NON-CONFORMING COUPLING WITH THE BOUNDARY ELEMENT METHOD IN TIME DOMAIN

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Abstract. *Wave propagation phenomena in large or unbounded media can be effectively analyzed with the Boundary Element Method (BEM). On the other hand, material discontinuities or the occurrence of non-linearities are always difficult to be tackled by BEM and require very sophisticated techniques. The use of domain decomposition with a BEM-BEM coupling or the combination with Finite Element Methods provides efficient means for such situations.*

Here, an approach is presented which incorporates the independent use of boundary and finite element methods for each sub-domain and, moreover, allows for non-conforming interface discretizations. This is basically achieved by using so-called Dirichlet-to-Neumann maps which map for instance the boundary displacements to boundary tractions. Such maps are well-established in static problems and will be here carried over to dynamic problems. The interface conditions will then be formulated by means of Lagrange multipliers in a weak sense and, therefore, allow the relaxation needed to handle non-conforming discretizations.

The method is validated by some small benchmark examples using the scalar wave equation and show a good performance in terms of the quality of the results.

Keywords: *Time domain BEM, Domain decomposition, Non-conforming interfaces*

1. INTRODUCTION

The Boundary Element Method (BEM) and the Finite Element Method (FEM) are both well developed methods for the numerical solution of (initial) boundary value problems as they occur in physical problems. It is often believed that these two methods are competitive but a closer look at their respective advantages and disadvantages shows that they are rather complementary and should be combined adaptively.

This has already been outlined by **?**, and a few examples are the efficient application to stress concentration problems and to domains of infinite size for BEM, whereas FEM is superior in cases of non-linear material behavior, large deformations, and in principle more enhanced concerning the solution procedure.

So far, the application of this idea has been mainly carried out by generating so-called boundary element stiffness matrices (**?**) which show good results but do not provide a structured system matrix for an efficient solution procedure. This idea has been extended to dynamic problems by **?**. Another common approach in dynamics are the staggering and related methods (**?**; **?**). These methods are very flexible, easy parallelizable, and efficient. Nevertheless, they usually employ empirical parameters in order to ensure convergence.

Another issue is that most coupling formulations require conforming interface discretizations. This means that not only the interface nodes have to coincide but also the approximation order has to be equal. Only if both restrictions are fulfilled, the interface conditions between the sub-domains can be fulfilled in a strong sense. A relaxation to these requirements is easily obtained by the introduction of Lagrange multipliers (**?**). Moreover, variational principles can be established with interface energy terms (**?**). In these formulations, the interface conditions are applied in a weak or integral sense such that the treatment of non-conforming interface conditions becomes feasible.

In the mathematical community, such methods are established for the analysis of elliptic boundary value problems. For the use of finite element methods the so-called *Mortar* finite element method (**?**) is well known, whereas **?** provides a full analysis of such methods with finite and boundary element discretizations. In the latter reference, the idea of the Dirichlet-to-Neumann maps with Steklov-Poincaré operator is proclaimed which led to the ideas of this work.

In the following, the methodology for static (i.e., elliptic) problems is reviewed mainly by help of **?**. Afterwards a straightforward extension to dynamic problems is proposed whose validity will then be shown by small test examples.

2. STATIC BOUNDARY VALUE PROBLEMS

Consider an elliptic boundary value problem of the type

$$\begin{aligned} \mathcal{L}\,u(x) &= f(x) && x \in \Omega \\ u_\Gamma(y) &= \bar{u}(y) && y \in \Gamma_D \\ q(y) := \mathcal{T}_y\,u(x) &= \bar{q}(y) && y \in \Gamma_N \end{aligned} \tag{1}$$

with some elliptic differential operator $\mathcal{L}$, a source term f, given Dirichlet and Neumann boundary conditions, $\bar{u}$ and $\bar{q}$ respectively, and the *traction* operator $\mathcal{T}$, which transforms displacements u to tractions q (or pressure to normal flux). Moreover, Ω is the computational domain with its disjoint boundary parts Γ_D and Γ_N. In the following, only homogeneous field equations are considered, i.e., $f = 0$, which poses no restriction to the

presented methodology but is merely done for simplicity. Note in Eq. (1) the subscript Γ is used in order to point out that the *boundary* trace of u has to equal the given Dirichlet data $\bar{u}$.

Assuming the existence of a *Steklov-Poincaré* operator $\mathcal{S}$ which maps the boundary values of u to q,

$$(\mathcal{S}u_\Gamma(y)) = q(y) \qquad y \in \Gamma\,, \tag{2}$$

the above boundary value problem can be reformulated in such a way that only boundary variables are involved

$$\begin{aligned} (\mathcal{S}u_\Gamma(y)) &= q(y) && y \in \Gamma \\ u_\Gamma(y) &= \bar{u}(y) && y \in \Gamma_D \\ q(y) &= \bar{q}(y) && y \in \Gamma_N\,. \end{aligned} \tag{3}$$

Now, a variational formulation is easily established by requiring the solution u to directly fulfill the essential boundary conditions $\bar{u}$ and requiring the first equation of Eq. (3) to be fulfilled in weighted sense

$$\int_{\Gamma_N} (\mathcal{S}u)\, v \,\mathrm{d}s = \int_{\Gamma_N} \bar{q}\, v \,\mathrm{d}s\,. \tag{4}$$

Here, v denotes the test function which only lives on the Neumann boundary Γ_N. Equation (4) states nothing but the weak fulfillment of the Neumann or natural boundary conditions.

Discretization of this variational principle directly yields the system of equations

$$\mathsf{S}\mathsf{u}_\Gamma = \mathsf{f}_\Gamma\,, \tag{5}$$

where S is the discretized Steklov-Poincaré operator, u_Γ the vector of boundary displacement or pressure unknowns, and f_Γ contains the nodal forces of the applied Neumann boundary conditions.

Unfortunately, this operator $\mathcal{S}$ is not easily obtained but from looking at its mapping properties it becomes clear, how to mimic its action on the discrete level. Therefore, finite and boundary element discretizations are considered, whereas further possibilities as, e.g., Trefftz methods are not outlined.

2.1 Finite element discretization

A standard finite element treatment of the static boundary value problem as given in Eq. (6) yields the typical system of equations

$$\mathsf{A}\mathsf{u} = \mathsf{f} \Leftrightarrow \begin{pmatrix} \mathsf{A}_{\mathrm{II}} & \mathsf{A}_{\mathrm{I}\Gamma} \\ \mathsf{A}_{\Gamma\mathrm{I}} & \mathsf{A}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathsf{u}_{\mathrm{I}} \\ \mathsf{u}_\Gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{f}_\Gamma \end{pmatrix} \tag{6}$$

with the stiffness matrix A. Here, the degrees of freedom have simply been sorted according to its location in the interior of the domain Ω (subscript I) or the boundary Γ.

Applying to this system the so-called *static condensation* or taking its Schur complement leads to the boundary based equation

$$\mathsf{S}^{\mathrm{FEM}}\mathsf{u}_\Gamma = [\mathsf{A}_{\Gamma\Gamma} - \mathsf{A}_{\Gamma\mathrm{I}}\mathsf{A}_{\mathrm{II}}^{-1}\mathsf{A}_{\mathrm{I}\Gamma}]\mathsf{u}_\Gamma = \mathsf{f}_\Gamma \tag{7}$$

and a finite element discretization of the Steklov-Poincaré operator is obtained. Furthermore, this $\mathsf{S}^{\mathrm{FEM}}$ will never be explicitly computed, which would be much to expensive due to the inversion of A_{II}, but instead Eq. (6) is solved in the common way and the action of $\mathsf{S}^{\mathrm{FEM}}$ takes place implicitly.

2.2 Boundary element discretization

In boundary element methods the basic unknowns of the equations are already boundary based. Moreover, the integral equations allow to formulate the operator $\mathcal{S}$ in a continuous setting. Taking the first integral equation in standard BEM

$$\begin{aligned} c(x)u_\Gamma(x) + \int_\Gamma Q^*(x,y)u(y)\,\mathrm{d}s_y &= \int_\Gamma U^*(x,y)q(y)\,\mathrm{d}s_y \quad x,y\in\Gamma \\ \Rightarrow (c\mathcal{I}+\mathcal{K})\,u_\Gamma &= \mathcal{V}q \end{aligned} \tag{8}$$

with U^* as the fundamental solution to the field equation in Eq. (1), and its traction kernel $Q^* := \mathcal{T}_y U^*$. Furthermore, in Eq. (8) the operators are shown: the single layer operator $\mathcal{V}$, the double layer operator $\mathcal{K}$ and the identity $\mathcal{I}$.

Obviously, the application of the inverse single layer operator to Eq. (8) will yield

$$\mathcal{S}u_\Gamma = \mathcal{V}^{-1}\,(c\mathcal{I}+\mathcal{K})\,u_\Gamma = q\,, \tag{9}$$

i.e., the continuous Steklov-Poincaré operator. Using a piece-wise polynomial approximation of the boundary functions

$$u_\Gamma^h(y) = \sum_{m=1}^{M}\varphi_m(y)u_m \quad \text{and} \quad q^h(y) = \sum_{n=1}^{N}\psi_n(y)q_n \tag{10}$$

and a collocation or Galerkin scheme, one arrives at the discrete system of equations

$$\begin{pmatrix} \mathsf{V} & -\left(\frac{1}{2}\mathsf{I}+\mathsf{K}\right) \\ \mathsf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{q} \\ \mathsf{u}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathsf{f}_D \\ \mathsf{f}_N \end{pmatrix} \tag{11}$$

obtained by strongly requiring the Dirichlet boundary conditions and setting the Neumann boundary conditions in a weak form. Here, V is the full discrete single layer operator, whereas K is the discrete double layer operator for the Neumann boundary only. The matrix B is basically the L_2 scalar product of the shape functions used for the unknowns u_Γ and q,

$$\mathsf{B}[k,\ell] = \int_{\Gamma_N}\varphi_k(y)\psi_\ell(y)\,\mathrm{d}s_y\,. \tag{12}$$

Note that this formulation uses the unknown values of u_Γ and all degrees of freedom of q for the vector of unknowns. The unnecessary redundancy in q causes the advantage of block structured system matrix and the use of different approximation orders for u_Γ and q. Ways to solve such a system efficiently have been analyzed by **?**.

The discrete Steklov-Poincaré operator is now obtained by

$$\mathsf{S}^{\mathrm{BEM}}\mathsf{u}_\Gamma = \mathsf{B}\mathsf{V}^{-1}\left(\frac{1}{2}\mathsf{I}+\mathsf{K}\right)\mathsf{u}_\Gamma = \mathsf{f}\,. \tag{13}$$

Again, as in the case of finite elements, this matrix $\mathsf{S}^{\mathrm{BEM}}$ does not have to be computed explicitly but the solution of Eq. (11) rather mimics the action of the discrete Steklov-Poincaré operator.

3. DYNAMIC INITIAL BOUNDARY VALUE PROBLEMS

Dynamic problems as elastic wave propagation or the acoustic pressure distribution in time domain are described by initial boundary value problems of the kind

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) - \mathcal{L}(x)u(x,t) &= f(x,t) \qquad x \in \Omega, \qquad t \in (0,\,T) \\
u(y,t) &= \bar{u}(y,t) \qquad y \in \Gamma_D, \quad t \in (0,\,T) \\
q(y,t) &= \bar{q}(y,t) \qquad y \in \Gamma_N, \quad t \in (0,\,T) \\
u(x,t) &= u_0(x) \qquad x \in \Omega, \qquad t = 0 \\
\frac{\partial}{\partial t}u(x,t) &= v_0(x) \qquad x \in \Omega, \qquad t = 0^+
\end{aligned} \tag{14}$$

for the time interval $(0,\,T)$. $\mathcal{L}$ is again an elliptic partial differential operator. In addition to the time dependent boundary conditions, initial conditions u_0 and v_0 are given for the unknown function u and its first temporal derivative. Again, a simplification is carried out which does not restrict generality by assuming f, u_0, and u_0 to be identical zero.

Contrary to the static case, a continuous Steklov-Poincaré operator $\mathcal{S}$ cannot be formulated. Nevertheless, the operations carried out on the discrete level as in Eqs. (7) and (13) can be performed in the discrete equations in dynamics equally well as shown below.

3.1 Finite element discretization

The classical FEM approach to a problem as given in Eq. (14) leads to the semi-discrete system of equations

$$\mathsf{M}\ddot{\mathsf{u}}(t) + \mathsf{A}\mathsf{u}(t) = \mathsf{f}(t) \tag{15}$$

which can be treated for instance with the Newmark time stepping algorithm (**?**)

$$\begin{aligned}
\left[\mathsf{M} + (\Delta t)^2\beta\mathsf{A}\right]\mathsf{u}_{n+1} = &\left[2\mathsf{M} - (\Delta t)^2\left(\frac{1}{2} - 2\beta + \gamma\right)\mathsf{A}\right]\mathsf{u}_n \\
&- \left[\mathsf{M} - (\Delta t)^2\left(\frac{1}{2} + \beta - \gamma\right)\mathsf{A}\right]\mathsf{u}_{n-1} \\
&+ (\Delta t)^2\left[\beta\mathsf{f}_{n+1} + \left(\frac{1}{2} - 2\beta + \gamma\right)\mathsf{f}_n + \left(\frac{1}{2} + \beta - \gamma\right)\mathsf{f}_{n-1}\right]
\end{aligned} \tag{16}$$

with the parameters β and γ of the Newmark method and the time step size Δt. The system in Eq. (16) can be abbreviated as

$$\mathsf{A}_0\mathsf{u}_{n+1} = \mathsf{A}_1\mathsf{u}_n + \mathsf{A}_2\mathsf{u}_{n-1} + \mathsf{f}_{n+1} \tag{17}$$

and ordering according to the degrees of freedom in the interior of Ω and on its boundary Γ, leads to a similar form as in Eq. (6). Applying a static condensation to such a system leads to

$$\mathsf{S}^{\mathrm{FEM}}_{n+1}\mathsf{u}_{\Gamma,n+1} = \mathsf{f}_{n+1}\,. \tag{18}$$

It thereby becomes clear that the solution of the system in Eq. (16) contains the operation of $\mathsf{S}^{\mathrm{FEM}}_{n+1}$, i.e., a time domain Steklov-Poincaré operator for the $n+1$-th time step.

3.2 Boundary element discretization

The application of boundary element methods to a dynamic problem starts from the time-domain boundary integral equation (**?**)

$$c(y)u_\Gamma(y,t) + \mathcal{K} * u(y,t) = \mathcal{V} * q(y,t)\,, \tag{19}$$

where $*$ denotes the time convolution $f * g = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$. Equation (**??**) assumes the existence of a time domain fundamental solution for the given problem which will not always be the case. One can also make use of Laplace domain fundamental solutions and use the convolution quadrature method as outlined by **?**. Independent of this, the final system of equation has the form

$$\left(\frac{1}{2}\mathsf{I} + \mathsf{K}_0\right)\mathsf{u}_{\Gamma,n+1} + \sum_{i=1}^{n+1}\mathsf{K}_i\mathsf{u}_{\Gamma,n+1-i} = \mathsf{V}_0\mathsf{q}_{n+1} + \sum_{i=1}^{n+1}\mathsf{V}_i\mathsf{q}_{n+1-i} \tag{20}$$

which can be solved in the form the matrix block system

$$\begin{pmatrix}\mathsf{V}_0 & -\left(\frac{1}{2}\mathsf{I} + \mathsf{K}_0\right)\\ \mathsf{B} & \mathsf{0}\end{pmatrix}\begin{pmatrix}\mathsf{q}_{n+1}\\ \mathsf{u}_{\Gamma,n+1}\end{pmatrix} = \begin{pmatrix}\mathsf{f}_{D,n+1} + \sum_{i=1}^{n+1}\mathsf{V}_i\mathsf{q}_{n+1-i} - \mathsf{K}_i\mathsf{u}_{n+1-i}\\ \mathsf{f}_{N,n+1}\end{pmatrix}. \tag{21}$$

Application of the same procedure as in Eq. (13) yields the dynamic discrete Steklov-Poincaré operator

$$\mathsf{S}^{\mathrm{FEM}}_{n+1}\mathsf{u}_{\Gamma,n+1} = \mathsf{f}_{n+1}\,, \tag{22}$$

which, again, will not be computed explicitly but only mimicked by Eq. (**??**).

4. DOMAIN DECOMPOSITION METHOD

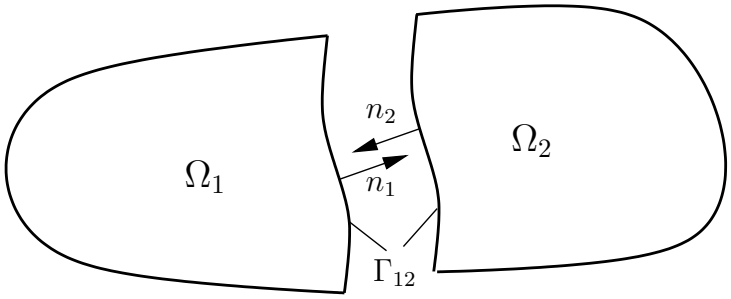

Figure 1 - Domain Ω subdivided in two sub-domains, Ω_1 and Ω_2.

The computational Ω is now subdivided in, for instance, 2 sub-domains Ω_1 and Ω_2 with respective boundaries Γ_1 and Γ_2, each of which possibly decomposed in a Dirichlet Γ_{Di} and a Neumann part Γ_{Ni}, $i = 1, 2$. The interface is set up by the intersection of the sub-domain boundaries, i.e., $\Gamma_{12} = \Gamma_1 \cap \Gamma_2$. This constellation is depicted in Fig. **??**.

On each sub-domain Ω_i a (initial) boundary value problem can be posed as it has been done in Eqs. (1) and (14), where the differential operators $\mathcal{L}_i$ do not have to be the same, as for example in soil-structure interaction problems. But, so far, the coupled problem is incomplete and the interface or transmission conditions have to be posed

$$u_1(y) = u_2(y) \quad \text{and} \quad q_1(y) + q_2(y) = 0 \quad y \in \Gamma_{12}\,. \tag{23}$$

Note, that the subscript Γ for the emphasis of the boundary trace of u will be omitted henceforth if not relevant.

The classical approaches work with a strict fulfillment of these conditions. In order to meet the conditions in Eq. (**??**) at every point $y \in \Gamma_{12}$, the nodes of the interface discretizations have to coincide and the order of approximation has to be equal. Such a severe restriction would be violated in case of independent mesh refinement, use of optimal mesh sizes, or simply when the sub-domain discretizations are obtained independently is it frequently occurs in the partitioned analysis large structures. Such cases are shown in Fig **??**. Even the use of triangles and quadrilaterals on opposite sides of the interface with coincident nodes forms a non-conforming situation.

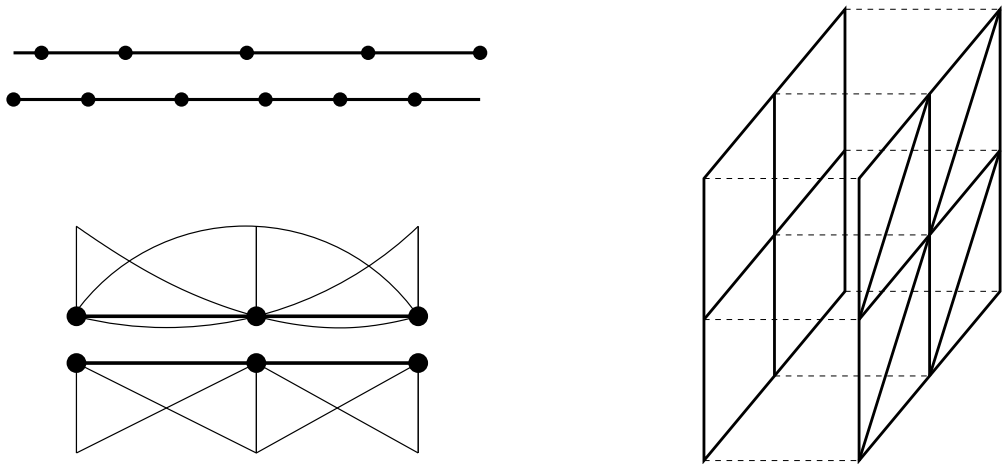

Figure 2 **-** Cases of non-conforming interface discretizations: non-coincident nodes (upper left), different interpolation orders in 2D (lower left) and 3D (right).

In order to circumvent these problems, a weak interface formulation is used which has the form

$$\int_{\Gamma_{12}} \lambda_{12}(u_1 - u_2)\,\mathrm{d}s = 0\,. \tag{24}$$

In Eq. (**??**) Lagrange multipliers λ_{12} have been introduced which are physically identical with the variable q (be it tractions or fluxes) on the interface. For the incorporation of non-conforming interfaces, a Neumann domain decomposition method is used (cf. **?** and references therein for a classification of these methods), where a strong fulfillment of the second part in Eq (**??**) is assumed, $\lambda_{12} = q_1 = -q_2$. This allows for the variational principle

$$\begin{aligned}
\int_{\Gamma_{N1}\cap\Gamma_{12}} (\mathcal{S}_1 u_1)\, v_1\,\mathrm{d}s - \int_{\Gamma_{12}} \lambda_{12}(v_1 - v_2)\,\mathrm{d}s &= \int_{\Gamma_{N1}} \bar{q}_1\, v_1\,\mathrm{d}s \\
\int_{\Gamma_{N2}\cap\Gamma_{12}} (\mathcal{S}_2 u_2)\, v_2\,\mathrm{d}s - \int_{\Gamma_{12}} \lambda_{12}(v_1 - v_2)\,\mathrm{d}s &= \int_{\Gamma_{N2}} \bar{q}_2\, v_2\,\mathrm{d}s \\
\int_{\Gamma_{12}} (u_1 - u_2)\mu_{12}\,\mathrm{d}s &= 0
\end{aligned} \tag{25}$$

with the test functions v_i and μ_{12} corresponding to u_i and λ_{12}, respectively. The first two equations state a weighted equilibrium in the sense that the outcome of the application

of $\mathcal{S}_i$ to u_i on the Neumann boundary part and the interface has to equal the interface values of $q_i = \pm\lambda_{12}$ and the given Neumann data $\bar{q}_i$. The third equation then enforces the weak interface conditions for the variables u_i.

The discretization of this variational principle leads to the system of equations

$$\begin{pmatrix} \mathsf{S}_1 & 0 & -\mathsf{C}_{12} \\ 0 & \mathsf{S}_2 & \mathsf{C}_{21} \\ \mathsf{C}_{12}^T & -\mathsf{C}_{21}^T & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \lambda_{12} \end{pmatrix} = \begin{pmatrix} \mathsf{f}_1 \\ \mathsf{f}_2 \\ 0 \end{pmatrix} \tag{26}$$

which actually comprises a saddle point problem. In this block system, the matrices S_i can then be replaced by the systems which define the action of $\mathcal{S}_i$ with either a finite or a boundary element discretization. As an example the use of finite elements in sub-domain Ω_1 and of boundary elements in Ω_2 leads to the system

$$\begin{pmatrix} \mathsf{A}_{\mathrm{II}} & \mathsf{A}_{\mathrm{I}\Gamma} & 0 & 0 & 0 \\ \mathsf{A}_{\Gamma\mathrm{I}} & \mathsf{A}_{\Gamma\Gamma} & 0 & 0 & -\mathsf{C}_{12} \\ 0 & 0 & \mathsf{V} & -\left(\frac{1}{2}\mathsf{I} + \mathsf{K}\right) & 0 \\ 0 & 0 & \mathsf{B} & 0 & \mathsf{C}_{12} \\ 0 & \mathsf{C}_{12}^T & 0 & -\mathsf{C}_{21}^T & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u}_{\mathrm{I},1} \\ \mathsf{u}_{\Gamma,1} \\ \mathsf{q}_2 \\ \mathsf{u}_{\Gamma,2} \\ \lambda_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{f}_{\Gamma,1} \\ \mathsf{f}_{D,2} \\ \mathsf{f}_{N,2} \\ 0 \end{pmatrix} . \tag{27}$$

The remaining point is the structure of the coupling matrices C_{ij} which are employed for this formulation. Therefore, it is necessary to label one of the sub-domains the *slave* and the other the *master*. Assuming that Ω_1 is the slave, it is determining the approximation of the Lagrange multipliers. This means that the shape functions for λ_{12} are defined on the interface side which belongs to the slave. Then the coupling matrices are

$$\mathsf{C}_{12}[k,\ell] = \int_{\Gamma_{12}} \varphi_k^1(y)\psi_\ell^1(y)\,\mathrm{d}s_y \quad \text{and} \quad \mathsf{C}_{21}[k,\ell] = \int_{\Gamma_{12}} \varphi_k^2(y)\psi_\ell^1(y)\,\mathrm{d}s_y\,, \tag{28}$$

where the superscripts have been used to indicate to which sub-domain the shape function belongs. Obviously, C_{12} has the same structure as B and provides no difficulty. C_{21} on the other hand contains the scalar product of shape functions which live on different meshes. This is the very point, where one has to pay for the use of non-conforming discretizations and special techniques have to be used to find the element intersections in 3D. The use of polygon clipping algorithms (**?**) from computer graphics seem to be a good choice for such task.

5. Numerical Example

The presented methodology is validated by a small benchmark problem. Therefore, the scalar wave equation is taken

$$\frac{\partial^2}{\partial t^2} u_i(x,t) - c_i^2 \Delta u_i(x,t) = 0 \tag{29}$$

on a domain $\Omega = (0,\,2) \times (0,\,1)$ which will be subdivided in two equal unit squares. with zero initial conditions and the boundary conditions as shown in Fig. **??**. Basically, the left end is fixed with zero pressure and on the right end a unit step function flux, $\bar{q}(t) = H(t)$, is applied. For the special case of $c_1 = c_2$ an analytical solution can be given (**?**). A BEM-BEM and a BEM-FEM coupling have been tested. For the first coupling,

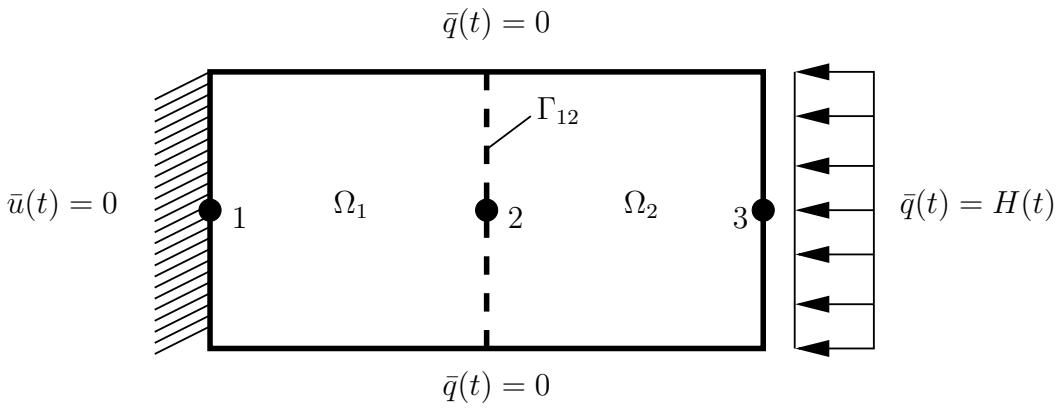

Figure 3 - Example problem. The filled circles with number show the positions at which pressure and flux have been evaluated.

the left sub-domain was discretized with 4 and the right with 5 linear elements per side. The choice of elements was simply to ensure non-coincident interface nodes. The time step was set to $\Delta t = 0.025$ and the wave velocity was set to $c_1 = c_2 = 1$. In case of the BEM-FEM coupling, 4 linear boundary elements per side and a mesh of 10×10 bilinear finite elements have been taken with a time step size of 0.025.

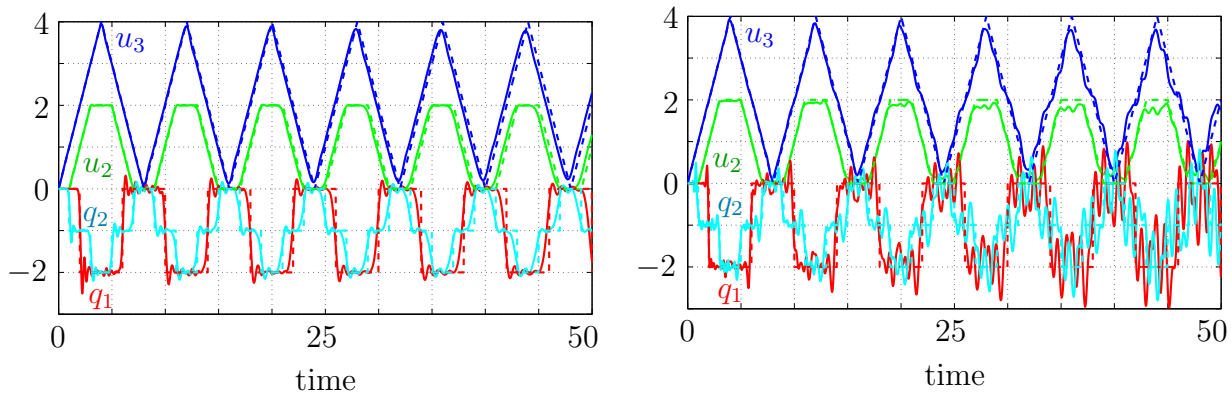

Figure 4 - Solution curves for a BEM-BEM coupling (left) and a BEM-FEM coupling (right). Pressure curves u_3 and u_2 for the right boundary and the interface and flux curves q_2 and q_1 for the interface and the left boundary.

Figure **??** shows the solution curves for both analyses, the BEM-BEM coupling on the left and the BEM-FEM coupling on the right side. In detail, the pressure $u(t)$ has been plotted for the side of the applied flux at node 3 and at the interface at node 2. These are the curves u_3 and u_2, respectively. The flux $q(t)$ has been plotted for the interface at node 2 and at the Dirichlet boundary at node 1 and is shown by the curves q_2 and q_1. The dashed lines in the plots represent the analytical solution. All plotted results show a good agreement with the analytical solution, especially when considering the gross discretization which has been used.

6. CONCLUSION

The aim of this work is to develop a method for the analysis of multi-physical problems such as soil-structure or fluid-structure interaction. The main emphasis lies on the

independent and optimal use of finite and boundary element methods and the treatment of non-conforming interface discretizations. Moreover, the static *and* dynamic analyses have to be carried out with this method. These criteria are fulfilled by the proposed coupling.

So far, an efficient solution procedure has not yet been analyzed. Nevertheless, the resulting system matrices are well structured and, therefore, pre-conditioning will be possible for an iterative solution. Having achieved this, a parallel solver will be straightforward.

Moreover, the proposed method has to be extended to vector problems (e.g., elasticity) and a way to handle more complicated transmission conditions as they occur in fluid-structure interaction will be sought. Additionally, the effective computation of the coupling matrices for the master side in 3D will be another important task.

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