53}$$

For a Dirac force in the fluid the respective relative fluid displacement solution is

$$\hat{U}_{ij}^{ff} = \frac{s^4 \rho_f^2 \phi^2}{4\pi r} \Bigg\{ \tag{54}$$
$$\frac{e^{-\lambda_1 s r}}{(\lambda_1^2 - \lambda_3^2)} \left[\frac{R_1}{\beta \lambda_1^2} (G\lambda_1^2 - \rho_M)(K + \frac{1}{3} G) \rho_f \left(\frac{\phi^2}{\beta} - \phi \right)^2 \right.$$
$$+ \delta_{ij}\, s^2 \frac{\phi^2}{\beta^2} \left((K + \frac{4}{3} G)\, G^2 \lambda_1^4 - \frac{\rho_{\beta M} \rho_M^2}{\lambda_1^2} \right.$$
$$- \frac{G}{\beta} \left(G\beta\rho + 2\rho_s(1-\phi)\beta(K + \frac{4}{3} G) - \beta^2 \rho_f (K + \frac{7}{3} G) + \rho_f \phi^2 (K + \frac{4}{3} G) \right) \lambda_1^2$$
$$\left. \left. + \frac{\rho_M}{\beta} \left(\rho_f \phi^2 (K + \frac{7}{3} G) - \phi \rho_f \beta (K + \frac{4}{3} G) - G\beta^2 \rho_f + \rho_s (1-\phi)\beta (K + \frac{10}{3} G) \right) \right) \right]$$
$$- \frac{e^{-\lambda_3 s r}}{(\lambda_1^2 - \lambda_3^2)} \left[\frac{R_3}{\beta \lambda_3^2} (G\lambda_3^2 - \rho_M)(K + \frac{1}{3} G) \rho_f \left(\frac{\phi^2}{\beta} - \phi \right)^2 \right.$$

$$
+ \delta_{ij}\, s^2 \frac{\phi^2}{\beta^2}\Bigg((K+\frac{4}{3}G)\, G^2 \lambda_3^4 - \frac{\rho_{\beta M}\rho_M^2}{\lambda_3^2}
$$

$$
- \frac{G}{\beta}\left(G\beta\rho + 2\rho_s(1-\phi)\beta(K+\frac{4}{3}G) - \beta^2\rho_f(K+\frac{7}{3}G) + \rho_f\phi^2(K+\frac{4}{3}G)\right)\lambda_3^2
$$

$$
+ \frac{\rho_M}{\beta}\left(\rho_f\phi^2(K+\frac{7}{3}G) - \phi\rho_f\beta(K+\frac{4}{3}G) - G\beta^2\rho_f + \rho_s(1-\phi)\beta(K+\frac{10}{3}G)\right)\Bigg)\Bigg]
$$

$$
- \frac{3r_{,i}r_{,j}-\delta_{ij}}{r^2}\frac{\rho_M}{\beta\lambda_1^2\lambda_3^2}(K+\frac{1}{3}G)\rho_f\left(\frac{\phi^2}{\beta}-\phi\right)^2 - \frac{\delta_{ij}\, s^2}{\beta\lambda_1^2\lambda_3^2}\frac{\phi^2}{\beta}\rho_{\beta M}\rho_M^2\Bigg\}\,.
$$

Appendix II. CONVOLUTION QUADRATURE METHOD

The 'Convolution Quadrature Method' developed by Lubich numerically approximates a convolution integral for $n = 0, 1, \ldots, N$

$$
y(t) = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau \quad \rightarrow \quad y(n\Delta t) = \sum_{k=0}^{n} \omega_{n-k}(\Delta t)\, g(k\Delta t)\,, \tag{55}
$$

by a quadrature rule whose weights are determined by the Laplace transformed function $\hat{f}$ and a linear multistep method. This method was originally published in Lubich (1988a) and (1988b). Application to the boundary element method may be found in Schanz and Antes (1997b). Here, a brief overview of the method is given.

In formula (55), the time t is divided in N equal steps Δt. The weights $\omega_n(\Delta t)$ are the coefficients of the power series

$$
\hat{f}\left(\frac{\gamma(z)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n(\Delta t)\, z^n \tag{56}
$$

with the complex variable z. The coefficients of a power series are usually calculated with Cauchy's integral formula. After a polar coordinate transformation, this integral is approximated by a trapezoidal rule with L equal steps $\frac{2\pi}{L}$. This leads to

$$
\omega_n(\Delta t) = \frac{1}{2\pi i}\int_{|z|=\mathscr{R}} \hat{f}\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1}\, \mathrm{d}z \approx \frac{\mathscr{R}^{-n}}{L}\sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma\left(\mathscr{R}e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) e^{-in\ell\frac{2\pi}{L}}\,, \tag{57}
$$

where $\mathscr{R}$ is the radius of a circle in the domain of analyticity of $\hat{f}(z)$.

The function $\gamma(z)$ is the quotient of the characteristic polynomials of the underlying multistep method, e.g., for a BDF 2, $\gamma(z) = \frac{3}{2} - 2z + \frac{1}{2}z^2$. The used linear multistep method must be $A(\alpha)$-stable and stable at infinity (Lubich 1988b). Experience shows that the BDF 2 is the best choice (Schanz 1999). Therefore, it is used in all calculations in this paper.

If one assumes that the values of $\hat{f}(z)$ in (57) are computed with an error bounded by ϵ, then the choice $L = N$ and $\mathscr{R}^N = \sqrt{\epsilon}$ yields an error in ω_n of size $\mathscr{O}(\sqrt{\epsilon})$ Lubich (1988a). Several tests conducted by the author lead to the conclusion that the parameter $\epsilon = 10^{-10}$ is the best choice for the kind of functions dealt with in this paper (Schanz and Antes 1997a). The assumption $L = N$ leads to a order of complexity $\mathscr{O}\left(N^2\right)$ for calculating the N coefficients $\omega_n(\Delta t)$. Due to the exponential function at the end of formula (57) this can be reduced to $\mathscr{O}(N \log N)$ using the technique of the Fast Fourier Transformation (FFT).

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