Qualitative Abstraction of Piecewise Affine Systems

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Abstract

Qualitative or symbolic abstractions of hybrid systems received considerable interest recently to solve problems of hybrid systems estimation, control and verification symbolically. To abstract a hybrid system one has to slice the continuously valued input/output/state-space into a (finite) set of partitions. The number of partitions potentially grows exponentially with the dimension of the space. As a consequence, one has to divide the spaces carefully in order to obtain a manageable abstraction. This paper presents a systematic procedure to partition the state-space of piecewise affine (PWA) systems into qualitatively distinct regions. As a consequence, we obtain a moderately large set of partitions that characterises the hybrid dynamics of the PWA system. The abstraction scheme helps also to keep the number of so called *spurious* behaviors of qualitative simulation small, in particular when compared to the typically used grid-based abstractions.

Introduction

Complexity in hybrid systems analysis, estimation and control arises from the close interaction between the system's mode-dependent continuous dynamics and discrete mode changes. Optimal hybrid estimation, for example, has to consider all possible hybrid trajectories that the system can exhibit and performs the associated numerical filtering process for each trajectory. Since the number of trajectories grows exponentially over time, it is easy to see that suboptimal and computationally efficient methods are key for any real-time operation of hybrid estimation.

A tempting approach is to use finite (qualitative) abstractions of the continuous dynamics together with the discrete dynamics of the hybrid system and re-formulate the hybrid estimation/control task in a pure discrete way. The rich toolset of Qualitative Reasoning and Model Checking can then be used to solve tasks for analysis, simulation, verification, estimation and control. However, there is no free lunch. A qualitative model for the continuously valued dynamics requires a finite abstraction of the input/output/state-space of the hybrid model. Input and output space partitions can arise naturally through quantisation. The state-space abstraction, however, is more demanding. In particular, as the number of partitions potentially grows exponentially with the dimension of the continuous state. Another difficulty is that qualitative models allow trajectories that the underlying real system cannot show. These so called *spurious behaviors* (Kuipers 1994) can significantly deteriorate the reasoning result, for example, in that one fails to prove stability of a hybrid control system.

This paper provides an approach for the qualitative abstraction of *piecewise affine (PWA)* systems that addresses both issues mentioned above. We propose a state-space abstraction scheme that uses distinct features of the system's continuous and discrete dynamics. As a result we obtain a separation that partitions the state-space in *qualitatively distinct regions* only and thus keeps the number of partitions moderate. Another benefit is that our partitioning scheme reduces the number of spurious behaviors compared to using a state-space abstraction through a hyper-dimensional grid.

Related Research

Qualitative or symbolic abstractions of dynamic systems are a major theme in Qualitative Reasoning (Weld & de Kleer 1990; Kuipers 1994), a fruitful branch of AI. Our work on qualitative abstraction of a system's state-space has its origins in the pioneering work of Yip, Zhao and Bailey-Kellogg (Yip 1991; Yip & Zhao 1996; Bailey-Kellogg, Zhao, & Yip 1996) that provide symbolic abstractions for complex non-linear dynamics. Whereas they use advanced reasoning methods for system's analysis, Lunze and coworkers (Lunze 1994; Lunze, Nixdorf, & Schröder 1999; Schröder 2003) build their qualitative abstraction upon the concept of stochastic automata and use them to mainly solve diagnosis problems.

Timed automata (Alur & Dill 1994), a specific class of hybrid systems, lead themself to a symbolic model and thus allow one to apply analysis and verification methods from computer science, such as bisimulation. This line of research received considerable interest, e.g. (Alur *et al.* 2000) and much effort was devoted to extending the applicability of symbolic approximations and bisimulation techniques to solve analysis, verification and control problems for other, more general, classes of hybrid systems (Tiwari 2003; Girard & Pappas 2006; Tabuada 2007). Most recent research limits its scope to systems with 'strong' stability properties so that the artifacts of discrete approximation, i.e. spurious behaviors, do not prevent one from applying bisimulation type analysis techniques. All of these techniques face also the curse of dimensionality. Discrete abstractions of continuous state-spaces lead to a number of domains for the state that grows exponentially with the state's dimension. We cannot fully avoid this difficulty for our proposed stochastic automata encoding of PWA systems, but provide an abstraction technique that slices the state-space carefully according to qualitative distinctions of the system's dynamics. This keeps the number of state partitions moderate and, as a nice side-effect, actively reduces spurious behaviors. To deal with complexity, we can further draw upon our work to efficiently encode stochastic automata in an OBDD-like fashion (Kleissl & Hofbaur 2005).

PWA Systems

A widely adopted and versatile class of hybrid systems are the so-called *piecewise affine (PWA)* systems. PWA systems specify a hybrid model with continuously valued state $\mathbf{x} = [x_1, \ldots, x_{n_x}]^T$, input $\mathbf{u} = [u_1, \ldots, u_{n_u}]^T$ and output $\mathbf{y} = [y_1, \ldots, y_{n_y}]^T$. The model specifies dynamics in discrete-time (sampling period T_d) through the affine discrete-time model

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k + \mathbf{f}_i & (1) \\ \mathbf{y}_k &= \mathbf{C}_i \mathbf{x}_k + \mathbf{D}_i \mathbf{u}_k + \mathbf{g}_i \ , & (2) \end{aligned}$$

where the subscript $i = 1, \ldots, l$ of the model-parameter $\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i, \mathbf{f}_i, \mathbf{g}_i$ stands for the *mode* or PWA dynamics that is valid in a particular domain $\mathcal{D}_i$ of the combined state/input space. More specifically, the state traverses at mode i whenever

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_k \end{bmatrix} \in \mathcal{D}_i \ .$$

To guarantee *uniqueness* for the PWA trajectories one ensures that the domains $\mathcal{D}_i$ specify a *non-overlapping separation* of the state/input space $\mathcal{D}$. A domain $\mathcal{D}_i$ is usually defined through a *polyhedral* partition of the combined state/input space that can be expressed through constraints of the form

$$\mathbf{G}_i^x \mathbf{x}_k + \mathbf{G}_i^u \mathbf{u}_k \leq \mathbf{G}_i^c \ .$$

For the scope of this paper we use a slightly weaker domain specification that allows us to abstract the continuous state and input space separately, more specifically, we select a mode i, whenever the two inequalities hold

$$\begin{aligned} \mathbf{x}_k \in \mathcal{D}_{x,i}: \quad \mathbf{G}_i^x \mathbf{x}_k &\leq \mathbf{G}_i^{cx} \\ \mathbf{u}_k \in \mathcal{D}_{u,i}: \quad \mathbf{G}_i^u \mathbf{u}_k &\leq \mathbf{G}_i^{cu} \ . \end{aligned} \quad (3)$$

Again, to guarantee uniqueness of PWA trajectories, the partitions $\mathcal{D}_i = \mathcal{D}_{x,i} \times \mathcal{D}_{u,i}$ do not overlap, i.e. $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset, i \neq j$.

Example

Figure 1 shows trajectories[1] for an autonomous PWA system with $l = 3$ modes and a 2-dimensional state-space that is limited to

$$-6 \leq x_1 \leq +6, \quad -3 \leq x_2 \leq +3 \ . \quad (4)$$

[1] As in many real-world applications of hybrid systems, we obtain the PWA model through sampling of the continuous time dynamics shown in Fig. 1.

Mode switching occurs at $x_1 = -2$ (between modes 1 and 2) and at $x_1 = +2$ (between modes 2 and 3). Our discrete-time PWA model operates at a sampling period $T_d = 0.1$ [sec.] and defines the dynamics as follows: PWA Mode 1 specifies dynamics with a stable equilibrium at $\mathbf{x}_r = [-3 \quad 0]^T$, eigenvalues $z_1 = z_2 = e^{-2T_d}$ and an eigenvector $\mathbf{p} = [1 \quad -1]^T$. The dynamics of mode 2 are characterised through an unstable equilibrium (saddle point) at the origin, eigenvalues $z_1 = e^{-T_d}, z_2 = e^{+T_d}$ and associated eigenvectors $\mathbf{p}_1 = [2 \ \ 1], \ \mathbf{p}_2 = [2 \quad -1]$. Finally, at mode 3 the system exhibits an undamped oscillatory behavior with equilibrium $\mathbf{x}_r = [2 \ \ 0]^T$ and frequency $\omega = 1$. We will use the same scalar measurement $y_k = x_{1,k} + x_{2,k}$ for all three PWA modes.

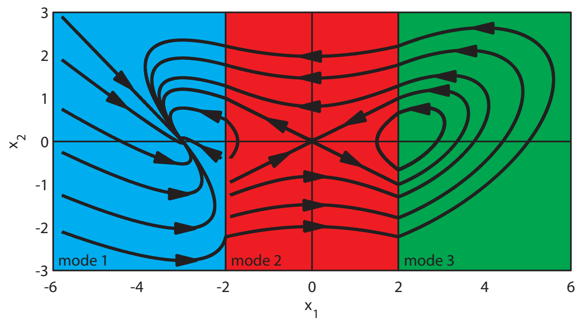


Figure 1: 3-mode PWA system

Qualitative PWA Model

A qualitative PWA model abstracts the continuously valued affine dynamics (1-2) symbolically. Thus, it merges the continuous dynamics with its discrete dynamics (mode changes) into one common discretely valued behavioral description. For this purpose, one defines *partitions* for the continuously valued state-, input- and output-space. The input- and output-space partitions can arise naturally through quantisation, for example, whenever one deals with real-world signals with low resolution (e.g. 4-bit A/D converter). In terms of the input-space, we only have to make sure, that the qualitative abstraction allows us to formulate the input inequality of the PWA guard condition (3). The state-space abstraction can either be derived recursively during qualitative reasoning (Kuipers 1994) or verification (Girard & Pappas 2006) or the partitions are specified explicitly prior compiling an automaton abstraction for the PWA model. Since we intend to use a qualitative model for fast on-line reasoning, we choose the second form and compute a so-called *stochastic automaton* from the PWA model that encodes the model's dynamics through a state machine with stochastic transition specification.

Stochastic Automaton PWA Model

A stochastic automaton (Lunze 1994; Bukharaev 1995; Schröder 2003) defines a tuple

$$\mathcal{A} = \langle \mathcal{X}, \mathcal{U}, \mathcal{Y}, P_{\mathcal{T}}, P_{\mathcal{O}} \rangle \ , \quad (5)$$

where $\mathcal{X} = \{X_1, \ldots, X_{N_x}\}$, $\mathcal{U} = \{U_1, \ldots, U_{N_u}\}$ and $\mathcal{Y} = \{Y_1, \ldots, Y_{N_y}\}$ denote the finite domains for the automaton state $\bar{x}$, input $\bar{u}$ and output $\bar{y}$, respectively. With $\bar{x}_k$, $\bar{u}_k$ and $\bar{y}_k$ we denote valuations of the state, input, and output at a particular *time-step* k. The behavior of the automaton is captured through the conditional *transition-* and *observation-probabilities*[2]

$$\begin{aligned} P_{\mathcal{T}}(\bar{x}_{k+1}, \bar{x}_k, \bar{u}_k) &= P(\bar{x}_{k+1}|\bar{x}_k, \bar{u}_k) \\ P_{\mathcal{O}}(\bar{y}_k, \bar{x}_k, \bar{u}_k) &= P(\bar{y}_k|\bar{x}_k, \bar{u}_k)\ . \end{aligned} \tag{6}$$

A stochastic automaton can be almost directly used as a *qualitative model* of our PWA system (1-3). The only entity that we have to add is a map $M : \mathcal{X} \times \mathcal{U} \rightarrow \{1, 2, \ldots, l\}$ that specifies the PWA mode for every qualitative state/input pair. This enables us to define the qualitative abstraction of a PWA system as an extended stochastic automaton through the tuple

$$\mathcal{A}_{pwa} = \langle \mathcal{X}, \mathcal{U}, \mathcal{Y}, P_{\mathcal{T}}, P_{\mathcal{O}}, M \rangle\ . \tag{7}$$

Automaton compilation: To compile a stochastic automaton $\mathcal{A}_{pwa}$ with pre-defined state/input/output-space partitions one has to compute the conditional probabilities $P_{\mathcal{T}}(\cdot) = P(X_i|X_j, U_\zeta)$ and $P_{\mathcal{O}}(\cdot) = P(Y_i|X_j, U_\zeta)$ for all triples $\{i, j, \zeta\}$ according to the PWA dynamics. For this purpose one assumes a uniform distribution for $\mathbf{x}_k \in X_j$ and $\mathbf{u}_k \in U_\zeta$ and computes the distribution on state and output space for $\mathbf{x}_{k+1}$ and $\mathbf{y}_k$. This can be done, for example, through sampling or hyper-box mapping (Schröder 2003). Compilation of the stochastic automaton is computationally expensive. However, once we have compiled a stochastic automaton $\mathcal{A}_{pwa}$ for the PWA system, we can use this automaton model to efficiently perform qualitative simulation, estimation or control.

State-Space Abstraction

Qualitative abstraction through stochastic automaton compilation requires us to divide the continuous state-space

$$\mathcal{D}_x = \bigcup_{i=1,\ldots,l} \mathcal{D}_{x,i} \subset \mathbb{R}^{n_x}$$

into a finite set of non-overlapping partitions $\{\mathcal{D}_{x,1}, \ldots, \mathcal{D}_{x,N_x}\}$ where each partition $\mathcal{D}_{x,i}$ represents a qualitative abstraction or *state* $X_i = [\mathcal{D}_{x,i}]$ of the stochastic automaton. This has to be done carefully, since the number N_x potentially increases exponentially with the dimension n_x of the continuous PWA dynamics. On the other hand, one has to provide sufficiently fine partitions to retain the characteristics of the continuous dynamics.

Grid-based abstraction

The simplest abstraction of a bounded domain $\mathcal{D}_x$ in state-space is to apply an n_x-dimensional grid with fixed or adaptive grid-size. A grid-based abstraction happens naturally, whenever one partitions each state-variable individually and obtains the overall partition of the state-space through the cross product. This often leads to unsatisfactory results since the number of partitions explodes as the dimension of the system increases and the associated qualitative models do not constrain possible behaviors well enough. Simulating such a model would predict an unnecessarily large number of *spurious behaviors*, i.e. behaviors that the original (PWA) system cannot show. Figure 2 illustrates this property for mode 2 of our autonomous PWA system. The eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ of the dynamic matrix A_2 uniquely partition the state-space, however, the grid is not conform with this separation. Take the three partitions X_1, X_2, X_3 of Fig. 2. The continuous behavior at mode 2 clearly allows transitions $\mathbf{x}_k \in X_1 \rightarrow \mathbf{x}_{k+1} \in X_2$ and $\mathbf{x}_k \in X_2 \rightarrow \mathbf{x}_{k+1} \in X_3$. A stochastic automaton would encode these facts in terms of the transition probabilities $P_{\mathcal{T}}(X_2, X_1) \neq 0$ and $P_{\mathcal{T}}(X_3, X_2) \neq 0$. As a consequence, a simulation on the basis of this stochastic automaton would predict the qualitative behavior $X_1 \rightarrow X_2 \rightarrow X_3$, a behavior that the original PWA system cannot show! Another difficulty with a grid-based abstraction is to select the appropriate grid-size. An upper bound for the number of partitions N_x will be most likely the limiting factor for grid-size selection. However, it is difficult to judge, whether the resulting abstraction is fine enough to capture the details of the mode's continuous dynamics unless one performs exhaustive simulation studies.

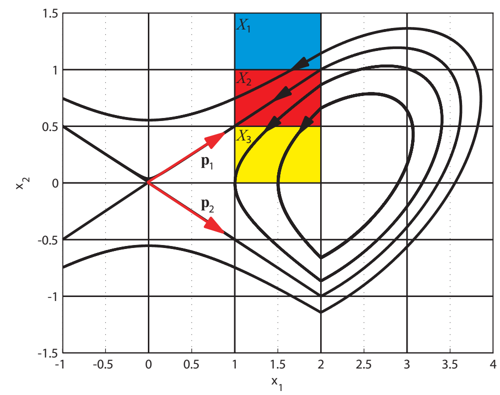


Figure 2: Boxed state-space abstraction

Besides abstracting the continuous dynamics, we have to make sure that the state-space partitions are also conform with the discrete dynamics of the PWA system. In detail, we have to ensure that the state-space abstraction allows us to formulate the state inequality of the PWA guard condition 3. PWA mode domains are in general polyhedral partitions, so that an abstraction through hyper-boxes in state-space can be inadequate.

These arguments illustrate that it is desirable to have a qualitative abstraction of the state-space that (1) respects the properties of the continuous dynamics as well as (2) captures the polyhedral specification of PWA mode domains.

[2] In general, one would define a *behavioral relation* $P_{\mathcal{Q}}(\bar{x}_{k+1}, \bar{y}_k, \bar{x}_k, \bar{u}_k) = P(\bar{x}_{k+1}, \bar{y}_k, |\bar{x}_k, \bar{u}_k)$ that defines $P_{\mathcal{T}}$ and $P_{\mathcal{O}}$ as its boundary distributions. However, the successor state $\mathbf{x}_{k+1}$ and output $\mathbf{y}_k$ of our PWA system are stochastically independent so that $P_{\mathcal{Q}}(\cdot) = P_{\mathcal{T}}(\cdot)P_{\mathcal{O}}(\cdot)$ holds and $P_{\mathcal{T}}$ and $P_{\mathcal{O}}$ represent the same information as $P_{\mathcal{Q}}$.

Qualitative abstraction of continuous dynamics

The example above indicates that we need a more general state-space separation technique that builds upon (non-overlapping) polyhedral partitions. We propose to use the *eigenvectors* or in general *hyper-planes* that are defined through the eigenvectors of the PWA dynamics (or their associated dynamic matrices $\mathbf{A}_i$) to partition the state-space into qualitatively distinct regions. For second order PWA dynamics at mode i that is characterised through the dynamic matrix $\mathbf{A}_i$ with eigenvalues $z_1, z_2, z_1 \neq z_2$ we can write

$$\mathbf{x}_{k+1} = \mathbf{A}_i \mathbf{x}_k + \mathbf{f}_i = \alpha_1 \mathbf{p}_1 z_1 + \alpha_2 \mathbf{p}_2 z_2 + \mathbf{f}_i \ ,$$

where $\mathbf{p}_j$ $(j = 1, 2)$ denotes the eigenvector for z_j and the parameters α_1 and α_2 are given through

$$\mathbf{x}_k = \alpha_1 \mathbf{p}_1 + \alpha_2 \mathbf{p}_2 \ .$$

It directly follows from Systems Theory (Hirsch & Smale 1974) that the eigenvectors, centred at the equilibrium point

$$\mathbf{x}_r = (\mathbf{I} - \mathbf{A}_i)^{-1} \mathbf{f}_i \ ,$$

partition the state-space into distinct regions. Each region is characterised through the signs of α_1 and α_2, clearly a qualitative distinction!

Eigenvector-partitioning works with eigenvectors for real-valued eigenvalues z_j, but not for complex-valued eigenvectors/eigenvalues as in mode 3 of our PWA example. In order to abstract these behaviors that spiral round the equilibrium point, we propose to partition the state-space into sectors differently. The complex-valued eigenvectors uniquely characterise the orientation of the elliptical behavior, whereas the eigenvalues determine its stability character. As a consequence, we can use the eigenvectors to determine the ellipse-axes for the spiralling behavior and use these axes to partition the state-space into 4 sectors. This abstraction enables us to uniquely characterise the direction of the behavior (clockwise or counter-clockwise) but not the mode's stability property. Ideally, one would want to use elliptical regions in state-space to specify a Lyapunov-function for an asymptotically stable equilibrium point, for example. However, the approximation of elliptical regions through polyhedral domains is impracticable. One way to overcome this difficulty is to use a combination of sectors and hyper-boxes to enclose the ellipsoids and use additional reasoning concepts (Hofbaur & Dourdoumas 2001) that go beyond the scope of standard stochastic automata theory.

An additional qualitative characterisation of the state-space is given through the sign of a state $\mathbf{x}_k$ or its components $x_{i,k}$, in particular. Consequently, we suggest to additionally partition the state-space according to the unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{n_x}$. Figure 3 shows the result of the combined sign, eigenvector and ellipse-axes based state-space separation for our PWA example along with possible trajectories.

Adding inputs and noise: Up to now, we used an autonomous PWA system. However, we intend to use a qualitative model for estimation and control and thus, we have to deal with a non-zero control input $\mathbf{u}_k = [u_{1,k}, \ldots, u_{n_u,k}]^T$

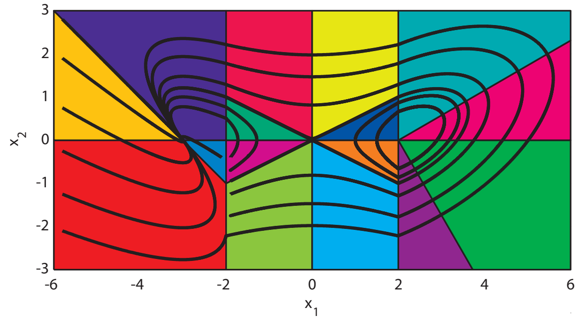


Figure 3: 3-mode PWA system with state-space partitions and system trajectories

and disturbances. We use a typical PWA notation (Kvasnica *et al.* 2006) and introduce bounded additive disturbances $\mathbf{w} = [w_1, \ldots, w_{n_x}]^T$ that act upon the state-variable through

$$\mathbf{x}_{k+1} = \mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k + \mathbf{f}_i + \mathbf{w}_k \ . \qquad (8)$$

We characterise the disturbance in terms of a bounding polytope $\mathcal{W}$, for example, a hyperbox that defines min/max values for each $w_i, i = 1, \ldots, n_x$.

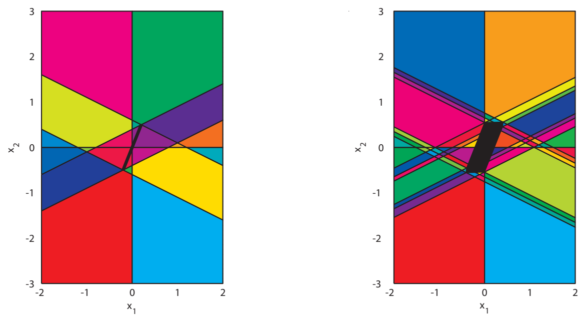
Let us deal with a scalar input u and an associated input vector $\mathbf{B}_i = [\mathbf{b}_i]$ for (8) first. A constant input $u_k = u^*$ shifts the equilibrium point $\mathbf{x}_r$ of mode i to

$$\mathbf{x}_r = (\mathbf{I} - \mathbf{A}_i)^{-1} (\mathbf{b}_i u_k + \mathbf{f}_i) \ . \qquad (9)$$

Therefore, an input $u_{min} \leq u_k \leq u_{max}$ at time-step k can be interpreted as shifting the origin for our eigenvalue/ellipse-axes based state-space separation according to (9). To partition the state-space into qualitatively distinct regions, we apply our separation scheme at the two extremal points of $\mathbf{x}_r$ ($u_k = u_{min}$ and $u_k = u_{max}$). Figure 4a shows the resulting separation for the PWA mode 2 with an input vector $\mathbf{b}_2 = [0.0992 \quad 0.0075]^T$ and $u_{min} = -1.0$, $u_{max} = 1.0$. The bar at the origin indicates the region for $\mathbf{x}_r$.

Dealing with multiple inputs is straightforward. Through combination of all min/max values for the inputs, we obtain a region (polytope) $\mathcal{X}_r$ in state-space for $\mathbf{x}_r$. We then perform the state-space separation at all extremal points of $\mathcal{X}_r$.

A disturbance $\mathbf{w} = [w_1, \ldots, w_{n_x}]^T$ can be handled in two ways. First, we can treat each disturbance w_j as an additional input (with input vector $\mathbf{e}_j$) and use the noise bounds to enlarge $\mathcal{X}_r$. This adds additional 2^{n_x} extremal points to $\mathcal{X}_r$ and thus introduces many additional partitions of the state-space. Figure 4b shows the resulting separation for bounded noise $|w_i| \leq 0.01, i = 1, 2$. The black region in the center indicates $\mathcal{X}_r$. The second way to deal with disturbances is to perform state-space separation for deterministic inputs only and include the effects of disturbances through the probability specifications of the resulting stochastic automaton. We prefer this approach since it keeps the number of state-space partitions small.



(a) Separation for $|u| \leq 1$

(b) Separation for $|u| \leq 1$ and $|w_i| \leq 0.01, i = 1, 2$

Figure 4: State-space separation for mode 2

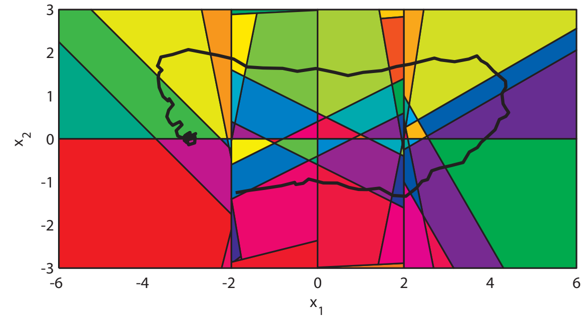


Figure 5: Overall state-space abstraction and PWA trajectory example

Abstraction of PWA mode changes

The PWA model operates on discrete-time. As a result, switching does not occur exactly on the mode-boundary but within its vicinity. Therefore, it is essential to capture the regions in state-space where switching can occur. Additionally, we observed that simulation and particularly PWA mode estimation results improve whenever one specifies also those regions in state-space from where we can reach the switching domains within one time-step. Both domains can be computed easily through a 1-step forward/backward reachability analysis. As before, we can include disturbances in two ways. Directly, in terms of an inclusive reachability analysis, or indirectly through the transition probabilities of the stochastic automaton. Again, we prefer the latter approach.

Combining behavior-based abstractions

The overall state-space abstraction combines the partitions from the dynamics-based separation scheme with the partitions for mode-change characterisation and computes a set of N_x non-overlapping polytopes $\{X_1, \ldots, X_{N_x}\}$ that partitions the continuous state-space into qualitatively distinct regions. Figure 5 shows the partitions for our non-autonomous PWA system with a scalar input u that acts through the input vectors

$$\mathbf{b}_1 = [0.0775\ 0.0585]^T,\ \mathbf{b}_2 = [0.0992\ 0.0075]^T,$$
$$\mathbf{b}_3 = [0.0515\ 0.0022]^T$$

along with a trajectory for $u_k = \sin(0.1k)$ and disturbance $|w_i| \leq 0.1, i = 1, 2$.

Qualitative Estimation

Our main application of the stochastic automaton model is to perform hybrid estimation which can be formulated through the following recursive belief update process

$$b^{(j)}_{k|k-1} = \sum_{X_i \in \mathcal{X}} P_{\mathcal{T}}(X_j, X_i, U_\zeta) b^{(i)}_{k-1} \quad (10)$$

$$b^{(j)}_{k} \propto P_{\mathcal{O}}(Y_\kappa | X_j, U_\xi) b^{(j)}_{k|k-1} \quad (11)$$

that computes the belief (or probability) $b^{(j)}_k$ for $\mathbf{x}_k \in X_j$, given the qualitatively abstracted input values and measurement, i.e. $\mathbf{u}_{k-1} \in U_\zeta,\ \mathbf{u}_k \in U_\xi$ and $\mathbf{y}_k \in Y_\kappa$.

For our example we use an input abstraction with 3 partitions, and a 4-bit measurement resolution ($2^4 = 16$ partitions). Figure 6 shows the PWA mode estimation result that we obtained through selecting the PWA mode with the maximum cumulated belief for the trajectory of Fig. 5 that starts at $\mathbf{x}_0 = [-1.9 \quad -1.25]^T$. We used the non-autonomous system with the input $u_k = \sin(0.1k)$ and disturbances $|w_i| \leq 0.1, i = 1, 2$. The estimation starts with no state knowledge, thus $b^{(j)}_0 = 1/N_x$ and requires some steps to focus. Estimation with our state-space abstraction (69 partitions) provides an estimation result that is similar to a grid-based abstraction with $9 \times 9 = 81$ partitions.

To judge estimation quality better, we performed additional experiments with random initial states. In order to highlight the effects of state-space abstraction and stochastic automaton estimation we used an idealised setting with perfect initial knowledge. This eliminates the focusing process at the beginning of an experiment. We generated 1000 random initial states and simulated the non-autonomous PWA system for 100 time-steps[3]. Hybrid estimation with our state-space separation scheme provides on average a PWA mode estimation error of 3.55%. In comparison, a boxed scheme provided 4.25% for a 6×6 grid, 3.10% for a 9×9 grid and 2.97% for a 12×12 grid.

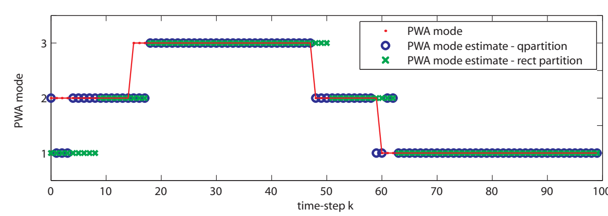


Figure 6: PWA-mode estimation result

[3]Most trajectories in our system get absorbed into the equilibrium point of mode 1 after about 100 time-steps.

Conclusion

The usual approach to abstract continuously-valued state-spaces is to use a grid-based abstraction of non-overlapping hyper-boxes. This requires one to select an appropriate grid-size that (a) is sufficiently fine to capture the system's dynamics and (b) is sufficiently coarse to keep the number of partitions manageable. To overcome this difficulty, we proposed a state-space abstraction scheme for PWA systems that uses *qualitative features* of the system's dynamics to partition the state-space into behavioral distinct regions. We present this abstraction for 2-dimensional systems to introduce the concepts concisely, however, we should note that it is equally well suited for higher order systems.

We used this abstraction to compile a stochastic automaton model for the PWA system and evaluated its estimation capabilities. We performed a random set of experiments and obtained evidence that our abstraction scheme leads to an estimation quality that is comparable with hyper-box abstractions that use a similar number of state-space partitions. However, in contrast to hyper-box approximations where one has do decide the grid resolution manually, we provide a quantisation scheme that *automatically selects an appropriate resolution* according to the system's dynamics. A more detailed analysis of qualitative simulation and estimation capabilities, in particular for higher order systems, is subject to ongoing research.

Our main motivation for a stochastic automata encoding is to formulate hybrid estimation and control schemes that use the discrete abstraction to quickly pre-select feasible and good estimation/control candidates. A consecutive numerical refinement can either validate an estimation/control candidate or identify spurious solutions and reject them. Our initial studies for such approaches (Kleissl & Hofbaur 2005; Kleissl 2006; Richter 2006) showed, that a good qualitative abstraction that avoids spurious behaviors is essential for this strategy. The results of this paper are an important step towards our proposed estimation and control schemes. However, they surely are also valuable for other works in hybrid systems analysis and verification.

Acknowledgement

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References

Alur, R., and Dill, D. 1994. A theory of timed automata. *Theoretical Computer Science* 126:183–235.

Alur, R.; Henzinger, T.; Lafferriere, G.; and Papas, G. 2000. Discrete abstractions of hybrid systems. *Proceedings of the IEEE* 88(7):971–984.

Bailey-Kellogg, C.; Zhao, F.; and Yip, K. 1996. Spatial aggregation: Language and applications. In *Proceedings of the 10th International Workshop on Qualitative Reasoning (QR96)*, 3–11.

Bukharaev, R. 1995. *Theorie der stochastischen Automaten*. Teubner Verlag.

Girard, A., and Pappas, G. 2006. Verification using simulation. In Hespanha, J., and Tiwari, A., eds., *Hybrid Systems: Computation and Control, HSCC 2006*, volume 3927 of *Lecture Notes in Computer Science*. Springer Verlag. 272–286.

Hirsch, M., and Smale, S. 1974. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press.

Hofbaur, M. W., and Dourdoumas, N. 2001. Lyapunov based reasoning methods. *IEEE Transactions on Systems, Man, and Cybernetics - Part A* 31(4):546–558.

Kleissl, W., and Hofbaur, M. W. 2005. A qualitative model for hybrid control. In *Proceedings of the 19th International Workshop on Qualitative Reasoning (QR05)*, 8–16.

Kleissl, W. 2006. *Control of Multi Component Hybrid Systems through Qualitative Pre-Selection*. PhD thesis, Faculty of Electrical Engineering, Graz University of Technology, Graz, Austria.

Kuipers, B. 1994. *Qualitative Reasoning: Modeling and Simulation with Incomplete Knowledge*. Cambridge, MA: MIT Press.

Kvasnica, M.; Grieder, P.; Baotic, M.; and Christophersen, F. 2006. Multi-parametric toolbox (mpt). Technical report, Institut für Automatic, ETH - Swiss Federal Intstitute of Technology Zürich.

Lunze, J.; Nixdorf, B.; and Schröder, J. 1999. Deterministic discrete-event representations of linear continuous-variable systems. *Automatica* 35(3):395–406.

Lunze, J. 1994. Qualitative modelling of linear dynamical systems with quantized state measurements. *Automatica* 30(3):417–431.

Richter, S. 2006. *Probabilistic Hybrid Estimation for PWA Systems through Qualitative Pre-Selection*. MSc thesis, Institute of Automation and Control, Graz University of Technology, Graz, Austria.

Schröder, J. 2003. *Modelling, State Observation and Diagnosis of Quantized Systems*, volume 282 of *LNCIS*. Springer Verlag.

Tabuada, P. 2007. Approximate simulation relations and finite abstractions of quantized control systems. In Bemporad, A.; Bicchi, A.; and Buttazzo, G., eds., *Hybrid Systems: Computation and Control, HSCC 2007*, volume 4416 of *Lecture Notes in Computer Science*. Springer Verlag. 529–542.

Tiwari, A. 2003. Approximate reachability for linear systems. In Maler, O., and Pnueli, A., eds., *Hybrid Systems: Computation and Control, HSCC 2003*, volume 2623 of *Lecture Notes in Computer Science*. Springer Verlag. 514–525.

Weld, D., and de Kleer, J., eds. 1990. *Qualitative Reasoning about Physical Systems*. Morgan Kaufmann.

Yip, K., and Zhao, F. 1996. Spatial aggregation: Theory and applications. *Journal of Artificial Intelligence Research* 5:1–26.

Yip, K. 1991. Understanding complex dynamics by visual and symbolic reasoning. *Artificial Intelligence* 51:179–221.