Galerkin-BEM for elastodynamics in the time domain based on the convolution quadrature method

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Abstract

The Boundary Element Method in time domain is well suited to treat wave propagation problems, especially, in unbounded domains. The time convolution in elastodynamics can be performed analytically or, as proposed here, with the convolution quadrature method. The spatial discretisation of the underlying time-dependent boundary integral equations is, in engineering, mostly done via the collocation method. Here, the symmetric Galerkin approach is used. As expected from the mathematical analysis for, e.g., the Helmholtz equation, this approach shows a more robust behavior compared to the collocation method. In the proposed presentation the formulation and several numerical studies will be shown.

Keywords: Symmetric Galerkin boundary element method, Convolution Quadrature Method, Time domain

1. Introduction

The Boundary Element Method (BEM) in time domain is well suited to treat wave propagation problems. Especially, the implicit fulfillment of the Sommerfeld radiation condition makes this method superior to other domain based discretization methods if unbounded domains are considered. The mathematical background of the underlying boundary integral equations for time dependent problems may be found in [1]. Nowadays in engineering, the spatial discretization of those time-dependent boundary integral equations is mostly done via the collocation method. But also Galerkin type approaches exists which are mostly used for time independent problems. For an overview see [2].

For the time discretization there exist in principle two approaches. Firstly, if time dependent fundamental solutions are available, the usage of ansatz functions with respect to time yields a time stepping procedure after an analytical time integration. This technique has been proposed by Mansur [3]. Secondly, the Convolution Quadrature Method (CQM) developed by Lubich [4], [5] can be used to establish the same time stepping procedure as obtained by a direct time integration, see [6]. Contrary to the approach from Mansur for this methodology only the Laplace domain fundamental solutions have to be used and the time integration is performed numerically. Hence, this approach can easily be extended to the inelastic [7] or anisotropic [8] case where the fundamental solutions are only available in the Laplace or Fourier domain.

Here, the CQM based approach will be used in a symmetric Galerkin BE formulation. This is motivated by the positive results of the Galerkin approach in statics as well as for the Helmholtz equation [9]. It will be studied whether the stability of the time stepping procedure is improved compared to the collocation based approach. Further, the quality of the results using the same spatial discretization will be compared to the collocation approach. These studies are a preparation to apply the CQM based Galerkin approach to visco- or poroelastic media.

2. Boundary integral equations

2.1. Problem statement

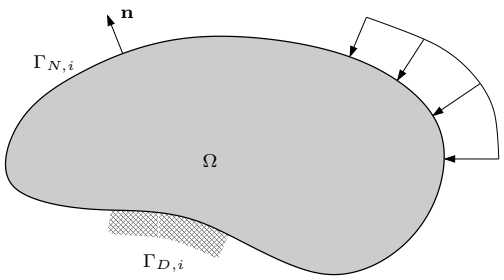

Figure 1: Problem statement

Let the time $T > 0$ be fixed and let $\Omega \subset \mathbb{R}^3$ be a domain with a compact boundary $\Gamma = \Gamma_{D,i} \cup \Gamma_{N,i}$ with the degrees of freedom $i = 1, 2, 3$ (Fig. 1). In $\Gamma_{D,i}$ and $\Gamma_{N,i}$ the notation $()_{,i}$ indicates that at each boundary point the type of boundary condition can be different in each direction i. Assuming further homogeneity and a linear elastic material (Young's modulus E, mass density ϱ, and Poisson's ratio ν) with a linear stress-strain relation, the dynamic behavior of a structure as well as the wave propagation in the domain $\Omega \times (0, T)$ is governed by the equation of motion

$$\left(c_1^2 - c_2^2\right) \operatorname{grad}\operatorname{div} \mathbf{u} + c_2^2 \, \Delta \mathbf{u} + \frac{\mathbf{b}}{\varrho} = \frac{\partial^2}{\partial t^2} \mathbf{u} \tag{1}$$

where Δ denotes, as usual, the Laplace operator. The vector $\mathbf{u} = \mathbf{u}(\mathbf{x}; t)$ describes the displacement field at some point $\mathbf{x} \in \mathbb{R}^3$ and time t. Furthermore, $\mathbf{b} = \mathbf{b}(\mathbf{x}; t)$ is the body force vector per unit mass, and c_1, c_2 are the dilatational and shear wave velocities, respectively

$$c_1^2 = \frac{E(1-\nu)}{\varrho(1-2\nu)(1+\nu)}\,, \qquad c_2^2 = \frac{E}{2\varrho(1+\nu)}\,. \tag{2}$$

Under the assumption of vanishing body forces as well as vanishing initial conditions and by taking into account some

component-wise prescribed Dirichlet data $g_{D,i}(\mathbf{x};t)$ on $\Gamma_{D,i}$ and Neumann data $g_{N,i}(\mathbf{x};t)$ on $\Gamma_{N,i}$, respectively, the above system reads as

$$\begin{aligned}(\mathcal{L}\mathbf{u})(\mathbf{x};t) &= \mathbf{0} && \forall\,(\mathbf{x};t)\in\Omega\times(0,T)\\ u_i(\mathbf{x};t) &= g_{D,i}(\mathbf{x};t) && \forall\,(\mathbf{x};t)\in\Gamma_{D,i}\times(0,T)\\ q_i(\mathbf{x};t) &= g_{N,i}(\mathbf{x};t) && \forall\,(\mathbf{x};t)\in\Gamma_{N,i}\times(0,T)\end{aligned} \tag{3}$$

with the linear (hyperbolic) partial differential operator $\mathcal{L} = \left(c_1^2-c_2^2\right)\operatorname{grad}\operatorname{div}+c_2^2\Delta-\partial^2/\partial t^2$. In Eqn (3), $q_i(\mathbf{x},t)$ denotes the i-th component of the traction vector $\mathbf{q}(\mathbf{x},t)=\boldsymbol{\sigma}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})$ defined by the product of the Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{x},t)$ and the outward normal vector $\mathbf{n}(\mathbf{x})$.

2.2. Time-dependent boundary integral equations

The derivation of boundary integral formulations starts with the weak form of Eqn (3)

$$\int_{\Omega\times(0,t)}\mathbf{U}^*(\mathbf{x},\mathbf{y};t-\tau)\;(\mathcal{L}\mathbf{u})(\mathbf{y};\tau)\;\mathrm{d}\mathbf{y}\,\mathrm{d}\tau=\mathbf{0}\;. \tag{4}$$

Beside the spatial integration over the domain Ω, Eqn (4) contains a time integration of convolution type with respect to τ over the time interval $(0,t)$. Further, $\mathbf{U}^*(\mathbf{x},\mathbf{y};t-\tau)$ denotes the fundamental solution which can be used as a weighting function to obtain a boundary integral representation. For an unbounded domain $\mathbb{R}^3$ the fundamental solution has the property

$$(\mathcal{L}\;\mathbf{U}^*)(\mathbf{x},\mathbf{y};t,\tau)=\boldsymbol{\delta}(\mathbf{x}-\mathbf{y};t-\tau) \tag{5}$$

with $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$ and $t,\tau\in(0,T)$. In Eqn (5), $\boldsymbol{\delta}(\mathbf{x}-\mathbf{y};t-\tau)$ denotes the Delta-distribution. It is obvious that the solution of Eqn (5) can only be given in a distributional sense.

If vanishing initial conditions are assumed the application of Green's theorem to Eqn (4) yields the representation formula

$$\begin{aligned}\mathbf{u}(\tilde{\mathbf{x}};t)=&\int_{\Gamma\times(0,t)}\mathbf{U}^*(\tilde{\mathbf{x}},\mathbf{y};t-\tau)\;\left(\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}\mathbf{u}\right)(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&-\int_{\Gamma\times(0,t)}\left(\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}\mathbf{U}^*\right)(\tilde{\mathbf{x}},\mathbf{y};t-\tau)\;\mathbf{u}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\end{aligned} \tag{6}$$

with $\tilde{\mathbf{x}}\in\Omega$, $\mathbf{y}\in\Gamma$ and $t\in(0,T)$. Equation (6) contains only spatial integrals on the boundary Γ. Moreover, causality implies integral equations of Volterra type in the time variable and time-invariance implies that they must be of convolution type in time.

The occurring operator $\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}$ is a trace operator related to the outward normal vector $\mathbf{n}(\mathbf{y})$. In elastodynamics it represents the stress-strain relation based on Hooke's law. Therefore, applying $\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}$ with respect to $\mathbf{y}$ to the displacement field $\mathbf{u}(\mathbf{y};t)$ yields the relation

$$\left(\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}\mathbf{u}\right)(\mathbf{y};t)=\mathbf{q}(\mathbf{y};t)=\boldsymbol{\sigma}(\mathbf{y};t)\cdot\mathbf{n}(\mathbf{y})\;. \tag{7}$$

Now, by taking in Eqn (6) the limit $\Omega\ni\tilde{\mathbf{x}}\to\mathbf{x}\in\Gamma$ the first integral equation

$$\begin{aligned}\mathbf{u}(\mathbf{x};t)=&\int_{\Gamma\times(0,t)}\mathbf{U}^*(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{q}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&-\int_{\Gamma\times(0,t)}\left(\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}\mathbf{U}^*\right)(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{u}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&+\frac{1}{2}\mathbf{u}(\mathbf{x};t)\end{aligned} \tag{8}$$

is achieved. In Eqn (8), the singular behavior of the kernel functions has to be considered. Hence, the first integral in Eqn (8) is weakly and the second one is strongly singular. A second integral equation can be derived by the application of the operator $\lim_{\Omega\ni\tilde{\mathbf{x}}\to\mathbf{x}\in\Gamma}\mathcal{T}_{\tilde{\mathbf{x}}}^{\mathbf{n}(\mathbf{x})}$ to Eqn (6)

$$\begin{aligned}\mathbf{q}(\mathbf{x};t)=&\int_{\Gamma\times(0,t)}\left(\mathcal{T}_{\mathbf{x}}^{\mathbf{n}(\mathbf{x})}\mathbf{U}^*\right)(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{q}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&-\int_{\Gamma\times(0,t)}\left(\mathcal{T}_{\mathbf{x}}^{\mathbf{n}(\mathbf{x})}\mathcal{T}_{\mathbf{y}}^{\mathbf{n}(\mathbf{y})}\mathbf{U}^*\right)(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{u}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&+\frac{1}{2}\mathbf{q}(\mathbf{x};t)\;.\end{aligned} \tag{9}$$

Note, that the limiting process to obtain Eqn (9) is the same as for the first boundary integral equation (8). Due to the singular kernels in Eqn (8) the first integral in Eqn (9) is strongly and the second one is hypersingular. Now, by introducing the operators

$$\begin{aligned}&(\mathcal{V}*\mathbf{w})_\Gamma(\mathbf{x};t):=\\&\quad\int_{\Gamma\times(0,t)}\mathbf{U}^*(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{w}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&(\mathcal{K}'*\mathbf{w})_\Gamma(\mathbf{x};t):=\\&\quad\int_{\Gamma\times(0,t)}\left(\mathcal{T}_{\mathbf{x}}^{n(\mathbf{x})}\mathbf{U}^*\right)(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{w}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&(\mathcal{K}*\mathbf{w})_\Gamma(\mathbf{x};t):=\\&\quad\int_{\Gamma\times(0,t)}\left(\mathcal{T}_{\mathbf{y}}^{n(\mathbf{y})}\mathbf{U}^*\right)(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{w}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\\&(\mathcal{D}*\mathbf{w})_\Gamma(\mathbf{x};t):=\\&\quad\int_{\Gamma\times(0,t)}\left(-\mathcal{T}_{\mathbf{x}}^{n(\mathbf{x})}\mathcal{T}_{\mathbf{y}}^{n(\mathbf{y})}\mathbf{U}^*\right)(\mathbf{x},\mathbf{y};t-\tau)\;\mathbf{w}(\mathbf{y};\tau)\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}\tau\end{aligned} \tag{10}$$

Eqns (8) and (9) can be written more compact

$$\left[\begin{pmatrix}\frac{1}{2}\mathcal{I}-\mathcal{K} & \mathcal{V}\\ \mathcal{D} & \frac{1}{2}\mathcal{I}+\mathcal{K}'\end{pmatrix}*\begin{pmatrix}\mathbf{u}\\ \mathbf{q}\end{pmatrix}\right]_\Gamma(\mathbf{x};t)=\begin{pmatrix}\mathbf{u}(\mathbf{x};t)\\ \mathbf{q}(\mathbf{x};t)\end{pmatrix}\;. \tag{11}$$

In Eqns (10) and (11) the $*$ denotes the convolution in time. In Eqn (10), the operators can sequentially be titled as single layer, adjoint double layer, and double layer potential. The last operator is the so-called hyper-singular integral operator.

2.3. Symmetric Galerkin formulation

To obtain a symmetric Galerkin formulation for mixed boundary value problems, the first boundary integral equation is used only on the Dirichlet part $\Gamma_{D,i}$ while the second one is used only on the Neumann part $\Gamma_{N,i}$. Note, because here a vectorized problem has to be solved on each boundary point different types of boundary data in each direction may be prescribed. This yields

$$(\mathcal{V}*q)_{\Gamma,i}(\mathbf{x};t)-(\mathcal{K}*u)_{\Gamma,i}(\mathbf{x};t)=\frac{1}{2}g_{D,i}(\mathbf{x};t) \tag{12}$$

$$\left(\mathcal{K}'*q\right)_{\Gamma,i}(\mathbf{x};t)+(\mathcal{D}*u)_{\Gamma,i}(\mathbf{x};t)=\frac{1}{2}g_{N,i}(\mathbf{x};t)\;. \tag{13}$$

Equation (12) is only valid for $\mathbf{x}\in\Gamma_{D,i}$ while for Eqn (13) $\mathbf{x}\in\Gamma_{N,i}$ holds. Now, the displacements $u_i(\mathbf{y};t)$ and the tractions $q_i(\mathbf{y};t)$ are decomposed into $u_i=\tilde{u}_i+\tilde{g}_{D,i}$ with $\tilde{u}_i=0,\tilde{g}_{D,i}=g_{D,i}$ for $\mathbf{y}\in\Gamma_{D,i}$ and $q_i=\tilde{q}_i+\tilde{g}_{N,i}$ with $\tilde{q}_i=0,\tilde{g}_{N,i}=g_{N,i}$ for $\mathbf{y}\in\Gamma_{N,i}$, respectively. Note that $\tilde{g}_{D,i}$ and $\tilde{g}_{N,i}$ are arbitrary but fixed extensions of the prescribed

Dirichlet- and Neumann-data. Inserting these decompositions into Eqn (12) and (13) result in

$$\begin{aligned}(\mathcal{V}*\tilde{q})_{\Gamma_{D,i}}(\mathbf{x};t)-(\mathcal{K}*\tilde{u})_{\Gamma_{N,i}}(\mathbf{x};t)&=\\ ((\frac{1}{2}\mathcal{I}+\mathcal{K})*\tilde{g}_D)_{\Gamma,i}(\mathbf{x};t)-(\mathcal{V}*\tilde{g}_N)_{\Gamma,i}(\mathbf{x};t)&\\ (\mathcal{K}'*\tilde{q})_{\Gamma_{D,i}}(\mathbf{x};t)+(\mathcal{D}*\tilde{u})_{\Gamma_{N,i}}(\mathbf{x};t)&=\\ ((\frac{1}{2}\mathcal{I}-\mathcal{K}')*\tilde{g}_N)_{\Gamma,i}(\mathbf{x};t)-(\mathcal{D}*\tilde{g}_D)_{\Gamma,i}(\mathbf{x};t)&\end{aligned} \tag{14}$$

with the unknown data $\tilde{u}_i$ on $\Gamma_{N,i}$ and $\tilde{q}_i$ on $\Gamma_{D,i}$, respectively.

By defining the bilinear form [1]

$$a(\tilde{\mathbf{q}},\tilde{\mathbf{u}};\mathbf{w},\mathbf{v})=\int_0^T\sum_{i=1}^{3}\Big[\langle\mathcal{V}*\tilde{q},w\rangle_{\Gamma_{D,i}}-\langle\mathcal{K}*\tilde{u},w\rangle_{\Gamma_{D,i}}+\big\langle\mathcal{K}'*\tilde{q},v\big\rangle_{\Gamma_{N,i}}+\langle\mathcal{D}*\tilde{u},v\rangle_{\Gamma_{N,i}}\Big]\,\mathrm{d}t \tag{15}$$

using appropriate test-functions $\mathbf{w},\mathbf{v}$ and by defining the linear form

$$F(\mathbf{w},\mathbf{v})=\int_0^T\sum_{i=1}^{3}\Big[\langle(\frac{1}{2}\mathcal{I}+\mathcal{K})*\tilde{g}_D-\mathcal{V}*\tilde{g}_N,w\rangle_{\Gamma_{D,i}}+\langle(\frac{1}{2}\mathcal{I}-\mathcal{K}')*\tilde{g}_N-\mathcal{D}*\tilde{g}_D,v\rangle_{\Gamma_{N,i}}\Big]\,\mathrm{d}t \tag{16}$$

the system of boundary integral equations in Eqn (14) is equivalent to the variational problem:

Find $(\tilde{\mathbf{q}},\tilde{\mathbf{u}})$ *such that*

$$a(\tilde{\mathbf{q}},\tilde{\mathbf{u}};\mathbf{w},\mathbf{v})=F(\mathbf{w},\mathbf{v}) \tag{17}$$

is satisfied for all test-functions $(\mathbf{w},\mathbf{v})$.

3. Boundary element formulation

3.1. Spatial discretization

For the approximate solution of Eqn (17) a triangulation of the boundary $\Gamma=\cup_{k=1}^{n}\tau_k$ is introduced, i.e., the boundary Γ is the union of n boundary elements τ_k. Further, it is assumed that in each boundary element τ_k the unknown boundary data belong either to $\Gamma_{D,i}$ or to $\Gamma_{N,i}$ in each direction i. With respect to this triangulation the subspaces

$$\begin{aligned}S^{\alpha}_{h,i}(\Gamma_{D,i})&=\operatorname{span}\{\varphi^{\alpha}_{i,k}\}_{k=1}^{n_i}\\ S^{\beta}_{h,i}(\Gamma_{N,i})&=\operatorname{span}\{\varphi^{\beta}_{i,k}\}_{k=1}^{m_i}\end{aligned} \tag{18}$$

are defined containing n_i polynomial shape functions φ^{α} of order α and m_i polynomials φ^{β} of order β. The subspaces' dimensions n_i and m_i correspond to the number of unknowns on $\Gamma_{D,i}$ and $\Gamma_{N,i}$. Therefore, the unknown Dirichlet data and the unknown Neumann data can be approximated by

$$\begin{aligned}\tilde{q}^{\alpha}_{h,i}(\mathbf{x};t)&=\sum_{k=1}^{n_i}q_{i,k}(t)\;\varphi^{\alpha}_{i,k}(\mathbf{x})\\ \tilde{u}^{\beta}_{h,i}(\mathbf{x};t)&=\sum_{k=1}^{m_i}u_{i,k}(t)\;\varphi^{\beta}_{i,k}(\mathbf{x})\;.\end{aligned} \tag{19}$$

In the following, Neumann data are approximated by constant functions and Dirichlet data by linear functions, i.e., $\alpha=0$, $\tilde{q}_h=\tilde{q}^0_h$ and $\beta=1$, $\tilde{u}_h=\tilde{u}^1_h$. Further, the shape functions approximating the geometry are chosen of the same order as the Dirichlet data, i.e., the geometry is also approximated with linear polynomials.

3.2. Time discretization

After the spatial discretization the system of boundary integral equations has to be discretized also in the time variable. Here, as a time stepping procedure the Convolution Quadrature Method (CQM) proposed by Lubich is chosen. This is advantageous due to the fact, that the CQM deals only with the Laplace transformed fundamental solutions, which makes it also attractive for handling problems where fundamental solutions are only known in Laplace domain.

The goal is the computation of the convolution integral

$$y(t)=f*g=\int_0^t f(t-\tau)\;g(\tau)\,\mathrm{d}\tau\;. \tag{20}$$

If the Laplace transform of the function $f(t)$ is known and by dividing the time into M intervals of equal step size Δt the convolution can be approximated for the time step $t_m=m\Delta t$ by

$$y(m\Delta t)=\sum_{k=0}^{m}\omega^{m-k}(\Delta t)\;g(k\Delta t) \tag{21}$$

with the integration weights ω^m. These integration weights can be determined by

$$\omega^m(\Delta t)=\frac{\mathscr{R}^{-m}}{L}\sum_{\ell=0}^{L-1}\hat{f}\left(\frac{\gamma(\mathscr{R}\mathrm{e}^{\mathrm{i}\ell\frac{2\pi}{L}})}{\Delta t}\right)\mathrm{e}^{-\mathrm{i}m\ell\frac{2\pi}{L}}\;. \tag{22}$$

From Eqn (22) it is obvious that the integration weights depend only on the Laplace transform of the function f denoted by $\hat{f}(s)=\mathscr{L}_s\{f\}$ with the complex Laplace variable s. The parameters $\mathscr{R}$ and L depend mainly on the number of time steps M and are in the following chosen as $\mathscr{R}=10^{-5/2(M-1)}$ and $L=M-1$. Moreover, this choice allows the computation of the integration weights via a technique similar to the Fast Fourier Transformation. Finally, the term $\gamma(\cdot)$ represents the characteristic function of the underlying multistep method, e.g., a BDF2. More details about the Convolution Quadrature Method and the choice of the used parameters can be found in [4], [5], and [6].

Now, the convolution quadrature in Eqn (21) with the definition of the integration weights given in Eqn (22) is applied to the bilinear form in Eqn (15) and the linear form in Eqn (16). For example, an entry corresponding to the single layer potential can be computed for a time t_m and some fixed directions i and j as

$$\begin{aligned}&\langle(\mathcal{V}*\tilde{q}_h)(\mathbf{x};t_m),w_h(\mathbf{x})\rangle_{\Gamma_{D,i}}\\ &=\sum_{k}^{n_i}\sum_{\ell}^{n_j}\int_0^{t_m}\int_{\operatorname{supp}(\varphi^0_k)}\varphi^0_{i,k}(\mathbf{x})\;\times\\ &\quad\int_{\operatorname{supp}(\varphi^0_\ell)}U^*_{ij}(\mathbf{x},\mathbf{y};t_m-\tau)\;q_{j,\ell}(\tau)\;\varphi^0_{j,\ell}(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}}\,\mathrm{d}\tau\\ &\approx\sum_{k}^{n_i}\sum_{\ell}^{n_j}\sum_{p=0}^{m}\omega^{m-p}_{ij,k\ell}(\hat{\mathbf{U}}^*,\Delta t)\;q_{j,\ell}(p\;\Delta t)\\ &=\sum_{k}^{n_i}\sum_{\ell}^{n_j}\sum_{p=0}^{m}\mathsf{V}^{m-p}_{ij}[k,\ell]\;q_{j,\ell}(p\;\Delta t)\end{aligned} \tag{23}$$

with the integration weights

$$\omega^n_{ij,k\ell}=\frac{\mathscr{R}^{-n}}{L}\int_{\operatorname{supp}(\varphi^0_k)}\int_{\operatorname{supp}(\varphi^0_\ell)}\varphi^0_{i,k}(\mathbf{x})\;\times\sum_{l=1}^{L-1}\hat{U}^*_{ij}\left(\mathbf{x},\mathbf{y};\frac{\gamma(\mathscr{R}\mathrm{e}^{\mathrm{i}l\frac{2\pi}{L}})}{\Delta t}\right)\mathrm{e}^{-\mathrm{i}nl\frac{2\pi}{L}}\;\varphi^0_{j,\ell}(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}}\;. \tag{24}$$

Note, that in Eqn (23) *as well as throughout the entire paper no summation convention has been used.*

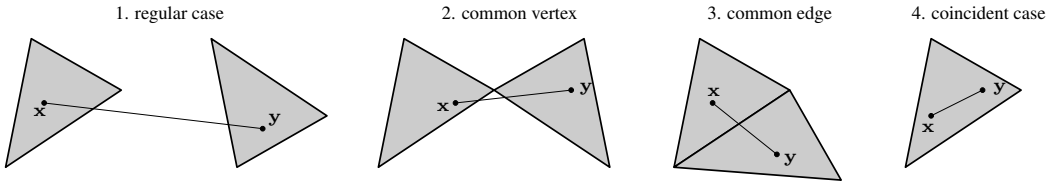

Figure 2: Singularities

3.3. Singular Integrals

The fundamental solution $\mathbf{U}^*(\mathbf{x},\mathbf{y};t-\tau)$ as well as the Laplace transformed fundamental solution $\hat{\mathbf{U}}^*(\mathbf{x},\mathbf{y};s)$ are in principle functions of type $1/r^n$ with $r=|\mathbf{x}-\mathbf{y}|$. The exponent n corresponds to the integration kernel and determines the kind of singularity. It is one in the single layer potential, two in the double layer potential, and three in the hypersingular operator, respectively. Therefore, attention must be paid to cases where $\mathbf{x}$ reaches $\mathbf{y}$, i.e., $r\to 0$. As shown in Fig. 2 in Galerkin methods four cases must be distinguished.

Obviously, in the regular case always $r\neq 0$ holds. So, this case can be treated without any modifications to the integral kernels and the integrations can be carried out with standard Gaussian quadrature rules. Without going too much into detail the other three cases require special integration techniques.

Basically, there exist two approaches to handle the occurring singularities. The first one is the analytical treatment of the singularity which demands a detailed knowledge about the integral kernel. The second approach is the one used here and was developed mainly by Sauter [10]. These formulas were developed using as less information of the kernels as possible to achieve formulas which are applicable to most integral operators. Finally, a numerical treatment of the singular kernels is possible. Therefore, this approach allows the use of a wide range of fundamental solutions in combination with different approximations for the geometry as well as for the Dirichlet and Neumann data within one implementation.

Nevertheless, it also must be mentioned that this approach is restricted to singular kernels which contain at most a Cauchy singularity, i.e., these integration rules can only be applied up to the double layer potentials but not to the hypersingular integral operator where the kernels are in fact non integrable.

To regularize the hypersingular bilinear form the elastostatic fundamental solution is used together with its regularized form. The regularization of the hypersingular elastostatic bilinear form is done via the Stoke's theorem and yields a bilinear form which contains only weak singularities [11]. In the following, $\hat{\mathcal{D}}_{ED}$ denotes the Laplace transformed elastodynamic hypersingular operator while $\mathcal{D}_{ES}$ represents the elastostatic one. Together with the operator $\mathcal{M}$ containing the *Günter-Derivatives* $\mathcal{M}_{ij}=n_j\partial_i-n_i\partial_j$, i.e.,

$$\mathcal{M}=\begin{pmatrix} 0 & n_2\partial_1-n_1\partial_2 & n_3\partial_1-n_1\partial_3\\ n_1\partial_2-n_2\partial_1 & 0 & n_3\partial_2-n_2\partial_3\\ n_1\partial_3-n_3\partial_1 & n_2\partial_3-n_3\partial_2 & 0\end{pmatrix} \tag{25}$$

the regularized form $\widetilde{\mathcal{D}}_{ES}$ of $\mathcal{D}_{ES}$ is given by

$$\begin{aligned}
&\langle\widetilde{\mathcal{D}}_{ES}\mathbf{u},\mathbf{w}\rangle_\Gamma = \\
&\int_\Gamma\int_\Gamma\left(\sum_{k=1}^{3}(\mathcal{M}_{k+2,k+1}\mathbf{w})(\mathbf{x})\cdot(\mathcal{M}_{k+2,k+1}\mathbf{u})(\mathbf{y})\right)\times\\
&\quad\frac{\mu}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}}+\\
&\int_\Gamma\int_\Gamma(\mathcal{M}\mathbf{w})^{\mathsf{T}}(\mathbf{x})\cdot\Big(\frac{\mu}{2\pi}\frac{\mathbf{I}}{|\mathbf{x}-\mathbf{y}|}-4\mu^2\,\mathbf{U}^*_{ES}(\mathbf{x},\mathbf{y})\Big)\cdot\\
&\quad(\mathcal{M}\mathbf{u})(\mathbf{y})\;\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}}+\\
&\int_\Gamma\int_\Gamma\sum_{i,j,k=1}^{3}(\mathcal{M}_{kj}w_i)(\mathbf{x})\,\frac{\mu}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}\times\\
&\quad(\mathcal{M}_{ki}u_j)(\mathbf{y})\;\mathrm{d}s_{\mathbf{y}}\,\mathrm{d}s_{\mathbf{x}}\;.
\end{aligned}\tag{26}$$

Note, that the commas in the first term of Eqn (26) are used just to separate the indices of the operator matrix $\mathcal{M}$. These indices are given with respect to modulo 3, i.e., 4 has to be identified with 1 and 5 with 2, respectively. Further, $\mathbf{I}$ represents the Identity matrix.

Due to the fact, that the hypersingularity is concentrated in the elastostatic part of the elastodynamic hypersingular operator a regularized form of the hypersingular elastodynamic bilinear form can be given by

$$\langle\hat{\mathcal{D}}^{\text{reg}}_{ED}\mathbf{u},\mathbf{w}\rangle_\Gamma=\langle\widetilde{\mathcal{D}}_{ES}\mathbf{u},\mathbf{w}\rangle_\Gamma+\langle(\hat{\mathcal{D}}_{ED}-\mathcal{D}_{ES})\;\mathbf{u},\mathbf{w}\rangle_\Gamma\;. \tag{27}$$

Equation (27) contains only the weak singularities from the regularized elastostatic hypersingular bilinear form $\langle\widetilde{\mathcal{D}}_{ES}\mathbf{u},\mathbf{w}\rangle_\Gamma$. The difference term is regular.

3.4. Linear system of equations

After introducing the spatial and time discretization, in Eqn (17) the discretized Galerkin variational problem now reads as:

Find $(\tilde{\mathbf{q}}_h^\alpha,\tilde{\mathbf{u}}_h^\beta)\in S_h^\alpha(\Gamma^D)\times S_h^\beta(\Gamma^N)$ *such that*

$$a(\tilde{\mathbf{q}}_h^\alpha,\tilde{\mathbf{u}}_h^\beta;\mathbf{w}_h^\alpha,\mathbf{v}_h^\beta)=F(\mathbf{w}_h^\alpha,\mathbf{v}_h^\beta) \tag{28}$$

is satisfied for all $(\mathbf{w}_h^\alpha,\mathbf{v}_h^\beta)\in S_h^\alpha(\Gamma^D)\times S_h^\beta(\Gamma^N)$.

In Eqn (28), the bilinear form a is given by

$$\begin{aligned}
a(\tilde{\mathbf{q}}_h^\alpha,\tilde{\mathbf{u}}_h^\beta;\mathbf{w}_h^\alpha,\mathbf{v}_h^\beta)=&\sum_{m=0}^{M-1}\sum_{i=1}^{3}\Big\langle(\mathcal{V}*\tilde{q}_h^\alpha)(\mathbf{x};t_m),w_h\Big\rangle_{\Gamma_{D,i}}\\
&-\Big\langle(\mathcal{K}*\tilde{u}_h^\beta)(\mathbf{x};t_m),w_h\Big\rangle_{\Gamma_{D,i}}\\
&+\Big\langle(\mathcal{K}'*\tilde{q}_h^\alpha)(\mathbf{x};t_m),v_h\Big\rangle_{\Gamma_{N,i}}\\
&+\Big\langle(\mathcal{D}*\tilde{u}_h^\beta)(\mathbf{x};t_m),v_h\Big\rangle_{\Gamma_{N,i}}
\end{aligned}\tag{29}$$

and the linear form by

$$F(\mathbf{w}_h, \mathbf{v}_h) = \sum_{m=0}^{M-1} \sum_{i=1}^{3} \left\langle ((\frac{1}{2}\mathcal{I} + \mathcal{K}) * \tilde{g}_D - \mathcal{V} * \tilde{g}_N)(\mathbf{x}; t_m), w_h \right\rangle_{\Gamma_{D,i}} + \left\langle ((\frac{1}{2}\mathcal{I} - \mathcal{K}') * \tilde{g}_N - \mathcal{D} * \tilde{g}_D)(\mathbf{x}; t_m), v_h \right\rangle_{\Gamma_{N,i}} . \tag{30}$$

This discretized variational problem can be written in a matrix notation as shown in Eqn (31).

$$\begin{bmatrix} \begin{bmatrix} \mathsf{V}_0 & & & \\ \mathsf{V}_1 & \mathsf{V}_0 & & \\ \vdots & & \ddots & \\ \mathsf{V}_{M-1} & \mathsf{V}_{M-2} & \dots & \mathsf{V}_0 \end{bmatrix} & - \begin{bmatrix} \mathsf{K}_0 & & & \\ \mathsf{K}_1 & \mathsf{K}_0 & & \\ \vdots & & \ddots & \\ \mathsf{K}_{M-1} & \mathsf{K}_{M-2} & \dots & \mathsf{K}_0 \end{bmatrix} \\ \begin{bmatrix} \mathsf{K}_0^\mathsf{T} & & & \\ \mathsf{K}_1^\mathsf{T} & \mathsf{K}_0^\mathsf{T} & & \\ \vdots & & \ddots & \\ \mathsf{K}_{M-1}^\mathsf{T} & \mathsf{K}_{M-2}^\mathsf{T} & \dots & \mathsf{K}_0^\mathsf{T} \end{bmatrix} & \begin{bmatrix} \mathsf{D}_0 & & & \\ \mathsf{D}_1 & \mathsf{D}_0 & & \\ \vdots & & \ddots & \\ \mathsf{D}_{M-1} & \mathsf{D}_{M-2} & \dots & \mathsf{D}_0 \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} \mathsf{q}_0 \\ \mathsf{q}_1 \\ \vdots \\ \mathsf{q}_{M-1} \end{bmatrix} \\ \begin{bmatrix} \mathsf{u}_0 \\ \mathsf{u}_1 \\ \vdots \\ \mathsf{u}_{M-1} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \bar{\mathsf{K}}_0 & & & \\ \mathsf{K}_1 & \bar{\mathsf{K}}_0 & & \\ \vdots & & \ddots & \\ \mathsf{K}_{M-1} & \mathsf{K}_{M-2} & \dots & \bar{\mathsf{K}}_0 \end{bmatrix} \cdot \begin{bmatrix} \mathsf{g}_0^D \\ \mathsf{g}_1^D \\ \vdots \\ \mathsf{g}_{M-1}^D \end{bmatrix} - \begin{bmatrix} \mathsf{V}_0 & & & \\ \mathsf{V}_1 & \mathsf{V}_0 & & \\ \vdots & & \ddots & \\ \mathsf{V}_{M-1} & \mathsf{V}_{M-2} & \dots & \mathsf{V}_0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{\mathsf{g}}_0^N \\ \tilde{\mathsf{g}}_1^N \\ \vdots \\ \tilde{\mathsf{g}}_{M-1}^N \end{bmatrix} \\ \begin{bmatrix} \bar{\mathsf{K}}_0' & & & \\ \mathsf{K}_1' & \bar{\mathsf{K}}_0' & & \\ \vdots & & \ddots & \\ \mathsf{K}_{M-1}' & \mathsf{K}_{M-2}' & \dots & \bar{\mathsf{K}}_0' \end{bmatrix} \cdot \begin{bmatrix} \mathsf{g}_0^N \\ \mathsf{g}_1^N \\ \vdots \\ \mathsf{g}_{M-1}^N \end{bmatrix} - \begin{bmatrix} \mathsf{D}_0 & & & \\ \mathsf{D}_1 & \mathsf{D}_0 & & \\ \vdots & & \ddots & \\ \mathsf{D}_{M-1} & \mathsf{D}_{M-2} & \dots & \mathsf{D}_0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{\mathsf{g}}_0^D \\ \tilde{\mathsf{g}}_1^D \\ \vdots \\ \tilde{\mathsf{g}}_{M-1}^D \end{bmatrix} \end{bmatrix} \tag{31}$$

In Eqn (31), the diagonal terms of the discrete double layer potential $\bar{\mathsf{K}}_0$ which correspond to the first time step is given by $\bar{\mathsf{K}}_0 = \frac{1}{2}\mathsf{I} + \mathsf{K}_0$ and, analogously, by $\bar{\mathsf{K}}_0' = \frac{1}{2}\mathsf{I} - \mathsf{K}_0'$ for the discrete adjoint double layer potential.

As it can be seen, the system matrices in Eqn (31) are composed of some kind of lower triangular *Toeplitz-block-matrices*. This property can be used to formulate the following recursion formula for the solution of Eqn (31) in the m-th time step. Finally, this yields

$$\begin{bmatrix} \mathsf{V}_0 & -\mathsf{K}_0 \\ \mathsf{K}_0^\mathsf{T} & \mathsf{D}_0 \end{bmatrix} \cdot \begin{bmatrix} \mathsf{q}_m \\ \mathsf{u}_m \end{bmatrix} = \begin{bmatrix} \mathsf{f}_m^D \\ \mathsf{f}_m^N \end{bmatrix} - \sum_{i=0}^{m-1} \begin{bmatrix} \mathsf{V}_{m-i} \cdot \mathsf{q}_i - \mathsf{K}_{m-i} \cdot \mathsf{u}_i \\ \mathsf{K}_{m-i}^\mathsf{T} \cdot \mathsf{q}_i + \mathsf{D}_{m-i} \cdot \mathsf{u}_i \end{bmatrix} \tag{32}$$

with the prescribed boundary data

$$\begin{bmatrix} \mathsf{f}_m^D \\ \mathsf{f}_m^N \end{bmatrix} = \begin{bmatrix} \bar{\mathsf{K}}_0 \cdot \mathsf{g}_m^D - \mathsf{V}_0 \cdot \tilde{\mathsf{g}}_m^N \\ \bar{\mathsf{K}}_0' \cdot \mathsf{g}_m^N - \mathsf{D}_0 \cdot \tilde{\mathsf{g}}_m^D \end{bmatrix} + \sum_{i=0}^{m-1} \begin{bmatrix} \mathsf{K}_{m-i} \cdot \mathsf{g}_i^D - \mathsf{V}_{m-i} \cdot \tilde{\mathsf{g}}_i^N \\ \mathsf{K}_{m-i}' \cdot \mathsf{g}_i^N - \mathsf{D}_{m-i} \cdot \tilde{\mathsf{g}}_i^D \end{bmatrix} . \tag{33}$$

From Eqn (32), it is obvious that the solution requires only the inversion of the matrix corresponding to the first time step. Therefore, the matrix V_0, which is symmetric as a result of the Galerkin discretization, is decomposed via a *Cholesky-factorization*. Afterwards, the *Schur-Complement-System* is computed by

$$\mathsf{S}_0 = \mathsf{K}_0^\mathsf{T} \mathsf{V}_0^{-1} \mathsf{K}_0 + \mathsf{D}_0 . \tag{34}$$

Due to the symmetry of V_0 and D_0 the Schur-Complement S_0 is also symmetric and can be decomposed by a Cholesky-factorization.

Hence, the displacement field u and the tractions q can be found by solving

$$\mathsf{S}_0 \mathsf{u}_m = \tilde{\mathsf{f}}_m^N - \mathsf{K}_0^\mathsf{T} \mathsf{V}_0^{-1} \tilde{\mathsf{f}}_m^D \tag{35}$$

and

$$\mathsf{q}_m = \mathsf{V}_0^{-1} \left(\tilde{\mathsf{f}}_m^D + \mathsf{K}_0 \mathsf{u}_m \right) \tag{36}$$

for every time step $m = 0, \dots, M-1$. In Eqns (35) and (36) the vectors $\tilde{\mathsf{f}}_m^D$ and $\tilde{\mathsf{f}}_m^N$ are given by

$$\begin{bmatrix} \tilde{\mathsf{f}}_m^D \\ \tilde{\mathsf{f}}_m^N \end{bmatrix} = \begin{bmatrix} \mathsf{f}_m^D \\ \mathsf{f}_m^N \end{bmatrix} - \sum_{i=0}^{m-1} \begin{bmatrix} \mathsf{V}_{m-i} \cdot \mathsf{q}_i - \mathsf{K}_{m-i} \cdot \mathsf{u}_i \\ \mathsf{K}_{m-i}^\mathsf{T} \cdot \mathsf{q}_i + \mathsf{D}_{m-i} \cdot \mathsf{u}_i \end{bmatrix} . \tag{37}$$

In Eqn (37), the usual structure of a BE time stepping technique is observed. The first term denotes the given boundary data of the actual time step m whereas in the second term the time history is stored.

4. Numerical Examples

4.1. Model problem

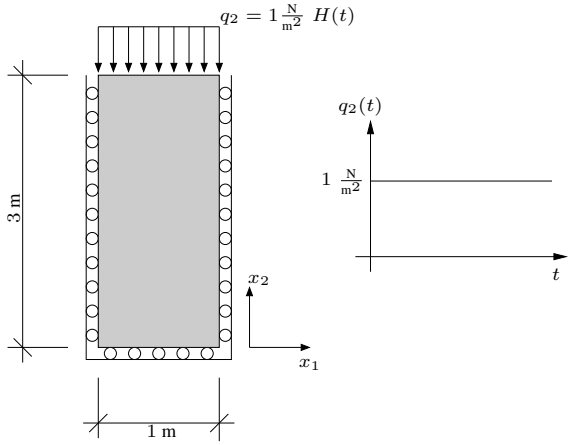

Figure 3: Boundary Conditions

In the following, a 1-d elastodynamic rod is considered as depicted in Fig. 3. The rod is fixed on one end, and excited by a pressure jump according to a unit step function $q_2(\mathbf{x}^\star; t) = -1\mathrm{N/m}^2\ H(t)$ on the other free end with $\mathbf{x}^\star = [\ x_1;\ 3;\ x_3\ ]^\mathsf{T}$. The remaining surfaces are traction free with respect to their tangential directions and blocked in the normal direction. The material data represent steel with Young's modulus $E = 2.11 \cdot 10^{11}\ \mathrm{N/m}^2$ and the density $\varrho = 7850\ \mathrm{kg/m}^3$. An exception has been made to the use of Poisson's ratio which is chosen to $\nu = 0$. This artificial value is taken in order to model the 1-d analytical solution.

Figure 4 shows the mesh for the described model problem. It is made up with 112 elements on 58 nodes. As mentioned before, the tractions are approximated with constant functions whereas the displacements are approximated linear.

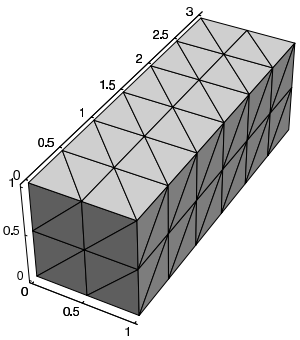

Figure 4: 112 elements, 58 nodes

In order to compare some results for different time and/or spatial discretizations the dimensionless value

$$\beta = \frac{c_1\ \Delta t}{r_e} \tag{38}$$

is introduced. Beside the velocity of the fast wave the value β depends on the step size Δt and the characteristic element length r_e. To determine the characteristic length is not easy due to the fact that in 3-d it is not clear which length should be chosen. Here, a characteristic length of $r_e = 0.5$ m is used.

Figure 5 compares results arising in collocation and in the proposed Galerkin method for the longitudinal displacements at point $\mathbf{x}_u^\star = [\ 0.5;\ 3;\ 0.5\ ]^\mathsf{T}$. The chosen time step size is $\Delta t = 2 \cdot 10^{-5}$ s. Together with the wave velocity $c_1 = \sqrt{E/\varrho}$, this yields $\beta = 0.2$.

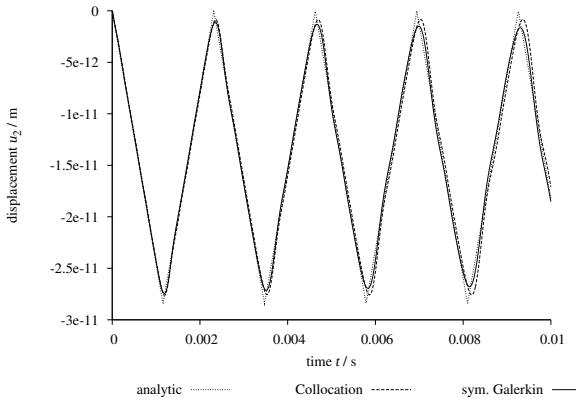

Figure 5: Collocation method vs sym. Galerkin: Longitudinal Displacements

As it can be seen, the results obtained from the symmetric Galerkin method are slightly better than the collocation results. The shifting of the phase is not as distinctive as it is for the results coming from collocation.

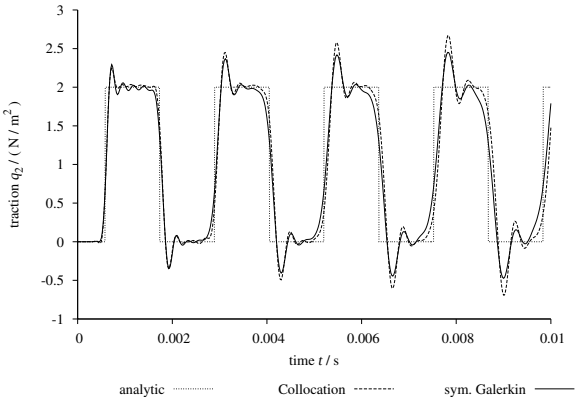

Figure 6: Collocation method vs sym. Galerkin: Tractions

Figure 6 shows the traction solution $q_2(\mathbf{x}_q^\star; t)$ for $\mathbf{x}_q^\star = [\ 0.5;\ 0;\ 0.5\ ]^\mathsf{T}$. Here, the collocation seems to approximate the analytical solution a little bit better than the Galerkin method does, especially the mapping of the discontinuities is better.

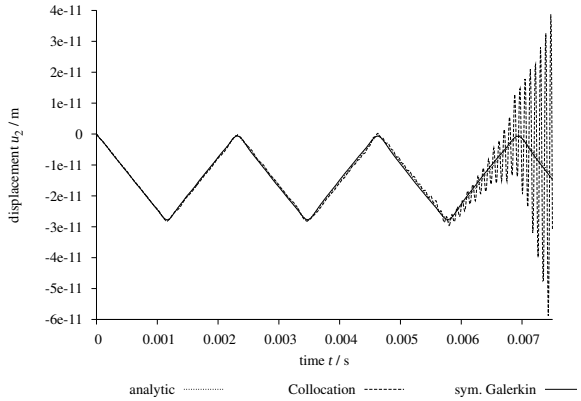

Figure 7: Longitudinal Displacements: Influence of time step size

In Fig. 7 a finer time step size of $\Delta t = 8 \cdot 10^{-6}$ s is chosen. Therefore, $\beta = 0.08$ is obtained. The longitudinal displacements $u_2(\mathbf{x}_u^\star; t)$ are more or less equal for both methods up to the time $t = 4.5 \cdot 10^{-3}$ s. But then, the collocation method gets unstable.

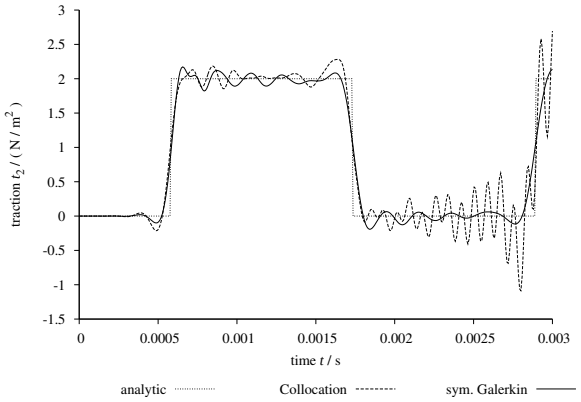

Figure 8: Tractions at the fixed end: Influence of time step size

This lost of stability is even more worse if the tractions $q(\mathbf{x}_q^\star; t)$ are considered. In Fig. 8 the tractions at the fixed end are depicted up to the time $t = 3 \cdot 10^{-3}$ s. On the other hand the results obtained by the symmetric Galerkin method are very stable up to the time $t = 0.01$ s, and even beyond. In Fig. 9 and Fig. 10 the longitudinal displacements and tractions show this behavior.

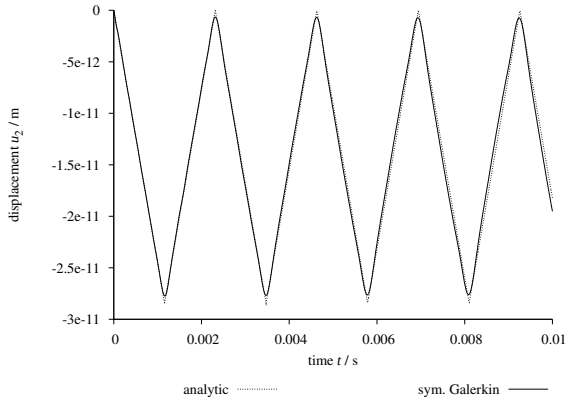

Figure 9: Longitudinal displacements at Point $\mathbf{x}_2^\star$

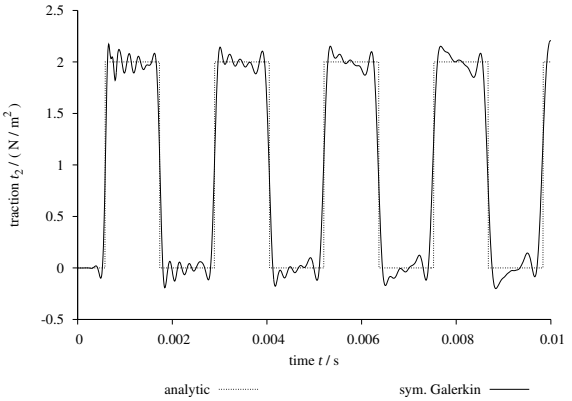

Figure 10: Tractions at point $\mathbf{x}_q^\star$

5. Conclusion

In the actual paper, a boundary element method for elastodynamics based on a Galerkin discretization in space and on the Convolution Quadrature Method in time has been presented. To obtain a symmetric, or to be more precise, a skew-symmetric formulation, also the usage of the second integral equation is required. After the prescribed boundary data are inserted in the system of boundary integral equations a variational form is introduced. Then, this variational form is discretized in space and time. While the spatial discretization is done by a standard Galerkin method the Convolution Quadrature Method is used as a time stepping procedure, which uses only the Laplace transform of the fundamental solutions. Finally, a well structured system of lower triangular block-matrices is obtained which can be solved recursively, i.e., only an update of the right hand-side has to be made due to the fact that just the factorizations for the system matrices of the first time step are required for the left hand-side.

The presented numerical examples show that this approach has obviously better numerical stability properties than the wider used collocation methods have. Therefore, the positive results already obtained in statics with the symmetric Galerkin methods continues also in time domain analysis. Nevertheless, it has to be mentioned that due to the variational form the computation of the matrix entries is more complex. Hence, at the moment the overall computation is more time consuming than the standard collocation method. In future, these computational costs should be reduced by, e.g, the application of some adaptive integration rules or the enhancement of fast-methods like ACA or $\mathcal{H}$-matrices to the time domain.

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