An alternative approach in deriving poroelastic plate theories

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Abstract

In the building industry often porous materials are used for sound insulation. Those structures have mostly a plate-like geometry. Thus, a plate theory which takes the porous character of the material into account seems to be promising. Known elastodynamic plate theories based on kinematic and kinetic assumptions can not be easily extended to poroelasticity because it is questionable how these assumptions can be transferred to the pore pressure. By using a series expansion in thickness direction for all unknowns, a priori assumptions are not needed. This results, finally, in a poroelastic Kirchhoff plate with a consistent treatment of the pore pressure.

Keywords: Poroelastic plate theory, poroelasticity, Biot, plate vibration

1. Introduction

Classical plate theories rely on engineering assumptions regarding the distribution of stress and strain quantities over the thickness. Up to now, poroelastic plate formulations where mainly developed by adopting those assumptions to the poroelastic case [1]. However, it is questionable how to treat the pore pressure. Fortunately, a series expansion of the primal variables with respect to the thickness also results in plate theories which are similar or even equal to the classical ones [2]. This concept can be extended to derive poroelastic plate theories of arbitrary order. Here, the first order theory is presented.

2. Biot's theory of poroelasticity

In this work, Biot's theory of poroelasticity is used [3]. An elastic skeleton with a statistical distribution of interconnected pores and full saturation is considered. The porosity $\phi = \frac{V^f}{V}$ is defined as the ratio of the fluid volume to the bulk volume. All equations are given in the frequency domain such that only the solid displacement field $\mathbf{u}$ and the pore pressure p are needed as independent variables [4]. The total stress tensor $\boldsymbol{\sigma}$ contains all stress components combining those of the solid and those of the fluid. Beside the pore pressure, two more quantities appear compared to the linear elasticity, namely a kind of fluid strain quantity ζ and the flux $\mathbf{q}$. The fluid transport in the interstitial space is modelled by a generalised Darcy's law.

Starting point for the derivation of the plate equations is the variation of the inner energy of a poroelastic continuum

$$\delta\Pi_I = \int_\Omega (\boldsymbol{\sigma} : \nabla\delta\mathbf{u} + \zeta\delta p)\,\mathrm{d}\,\Omega \tag{1}$$

in the domain Ω. In (1), the variation δ is applied on the primal variables $\mathbf{u}$ and p. Using a partial integration (1), yields

$$\begin{aligned}\int_\Omega (\boldsymbol{\sigma} : \nabla\delta\mathbf{u} + \zeta\delta p)\ \mathrm{d}\,\Omega &= \int_\Gamma \left(\mathbf{t}\cdot\delta\mathbf{u} - \frac{1}{\mathrm{i}\omega}q_n\delta p\right)\ \mathrm{d}\,\Gamma \\ &- \int_\Omega \left((\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u} - \frac{1}{\mathrm{i}\omega}\mathbf{q}\cdot\nabla\delta p\right)\ \mathrm{d}\,\Omega\end{aligned} \tag{2}$$

where ζ has been replaced by means of the continuity equation $\zeta = -\frac{1}{\mathrm{i}\omega}\nabla\cdot\mathbf{q}$. Moreover, the total stress vector $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$ and the normal flux $q_n = \mathbf{q}\cdot\mathbf{n}$ appear, where $\mathbf{n}$ denotes the outward normal component. Eq.(2) is the basis for obtaining the partial differential equations and also the basis for a finite element formulation.

3. Poroelastic plates

The essential feature of a plate is a small third dimension compared to the other two dimensions. Here, the plate midsurface A lies in the (x_1, x_2)-plane and the x_3-direction will be the plate thickness as depicted in Fig. 1.

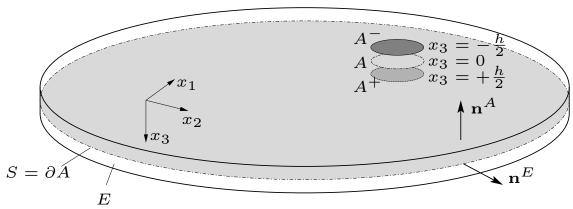

Figure 1: Geometry of the plate

The solid displacement field $\mathbf{u}$ and the pore pressure p, as well as the respective variations of those functions, are developed in a power series with respect to the thickness

$$\mathbf{u}(x_1, x_2, x_3) = \sum_{k=0}^{\infty} \overset{\mathrm{k}}{\mathbf{u}}(x_1, x_2)\, {x_3}^{\mathrm{k}} \tag{3a}$$

$$p(x_1, x_2, x_3) = \sum_{k=0}^{\infty} \overset{\mathrm{k}}{p}(x_1, x_2)\, {x_3}^{\mathrm{k}}\ . \tag{3b}$$

With the series expansions (3) inserted into (2), the integration over the thickness variable x_3 can be performed. After investigating the physical meaning of the coefficients in (3), it turns out that only terms of even order in $\overset{\mathrm{k}}{u}_3$ and odd order in $\overset{\mathrm{k+1}}{u}_1$, $\overset{\mathrm{k+1}}{u}_2$ and $\overset{\mathrm{k+1}}{p}$ are related to the plate problem. The others are quantities describing the in-plane problem. The function $\overset{0}{u}_3 := \overset{0}{w}$ corresponds to the vertical deflection of the plates mid-surface. $\overset{1}{\boldsymbol{\psi}}$ is a vector containing the functions $\overset{1}{u}_1, \overset{1}{u}_2$ which are identified as the rotations. The other higher order terms are related to warping effects, for details see [2]. The coefficients of the pore pressure with a order $k > 0$ can hardly be explained which is due to the scalar nature of the pore pressure.

As depicted in Fig.1, the boundary of the plate is given as $\Gamma = A^+ \cup A^- \cup E$ and the boundary integral in (2) must be decomposed accordingly.

The body forces $\mathbf{f}^s$ and $\mathbf{f}^f$ have to be approximated by a series expansion as well. However, it seems reasonable to assume them to be constant and hence only the constant term of the expansion is considered further.

Obviously, the series have to be truncated and, therefore, the plate parameter $c^2 = \frac{h^2}{12}$ is introduced. The truncation is chosen such that only terms multiplied by a specific order of the plate parameter $\mathcal{O}\left(c^2\right)$ are considered and the rest is neglected.

3.1. *Zeroth order*

For the zeroth order approximation, only terms multiplied by $\left(c^2\right)^0 = 1$, are considered. This means that the value *zero* is synonymous to the statement *'is of higher order'*.

The variational form (2) then only contains the functions $\overset{0}{w}, \overset{1}{\boldsymbol{\psi}}, \overset{1}{p}$ together with the variations of those functions. As mentioned in [2] for the elastostatic case, loadings are connected to a higher order. The elastostatic zeroth order theory then describes only rigid body motions.

The same applies to the poroelastic case, whereas inertia terms and terms multiplied by the frequency dependent poroelastic factor β (see [5]) are found to be connected to a higher order as well and have to be neglected.

Assuming a statically determined or undetermined supported plate, the poroelastodynamic theory of zeroth order ends up with the trivial solution $\overset{0}{w} = \overset{1}{\boldsymbol{\psi}} = \overset{1}{p} = 0$.

3.2. *First order*

First order means that all terms multiplied by $\left(c^2\right)^1 = c^2$ or lower are considered and the rest is neglected. This leads to a variational form (2) containing the variables $\overset{0}{w}, \overset{1}{\boldsymbol{\psi}}, \overset{1}{p}$ and $\overset{2}{w}, \overset{3}{\boldsymbol{\psi}}, \overset{3}{p}$.

The first order approximation leads to a system of six partial differential equations (PDE's). Within this context, the full operator is not given explicitly, however, it is basically an extension of the zeroth order operator.

The insights gained in the previous section concerning some quantities being connected to higher orders, must be transferred to this system as well. Thus, all loadings and inertia terms as well as β which are multiplied by c^2 must be neglected.

In due consideration of those specifications, the whole system can be reduced to two coupled PDE's, with the first given as

$$D\,\Delta\Delta\overset{0}{w} - h\omega^2 \varrho_\beta\,\overset{0}{w} + hB_1\,\Delta\overset{1}{p} - h\beta\,\overset{1}{p} = t_3 + hf_3 \qquad \text{(4a)}$$

and the second as

$$\mathrm{i}\omega\, B_1 \Delta\overset{0}{w} - \mathrm{i}\omega\beta\,\overset{0}{w} - B_2\,\overset{1}{p} = \frac{1}{2} q_3\ . \qquad \text{(4b)}$$

In the above equations, $D = \frac{Eh^3}{12(1-\nu^2)}$ is the plate stiffness with E as the Young's modulus and ν as Poisson's ratio. B_1 and B_2 are some poroelastic constants and $\varrho_\beta = (1-\phi)\,\varrho^s + (\phi - \beta)\,\varrho^f$ is the frequency dependent density of the bulk volume.

With (4) a coupled system of PDE's is given describing the dynamics of thin poroelastic plates. The pore pressure is linearly distributed over the thickness since only $\overset{1}{p}$ appears.

4. Numerical results

The weak form of the system (4) is solved using a finite element formulation. To assure the C^1 inter-element continuity, rectangular Bogner-Fox-Schmit elements are used for $\overset{0}{w}$ and linear elements for $\overset{1}{p}$.

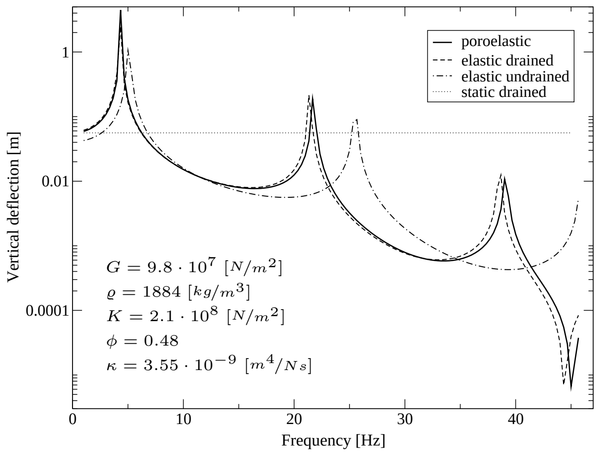

Figure 2: Elastic-drained, poroelastic and elastic-undrained response of a rectangular poroelastic plate

The undrained and drained case assume a strong and a less influence of the interstitial fluid, respectively. They can be seen as lower and upper bounds for the poroelastic material. As depicted in Fig. 2, the poroelastic plate almost behaves like the elastic-drained. For stiffer plates and higher viscosities, the difference between the poroelastic and elastic-drained plate is even smaller.

5. Conclusion

A poroelastic plate theory has been derived from the governing equations of 3-dimensional poroelasticity without assumptions regarding the primal variables. A comparison to [1] shows similarities, yet some terms differ or are missing which is due to the allocation of those terms to higher orders. The underlying theory implies that the interstitial fluid has no strong influence when dealing with thin plates. For thick plates an extension to the second order seems indispensable. This will be the matter of forthcoming studies.

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