ON THE EQUIVALENCE OF THE LINEAR BIOT'S THEORY AND THE LINEAR THEORY OF POROUS MEDIA

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ABSTRACT

Assuming a geometrically linear description (small displacements and small deformation gradients) and linear constitutive equations (Hooke's law) the governing equations are derived for two poroelastic theories, Biot's theory (BT) and the Theory of Porous Media (TPM – mixture theory extended by the concept of volume fractions). In both cases, the solid displacements and the pore pressure are the primary unknowns. Note that this is only possible in the Laplace domain leading to the same structure of the coupled differential equations for both approaches. But the differential equations arising in BT and TPM possess different coefficients with different physical interpretations. Correlating these coefficients to each other leads to the well-known problem of Biot's 'apparent mass density'. Furthermore, some inconsistencies are observed if Biot's stress coefficient is correlated to the structure arising in TPM.

In addition to the comparison of the governing equations and the identification of the model parameters, the displacement and pressure solutions of both theories are presented for a one-dimensional column. The results show good agreement between both approaches in case of incompressible constituents whereas in case of compressible constituents large differences appear.

Keywords: Biot, TPM, porous media, linear theory

INTRODUCTION

For a wide range of fluid infiltrated materials, such as water saturated soils, oil impregnated rocks, or air filled foams, the elastic as well as a viscoelastic description of the material behavior is a crude approximation for the investigation of wave propagation in such media. Due to their porosity and due to the interaction of the skeleton and the pore content, a different theoretical approach is necessary to describe the observed effects like the second compressional wave.

A historical review on the subject of multiphase continuum mechanics identifies two theories which have been developed and are used nowadays, namely the Biot theory (BT) and the Theory of Porous Media (TPM). For more details, the reader is directed to the work of de Boer and Ehlers (1988, 1990) or to the recently published monograph (de Boer 2000). The early works on porous media are attributed to Fillunger (1913). In this paper and in subsequent ones, Fillunger was concerned with the question of buoyancy of barrages. At the same time, a more intuitively theory has been developed by von Terzaghi (1923). These two basic works form the basis for the two different theories used up to day.

Based on the work of von Terzaghi, a theoretical description of porous materials saturated by a viscous fluid was presented by Biot (1941). This was the starting point of the theory of poroelasticity or the BT. In the following years, Biot extended his theory to anisotropic

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cases (Biot 1955) and also to poroviscoelasticity (Biot 1956a). The dynamic extension of Biot's theory was published in 1956 in two papers, one covering the low frequency range (Biot 1956b) and the other one covering the high frequency range (Biot 1956c). One of the significant findings in these papers was the identification of three different wave types for a 3-d continuum, namely two compressional waves and one shear wave. The additional compressional wave is also known as the slow wave and has been experimentally confirmed (Plona 1980). In Biot's original approach a fully saturated material was assumed. The extension to a nearly saturated (partially saturated) poroelastic solid was presented by Vardoulakis and Beskos (1986).

On the other hand, based on the work of Fillunger, a different approach, namely the Theory of Porous Media, has been developed. This theory is based on the axioms of continuum theories of mixtures (Truesdell and Toupin 1960; Bowen 1976) extended by the concept of volume fractions by Bowen (1980, 1982) and others (de Boer and Ehlers 1986; Ehlers 1989; Ehlers 1993b; Ehlers 1993a; Diebels 2000). Thus the TPM proceeds from the assumption of immiscible and superimposed continua with internal interactions.

Remarks on the equivalence of both theories are found in the work of Bowen (1982). In this paper, he showed that the BT is a special case of a linearized theory of mixtures with constant volume fractions. Bowen called this the case of 'frozen volume fraction'. To achieve equivalence between both approaches the parameter Q introduced by Biot has to be zero, which means that the interaction between both constituents is neglected. Furthermore, Ehlers and Kubik (1994) compared and discussed the linear versions of both theories claiming that they are equivalent if Biot's apparent mass density is assumed to be zero. This density is introduced into BT to describe the dynamic interaction of the constituents. As a consequence of the work by Bowen (1982) and by Ehlers and Kubik (1994), it may be stated that even if both approaches are similar the theories are mainly different in the way how the solid-fluid interaction is modeled.

In both papers, the authors used solid displacements, seepage velocity, and pore pressure as unknowns. In the following, a two-phase material consisting of an elastic solid skeleton and an interstitial viscous fluid is assumed. Furthermore, the assumption of full saturation is made, e.g., the whole pore space is filled with fluid. For such materials the governing equations are given based on the TPM (de Boer 2000; Ehlers 1989) in section 3 and based on BT (Biot 1941; Biot 1956b; Biot 1956c) in section 4. In the present contribution, contrary to the comparisons mentioned above, the governing equations are formulated using only solid displacements and pore pressure as unknowns which is only possible in the Laplace domain. Bonnet and Auriault (1987) have shown that this choice is sufficient to describe a poroelastic continuum.

The main focus of the paper is on wave propagation problems. Therefore, a linear description of the geometry in terms of small displacements and small deformation gradients is assumed. Furthermore, we restrict ourselves to linear constitutive equations. The combination of both assumptions leads to a set of linear differential equations which is transformed into the Laplace domain (concerning the linearization process see (Schanz and Diebels 2003)). The two sets of equations arising in BT and TPM, respectively, are compared not only term by term but also an analytical solution for the displacement and pressure results of a one-dimensional column are compared for both theories.

In a two-phase material not only each constituent, the solid and the fluid, may be compressible on a microscopic level but also the skeleton itself possesses a structural compressibility. If the compression modulus of one constituent is much larger on the microscale than the compression modulus of the bulk material this constituent is assumed to be materially incompressible. A common example for an materially incompressible solid constituent is soil. In this case, the individual grains are much stiffer than the skeleton itself. In the following, the governing

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equations are given for materially compressible and incompressible constituents, respectively, cf. (Bowen 1980; Bowen 1982; Ehlers 1993b; Ehlers 1993a; Diebels 2000). Beside these two extreme cases, there exists so-called hybrid models where only one of the constituents is modeled as incompressible and the other one as compressible (Ehlers 1993b; Ehlers 1993a; Diebels 2000). As these intermediate cases can be simply deduced from the equations given next they are not considered here in detail.

Throughout this paper, the summation convention is applied over repeated indices and Latin indices receive the values 1,2, and 1,2,3 in two-dimensions (2-d) and three-dimensions (3-d), respectively. Commas $()_{,i}$ denote spatial derivatives and primes $()'$ denote the material time derivative with respect to the moving skeleton.

THEORY OF POROUS MEDIA

In order to describe the two different phases of the material the concept of volume fractions is introduced (Bowen 1980; Bowen 1982; Ehlers 1989). Therefore, the given volume element V is divided in two fractions V^S and V^F occupied by the solid skeleton (index S) and the interstitial fluid (index F), respectively. If the whole space is filled with matter, the saturation condition requires $V = V^S + V^F$. The volume fraction of each constituent is defined by

$$n^k = \frac{V^k}{V} \qquad \text{with} \quad k = F, S. \tag{1}$$

The partial densities ρ^k of both constituents relate the mass element of the constituents to the volume element V of the mixture while the effective densities ρ^{kR} relate the same element of mass to the volume element occupied by the constituent. Therefore, the partial densities are obtained by the product of the volume fraction and the respective effective density

$$\rho^k = \rho^{kR} n^k \qquad \text{with} \quad k = F, S. \tag{2}$$

Changes of the partial density are therefore possible due to changes of the effective density and of the volume fraction, i.e., the material itself as well as its porous structure allow for a compressibility.

Compressible constituents

Within the general framework of compressible constituents, the effective densities are state variables and the volume fractions are internal variables (Bowen 1980; Bowen 1982; Ehlers 1989) which may be transformed to state variables under certain conditions. A consequent linearization, however, shows that in the linear case the volume fraction has to be constant, i.e., the initial solid volume fraction n_0^S does not change (Schanz and Diebels 2003). Subsequently, due to the saturation condition $1 = n^S + n^F$, both volume fractions are assumed to be constant

$$n^S \approx n_0^S \qquad \Rightarrow \quad n^F \approx n_0^F = 1 - n_0^S. \tag{3}$$

This corresponds to the case called 'frozen volume fractions' by Bowen (1982).

The balance equations of momentum of a two-phase continuum give the basis for the theoretical description within the TPM. They can either be given for both constituents separately or one of the individual balances may be replaced by the balance of momentum of the mixture as discussed in detail in (Diebels and Ehlers 1996). In the present contribution, the mixture balance of momentum is used in combination with the fluid momentum balance. The linear balances of momentum are

1. for the mixture

$$n_0^S \rho^{SR} u_i'' + n_0^F \rho^{FR} \left[u_i'' + w_i' \right] = T_{ij,j}^S + T_{ij,j}^F + \rho b_i \tag{4}$$

2. for the fluid

$$n_0^F \rho^{FR} \left[u_i'' + w_i' \right] = T_{ij,j}^F + p_i^* + n_0^F \rho^{FR} b_i^F \ , \tag{5}$$

where no distinction between the partial time derivative and the material time derivative have to be made (for details on the linearization see Schanz and Diebels (2003)). In Eqs. (4) and (5), u_i denotes the solid displacement and w_i denotes the seepage velocity defined as the relative velocity of the fluid with respect to the deforming solid skeleton. The stress tensor is given by T_{ij}^k with $k = S$ for the solid skeleton and $k = F$ for the fluid, respectively. The body force density in the fluid and in the solid is $n_0^F \rho^{FR} b_i^F$ and $n_0^S \rho^{SR} b_i^S$, respectively, and, additionally, the bulk body force ρb_i with $\rho = n_0^S \rho^{SR} + n_0^F \rho^{FR}$ is introduced. The force density p_i^* results from a momentum production representing the interaction between both constituents. Therefore, it is obviously not present in the equation for the mixture (4). Additionally, the balance of momentum of momentum is fulfilled if the stress tensors are symmetric.

Furthermore, constitutive assumptions must be specified which link the stress tensors and the momentum production term to kinematic quantities. Neglecting the fluid extra-stress (Diebels et al. 2001), the stress tensor of the fluid is governed by the pore pressure p

$$T_{ij}^F = -n^F p \delta_{ij} \quad \text{and accordingly} \quad T_{ij,j}^F = \left(-n^F p \delta_{ij} \right)_{,j} = -p n_{,i}^F - n^F p_{,i} \ , \tag{6}$$

where δ_{ij} denotes the Kronecker delta. Furthermore, the viscosity of the fluid is taken into account by the momentum production or by the interaction force between the solid and the fluid which is given by the linear relation (Ehlers 1989)

$$p_i^* = p n_{,i}^F - \frac{\left(n^F \right)^2}{\kappa^T} w_i \tag{7}$$

with the permeability κ^T, $()^T \hat{=} \text{TPM}$. This permeability depends on the intrinsic permeability k^S and on the fluid viscosity μ^F according to the relation $\kappa^T = k^S / \mu^F$ (see, e.g., Diebels et al. (2001)). Note, independent of the assumption of constant volume fractions the gradient $n_{,i}^F$ cancels itself in the equilibrium (5) due to the sum $T_{ij,j}^F + p_i^*$.

For the solid skeleton Hooke's law is taken into account assuming a linear elastic behavior. Hence, with the extra stress

$$\left(\sigma_E^S \right)_{ij} = G \left(u_{i,j} + u_{j,i} \right) + \left(K - \frac{2}{3} G \right) \delta_{ij} u_{k,k} \tag{8}$$

the stress tensor of the solid skeleton is given by (Diebels 2000)

$$
\begin{aligned}
T_{ij}^S &= -z^S n_0^S p \delta_{ij} + \left(\sigma_E^S \right)_{ij} \\
&= G \left(u_{i,j} + u_{j,i} \right) + \left(\left(K - \frac{2}{3} G \right) u_{k,k} - z^S n_0^S p \right) \delta_{ij} \ ,
\end{aligned}
\tag{9}
$$

if a linear strain-displacement relation $\varepsilon_{ij} = 1/2 \left(u_{i,j} + u_{j,i} \right)$ holds. The shear modulus G and the compression modulus K are introduced in the constitutive equations. These material constants refer to the bulk material and, therefore, the compression modulus includes also the compressibility of the skeleton structure. Furthermore, the state variable z^S was introduced in Diebels (2000) to separate effects related to material and structural compressibilities, respectively.

The balance of mass of the solid is formulated for the partial density $\rho^S = n^S \rho^{SR}$ and is split into two parts by the introduction of the arbitrary function $0 \leq z^S \leq 1$ (Diebels 2000)

$$\left(\rho^S\right)' + \rho^S u'_{i,i} = 0 \qquad \Rightarrow$$
$$\rho^{SR}\left(\left(n^S\right)' + z^S n^S u'_{i,i}\right) + n^S\left(\left(\rho^{SR}\right)' + \left(1 - z^S\right)\rho^{SR} u'_{i,i}\right) = 0 \,. \tag{10}$$

For arbitrary values of z^S, (10) is fulfilled if each part of the sum is equal to zero

$$\left(n^S\right)' = -z^S n^S u'_{i,i} \qquad \text{and} \qquad \left(\rho^{SR}\right)' = \left(1 - z^S\right)\rho^{SR} u'_{i,i} \,. \tag{11}$$

Based on a micro mechanical investigation, Diebels (2000) has proposed the dependence $z^S = 1 - K^S/K^{SR}$ relating z^S to the compression modulus of the structure K^S and the compression modulus of the solid grains K^{SR}. He showed that this choice is thermodynamically admissible and that in the case of an incompressible solid skeleton the limit $z^S = 1$ transforms (10) into the well known volume balance $n^S = n_0^S \det \mathbf{F}_S^{-1}$.

Finally, an equation of state for the fluid must be prescribed because in (6) for the fluid extra stress tensor no constitutive assumption was given. Within the framework of a linear theory, the simplest case of the ideal gas equation is applied

$$\rho^{FR}(p) = \frac{p}{R\vartheta} \,, \tag{12}$$

with the absolute temperature ϑ and the specific gas constant R. More complex laws to describe the volumetric behavior of the fluid could be included here, however, the linearization neglects additional effects. Combining the ideal gas equation with the linearized form of the continuity equation for the fluid with respect to the moving solid reference system yields

$$n_0^F \frac{p'}{R\vartheta} + \rho_0^{FR}\left(n_0^F + z^S n_0^S\right)u'_{i,i} + n_0^F \rho_0^{FR} w_{i,i} = 0 \,. \tag{13}$$

According to Eq. (3), also constant volume fractions are inserted in (13).

Gathering all above given linear balances the following set of coupled differential equations is obtained

$$\rho_0 u_i'' + n_0^F \rho_0^{FR} w_i' = G u_{i,jj} + \left(K + \frac{1}{3}G\right)u_{j,ji} - \left(n_0^F + z^S n_0^S\right)p_{,i} + \rho b_i \,, \tag{14a}$$

$$n_0^F \rho_0^{FR}\left[u_i'' + w_i'\right] = -n_0^F p_{,i} - \frac{\left(n_0^F\right)^2}{\kappa^T}w_i + n_0^F \rho^{FR} b_i^F \,, \tag{14b}$$

$$n_0^F \frac{p'}{R\vartheta} + \rho_0^{FR}\left(n_0^F + z^S n_0^S\right)u'_{i,i} + n_0^F \rho_0^{FR} w_{i,i} = 0 \,. \tag{14c}$$

The primary variables in (14) are the solid displacement u_i, the seepage velocity w_i, and the pore pressure p. Note that in Eqs. (14), due to the linearization, constant densities ρ_0^{SR}, ρ_0^{FR} and $\rho_0 = n_0^S \rho_0^{SR} + n_0^F \rho_0^{FR}$ are used with the exception of the body force terms (Boussinesq approximation) where a linear approximation of the density is inserted.

From a physical point of view it is sufficient to describe the problem with only two primary variables namely the solid displacement u_i and pore pressure p instead of three variables as discussed in Bonnet (1987) and Lewis and Schrefler (1998).

In the quasi-static case, i.e., $u_i'' \approx 0, w_i' \approx 0$, the balance of momentum of the fluid (14b) can be rearranged to express the seepage velocity in terms of the pore pressure gradient. In this

case, Darcy's law is obtained. Inserting this expression into Eqs. (14a) and (14c) eliminates the seepage velocity as primary variable from the set of the governing equations. Since in the dynamic case, w_i is given as time derivative in (14b), this procedure is only possible in Laplace domain. Under the assumption of vanishing initial conditions the transformed Eqs. (14) are

$$\rho_0 s^2 \hat{u}_i + n_0^F \rho_0^{FR} s \hat{w}_i = G \hat{u}_{i,jj} + \left(K + \frac{1}{3} G \right) \hat{u}_{j,ji} - \left(n_0^F + z^S n_0^S \right) \hat{p}_{,i} + \rho \hat{b}_i , \qquad (15a)$$

$$n_0^F \rho_0^{FR} \left[s^2 \hat{u}_i + s \hat{w}_i \right] = -n_0^F \hat{p}_{,i} - \frac{\left(n_0^F \right)^2}{\kappa^T} \hat{w}_i + n_0^F \rho^{FR} \hat{b}_i^F , \qquad (15b)$$

$$n_0^F \frac{s \hat{p}}{R \vartheta} + \rho_0^{FR} \left(n_0^F + z^S n_0^S \right) s \hat{u}_{i,i} + n_0^F \rho_0^{FR} \hat{w}_{i,i} = 0 \qquad (15c)$$

where $\hat{()}$ indicates the Laplace transform and s is the complex Laplace variable.

Rearranging the Laplace transformed balance of momentum for the fluid (15b) the seepage velocity is obtained

$$\hat{w}_i = -\frac{\beta^T}{s n_0^F \rho_0^{FR}} \left[\hat{p}_{,i} + s^2 \rho_0^{FR} \hat{u}_i - \rho^{FR} \hat{b}_i^F \right] \qquad \text{with} \quad \beta^T = \frac{s n_0^F \rho_0^{FR} \kappa^T}{n_0^F + s \kappa^T \rho_0^{FR}} . \qquad (16)$$

Eliminating the seepage velocity $\hat{w}_i$ from the remaining balances (15a) and (15c) by use of (16), finally, the balance of momentum for the mixture

$$s^2 \left(\rho_0 - \beta^T \rho_0^{FR} \right) \hat{u}_i - \beta^T \left[\hat{p}_{,i} - \rho^{FR} \hat{b}_i^F \right] = G \hat{u}_{i,jj} + \left(K + \frac{1}{3} G \right) \hat{u}_{j,ji} - \left(n_0^F + z^S n_0^S \right) \hat{p}_{,i} + \rho \hat{b}_i \quad (17)$$

and the mass balance of the fluid

$$n_0^F \frac{s \hat{p}}{R \vartheta} + \rho_0^{FR} \left(n_0^F + z^S n_0^S - \beta^T \right) s \hat{u}_{i,i} - \frac{\beta^T}{s} \hat{p}_{,ii} + \frac{\beta^T \rho^{FR}}{s} \hat{b}_{i,i}^F = 0 \qquad (18)$$

are achieved. These operations establish a system of coupled partial differential equations for the unknowns solid displacement $\hat{u}_i$ and pore pressure $\hat{p}$

$$G \hat{u}_{i,jj} + \left(K + \frac{1}{3} G \right) \hat{u}_{j,ji} - \left(n_0^F + z^S n_0^S - \beta^T \right) \hat{p}_{,i} - s^2 \left(\rho_0 - \beta^T \rho_0^{FR} \right) \hat{u}_i = \beta^T \rho^{FR} \hat{b}_i^F - \rho \hat{b}_i ,$$

$$(19)$$

$$\hat{p}_{,ii} - n_0^F \frac{s^2}{\beta^T R \vartheta} \hat{p} - \frac{s^2 \rho_0^{FR}}{\beta^T} \left(n_0^F + z^S n_0^S - \beta^T \right) \hat{u}_{i,i} = \rho^{FR} \hat{b}_{i,i}^F . \qquad (20)$$

An analytical representation of Eqs. (19) and (20) in time domain is only possible for a constant value β^T. This is only achieved in the limit $\kappa^T \to \infty$, i.e., $\mu^F \to 0$. Consequently, the interaction force p_i^* between the solid and the fluid is proportional to the pore pressure $p_i^* \approx p n_{,i}^F$ and the influence of the seepage velocity on the momentum exchange vanishes. Evidently, this is only valid under equilibrium conditions where no fluid motion takes place.

Incompressible constituents

Naturally, the linearized balances of momentum for the mixture (4) and for the fluid (5) are not changed due to the assumption of incompressible constituents. On the other hand, the continuity equation of the solid (10) reduces to a balance of volume. As stated above, the incompressible case is included in the general framework by the choice $z^S = 1$. The physical

interpretation is obviously a constant density ρ^{SR} resulting in the well-known balance of volume (Ehlers 1993b; Ehlers 1993a; Diebels 2000)

$$\left(n^S\right)' + n^S u_{i,i} = 0 \ . \tag{21}$$

Assuming both constituents as materially incompressible and inserting the assumptions $\rho^{FR} =$ const. and $z^S = 1$ into the continuity equation of the fluid (13) yields

$$\rho^{FR}\left(n_0^F w_i + u_i'\right)_{,i} = 0 \ . \tag{22}$$

The constitutive equations for the incompressible solid and incompressible fluid can also easily be achieved. The stress tensor of the fluid (6) and the interaction force (7) are not changed yielding the well known principle of effective stress, but note that the pore pressure becomes a Lagrangian multiplier in this case which ensures the assumption of constant density. There is no longer an equation of state linking the density to the pressure. Finally, the divergence of the total stress is obtained by these assumptions in combination with the saturation condition $n^S + n^F = 1$

$$T_{ij,j}^S + T_{ij,j}^F = G u_{i,jj} + \left(K + \frac{1}{3}G\right) u_{j,ji} - p_{,i} \ . \tag{23}$$

As in the compressible case, the incompressible model results in three equations for the three variables solid displacement u_i, pore pressure p, and the seepage velocity w_i

$$\rho_0 u_i'' + n_0^F \rho_0^{FR} w_i' = G u_{i,jj} + \left(K + \frac{1}{3}G\right) u_{j,ji} - p_{,i} + \rho b_i \ , \tag{24a}$$

$$n_0^F \rho_0^{FR}\left[u_i'' + w_i'\right] = -n_0^F p_{,i} - \frac{\left(n^F\right)^2}{\kappa^T} w_i + n_0^F \rho^{FR} b_i^F \ , \tag{24b}$$

$$\left(n_0^F w_i + u_i'\right)_{,i} = 0 \ . \tag{24c}$$

Because the balance of momentum of the fluid Eq. (24b) is equal to Eq. (15b) of the compressible case, an extraction of the seepage velocity is only possible in Laplace domain. The transformation of Eq. (24b) leads to the same expression as given in (16). Eliminating the seepage velocity from the balance of momentum (24a) and from the balance of volume (24c) results in the set of coupled differential equations for the unknowns solid displacement $\hat{u}_i$ and pore pressure $\hat{p}$

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ji} - \left(1 - \beta^T\right)\hat{p}_{,i} - s^2\left(\rho_0 - \beta^T \rho_0^{FR}\right)\hat{u}_i = \beta^T \rho_0^{FR}\hat{b}_i^F - \rho\hat{b}_i \ , \tag{25}$$

$$\hat{p}_{,ii} - \frac{s^2 \rho_0^{FR}}{\beta^T}\left(1 - \beta^T\right)\hat{u}_{i,i} = \rho^{FR}\hat{b}_{i,i}^F \ . \tag{26}$$

As in the compressible case, an analytical representation in time domain is only possible for $\kappa^T \to \infty$.

BIOT'S THEORY

In this section, Biot's model of a poroelastic continuum is presented. Using different notation for all variables which are not definitely the same as in the TPM approach allows for an *a posteriori* comparison between both approaches including an identification of the individual terms and of their physical meaning.

Compressible constituents

Following Biot's approach to model the behavior of porous media, an elastic skeleton with a statistical distribution of interconnected pores is considered (Biot 1955). The porosity is denoted by

$$\phi = \frac{V^F}{V} , \tag{27}$$

where V^F is the volume of the interconnected pore space contained in a sample of bulk volume V. Contrary to these connected pores the sealed pores are considered as part of the solid. Therefore, $\phi = n^F$ is only valid if all pores are interconnected. As mentioned above, full saturation is assumed leading to $V = V^F + V^S$ with V^S the volume of the solid including sealed pores.

If the constitutive equations are formulated with the total stress $\sigma_{ij} = \sigma_{ij}^S + \sigma^F \delta_{ij}$ the constitutive equation (Biot 1941)

$$\sigma_{ij} = G(u_{i,j} + u_{j,i}) + \left(\left(K - \frac{2}{3}G \right) u_{k,k} - \alpha p \right) \delta_{ij} \tag{28}$$

is given using Biot's effective stress coefficient α and the solid strain is replaced by the common linear strain-displacement relation $\varepsilon_{ij}^S = 1/2 (u_{i,j} + u_{j,i})$. In addition to the total stress σ_{ij}, the variation of fluid volume ζ per unit reference volume is introduced as a second constitutive equation

$$\zeta = \alpha u_{i,i} + \frac{\phi^2}{R^B} p . \tag{29}$$

In Eq. (29), R^B is an additional material parameter characterizing the coupling between the solid and the fluid. The variation of fluid content ζ is governed by the mass balance, i.e., by the continuity equation

$$\frac{\partial \zeta}{\partial t} + q_{i,i} = 0 \tag{30}$$

with the specific flux q_i. This flux is identified with the filter velocity $q_i = \phi w_i$. A time integrated form of (30) identifies ζ as a kind of volumetric strain describing the motion of the fluid relative to the solid as discussed in Detournay and Cheng (1993).

Additional to (30), the balance of momentum for the bulk material must be fulfilled. The dynamic equilibrium is given by

$$\sigma_{ij,j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} + \phi \rho_F \frac{\partial w_i}{\partial t} , \tag{31}$$

with the bulk body force per unit volume $f_i = (1 - \phi) f_i^S + f_i^F$, and the bulk density $\rho = \rho_S (1 - \phi) + \phi \rho_F$. For these densities the subscript is used instead of the superscript in order to distinct them from the quantities arising in TPM.

Furthermore, the fluid transport in the interstitial space in terms of the specific flux q_i is modeled with a generalized Darcy's law

$$\phi w_i = q_i = -\kappa^B \left(p_{,i} + \rho_F \frac{\partial^2 u_i}{\partial t^2} + \frac{\rho_a + \phi \rho_F}{\phi} \frac{\partial w_i}{\partial t} - f_i^F \right) \tag{32}$$

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which is given constitutively. Here, $\kappa^B = k^S/\mu^F$ denotes the permeability defined by the intrinsic permeability k^S and the viscosity of the fluid μ^F. The superscript B is chosen to distinct this permeability from the permeability in the TPM. Furthermore, f_i^F is the fluid body force per unit volume. In Eq. (32), an additional density, the so-called apparent mass density ρ_a was introduced by Biot (1956b). The apparent mass describes the dynamic interaction between fluid phase and solid skeleton. Typically it is written as $\rho_a = C\phi\rho_F$ where C is a factor depending on the geometry of the pores and the frequency of excitation. At low frequencies, Bonnet and Auriault (1985) measured $C = 0.66$ for a sphere assembly of glass beads. In higher frequency ranges, a certain functional dependence of C on the frequency has been proposed based on conceptual porosity structures (Biot 1956c; Bonnet and Auriault 1985). In the following, $C = 0.66$ is assumed.

The equation of motion for the poroelastic model is obtained from the above balance laws and constitutive equations. As shown in Bonnet (1987), it is sufficient to use the solid displacements u_i and the pore pressure p as basic variables to describe a poroelastic continuum. Therefore, the above equations are reduced to these two primary variables as was already done for the TPM model. First, Darcy's law (32) is transformed into Laplace domain and rearranged to obtain

$$\hat{w}_i = -\underbrace{\frac{\kappa^B\rho_F\phi^2 s}{\phi^2 + s\kappa^B\left(\rho_a + \phi\rho_F\right)}}_{\beta^B} \frac{1}{s\phi\rho_F}\left(\hat{p}_{,i} + s^2\rho_F\hat{u}_i - \hat{f}_i^F\right) .\tag{33}$$

In Eq. (33), the abbreviation β^B is defined for further usage and the superscript B is chosen to distinct from the similar abbreviation in TPM. Moreover, as in TPM vanishing initial conditions for u_i, p, and w_i are assumed. Now, the final set of differential equations for the displacement $\hat{u}_i$ and the pore pressure $\hat{p}$ is obtained by inserting the constitutive Eqs. (28) and (29) in the Laplace transformed dynamic equilibrium (31) and into the continuity equation (30). Taking into account $\hat{w}_i$ according to (33) leads to

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - \left(\alpha - \beta^B\right)\hat{p}_{,i} - s^2\left(\rho - \beta^B\rho_F\right)\hat{u}_i = \beta^B\hat{f}_i^F - \hat{f}_i ,\tag{34}$$

$$\hat{p}_{,ii} - \frac{\phi^2 s^2\rho_F}{\beta^B R^B}\hat{p} - \left(\alpha - \beta^B\right)\frac{s^2\rho_F}{\beta^B}\hat{u}_{i,i} = \hat{f}_{i,i}^F .\tag{35}$$

This set of coupled differential equations describes the behavior of a poroelastic continuum on the basis of BT. As in TPM, an analytical representation in time domain is only possible for $\kappa^B \to \infty$. This case represents negligible friction between solid and interstitial fluid.

Incompressible constituents

To find the respective governing equations for incompressible constituents the material parameters α and R^B have to be rewritten in a different way. Considerations of constitutive relations at micro mechanical level as given in Detournay and Cheng (1993) lead to a more rational model for this purpose

$$\alpha = 1 - \frac{K}{K_S} \qquad \text{and}\tag{36a}$$

$$R^B = \frac{\phi^2 K_F K_S^2}{K_F\left(K_S - K\right) + \phi K_S\left(K_S - K_F\right)} ,\tag{36b}$$

where K_S denotes the compression modulus of the solid grains and K_F the compression modulus of the fluid. Based on these expressions materially incompressible behavior of the constituents may be described by the above given constitutive assumptions. Note that material incompressibility means that the compression modulus of each individual constituent is much larger than that one of the bulk material. The respective conditions are (Detournay and Cheng 1993)

$$\frac{K}{K_S} \ll 1 \quad \text{incompressible solid,} \qquad \frac{K}{K_F} \ll 1 \quad \text{incompressible fluid .} \tag{37}$$

The corresponding limit process shows (Detournay and Cheng 1993)

$$\alpha \approx 1 \qquad \text{and} \quad R^B \to \infty . \tag{38}$$

According to (29) $\zeta \approx u_{i,i}$, i.e., the fluid is influenced only by the solid volumetric strain. With these considerations at hand the set of governing differential equations reduces to

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G \right) \hat{u}_{j,ij} - \left(1 - \beta^B \right) \hat{p}_{,i} - s^2 \left(\rho - \beta^B \rho_F \right) \hat{u}_i = \beta^B \hat{f}_i^F - \hat{f}_i , \tag{39}$$

$$\hat{p}_{,ii} - \left(1 - \beta^B \right) \frac{s^2 \rho_F}{\beta^B} \hat{u}_{i,i} = \hat{f}_{i,i}^F . \tag{40}$$

COMPARISON OF BOTH THEORIES

Both theories, namely BT and TPM, describe the same physical behavior of fluid saturated porous media. In the sections 3 and 4, the governing equations for each theory are given for compressible as well as incompressible constituents. Now, a comparison of these equations is performed in order to identify the the physical interpretation of the parameters of both approaches and to show whether there are discrepancies between the theories even if the underlying structure of the governing equations is the same.

Preliminary for this comparison is the (evident) assumption that the independent variables solid displacement u_i and pore pressure p have the same physical meaning in both approaches. Furthermore, from the constitutive equations Eqs. (8) and (28) it can be concluded that the shear modulus G and the compression modulus K are macroscopic moduli valid for the porous skeleton, i.e., solid material including its structure. In order to compare the other parameters arising in the governing equations the model equations are recalled, first, for incompressible constituents:

- Displacement, TPM (25) and BT (39)

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G \right) \hat{u}_{j,ji} - \left(1 - \beta^T \right) \hat{p}_{,i} - s^2 \left(\rho_0 - \beta^T \rho_0^{FR} \right) \hat{u}_i = \beta^T \rho_0^{FR} \hat{b}_i^F - \rho \hat{b}_i$$

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G \right) \hat{u}_{j,ij} - \left(1 - \beta^B \right) \hat{p}_{,i} - s^2 \left(\rho - \beta^B \rho_F \right) \hat{u}_i \quad = \beta^B \hat{f}_i^F \qquad - \hat{f}_i$$

- Pore pressure, TPM (26) and BT (40)

$$\hat{p}_{,ii} - \frac{s^2 \rho_0^{FR}}{\beta^T} \left(1 - \beta^T \right) \hat{u}_{i,i} = \rho^{FR} \hat{b}_{i,i}^F$$

$$\hat{p}_{,ii} - \frac{s^2 \rho_F}{\beta^B} \left(1 - \beta^B \right) \hat{u}_{i,i} \quad = \hat{f}_{i,i}^F$$

TABLE 1. Material data of Berea sandstone (rock)

	K $\left(\frac{N}{m^2}\right)$	G $\left(\frac{N}{m^2}\right)$	$\rho\left(\frac{kg}{m^3}\right)$	ϕ	$K_S\left(\frac{N}{m^2}\right)$	$\rho_F\left(\frac{kg}{m^3}\right)$	$K_F\left(\frac{N}{m^2}\right)$	$\kappa\left(\frac{m^4}{Ns}\right)$
rock	$8\cdot10^9$	$6\cdot10^9$	2458	0.19	$3.6\cdot10^{10}$	1000	$3.3\cdot10^9$	$1.9\cdot10^{-10}$

Comparing the densities it is found that

$$\rho_0^{FR} \overset{!}{=} \rho_F \quad \text{and} \quad \rho_0 = n_0^S \rho_0^{SR} + n_0^F \rho_0^{FR} \overset{!}{=} \rho = \rho_S(1-\phi) + \phi\rho_F , \tag{41}$$

i.e., in BT the densities correspond to the effective densities introduced in TPM. Furthermore, the porosity can be identified with the initial fluid volume fraction $n_0^F = \phi$ which is also indicated by their initial definitions (1) and (27). However, as mentioned in section 4, this conclusion requires that all pores are interconnected.

With theses identifications in mind the body forces can be compared. While in TPM the body force density b_i^k is defined as force per mass, in BT f_i^k is defined as force per volume ($k = S, F$). According to the definition of the densities the identification $f_i^k = \rho^{kR} b_i^k$ is obtained. Clearly, this difference is due to the different underlying definitions of the body forces.

Finally, the governing equations of both theories become identical if $\beta^B = \beta^T$ holds. Comparing the definitions (33) and (16) the following identity has to hold

$$\beta^B = \frac{\kappa^B \rho_F \phi^2 s}{\phi^2 + s\kappa^B(\rho_a + \phi\rho_F)} \overset{!}{=} \beta^T = \frac{sn_0^F \rho_0^{FR} \kappa^T}{n_0^F + s\kappa^T \rho_0^{FR}} . \tag{42}$$

Before evaluating (42) the different expressions for the permeabilities have to be discussed. In BT, the permeability is defined by the quotient of the intrinsic permeability k^S and the viscosity μ^F of the fluid, i.e., $\kappa^B = k^S/\mu^F$. Note that the intrinsic permeability k^S describes only the pore structure. As discussed in section 3, the permeability κ^T in TPM has the same physical meaning as κ^B. Taking this equality into account from Eq. (42) it follows that

$$n_0^F \rho_0^{FR} \overset{!}{=} \rho_a + \phi\rho_F \qquad \Rightarrow \qquad \rho_a \equiv 0 . \tag{43}$$

However, this can only be achieved if the apparent mass density ρ_a vanishes, which is in accordance with Ehlers and Kubik (1994). Therefore, for the incompressible case it can be concluded that the linearized governing equations of both theories are identical if condition (43) holds.

This equivalence of both approaches is also verified using the solution for a one dimensional column of length ℓ as sketched in Fig. 1. The side walls and the bottom are assumed to be rigid, frictionless, and impermeable. Hence, the displacements normal to the surface are blocked and, on the other hand, the column is free to slide parallel to the wall. At the top, the stress $\sigma_y(y = \ell)$ is prescribed while the pressure vanishes, i.e., $p(y = \ell) = 0\,\text{N/m}^2$. Therefore, the surface is ideally drained. At the bottom the column is fixed and impermeable, i.e., $u_y(y = 0) = 0\,\text{m}$ and $q_y(y = 0) = 0\,\text{m/s}$. Due to these restrictions only the displacement component u_y in vertical direction and the pore pressure p remain as degrees of freedom. Semi-analytical solutions are found for the compressible case in Schanz and Cheng (2000) and for the incompressible case in Schanz and Diebels (2003).

In Fig. 2, the displacement at the top of the 1-d column (see, Fig. 1) is plotted versus time for both theories. The material data are those of a rock as given in Table 1. The agreement of both results is perfect as expected due to the identification of the parameters.
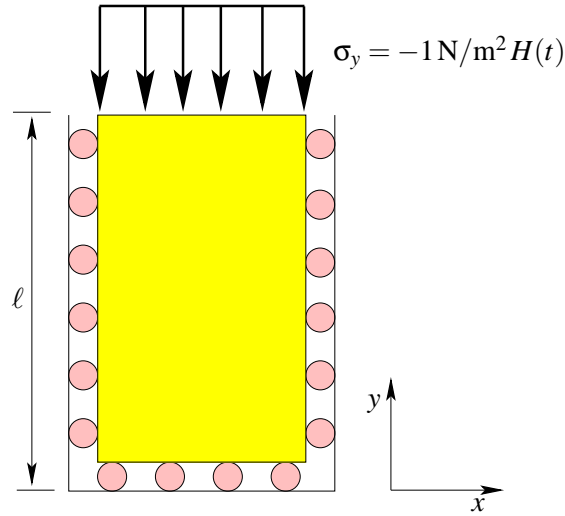
FIG. 1. Geometry and dynamic loading of a one-dimensional column

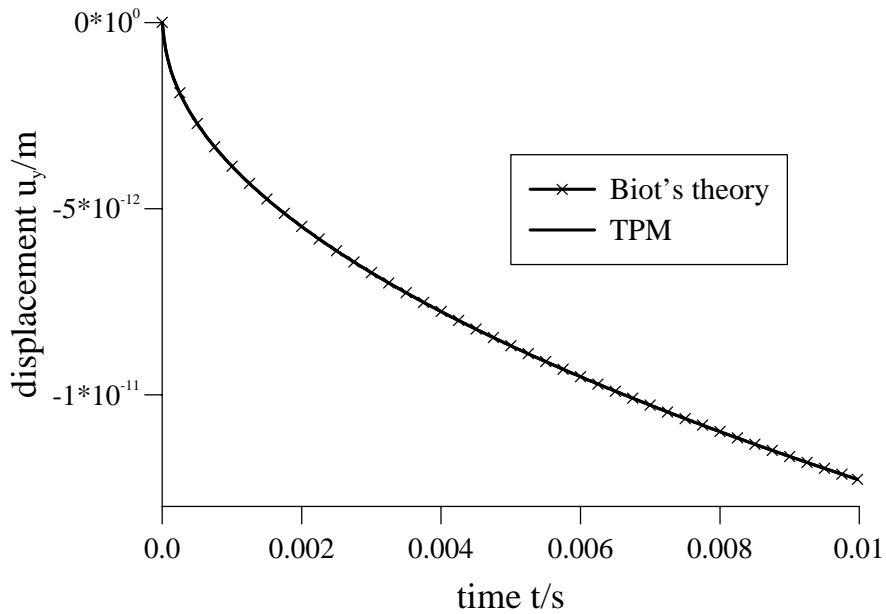$$\sigma_y = -1\,\mathrm{N/m}^2\,H(t)$$

FIG. 2. Displacement at the top of the column

As stated above, the perfect agreement of the results shown in Fig. 2 can only be achieved for vanishing apparent mass density. So, the question is which influence has the apparent mass density in BT and what differences appear if condition (43) is not fulfilled? To answer this, numerical tests have shown that the apparent mass density has no influence on the results for the given set of material data according to Table 1. However, if the permeability $\kappa = \kappa^B = \kappa^T$ is increased or if the viscosity of the fluid is decreased, differences appear depending on the apparent mass. As already reported by Schanz and Cheng (2000), the second slow compressional wave becomes visible for increased permeabilities.

To study this effect, the pressure 5 m behind excitation of a long column ($\ell = 1000\,$m) is
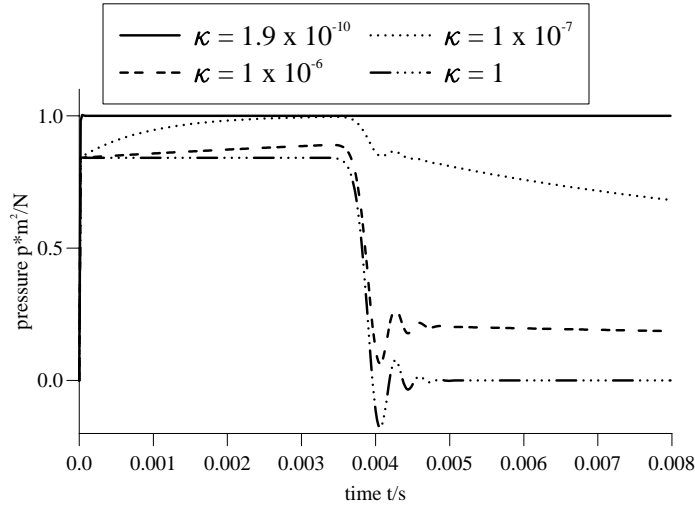
(a) $\rho_a = 0.66\phi\rho_F$
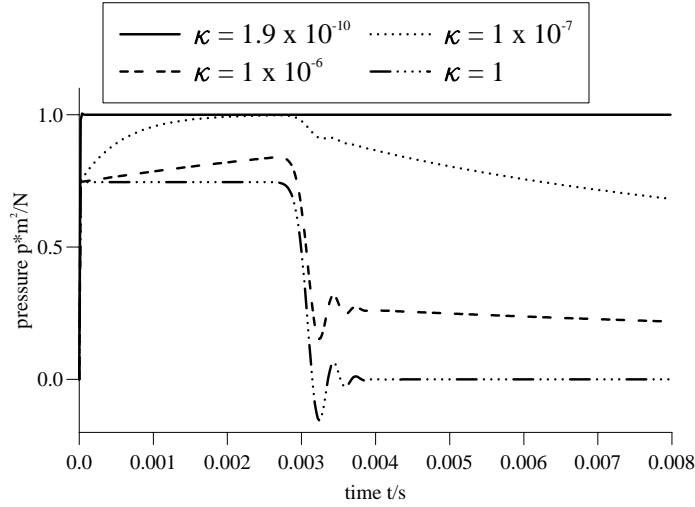
(b) $\rho_a \equiv 0$

FIG. 3. Pressure $5\,\mathrm{m}$ behind excitation of an infinite column: Different apparent mass densities

depicted versus time for different values of κ in Fig. 3. The smallest value of the permeability $\kappa = 1.9 \cdot 10^{-10}\,\mathrm{m}^4/(\mathrm{Ns})$ represents the realistic case. In this case, both graphs show no difference. Furthermore, two effects are observed. Firstly, an initial jump indicates that the fast compressional wave travels with infinite wave speed. This is due to the incompressible model. Secondly, by increasing κ a second jump becomes visible corresponding to the highly damped second compressional wave (Schanz and Cheng 2000). This slow compressional wave is of negative amplitude because it represents the out-of-phase movement of fluid and solid. The arrival time and, therefore, the wave speed of this wave depends on the chosen value of the

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apparent mass density. Additionally, the pressure level in case of $\rho_a \equiv 0$ is smaller than that of $\rho_a = 0.66\phi\rho_F$. These results show that the interaction between solid and fluid is influenced by the apparent mass density according to the structure of the BT, where ρ_a was introduced as 'mass coupling parameter' (Biot 1956b). Therefore, it is concluded that TPM and BT model the interaction between solid and fluid in a different way, however, for the test data this has no significant influence.

In the next step, the comparison of the compressible models is performed. For this purpose, the governing equations are recalled

- for the displacements, TPM (19) and Biot (34)

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ji} - \left(n_0^F + z^S n_0^S - \beta^T\right)\hat{p}_{,i} - s^2\left(\rho_0 - \beta^T \rho_0^{FR}\right)\hat{u}_i = \beta^T \rho^{FR}\hat{b}_i^F - \rho\hat{b}_i$$

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - \left(\alpha - \beta^B\right)\hat{p}_{,i} \qquad - s^2\left(\rho - \beta^B \rho_F\right)\hat{u}_i \quad = \beta^B \hat{f}_i^F \qquad - \hat{f}_i$$

- for the pore pressure, TPM (20) and Biot (35)

$$\hat{p}_{,ii} - n_0^F \frac{s^2}{\beta^T R\vartheta}\hat{p} - \frac{s^2\rho_0^{FR}}{\beta^T}\left(n_0^F + z^S n_0^S - \beta^T\right)\hat{u}_{i,i} = \rho^{FR}\hat{b}_{i,i}^F$$

$$\hat{p}_{,ii} - \frac{\phi^2 s^2 \rho_F}{\beta^B R^B}\hat{p} \quad - \frac{s^2\rho_F}{\beta^B}\left(\alpha - \beta^B\right)\hat{u}_{i,i} \qquad = \hat{f}_{i,i}^F$$

Obviously, if the equivalences found in case of incompressible constituents are taken into account only two additional parameters have to be identified. Firstly, the comparison of the pressure term yields the not expected result

$$\frac{\rho_F \phi}{R^B} = \frac{1}{R\vartheta} \ . \tag{44}$$

On the left hand side of Eq. (44), the material parameter R^B (36b) depending on the fluid as well as on the solid properties is compared with the gas constant R and the absolute temperature ϑ on the right hand side characterizing only the fluid. The same appears in identifying the last parameter α. If both expressions are the same in both theories it must hold that

$$\alpha = 1 - \frac{K}{K_S} \overset{!}{=} n_0^F + z^S n_0^S \ . \tag{45}$$

Inserting the identification given by Diebels (2000) into (45)

$$z^S = 1 - \frac{K^S}{K^{SR}} \tag{46}$$

leads to a contradiction. It should be kept in mind that K and K_S have the same physical interpretation in Biot's theory as K^S and K^{SR} have in TPM, respectively. Therefore, the constitutive relation derived on micro mechanical models for both theories are not in agreement to each other.

This contradiction becomes obvious in Fig. 4 where the displacement at the top of the column is given versus time. While in the incompressible case no difference between the displacement solutions of both approaches are visible, here, large differences appear. To compute

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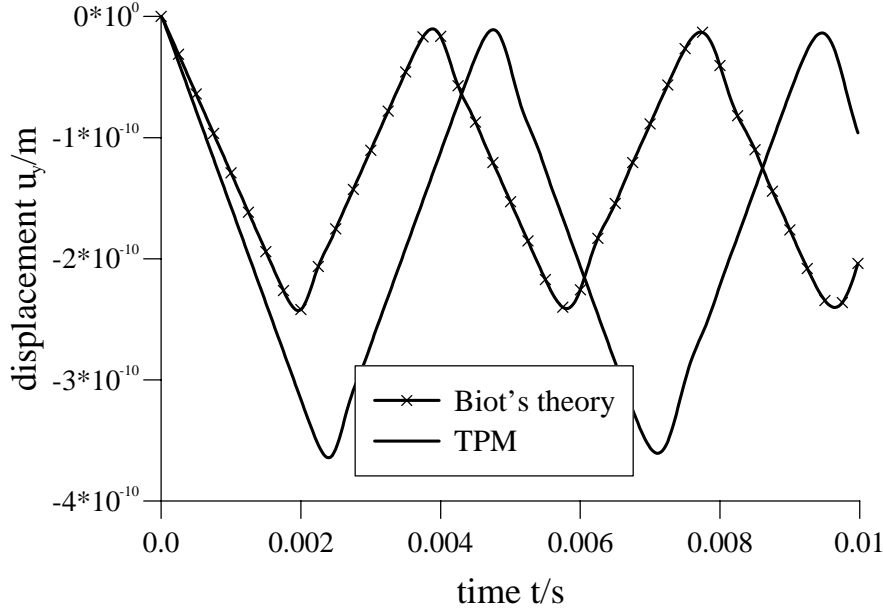
**FIG. 4.   Displacement at the top of the column: Compressible constituents**

these results the specific gas constant of water $R = 461,61\,\mathrm{Nm}/(\mathrm{kgK})$ at the absolute temperature $\vartheta = 293\,\mathrm{K}$ is used.

Finally, to close the comparison the influence of the apparent mass density in case of compressible constituents is studied. As before in the incompressible case, the pressure in a long column is presented for both theories. The pressure result $5\,\mathrm{m}$ behind excitation is depicted versus time for different values of $\kappa = \kappa^B = \kappa^T$ in Fig. 5. Additionally to the observations in the incompressible case, in the compressible case the fast compressional wave is observed as a first jump. This appears in BT at $t \approx 0.0017\,\mathrm{s}$ and in TPM at $t \approx 0.0022\,\mathrm{s}$. The arrival time of the slow compressional wave in BT is, as in the incompressible case, different whichever the apparent mass density is zero or not, i.e., at $t \approx 0.004\,\mathrm{s}$ or at $t \approx 0.005\,\mathrm{s}$. The arrival time of the slow compressional wave in TPM is $t \approx 0.013\,\mathrm{s}$ and does not coincide with one of Biot's models. Furthermore, the pressure level in TPM is much smaller than in BT. This is in accordance with the different displacement levels as shown in Fig. 4.

CONCLUSIONS

In the present article, Biot's theory for both compressible and incompressible constituents is recalled. Additionally, under the assumption of a linear theory, the dynamic equations for the mixture theory based Theory of Porous Media (TPM) are presented. Both theories model a two-phase continuum consisting of a porous solid skeleton saturated with an interstitial pore fluid. A comparison of the governing equations as well as wave propagation results for an one-dimensional poroelastic column are presented.

Summarizing the results of the comparison, the structure of the governing differential equations in BT and in TPM is the same. So, the wave forms predicted by both theories are equal. As a side effect, it was shown that Darcy's law results naturally from the balance of momentum in the fluid. Due to this, also the generalized version of Darcy's law, which takes the inertia

15
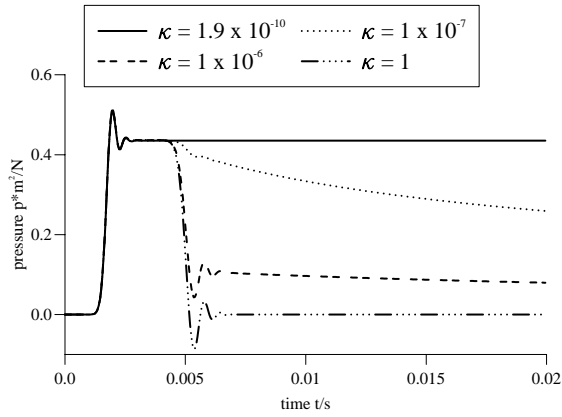
(a) Biot's theory $\rho_a = 0.66\phi\rho_F$
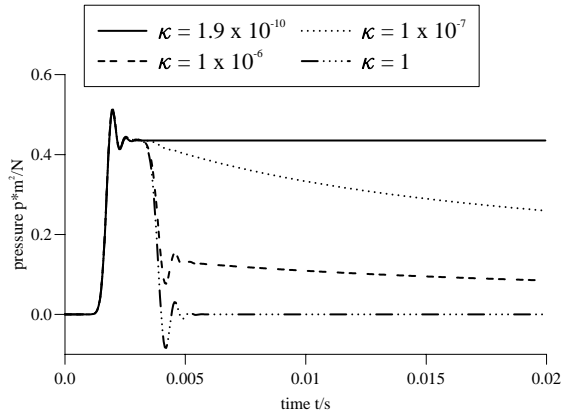
(b) Biot's theory $\rho_a \equiv 0$
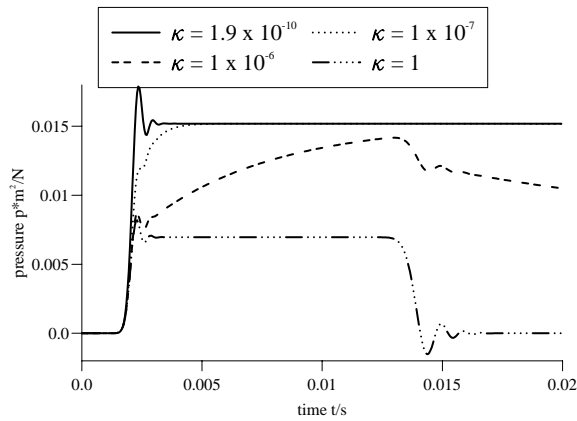
(c) TPM

FIG. 5.   Pressure $5\,\mathrm{m}$ behind excitation of an infinite column: Compressible model

terms into account, is a consequence of the fluid balance of momentum.

In case of incompressible constituents, the model parameters are identified in a way that the governing equations are the same in both theories if the apparent mass density is set to zero. The equivalence between both approaches is also verified numerically. On the other hand, in case of compressible constituents neither the identification procedure nor the numerical results match. This is related to the definition of Biot's stress coefficients α and R^B and the identification of the state variable z^S in TPM, respectively. These parameters are motivated by micro mechanical considerations in both theoretical approaches. The micro mechanical motivation for the parameters is not mandatory and should be critically surveyed.

REFERENCES

Biot, M. (1941). "General theory of three-dimensional consolidation." *Journal of Applied Physics*, 12, 155–164.

Biot, M. (1955). "Theory of elasticity and consolidation for a porous anisotropic solid." *Journal of Applied Physics*, 26, 182–185.

Biot, M. (1956a). "Theory of deformation of a porous viscoelastic anisotropic solid." *Journal of Applied Physics*, 27(5), 459–467.

Biot, M. (1956b). "Theory of propagation of elastic waves in a fluid-saturated porous solid.I. low-frequency range." *Journal of the Acoustical Society of America*, 28(2), 168–178.

Biot, M. (1956c). "Theory of propagation of elastic waves in a fluid-saturated porous solid.II. higher frequency range." *Journal of the Acoustical Society of America*, 28(2), 179–191.

Bonnet, G. (1987). "Basic singular solutions for a poroelastic medium in the dynamic range." *Journal of the Acoustical Society of America*, 82(5), 1758–1762.

Bonnet, G. and Auriault, J.-L. (1985). "Dynamics of saturated and deformable porous media: Homogenization theory and determination of the solid-liquid coupling coefficients." *Physics of Finely Divided Matter*, N. Boccara and M. Daoud, eds., Springer Verlag, Berlin, 306–316.

Bowen, R. (1976). "Theory of mixtures." *Continuum Physics*, A. Eringen, ed., Vol. III, Acadenic Press, New York, 1–127.

Bowen, R. (1980). "Incompressible porous media models by use of the theory of mixtures." *International Journal of Engineering Science*, 18, 1129–1148.

Bowen, R. (1982). "Compressible porous media models by use of the theory of mixtures." *International Journal of Engineering Science*, 20(6), 697–735.

de Boer, R. (2000). *Theory of Porous Media*. Springer-Verlag, Berlin.

de Boer, R. and Ehlers, W. (1986). "Theorie der Mehrkomponentenkontinua mit Anwendungen auf bodenmechanische Probleme, Teil I." *Forschungsbericht aus dem Fachbereich Bauwesen 40*, Universität - GH Essen.

de Boer, R. and Ehlers, W. (1988). "A historical review of the formulation of porous media theories." *Acta Mechanica*, 74, 1–8.

de Boer, R. and Ehlers, W. (1990). "The development of the concept of effective stresses." *Acta Mechanica*, 83, 77–92.

Detournay, E. and Cheng, A. H.-D. (1993). *Fundamentals of Poroelasticity*, Vol. II of *Comprehensive Rock Engineering: Principles, Practice & Projects*, chapter 5, 113–171. Pergamon Press.

Diebels, S. (2000). "Mikropolare zweiphasenmodelle: Formulierung auf der basis der theorie poröser medien." *Bericht Nr. II-4*, Universität Stuttgart, Institut für Mechanik, Lehrstuhl II.

Diebels, S. and Ehlers, W. (1996). "Dynamic analysis of a fully saturated porous medium ac-

counting for geometrical and material non-linearities." *International Journal for Numerical Methods in Engineering*, 39, 81–97.

Diebels, S., Ehlers, W., and Markert, B. (2001). "Neglect of the fluid-extra stresses in volumetrically coupled solid-fluid problems." *Zeitschrift für angewandte Mathematik und Mechanik*, 81, S521–S522.

Ehlers, W. (1989). "Poröse Medien – ein kontinuumsmechanisches Modell auf der Basis der Mischungstheorie." *Forschungsbericht aus dem Fachbereich Bauwesen 47*, Universität - GH Essen.

Ehlers, W. (1993a). "Compressible, incompressible and hybrid two-phase models in porous media theories." *ASME: AMD-Vol.*, 158, 25–38.

Ehlers, W. (1993b). "Constitutive equations for granular materials in geomechanical context." *Continuum Mechanics in Environmental Sciences and Geophysics*, K. Hutter, ed., CISM Courses and Lecture Notes, No. 337, Springer-Verlag, Wien, 313–402.

Ehlers, W. and Kubik, J. (1994). "On finite dynamic equations for fluid-saturated porous media." *Acta Mechanica*, 105, 101–117.

Fillunger, P. (1913). "Der Auftrieb von Talsperren, teil I-III." *Österr. Wochenschrift für den öffentlichen Baudienst*, 532–570.

Lewis, R. and Schrefler, B. (1998). *The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media*. John Wiley and Sons, Chichester.

Plona, T. (1980). "Observation of a second bulk compressional wave in porous medium at ultrasonic frequencies." *Applied Physics Letters*, 36(4), 259–261.

Schanz, M. and Cheng, A.-D. (2000). "Transient wave propagation in a one-dimensional poroelastic column." *Acta Mechanica*, 145, 1–18.

Schanz, M. and Diebels, S. (2003). "A comparative study of Biot's theory and the linear Theory of Porous Media for wave propagation problems." *Acta Mechanica*, (in Press).

Truesdell, C. and Toupin, R. (1960). "The classical field theories." *Handbuch der Physik*, S. Flügge, ed., Vol. III/1, Springer-Verlag, Berlin, 226–793.

Vardoulakis, I. and Beskos, D. (1986). "Dynamic behavior of nearly saturated porous media." *Mechanics of Composite Materials*, 5, 87–108.

von Terzaghi, K. (1923). "Die Berechnung der Durchlässigkeit des Tones aus dem Verlauf der hydromechanischen Spannungserscheinungen." *Sitzungsbericht der Akademie der Wissenschaften (Wien): Mathematisch–Naturwissenschaftlichen Klasse*, 132, 125–138.