Fundamental Solutions for Incompressible Modeled Poroelastic Solids

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Summary

Some poroelastic materials can be approximated by the assumption of incompressible constituents with Biot's theory, i.e., the only compressibility is based on the structural behavior. For those cases no fundamental solutions and, consequently, no Boundary Element formulations are available. Here, those fundamental solutions are deduced. The incompressible time-dependent BE formulation is achieved by changing the fundamental solutions of the corresponding compressible BE formulation. Clearly, such a formulation can also be achieved by using the material parameters for incompressible constituents in the compressible formulation, however the incompressible formulation is computationally faster. This is due to less operations needed for the calculation of the incompressible fundamental solutions. The proposed formulation is applied to wave propagation in poroelastic half space showing that not all materials allow an incompressible approximation. Further, it is demonstrated that the fast compressional wave and not the slow compressional wave vanishes.

Introduction

The efficiency of BEM in dealing with semi-infinite domain problems, e.g., soil-structure interaction, have long been recognized by researchers and engineers. For soil, a fluid saturated material, a poroelastic constitutive model should be used in connection with a time-dependent BE formulation to model wave propagation problems correctly. Dynamic poroelastic BE formulations are published in frequency domain, e.g., [5], in Laplace domain, e.g., [3], and in time domain [3, 9]. In all of the above mentioned formulations, Biot's theory is used assuming compressible constituents. Beside the compressibility of the constituents also a structural compressibility exists and is modeled in Biot's theory. For some materials, e.g., soil, the compression modulus of the constituents itself is much larger than the compressibility of the structure. In these cases, it is sufficient to approximate both the fluid and solid constituents as incompressible, i.e., only the structural compressibility remains.

Here, Biot's model for this special case is given. Subsequently the novel fundamental solutions for this case are derived using the method of Hörmander [7]. As the unknowns in the incompressible model are the same as in the compressible model, i.e., the solid displacement and the pore pressure, and, further, the principal structure of the set of governing equations is similar, all the BE formulation mentioned above can be used. Aiming on wave propagation problems, the time-dependent BE formulation based on the Convolution Quadrature Method as proposed by Schanz [10] is used here.

To demonstrate the limits of the incompressible approximation for a poroelastic medium, two extreme cases of materials, a rock and a soft sediment, are used. The results of the incompressible modeling are compared with the compressible modeling at the example of a half space.

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Poroelastic boundary integral formulation

The assumption of incompressible constituents in a two-phase material is given if $\frac{K}{K_s} << 1$ for an incompressible solid and $\frac{K}{K_f} << 1$ for an incompressible fluid is fulfilled [6] with K_s and K_f denoting the compression modulus of the solid grains and the fluid, respectively, and K denoting the bulk compression modulus. Considerations of constitutive relation at micro mechanical level [6] for Biot's effective stress coefficient α and R yield the conditions

$$\alpha = 1 - \frac{K}{K_s} \approx 1 \quad \text{and} \quad R = \frac{\phi^2 K_f K_s^2}{K_f (K_s - K) + \phi K_s (K_s - K_f)} \to \infty \tag{1}$$

if both constituents are modeled incompressible. Now, following Biot's theory [1], solid displacements u_i and the pore pressure p are a sufficient set of independent variables [2] to model the constitutive behavior of a poroelastic continuum. However, the dynamic governing differential equations using them can only be formulated in Laplace- or Fourier domain. Here, a representation in Laplace domain (denoted by $\hat{f}(s)$) is used which in matrix form yields

$$\mathbf{B}^* \begin{bmatrix} \hat{u}_i \\ \hat{p} \end{bmatrix} + \begin{bmatrix} \hat{F}_i \\ \hat{a} \end{bmatrix} = \mathbf{0}\,, \qquad \mathbf{B}^* = \begin{bmatrix} G\partial_{jj} + \left(K + \frac{1}{3}G\right)\partial_{ij} - s^2(\rho - \beta\rho_f) & -(1-\beta)\partial_i \\ -(1-\beta)s\partial_j & \dfrac{\beta}{\rho_f s}\partial_{ii} \end{bmatrix} \tag{2}$$

with the not self adjoint operator $\mathbf{B}^*$ taking the incompressibility into account. In Eq. (2), the abbreviation $\beta = \phi^2 s\kappa\rho_f / \left(\phi^2 + s\kappa(\rho_a + \phi\rho_f)\right)$ and $\partial_i = ()_{,i}$ is used. The bulk material is defined by the compression modulus K, the shear modulus G, and the porosity ϕ. Further, the bulk density $\rho = \rho_s(1-\phi) + \phi\rho_f$ with the densities of the solid ρ_s and the fluid ρ_f is used. The apparent mass density ρ_a introduced by Biot is assumed to be $\rho_a = 0.66\,\phi\,\rho_f$ [5]. The load vector consists of forces in the solid $\hat{F}_i$ and sources in the fluid $\hat{a}$.

The so-called fundamental solution matrix $\mathbf{G}$ of the adjoint system to (2), i.e., the solutions due to a unit impulsive point force $\hat{F}_{ij}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\delta_{ij}$ and to a unit impulsive source $\hat{a}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$, respectively, at the point $\mathbf{y}$, is defined by

$$\mathbf{BG} + \mathbf{I}\delta(\mathbf{x}-\mathbf{y}) = \mathbf{0} \quad \text{with} \quad \mathbf{G} = \begin{bmatrix} \hat{U}^s_{ij} & \hat{U}^f_i \\ \hat{P}^s_j & \hat{P}^f \end{bmatrix} \tag{3}$$

using the adjoint operator $\mathbf{B}$. Following the ideas of Hörmander [7], the fundamental solutions are found based on a scalar function φ via the ansatz $\mathbf{G} = \mathbf{B}^{co}\varphi$ with the matrix of cofactors $\mathbf{B}^{co}$ of the operator matrix $\mathbf{B}$. Using this ansatz in Eq. (3) and taking the definition of the inverse matrix $\mathbf{B}^{-1} = \frac{\mathbf{B}^{co}}{\det(\mathbf{B})}$ into account, a more convenient form of the governing equation is achieved

$$\mathbf{BB}^{co}\varphi + \mathbf{I}\delta(\mathbf{x}-\mathbf{y}) = \det(\mathbf{B})\,\mathbf{I}\varphi + \mathbf{I}\delta(\mathbf{x}-\mathbf{y}) = \mathbf{0}\,. \tag{4}$$

Now, with the three roots of $\det(\mathbf{B}) = 0$

$$\lambda_1^2 = \frac{s^2}{K + \frac{4}{3}G}\left(\rho - \beta\rho_f + \rho_f\frac{(1-\beta)^2}{\beta}\right) \qquad \lambda_2^2 = 0 \qquad \lambda_3^2 = \frac{s^2(\rho - \beta\rho_f)}{G} \tag{5}$$

the problem is reduced to find a solution of the iterated Helmholtz operator

$$\psi\nabla^2\left(\nabla^2-\lambda_3^2\right)\left(\nabla^2-\lambda_1^2\right)+\delta(\mathbf{x}-\mathbf{y})=0 \qquad \text{with} \quad \psi=\frac{G^2\beta}{s\rho_f}\left(K+\frac{4}{3}G\right)\left(\nabla^2-\lambda_3^2\right)\varphi \quad (6)$$

This solution can be deduced from the known solution in the case of $\lambda_2 \neq 0$, here for the case of a 3-d continuum (see, e.g., [4])

$$\psi=\frac{1}{4\pi r}\left[\frac{e^{-\lambda_1 r}}{\lambda_1^2\left(\lambda_1^2-\lambda_3^2\right)}+\frac{1}{\lambda_1^2\lambda_3^2}+\frac{e^{-\lambda_3 r}}{\lambda_3^2\left(\lambda_3^2-\lambda_1^2\right)}\right]. \quad (7)$$

The remaining step is to apply the operator $\mathbf{B}^{co}$ to the scalar function φ to obtain the fundamental solutions

$$\hat{U}_{ij}^s=\frac{1}{4\pi r\rho_1 s^2}\left[\frac{R_1\rho_1\beta e^{-\lambda_1 r}+R_2\rho_f(1-\beta)^2}{\rho_1\beta+\rho_f(1-\beta)^2}+\left(\delta_{ij}\lambda_3^2-R_3\right)e^{-\lambda_3 r}\right] \quad (8a)$$

$$\hat{U}_i^f=\frac{(1-\beta)\rho_f r_{,j}}{4\pi r\left(\beta\rho_1+(1-\beta)^2\rho_f\right)}\left[\lambda_1 e^{-\lambda_1 r}+\frac{e^{-\lambda_1 r}-1}{r}\right]=s\hat{P}_i^s \quad (8b)$$

$$\hat{P}^f=\frac{s\rho_f}{4\pi r\beta}\,\frac{(1-\beta)^2\rho_f e^{-\lambda_1 r}+\beta\rho_1}{(1-\beta)^2\rho_f+\beta\rho_1} \quad (8c)$$

with $R_k=\left(3r_{,i}r_{,j}-\delta_{ij}\right)/r^2+\lambda_k\left(3r_{,i}r_{,j}-\delta_{ij}\right)/r+\lambda_k^2 r_{,i}r_{,j}$ and $\rho_1=\rho-\beta\rho_f$. It can be shown that the same can also be achieved by introducing the conditions (1) into the fundamental solutions of the compressible case.

The singular behavior is studied using the series representation of the exponential function. To do so, it is easily obtained that the above given fundamental solutions have the same singular behavior as the compressible solutions, i.e., the displacement solution (8a) has a weak singularity equal to the elastostatic one and the pressure (8c) is also weakly singular yielding the same expression as in the compressible case. The other two fundamental solutions in (8b) are regular. Hence, also the derivatives of the above fundamental solutions necessary for the singular integral equation behave like in the compressible case. The derivation of the integral equation exactly follows the procedure for the compressible case and, therefore, is not explained here. Further, the time stepping procedure is achieved, also as in case of compressible constituents, using the Convolution Quadrature Method. Details on this BE formulation may be found in [10].

Wave propagation in a half space

In order to demonstrate the effect of modeling the constituents incompressible, wave propagation phenomenon in a poroelastic half space is considered. Results obtained by the incompressible model are compared to those of the compressible model. For the comparison, a long strip ($6\,\mathrm{m}\times 33\,\mathrm{m}$) is discretized with 396 triangular linear elements on 242 nodes (see Fig. 1). The time step size used is $\Delta t=0.00008\,\mathrm{s}$ in case of rock and $\Delta t=0.0015\,\mathrm{s}$ in case of sediment. The modeled half space is loaded on area A $(1\,\mathrm{m}^2)$ by a vertical total

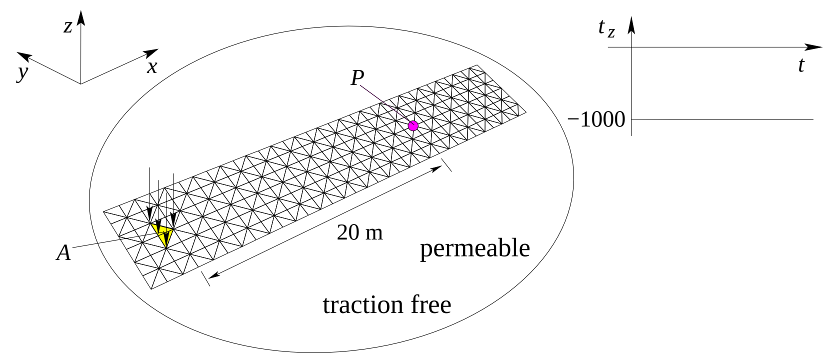


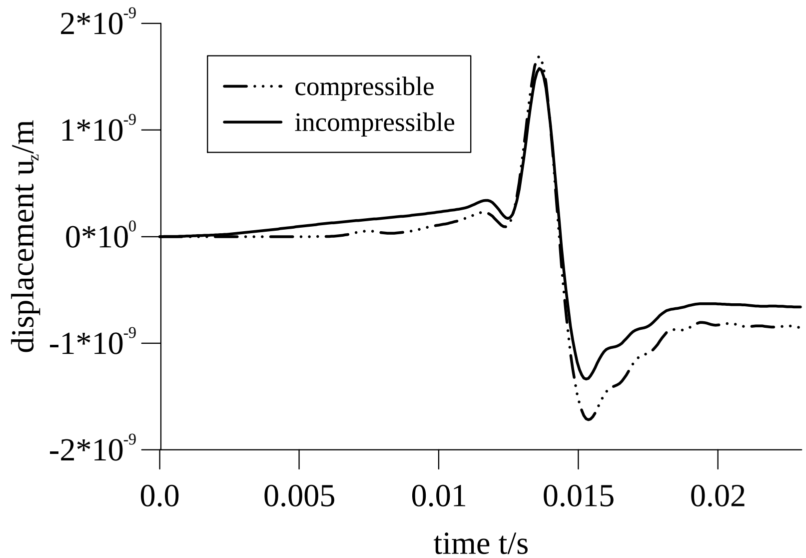
Figure 1: Half space under vertical load: Discretization and load history

stress vector $t_z = -1000\,\mathrm{N/m^2}\,H(t)$ (shaded area in Fig. 1) and the remaining surface is traction free. The pore pressure is assumed to be zero all over the surface, i.e., the surface is permeable. The material properties are those corresponding to a rock (Berea sandstone) and a soft sediment (mud) given in Tab. 1. The interstitial fluid in the rock is water and in the sediment salty water.

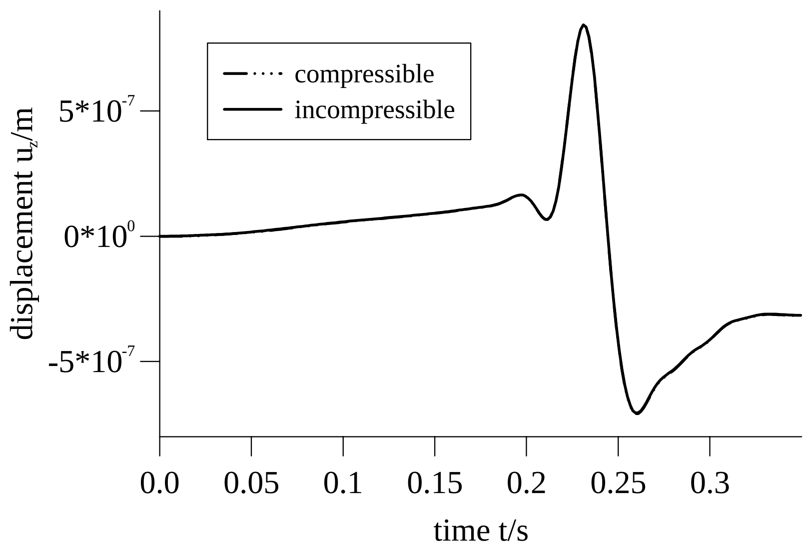
Before looking at the results, it may be convenient to look at the ratios of the compression moduli. For rock there is: $K/K_s = 0.22, K/K_f = 2.42$ and for sediment $K/K_s = 0.001, K/K_f = 0.016$. Hence, it can be expected that the incompressible modeling for the rock fails and give good results for the sediment. Exactly, this is confirmed by the results given in Fig. 2a for the rock and Fig. 2b for the sediment. There, the horizontal displacement at the marked point on the mesh, 20 m from the margin of the loaded area, for an incompressible as well as compressible model is plotted versus time t. Clearly, for the rock there are large differences whereas for sediment both results are indistinguishable. Also, from the incompressible rock results it can be observed that the arrival of the fast compressional wave, the first deviation from zero, tends to zero. The Rayleigh wave, the large amplitude at $t \approx 0.015\,\mathrm{s}$, is not affected by the different modelings. This is in accordance with the theory where one wave vanishes, i.e., one of the roots in Eq. (5) is zero, and the third root, corresponding to the shear wave, is not influenced. Finally, it should be remarked that the usage of the incompressible fundamental solutions needs about 20 % less CPU time.

Conclusions

Here, novel poroelastodynamic fundamental solutions in Laplace domain for incompressible constituents are presented using Biot's theory. The incompressible time-dependent BE formulation is achieved by changing the fundamental solutions of the corresponding compressible BE formulation based on the "Convolution Quadrature Method" of Lubich [8]. Finally, wave propagation in a poroelastic half space is studied using two extreme cases of materials, a rock and a soft sediment. Comparing wave propagation results using a compressible theory and a incompressible theory shows clearly that in case of rock the incompressible approximation fails whereas in case of sediment there is no difference. In



(a) Berea sandstone (rock)



(b) Soft sediment (mud)

Figure 2: Vertical displacement versus time

Table 1: Material data of Berea sandstone (rock) and a sediment (mud)

	$K\left(\frac{N}{m^2}\right)$	$G\left(\frac{N}{m^2}\right)$	$\rho\left(\frac{kg}{m^3}\right)$	ϕ	$K_s\left(\frac{N}{m^2}\right)$	$\rho_f\left(\frac{kg}{m^3}\right)$	$K_f\left(\frac{N}{m^2}\right)$	$\kappa\left(\frac{m^4}{Ns}\right)$
rock	$8\cdot 10^9$	$6\cdot 10^9$	2458	0.19	$3.6\cdot 10^{10}$	1000	$3.3\cdot 10^9$	$1.9\cdot 10^{-10}$
mud	$2\cdot 10^7$	$1.2\cdot 10^7$	1396	0.76	$3.6\cdot 10^{10}$	1000	$2.25\cdot 10^9$	$1\cdot 10^{-8}$

this case, the incompressible model has to be preferred because of lower computational costs.

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References

1. Biot, M. (1956): “Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. I. Low-Frequency Range, II. Higher Frequency Range”, *Journal of the Acoustical Society of America*, Vol. 28, pp. 168–191.
2. Bonnet, G. (1987): “Basic Singular Solutions for a Poroelastic Medium in the Dynamic Range”, *Journal of the Acoustical Society of America*, Vol. 82, pp. 1758–1762.
3. Chen, J., and Dargush, G. (1995): “Boundary Element Method for Dynamic Poroelastic and Thermoelastic Analysis”, *International Journal of Solids and Structures*, Vol. 32, pp. 2257–2278.
4. Cheng, A.-D., and Antes, H. (1991): “On Free Space Green’s Function for High Order Helmholtz Equations”, In S. Kobayashi and N. Nishimura (Eds.), *Boundary Element Methods: Fundamentals and Applications*, Berlin, pp. 67–71. Springer-Verlag.
5. Cheng, A. H.-D., Badmus, T., and Beskos, D. (1991): “Integral Equations for Dynamic Poroelasticity in Frequency Domain with BEM Solution”, *Journal of Engineering Mechanics ASCE*, Vol. 117, pp. 1136–1157.
6. Detournay, E., and Cheng, A. H.-D. (1993): *Fundamentals of Poroelasticity*, Vol. II of *Comprehensive Rock Engineering: Principles, Practice & Projects*, Chapter 5, pp. 113–171 Pergamon Press.
7. Hörmander, L. (1963): *Linear Partial Differential Operators*, Springer-Verlag.
8. Lubich, C. (1988): “Convolution Quadrature and Discretized Operational Calculus. I/II.”, *Numerische Mathematik*, Vol. 52, pp. 129–145/413–425.
9. Schanz, M. (2001): “Application of 3-d Boundary Element Formulation to Wave Propagation in Poroelastic Solids”, *Engineering Analysis with Boundary Elements*, Vol. 25, pp. 363–376.
10. Schanz, M. (2001): *Wave Propagation in Viscoelastic and Poroelastic Continua: A Boundary Element Approach*, Lecture Notes in Applied Mechanics. Springer-Verlag, Berlin, Heidelberg, New York.