A CQM based symmetric Galerkin Boundary Element Formulation for semi-infinite domains

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Abstract. The present work deals with the problem of modelling wave propagation phenomena within a 3-d elastodynamic halfspace. While there exist several Boundary Element formulations based on the more common collocation method, the development of Symmetric Galerkin Boundary Element Methods in this field is at the very beginning and rather challenging. On that score the present work should be understood as a first step.

Here, the time discretization of the underlying Boundary Integral Equations (BIEs) is done via the Convolution Quadrature Method (CQM) proposed by Lubich. After the time discretization, a variational formulation is established resulting in a Galerkin based method in space. Moreover, to obtain a symmetric Galerkin Boundary Element formulation the 2nd BIE is required. This BIE involves hypersingular kernel functions which must be treated carefully in the numerical implementation. Hence, a regularization based on integration by parts of the elastodynamic fundamental solution is presented which, finally, results in a Boundary Element formulation containing at least only weakly singular kernel functions.

In Boundary Element Methods semi-infinite domains are commonly approximated in space by considering just a sufficiently large enough region. Unfortunately, applying this procedure to the symmetric formulation implies the evaluation of additional terms on the truncated surface's boundary due to the regularization of the involved kernel functions.

Therefore, a methodology based on infinite elements intended to overcome this drawback will be presented. The numerical tests done so far show that this approach might be capable of treating semi-infinite domains also within a symmetric Galerkin scheme.

Initial boundary value problem for elastodynamics

By definition of the Lamé-Navier operator

$$\mathcal{L} := -(\lambda + \mu)\nabla\nabla \cdot -\mu\nabla \cdot \nabla \tag{1}$$

in terms of the Nabla operator ∇ and Lamé's constants λ, μ the initial boundary value problem for some linear elastic solid occupying the domain $\Omega \subset \mathbb{R}^3$ with its boundary $\Gamma = \partial\Omega$ reads as

$$\begin{aligned} (\mathcal{L}\mathbf{u})(\tilde{\mathbf{x}};t) + \varrho\frac{\partial^2}{\partial t^2}\mathbf{u}(\tilde{\mathbf{x}};t) &= \mathbf{b}(\tilde{\mathbf{x}};t) \qquad &(\tilde{\mathbf{x}};t) \in \Omega \times (0,\infty) \\ \mathbf{u}_\Gamma(\mathbf{y};t) &= \mathbf{g}_D(\mathbf{y};t) \qquad &(\mathbf{y};t) \in \Gamma_D \times (0,\infty) \\ \mathbf{t}(\mathbf{y};t) &= \mathbf{g}_N(\mathbf{y};t) \qquad &(\mathbf{y};t) \in \Gamma_N \times (0,\infty)\ . \end{aligned} \tag{2}$$

In (2), the unknown displacement field $\mathbf{u}(\tilde{\mathbf{x}};t)$ depends on the location $\tilde{\mathbf{x}} \in \Omega$ and the time $t \in (0,\infty)$. Furthermore, $\mathbf{u}_\Gamma$ and $\mathbf{t}$ are the boundary displacements and tractions for which the

Dirichlet data and Neumann data, $\mathbf{g}_D$ and $\mathbf{g}_N$ are prescribed on the boundary parts Γ_D and Γ_N, respectively. The body's mass density is denoted by ϱ and $\mathbf{b}(\tilde{\mathbf{x}};t)$ is a given body force per unit volume. For simplicity this body force is assumed to be absent in the following. Finally, homogeneous initial conditions are considered, i.e., $\mathbf{u}(\tilde{\mathbf{x}};0)=\mathbf{0}$ and $\frac{\partial}{\partial t}\mathbf{u}(\tilde{\mathbf{x}};0)=\mathbf{0}$ for all $\tilde{\mathbf{x}}\in\Omega$.

Boundary integral equations. To obtain a boundary element formulation of the stated problem, first, an appropriate boundary integral representation of the given system (2) is introduced [1]

$$\begin{aligned}\mathbf{u}(\tilde{\mathbf{x}};t) = &\int_0^t \int_\Gamma \mathbf{U}(\mathbf{y}-\tilde{\mathbf{x}};t-\tau)\cdot(\mathcal{T}_\mathbf{y}\mathbf{u})(\mathbf{y};\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau \\ &- \int_0^t \int_\Gamma [(\mathcal{T}_\mathbf{y}\mathbf{U})(\mathbf{y}-\tilde{\mathbf{x}};t-\tau)]^\top\cdot\mathbf{u}(\mathbf{y};\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau \quad \forall\,\tilde{\mathbf{x}}\in\Omega,\mathbf{y}\in\Gamma,t\in(0,T)\end{aligned} \tag{3}$$

containing the fundamental solution $\mathbf{U}(\mathbf{y}-\tilde{\mathbf{x}};t-\tau)$. In (3), $\mathcal{T}_\mathbf{y}=\mathcal{T}(\partial_\mathbf{y},\mathbf{n}(\mathbf{y}))$ denotes the stress operator based on Hooke's law

$$(\mathcal{T}_\mathbf{y}\mathbf{u})(\mathbf{y};t)=\mathbf{t}(\mathbf{y};t)=\boldsymbol{\sigma}(\mathbf{y};t)\cdot\mathbf{n}(\mathbf{y}) \tag{4}$$

where $\boldsymbol{\sigma}(\mathbf{y};t)$ is the Cauchy stress tensor and $\mathbf{n}(\mathbf{y})$ is the outward normal vector. The first boundary integral equation is obtained by applying a limiting process $\Omega\ni\tilde{\mathbf{x}}\to\mathbf{x}\in\Gamma$ onto the representation formula (3). Using operator notation, this boundary integral equation reads for a sufficiently smooth boundary Γ

$$(\mathcal{V}*\mathbf{t})(\mathbf{x};t)=((\tfrac{1}{2}\mathcal{I}+\mathcal{K})*\mathbf{u})(\mathbf{x};t) \qquad \forall\,\mathbf{x}\in\Gamma\,. \tag{5}$$

The introduced operators are the single layer operator $\mathcal{V}$, the identity operator $\mathcal{I}$, and the double layer operator $\mathcal{K}$ which are defined by

$$(\mathcal{V}*\mathbf{t})(\mathbf{x},t)=\int_0^t\int_\Gamma \mathbf{U}(\mathbf{y}-\mathbf{x},t-\tau)\cdot\mathbf{t}(\mathbf{y},\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau \tag{6a}$$

$$(\mathcal{I}*\mathbf{u})(\mathbf{x},t)=\int_0^t\int_\Gamma \delta(\mathbf{y}-\mathbf{x};t-\tau)\mathbf{I}\cdot\mathbf{u}(\mathbf{y};\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau \tag{6b}$$

$$(\mathcal{K}*\mathbf{u})(\mathbf{x},t)=\lim_{\varepsilon\to 0}\int_0^t\int_{\Gamma\setminus B_\varepsilon(\mathbf{x})}(\mathcal{T}_\mathbf{y}\mathbf{U})^\top(\mathbf{y}-\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{y},\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau\,. \tag{6c}$$

In these expressions, $B_\varepsilon(\mathbf{x})$ denotes a ball of radius ε centered at the point $\mathbf{x}$. Note that the single layer operator (6a) involves a weakly singular integral and that the integration of the double layer operator (6c) has to be understood in the sense of a Cauchy principal value. Moreover, in the definition (6b) $\mathbf{I}$ denotes the identity matrix, and δ is the Delta-distribution.

To obtain a symmetric formulation, additionally the second boundary integral formula is needed. The application of the traction operator $\mathcal{T}_\mathbf{x}$ to the dynamic representation formula (3) with a subsequent limit $\Omega\ni\tilde{\mathbf{x}}\to\mathbf{x}\in\Gamma$ yields

$$(\mathcal{D}*\mathbf{u})(\mathbf{x},t)=((\tfrac{1}{2}\mathcal{I}-\mathcal{K}')*\mathbf{t})(\mathbf{x},t) \qquad \forall\,\mathbf{x}\in\Gamma\,. \tag{7}$$

The newly introduced operators are the hypersingular operator $\mathcal{D}$ and the adjoint double layer operator $\mathcal{K}'$

$$(\mathcal{D}*\mathbf{u})(\mathbf{x},t)=-\lim_{\varepsilon\to 0}\int_0^t \mathcal{T}_\mathbf{x}\int_{\Gamma\setminus B_\varepsilon(\mathbf{x})}(\mathcal{T}_\mathbf{y}\mathbf{U})^\top(\mathbf{y}-\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{y},\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau \tag{8a}$$

$$(\mathcal{K}'*\mathbf{t})(\mathbf{x},t)=\lim_{\varepsilon\to 0}\int_0^t\int_{\Gamma\setminus B_\varepsilon(\mathbf{x})}(\mathcal{T}_\mathbf{x}\mathbf{U})(\mathbf{y}-\mathbf{x},t-\tau)\cdot\mathbf{t}(\mathbf{y},\tau)\,\mathrm{d}s_\mathbf{y}\,\mathrm{d}\tau\,. \tag{8b}$$

The application of the hypersingular operator has to be understood in the sense of a finite part.

Symmetric formulation. For the solution of the initial boundary value problem (2) the symmetric formulation as proposed in [3, 11] using both boundary integral equations (5) and (7) is considered. While the first integral equation (5) is used only on the Dirichlet part Γ_D of the boundary the second one (7) is evaluated on the Neumann part Γ_N

$$\begin{aligned} (\mathcal{V} * \mathbf{t})(\mathbf{x};t) - (\mathcal{K} * \mathbf{u})(\mathbf{x};t) &= (\tfrac{1}{2}\mathcal{I} * \mathbf{g}_D)(\mathbf{x};t) \qquad && (\mathbf{x},t) \in \Gamma_D \times (0,\infty) \\ (\mathcal{K}' * \mathbf{t})(\mathbf{x};t) + (\mathcal{D} * \mathbf{u})(\mathbf{x};t) &= (\tfrac{1}{2}\mathcal{I} * \mathbf{g}_N)(\mathbf{x};t) \qquad && (\mathbf{x},t) \in \Gamma_N \times (0,\infty)\ . \end{aligned} \tag{9}$$

Further, the Cauchy data $\mathbf{u}, \mathbf{t}$ are decomposed into

$$\mathbf{u} = \tilde{\mathbf{u}} + \tilde{\mathbf{g}}_D \quad \text{and} \quad \mathbf{t} = \tilde{\mathbf{t}} + \tilde{\mathbf{g}}_N\ . \tag{10}$$

In these decompositions, arbitrary but fixed extensions, $\tilde{\mathbf{g}}_D$ and $\tilde{\mathbf{g}}_N$, of the given Dirichlet and Neumann data, $\mathbf{g}_D$ and $\mathbf{g}_N$, are introduced such that

$$\begin{aligned} \tilde{\mathbf{g}}_D(\mathbf{x};t) &= \mathbf{g}_D(\mathbf{x};t) \qquad && (\mathbf{x},t) \in \Gamma_D \times (0,\infty) \\ \tilde{\mathbf{g}}_N(\mathbf{x};t) &= \mathbf{g}_N(\mathbf{x};t) \qquad && (\mathbf{x},t) \in \Gamma_N \times (0,\infty) \end{aligned} \tag{11}$$

holds. Note that the extension $\tilde{\mathbf{g}}_D$ of the given Dirichlet datum has to be continuous due to regularity requirements [12].
Inserting the decompositions (10) into (9) leads to the symmetric formulation for the unknown Cauchy data $\tilde{\mathbf{u}}, \tilde{\mathbf{t}}$

$$\begin{aligned} \mathcal{V} * \tilde{\mathbf{t}} - \mathcal{K} * \tilde{\mathbf{u}} &= (\tfrac{1}{2}\mathcal{I} + \mathcal{K}) * \tilde{\mathbf{g}}_D - \mathcal{V} * \tilde{\mathbf{g}}_N \qquad && (\mathbf{x},t) \in \Gamma_D \times (0,\infty) \\ \mathcal{K}' * \tilde{\mathbf{t}} + \mathcal{D} * \tilde{\mathbf{u}} &= (\tfrac{1}{2}\mathcal{I} - \mathcal{K}') * \tilde{\mathbf{g}}_N - \mathcal{D} * \tilde{\mathbf{g}}_D \qquad && (\mathbf{x},t) \in \Gamma_N \times (0,\infty)\ . \end{aligned} \tag{12}$$

Variational principles. Using the inner product $\langle f, g\rangle_\Gamma = \int_\Gamma f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}s_{\mathbf{x}}$ a variational formulation is introduced to find $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ such that

$$\begin{aligned} \langle \mathcal{V} * \tilde{\mathbf{t}}, \mathbf{w}\rangle_{\Gamma_D} - \langle \mathcal{K} * \tilde{\mathbf{u}}, \mathbf{w}\rangle_{\Gamma_D} &= \langle (\tfrac{1}{2}\mathcal{I} + \mathcal{K}) * \tilde{\mathbf{g}}_D - \mathcal{V} * \tilde{\mathbf{g}}_N, \mathbf{w}\rangle_{\Gamma_D} \\ \langle \mathcal{K}' * \tilde{\mathbf{t}}, \mathbf{v}\rangle_{\Gamma_N} + \langle \mathcal{D} * \tilde{\mathbf{u}}, \mathbf{v}\rangle_{\Gamma_N} &= \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}') * \tilde{\mathbf{g}}_N - \mathcal{D} * \tilde{\mathbf{g}}_D, \mathbf{v}\rangle_{\Gamma_N} \end{aligned} \tag{13}$$

holds for all test functions $\mathbf{w}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$. Note, as this Galerkin scheme is used only for the spatial integrations the test functions $\mathbf{w}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ exhibit no time dependency.

Temporal discretization. The time discretization of (13) can be done in several ways. Here, the Convolution Quadrature Method (CQM) developed by Lubich [8] is chosen. Thereby, the time convolution integrals of the form

$$y(t) = f * g = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau\ . \tag{14}$$

are approximated by a quadrature rule which forms a discrete convolution

$$y_m = y(m\,\Delta t) \approx \sum_{k=0}^{m} \omega_{m-k}(\hat{f}, \Delta t) g(k\,\Delta t)\,. \tag{15}$$

In (15), the quadrature weights ω_{m-k} depend only on the time step size Δt and the Laplace transform $\hat{f}$ of the function f. Confer to [10] for more details about the application of the CQM in boundary element methods.

Finally, applying this time stepping scheme to (13) yields a semi-discrete variational form

$$\sum_{k=0}^{m}\begin{bmatrix}\langle\hat{\mathcal{V}}_{m-k}\tilde{\mathbf{t}}_k,\mathbf{w}\rangle_{\Gamma_D}-\langle\hat{\mathcal{K}}_{m-k}\tilde{\mathbf{u}}_k,\mathbf{w}\rangle_{\Gamma_D}\\ \langle\hat{\mathcal{K}}'_{m-k}\tilde{\mathbf{t}}_k,\mathbf{v}\rangle_{\Gamma_N}+\langle\hat{\mathcal{D}}_{m-k}\tilde{\mathbf{u}}_k,\mathbf{v}\rangle_{\Gamma_N}\end{bmatrix}=\sum_{k=0}^{m}\begin{bmatrix}\langle(\frac{1}{2}\hat{\mathcal{I}}+\hat{\mathcal{K}})_{m-k}\tilde{\mathbf{g}}_{D_k}-\hat{\mathcal{V}}_{m-k}\tilde{\mathbf{g}}_{N_k},\mathbf{w}\rangle_{\Gamma_D}\\ \langle(\frac{1}{2}\hat{\mathcal{I}}-\hat{\mathcal{K}}')_{m-k}\tilde{\mathbf{g}}_{N_k}-\hat{\mathcal{D}}_{m-k}\tilde{\mathbf{g}}_{D_k},\mathbf{v}\rangle_{\Gamma_N}\end{bmatrix} \tag{16}$$

where $(\cdot)_{m-k}$ denotes the weight $\omega_{m-k}((\cdot),\Delta t)$ depending on the respective Laplace transformed integral operator.

A Boundary Element formulation for semi-infinite domains

The typical problem statement for an elastodynamic halfspace where the domain $\Omega=\{\tilde{\mathbf{x}}\,|\,\tilde{\mathbf{x}}\in\mathbb{R}^3\wedge\tilde{x}_3<0\}$ as well as the boundary $\Gamma=\{\mathbf{y}\,|\,\mathbf{y}\in\mathbb{R}^3\wedge y_3=0\}$ are unbounded reads as

$$\begin{aligned}((\mathcal{L}+\varrho\frac{\partial^2}{\partial t^2})\mathbf{u})(\tilde{\mathbf{x}};t)&=\mathbf{0} &\quad (\tilde{\mathbf{x}},t)&\in\Omega\times(0,\infty)\\ \mathbf{t}(\mathbf{y};t)&=\mathbf{g}(\mathbf{y};t) &\quad (\mathbf{y},t)&\in\Gamma\times(0,\infty)\end{aligned} \tag{17}$$

with homogeneous initial conditions $\mathbf{u}(\tilde{\mathbf{x}};0)=\mathbf{0}$ and $\frac{\partial}{\partial t}\mathbf{u}(\tilde{\mathbf{x}};0)=\mathbf{0}$ ensuring that the solution of (17) fulfill the Sommerfeld radiation condition [5].

Since $\Gamma_D=\{\emptyset\}$ and $\Gamma_N=\Gamma$ the semi-discrete variational form (16) reduces to

$$\sum_{k=0}^{m}\langle\hat{\mathcal{D}}_{m-k}\tilde{\mathbf{u}}_k,\mathbf{v}\rangle_\Gamma=\sum_{k=0}^{m}\langle(\tfrac{1}{2}\hat{\mathcal{I}}-\hat{\mathcal{K}}')_{m-k}\mathbf{g}_k,\mathbf{v}\rangle_\Gamma \tag{18}$$

which is an appropriate Galerkin formulation of (17) in terms of boundary integral equations.

Regularization of the hypersingular operator. Concerning the numerical evaluation of the bilinear form in (18) the involved hypersingularity makes a direct evaluation rather impossible. Therefore, a regularization based on Stokes theorem is used to transfer the hypersingular bilinear form to a weak one. This regularization is given in detail in [7] and based itself mainly on the work of Han [6]. Additionally, the double layer potential is also transfered to a weak form using the same techniques as for the hypersingular bilinear form.

Within this work it is sufficient to mention that the regularization process demands either a closed boundary surface Γ or vanishing integral kernels at infinity. Since the involved kernels fulfill the Sommerfeld radiation condition the last constraint is satisfied and the regularization holds also for the elastodynamic halfspace.

Nevertheless, problems might occur on a discrete level. There, it is a common practice to model just a truncated part of the infinite geometry. Unfortunately, the emerging truncation's borderline represents the surface's boundary such that Γ is neither closed anymore nor that the integral kernels can be assumed to vanish. Therefore, it must be ensured that the discretized area is closed or modelling the infinite surface.

Spatial discretization. Figure 1(a) illustrates the discretization approach of an unbounded domain. Thereby, the boundary Γ is represented in the computation by an approximation Γ_h which is the union of two sets of different geometrical elements

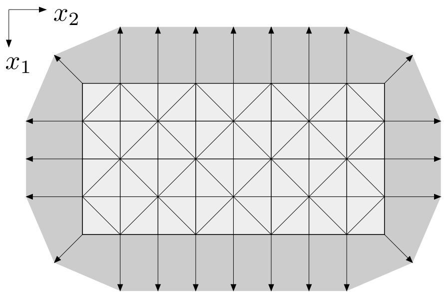
$$\Gamma_h=\bigcup_{\ell=0}^{N_e^f}\tau_\ell^f\;\cup\;\bigcup_{m=0}^{N_e^i}\tau_m^i\,. \tag{19}$$

In (19), τ^f denotes standard linear finite elements, e.g., surface triangles, and N_e^f is their total number. Additionally, the boundary's far-field is represented by N_e^i infinite boundary elements τ^i whose configuration is depicted in Fig. 1(b). For the concept of infinite elements refer to [2] and the references cited there.

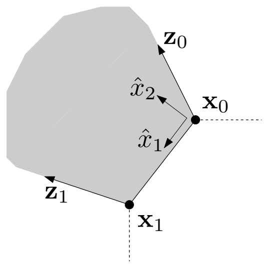
Further, the boundary functions $\tilde{\mathbf{u}}$ and $\mathbf{g}$ are approximated by the separation of variables with trial functions φ_i and ψ_j, which are defined with respect to the geometry partitioning (19), and time dependent coefficients $\mathbf{u}_i$ and $\mathbf{g}_j$

$$\tilde{\mathbf{u}}(\mathbf{x}) \approx \sum_{i=1}^{N} \mathbf{u}_i(t)\varphi_i(\mathbf{x}) \quad \text{and} \quad \mathbf{g}(\mathbf{x}) \approx \sum_{j=1}^{M} \mathbf{g}_j(t)\psi_j(\mathbf{x}) \,. \tag{20}$$

In case of finite boundary elements the functions φ_i are chosen to be equivalent to those shape functions forming the geometry approximation.



(a) Halfspace with infinite elements



(b) Infinite element

Figure 1: Discretized halfspace and infinite mapping

By introducing local coordinates $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2]^\top \in [0,1] \times [0,1)$ the mapping $\chi_\tau : \hat{\tau} \to \tau$ from the reference element $\hat{\tau}$ to an infinite element τ reads as

$$\mathbf{x} = \chi_\tau(\hat{\mathbf{x}}) = \langle \boldsymbol{\phi}(\hat{x}_1), \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} \rangle + \frac{\hat{x}_2}{1-\hat{x}_2} \langle \boldsymbol{\phi}(\hat{x}_1), \alpha \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_1 \end{bmatrix} \rangle \tag{21}$$

where $\mathbf{x}_0$ and $\mathbf{x}_1$ denote the two fixed vertex nodes, and $\mathbf{z}_0$ and $\mathbf{z}_1$ represent two direction vectors with the property $\langle \mathbf{z}_k, \mathbf{z}_k \rangle = 1$. The scalar value $\alpha > 0$ is a scaling factor. The function $\boldsymbol{\phi}$ within the dot product $\langle \cdot, \cdot \rangle$ describes the approximation for the finite extent and is given by $\boldsymbol{\phi}(\hat{x}_1) = [1 - \hat{x}_1, \hat{x}_1]^\top$.

Since the integral kernels depend mostly on the distance $r = |\mathbf{y} - \mathbf{x}|$ between two points $\mathbf{y}$ and $\mathbf{x}$ it is preferable to note that for infinite elements the asymptotic behavior of the distance is of order $\mathcal{O}((1-\hat{x}_2)^{-1})$. Moreover, the transformation of the integral kernels to the reference element demands the computation of the Gram determinant which itself can be expressed via the Jacobi matrix

$$\mathbf{J}_{\tau^i}(\hat{\mathbf{x}}) = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \hat{x}_1} & \frac{\partial \mathbf{x}}{\partial \hat{x}_2} \end{bmatrix} =: \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{O}(r) & \mathcal{O}(r^2) \end{bmatrix} \tag{22}$$

and reads as

$$g_{\tau^i}(\hat{\mathbf{x}}) = \sqrt{\det(\mathbf{J}_{\tau^i}^\top \mathbf{J}_{\tau^i})} = \sqrt{\langle \mathbf{J}_1, \mathbf{J}_1 \rangle \langle \mathbf{J}_2, \mathbf{J}_2 \rangle - \langle \mathbf{J}_1, \mathbf{J}_2 \rangle^2} = \mathcal{O}(r^3) \,. \tag{23}$$

For some kernel function $k(\mathbf{x}(\hat{\mathbf{x}}), \mathbf{y}(\hat{\mathbf{y}})) = \mathcal{O}(r^{-1})$ the integrand in a Galerkin scheme takes the form

$$I[\ell, m] = \int_{\tau_\ell} \int_{\tau_m} g_{\tau_\ell}(\hat{\mathbf{x}}) g_{\tau_m}(\hat{\mathbf{y}})\, k(\mathbf{x}(\hat{\mathbf{x}}), \mathbf{y}(\hat{\mathbf{y}}))\, \hat{\varphi}_\ell(\hat{\mathbf{x}}) \hat{\varphi}_m(\hat{\mathbf{y}})\, \mathrm{d}\hat{\mathbf{x}}\, \mathrm{d}\hat{\mathbf{y}} \,. \tag{24}$$

From (24), it is obvious that the trial function for an infinite element has to be of order $\mathcal{O}(r^{-3})$ to guarantee that the integral is finite. Therefore, the trial and test functions φ_i of an infinite element are chosen as

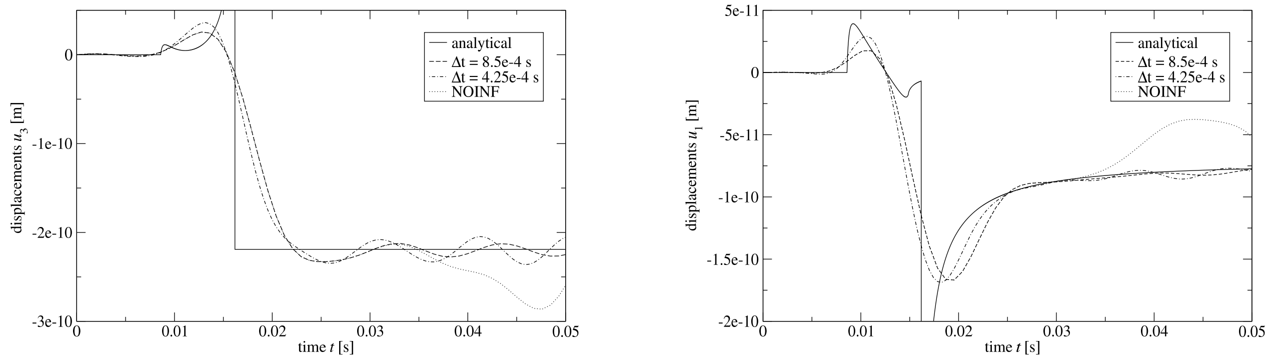
$$\varphi_i(\mathbf{x}) \circ \hat{\varphi}(\hat{\mathbf{x}}) = \boldsymbol{\phi}(\hat{x}_1)\,(1-\hat{x}_2)^3 \tag{25}$$

where $\boldsymbol{\phi}(\hat{x}_1)$ is identical to the function used for the geometry approximation (21).

Finally, a comment concerning the singular integrals must be made. All integral operators used in the present work are weakly singular. They are treated completely numerical based on quadrature rules developed by Sauter and Erichsen [4].

Numerical examples

Now, numerical results for the present Boundary Element formulation are given. The material data represents soil with Lamé's constants $\lambda = \mu = 1.3627 \cdot 10^8\,{}^{\mathrm{N}}\!/_{\mathrm{m}^2}$, and mass density $\varrho = 1884\,{}^{\mathrm{kg}}\!/_{\mathrm{m}^3}$. The discretization of the infinite domain consists of 800 regular linear triangles and 80 infinite elements with a scaling factor of $\alpha = 1$. The triangles occupy a total area of $20\mathrm{m} \times 20\mathrm{m}$. Moreover, at the mesh's center an area of $A = 2\mathrm{m}^2$ is excited by a traction jump $\mathbf{g} = [0, 0, -1]^\top H(t)\,{}^{\mathrm{N}}\!/_{\mathrm{m}^2}$ according to the unit step function $H(t)$. The remaining surface is traction free.



(a) Vertical displacements u_3

(b) Radial displacements u_1

Figure 2: Vertical and radial displacements at the observation point $\mathbf{x}^\star$

Figure 2 depicts the solution for an observation point $\mathbf{x}^\star$ on the surface in 4m distance to the center of the loading. The first and the second numerical solution vary in the chosen time step size but reveal in general the same behavior. Compared to the analytical solution [9] both displacement solutions exhibit oscillations for larger times which are due to artificial reflections at the crossing of finite and infinite boundary elements. But beside these effects, both numerical solutions show approximately the characteristics of the analytical solution. Contrary, a computation without infinite elements titled as 'NOINF' but with the same time step size as the first depicted numerical solution namely $\Delta t = 8.5 \cdot 10^{-4}\,\mathrm{s}$ yields a defective result for times larger than 0.034s. This is exactly the time the compression wave with the velocity $c_1 = \sqrt{{}^{\lambda+2\mu}\!/_{\varrho}} = 465.8\,{}^{\mathrm{m}}\!/_{\mathrm{s}}$ has to travel from the center of loading to the truncated boundary and back to the observation point $\mathbf{x}^\star$. From this it can be stated that the infinite element approach for the treatment of semi-infinite domains by symmetric Boundary Element Methods is expedient but requires further research.

Conclusions

A boundary element method for elastodynamics based on a Galerkin discretization in space and on the Convolution Quadrature Method in time was presented. To obtain a symmetric formulation, also the usage of the second boundary integral equation is required which demands the computation of hypersingular kernel functions. Therefore, a regularization of the elastodynamic hypersingular integral operator is used leading to a weakly singular bilinear form. Since the regularization is based on integration by parts it is not suitable for treating problems where the mesh of a halfspace is truncated.

To overcome this drawback the concept of infinite elements was used and adopted to the present boundary element formulation. Unfortunately, the numerical results obtained so far are not completely satisfactory since they feature oscillations for larger times. To reduce these oscillations or, better, to eliminate them further investigations are essential. Nevertheless, the infinite element approach is advantageous compared to computations without any infinite elements as it reaches the static limit for larger times.

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