# Sensitivity and robustness of designs for repeated measures accelerated degradation tests

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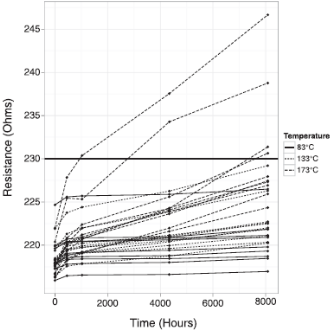
# Contents

Weaver and Meeker (2014): Methods for planning repeated measures accelerated degradation tests [WM2014]

- Optimal design to estimate $p-$quantile of the time $t_p$ until the degradation measure exceeds degradation limit
- Haselgruber and Stadlober's (2014): discussion paper on sensitivity and robustness of optimal design



# WM2014: Degradation Model with Random Effects

Degradation $Y$ as a stochastic function of stress

$$Y = x_1\gamma_1 + x_2\gamma_2\tau + b_0 + b_1\tau + \epsilon \quad \text{with } \epsilon \overset{iid}{\sim} N(0,\sigma^2)$$

- Unit-to-unit variability described by $b_0$ and $b_1$
  - $(b_0, b_1)^T \sim N(\boldsymbol{\beta}, \mathbf{V})$
  - mean $\boldsymbol{\beta} = (\beta_0, \beta_1)^T$ and covariance matrix $\mathbf{V}$
  - $(b_0, b_1)^T$ is independent of $\epsilon$
  - $\epsilon$ is independent over time
- Estimation of $\mathbf{V}$ based on the Fisher information matrix $\mathbf{I}$

# WM2014: Optimal Design

- Optimal and compromise designs in the style of Nelson (1990)
- Asymptotic variance determined by delta method is minimized

$$\mathrm{AVar}(\hat{t}_p) = \mathbf{a}^T \widehat{\mathbf{C}} \mathbf{a}$$

  - $\widehat{\mathbf{C}}$ is large sample approximation of covariance matrix of MLE
  - $a_i = \partial t_p / \partial \theta_i$, $\boldsymbol{\theta}$ contains unknown model parameters
- Directional derivative according to Whittle (1973)

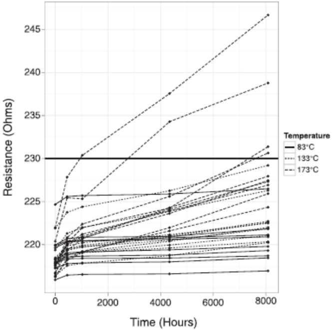
$$\Lambda(\eta, \nu) = \mathbf{a}^T (\mathbf{I}(\eta))^{-1} \mathbf{I}(\nu) (\mathbf{I}(\eta))^{-1} \mathbf{a} - \mathbf{a}^T (\mathbf{I}(\eta))^{-1} \mathbf{a}$$

.

- Design $\eta$ has smaller $\mathrm{AVar}(\hat{t}_p)$ than design $\nu$ if $\Lambda(\eta, \nu) > 0$.

# WM2014: Optimal Design for Examples 1 & 2

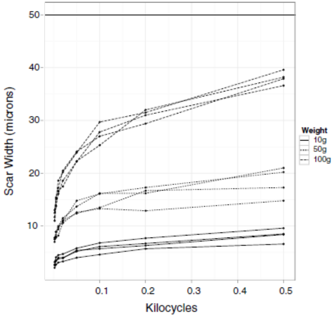
**Example 1**




Carbon resistors, Shiomi & Yanagisawa (1979)

| $x$ | $\pi$ | $\sqrt{AVar(\hat{t}_{0.1})}$ |
|---|---|---|
| 50 | 0.050 | |
| 83 | 0.711 | 21808 |
| 173 | 0.239 | |

**Example 2**




Metal wear resistance, Meeker & Escobar (1998)

| $x$ | $\pi$ | $\sqrt{AVar(\hat{t}_{0.1})}$ |
|---|---|---|
| 10 | 0.05 | 1795 |
| 100 | 0.95 | |

# Sensitivity and Robustness

- Optimal designs depend on model and on observed data
- Sensitivity investigated by comparing a grid of candidate designs with the optimal design
- Robustness of optimal solutions regarding the influence of the input data in using bootstrap simulations
    - Collection of continuous designs $\nu = (\mathbf{x}, \boldsymbol{\pi})$
    - Fixed boundary stress levels $x_L$ (low) and $x_H$ (high)
    - Fixed proportions at the bounds, $\pi_L$ and $\pi_H$
    - Inner level $x_M$ varied so that $x_L \leq x_M \leq x_H$ and $\sum_{j \in \{L,M,H\}} \pi_j = 1$ with $0 \leq \pi_j < 1 \; \forall \; j$.

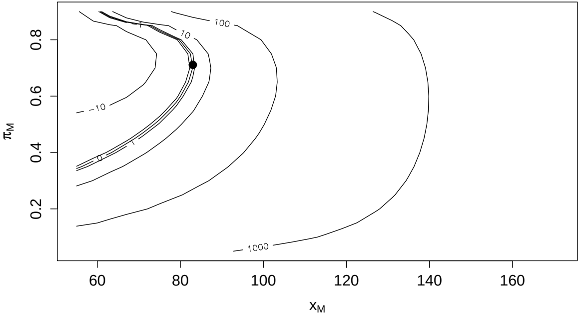
# Sensitivity Analysis of Example 1 (1)

- Degradation as resistance in Ohms

$$y_{ijk} = \gamma_2 x_k \tau_{ij} + b_{0i} + b_{1i}\tau_{ij} + \epsilon_{ijk}, \;\; \epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$$

  - Unit $i = 1..29$, time point $j = 1..5$, acceleration level $k = 1..3$
  - Time $\tau = \sqrt{\text{Time in hours}}$
  - Arrhenius transformed stress $x = -11605/(\text{T in } ^oC + 273.15)$
- MLE with `R` function `lme()`
- Optimal 3-level design $\eta$ with $\mathbf{x} = (50, 83, 173)$ and $\boldsymbol{\pi} = (0.05, 0.711, 0.239)$
- Designs $\nu = (\mathbf{x}, \boldsymbol{\pi}) = ((50, x_M, 173), (0.05, \pi_M, 0.95 - \pi_M))$ with $x_M \in (50, 173)$ and $\pi_M \in [0.05, 0.9]$ investigated

# Sensitivity Analysis of Example 1 (2)




- • is WM2014 optimal design $\eta$ with $(x_M, \pi_M) = (83, 0.711)$.
- $\eta$ has highest $x_M$ among all designs on isoline $\Lambda(\eta, \nu) = 0$
- $\forall\ x_M \in [63, 83)\ \exists$ two optimal designs with different $\pi_M$
- Equivalent 2-level design is $\mathbf{x} = (50, 173), \boldsymbol{\pi} = (0.366, 0.634)$

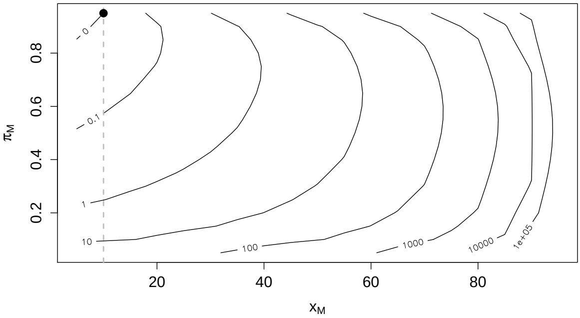
# Sensitivity Analysis of Example 2 (1)

- Degradation of log microns as

$$y_{ijk} = \gamma_1 x_{1k} + \gamma_2 x_{2k} \tau_{ij} + b_{0i} + b_{1i}\tau_{ij} + \epsilon_{ijk}, \;\; \epsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$$

  - Unit $i = 1..12$, time point $j = 1..8$, acceleration level $k = 1..3$
  - Time $\tau = \log(\text{Time in kilocycles})$
  - Stress $x_1 = x_2 = x$ as weight in grams
- MLE with `R` function `lme()`
- Optimal 2-level design $\eta$ with $\mathbf{x} = (10, 100), \;\; \boldsymbol{\pi} = (0.95, 0.05)$
- Designs $(\mathbf{x}, \boldsymbol{\pi}) = ((x_M \geq 5, 100), (\pi_M \geq 0.05, 1 - \pi_M))$ with $\pi_L = 0$ investigated

# Sensitivity Analysis of Example 2 (1)

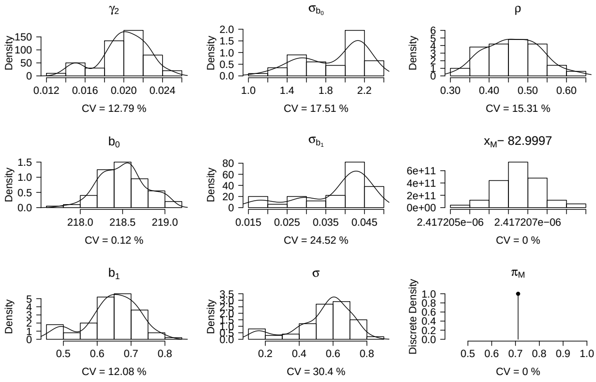



- • is optimal design $\eta$ with $x_M = 10$, $\pi_M = 0.95$ ($\pi_L = 0$)
- Equivalent optimal designs between $\mathbf{x} = (5, 100)$, $\boldsymbol{\pi} = (0.853, 0.147)$ and $\mathbf{x} = (10, 100)$, $\boldsymbol{\pi} = (0.95, 0.05)$
- Given $x_L = 10$ provides unique optimum $(x_M, \pi_M) = (10, 0.95)$

# Robustness of the Designs

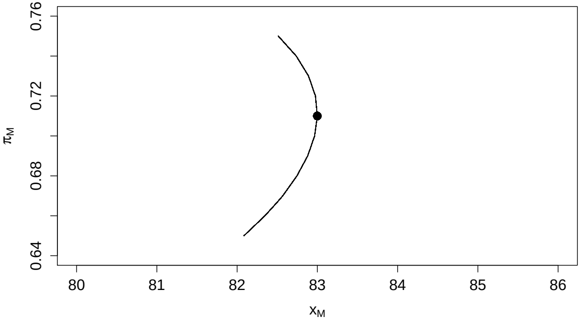
- $\mathrm{AVar}(\hat{t}_p)$ strongly depends on the input data.
- Robustness checked by bootstrap simulation
  - Select randomly $m_k$ data records with replication on each stress level $k$ where $m_k$ is the number of specimens tested at stress level $k$ in the original data set.
  - MLE with generated data
  - Estimate $\mathrm{AVar}(\hat{t}_p)$ for each candidate design $(\mathbf{x}, \boldsymbol{\pi})$ on the grid
  - Check the directional derivative on the grid with respect to the optimal design.

# Robustness of Example 1 (1)



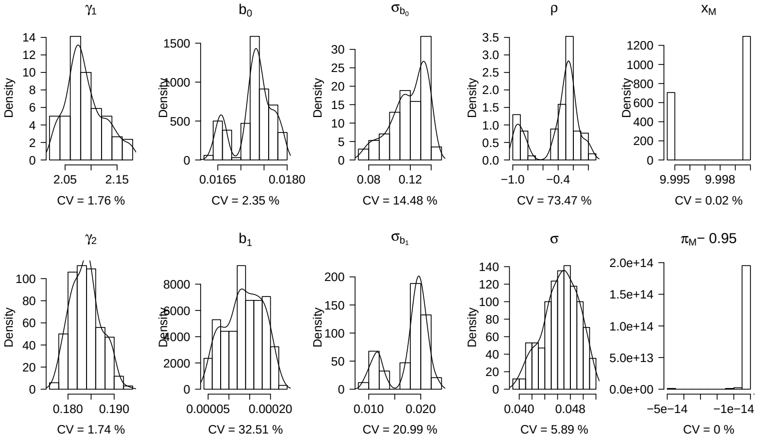

- $b_0$ was estimated very precisely
- CV $(:= 100\% * \text{standard error}/\text{estimate}) = 0$ for $x_M$ and $\pi_M$
- CV for other parameters between 10% and 30%

# Robustness of Example 1 (2)



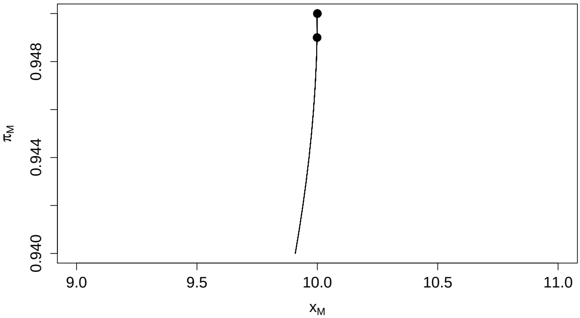

All bootstrap simulation runs provided the identical optimal design which indicates high robustness against the input data

# Robustness of Example 2 (1)




- Fixed effects $\gamma_1$, $\gamma_2$ and $b_0$ estimated precisely (CV $\approx$ 2%)
- CV for variance related parameters between 5% and 72%

# Robustness of Example 2 (2)




Optimal designs found over all bootstrap simulations between $\pi_{M_1} = \pi_{M_2} = 0.95$ and $x_{M_1} = 9.995$, $x_{M_2} = 10$ (tight range, maybe numerical uncertainty)

# Summary

- Random effects reduce error in degradation model
- Optimal design is robust against variability in input data
- Sensitivity analyses provide useful overview of equivalent optimal alternatives even among 2- and 3-level designs

# References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