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Anisotropic PML in a Finite-Element T, Phi Formulation

Introduction

Perfectly Matched Layers (PMLs) are used for reflectionless truncation of the problem boundaries in FEM applications. The basic concept behind PMLs for the method of finite-elements is to create an artificial material with a complex and anisotropic permittivity and permeability. For the **A**,*V* formulation [1] PML layers are well known. The field quantities for the **A**,*V* formulation are derived from the potentials as

$$\begin{aligned} \mathbf{B} &= curl\ \mathbf{A}, \\ \mathbf{E} &= -j\omega\mathbf{A} - j\omega \operatorname{grad} V \end{aligned} \tag{1}$$

and satisfy the constitutive relations: $\mathbf{B} = \left[\underline{\mu}\right]\mathbf{H}$, $\mathbf{D} = \left[\underline{\varepsilon}\right]\mathbf{E}$ and $\mathbf{J} = \left[\underline{\sigma}\right]\mathbf{E}$ where $\left[\underline{\mu}\right], \left[\underline{\varepsilon}\right]$ and $\left[\underline{\sigma}\right]$ are the tensors of permeability, permittivity and conductivity.

The complex material tensors of permittivity and of permeability in PMLs for the **A**,*V* formulation [2] can be written as

$$\frac{\left[\underline{\varepsilon}\right]}{\varepsilon_0} = \frac{\left[\underline{\mu}\right]}{\mu_0} = \left\langle \underline{C}\ \underline{C}\ \frac{1}{\underline{C}} \right\rangle. \tag{2}$$

The complex PML constant $\underline{C} = a - jb$ with $a = b = e$ is suggested to be selected for the **A**,*V* formulation in [2] to satisfy:

$$\frac{-\ln \rho_{\text{ref}}}{\left(\frac{1}{r_{\max}} + \beta\right)2nh} \le \quad e \quad \le \frac{-\ln d}{\left(\frac{1}{r_{\min}} + \beta\right)2h} \quad . \tag{3}$$

Therein, *n* is the number of PML layers, *h* is the thickness of one layer, ρ_{ref} is the maximal reflection coefficient and *d* is the maximal decay characterizing the element.

In this paper the method of perfectly matched layers is extented to the **T**,Φ formulation. To evaluate the accuracy of the suggested method, results have been compared with the solution obtained by a 3D edge finite-element **A**,*V* formulation.

Method

An artificial anisotropic lossy material has been applied to a 3D edge finite-element **T**,Φ formulation to act as perfectly matched layers. This material fully absorbs the electromagnetic field impinging on it without reflections on the PML-air interface [2]. These artificial material properties are implemented in a joint vector and scalar formulation (**T**, Φ ;**T**: current vector potential, Φ : magnetic scalar potential) realized by edge and nodal finite elements [1]. The field quantities for the **T**, Φ formulation are derived from the potentials as

$$\begin{aligned} \mathbf{H} &= \mathbf{T} - grad\Phi \\ \mathbf{J} &= curl\mathbf{T} \end{aligned} \tag{4}$$

and satisfy the constitutive relations: $\mathbf{B} = \left[\underline{\mu}\right]\mathbf{H}$, $\mathbf{D} = \left[\underline{\varepsilon}\right]\mathbf{E}$ and $\mathbf{E} = \left[\underline{\rho}\right]\mathbf{J}$ where $\left[\underline{\mu}\right]$, $\left[\underline{\varepsilon}\right]$ and $\left[\underline{\rho}\right]$ are the tensors of permeability, permittivity and resistivity.

The complex material tensors of permittivity and of permeability as well as the resulting specific electric resistance are the same as for the **A**, *V* formulation and can be written as

$$\frac{\left[\underline{\varepsilon}\right]}{\varepsilon_0} = \frac{\left[\underline{\mu}\right]}{\mu_0} = \left\langle \underline{C}\ \underline{C}\ \frac{1}{\underline{C}} \right\rangle \ ; \ \left[\underline{\rho}\right] = \left(j\omega\left[\underline{\varepsilon}\right]\right)^{-1} \ . \tag{5}$$

No additional conductivity is used in the PML layer. In the case when the PML layer borders a region with a permittivity or permeability other then 1, the complex material tensors (5) change to

$$\frac{\left[\underline{\varepsilon}\right]}{\varepsilon_0\varepsilon_{\mathrm{r}}} = \frac{\left[\underline{\mu}\right]}{\mu_0\mu_{\mathrm{r}}} = \left\langle \underline{C}\ \underline{C}\ \frac{1}{\underline{C}} \right\rangle \ . \tag{6}$$

The complex PML constant $\underline{C} = a - jb$ with $a = b = e$ is changed as compared to for the **A**,*V* formulation in (3) to differ by a factor k_n depending on the number of PML layers:

$$\frac{-\ln \rho_{\mathrm{ref}}}{k_n\left(\dfrac{1}{r_{\max}} + \beta\right)2nh} \leq \ e \ \leq \frac{-\ln d}{k_n\left(\dfrac{1}{r_{\min}} + \beta\right)2h} \ . \tag{7}$$

The factor k_n varies between 1 and $\sqrt{2}$. Equation (7) approximates the PML parameter *e* at which the back reflection remains below the specified bound.
Typical values for the second order finite-elements used are $\rho_{ref} = 10^{-4}$ and $d = 3\cdot 10^{-3}$. The dependence of the factor k_n on the number of PML layers used is:

$$k_n = \begin{cases} 1 & n \geq 4 \\ 1.2 & n = 3 \\ \sqrt{2} & n = 2 \end{cases} \tag{8}$$

As illustrated in the following section, for the **T**, Φ formulation it is adequate to use one

PML layer only. The PML coefficient is suggested then to be calculated as

$$e = \frac{-\ln d}{\left(\frac{1}{r_{\min}} + \beta\right)2h} \quad . \tag{9}$$

In the cases where n is greater than 1, it will also be shown by the numerical examples that the reflection of the PML layers and the number of iterations are better if the lower bound of (7) is chosen as PML coefficient.

Numerical Examples

Magic-T

As a first example, a waveguide hybrid junction known as Magic-T [3] has been investigated. The frequency range of the waveguide has been set to the X-Band ranging from 8.2 GHz to 12.4 GHz. The dimensions of the waveguide are: a=22.86mm, b=10.16mm.

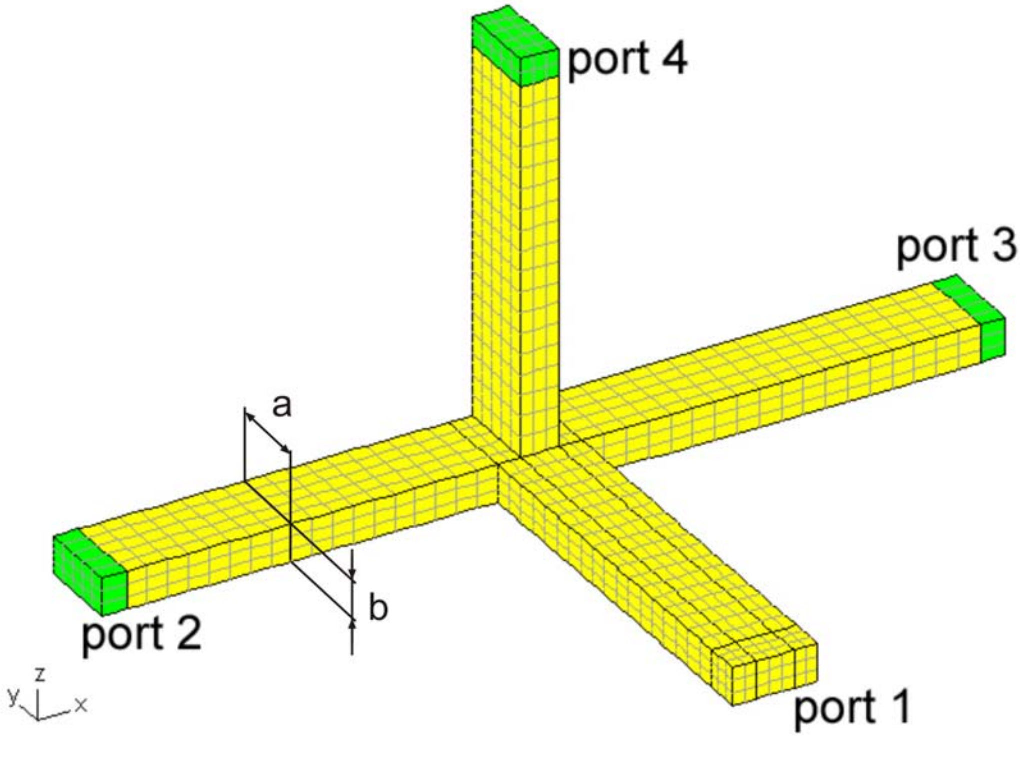

Figure 1: Waveguide hybrid junction, known as Magic-T. The PML layer is shown in green.

When a TE_{10} mode is incident in port 1 the electric field cannot excite the TE_{10} mode in port 4. Thus there is no coupling between these two ports. Similary to port 4, port 2 and port 3 are truncated with PMLs. If the PMLs do not work properly, there are reflections on port 2 and port 3.

These reflected waves can excite the TE_{10} mode in arm 4 and can be calculated from the Poynting vector [4].

$$P + jQ = \int_{port4} \underline{\mathbf{S}} \cdot \mathbf{n}\, d\Gamma = \frac{1}{2} \int_{port4} (\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{n}\, d\Gamma \tag{10}$$

The apparent power through port 4 is the sum of the reflected waves from port 2 and port 3. The length of port 1 , port 2 and port 4 are 0.1m. To maximise the signal strength on port 4, the length of port 3 has been altered to 0.1m + $\lambda/4$, where λ is the wave length. The walls of the waveguide are modeled as perfect conductors.
The problem was investigated at 10 GHz. For the **A**,*V* formulation and for the **T**,Φ formulation the PML parameters were set to the following values: *a = b = e = 2, h=27,43mm,* $r_{min} = r_{max} = 0.2m$, $\beta = 209.44$, $\rho_{ref} = 10^{-4}$ and $d = 3*10^{-3}$. To investigate the accuracy of the PML, different numbers of PML layers *n* have been used. The thickness *h* of one PML layer was calculated for the **A**,*V* formulation and for the **T**,Φ formulation using (7), to evaluate different sets of the reduction factor k_n for every number *n* of the PML layers.
The reflection coefficient r_T at port 4 is defined for this example as the ratio of the apparent power at port 4 to the apparent power at port 2. Table 1 shows the results for different numbers *n* of the PML layers and different thicknesses *h* of one PML layer.

n	h	r_T (**A**,*V*)	r_T (**T**,Φ)
3	h_{min}	2.34E-06	1.58E-05
	h_{max}	6.22E-04	5.13E-04
	$h_{min}/1.2$	9.85E-06	7.01E-06
	$h_{min}/1.414$	9.33E-05	4.21E-05
	$h_{min}/2$	1.64E-03	1.06E-03
2	h_{min}	1.07E-04	1.44E-04
	h_{max}	6.25E-04	5.20E-04
	$h_{min}/1.2$	2.30E-05	4.41E-05
	$h_{min}/1.414$	1.10E-04	2.23E-05
	$h_{min}/2$	1.68E-03	1.03E-03
1	h_{min}	6.51E-03	5.51E-03
	h_{max}	1.34E-03	8.48E-05
	$h_{min}/1.414$	1.34E-03	6.01E-04
	$h_{min}/2$	2.42E-03	4.90E-04
	$h_{max}/2$	2.17E-02	1.25E-02

Table 1: Reflection coefficient at port 4 for **A**,*V* formulation and for **T**,Φ formulation

The values of h_{min} and h_{max} were calculated from (7) as follows:

$$h_{min} \geq \frac{-\ln \rho}{k_n \left(\frac{1}{r_{max}} + \beta \right) 2 \cdot n \cdot e} = \frac{10.74E-3}{k_n \cdot n} [m]$$

$$h_{max} \leq \frac{-\ln d}{k_n \left(\frac{1}{r_{min}} + \beta \right) 2 \cdot e} = \frac{6.77E-3}{k_n} \; [m] \tag{11}$$

As can be seen it is not possible to use only one PML layer with the $\mathbf{A}, V$ formulation. Values with dark background show that the reflection is too high for the junction to work properly. The best values were marked with a light gray background. To demonstrate the difference between the $\mathbf{T}, \Phi$ formulation and the $\mathbf{A}, V$ formulation with one PML layer Fig. 2 has been plotted. Fig. 2a shows the electric field in the Magic-T with the $\mathbf{T}, \Phi$ formulation. As can be seen the PML layer works properly, there is no visible field at port 4. Figure 2b shows the electric field in the Magic-T with the $\mathbf{A}, V$ formulation. There is a visible electric field at port 4 showing that the reflection at port 2 and port 3 is to high.

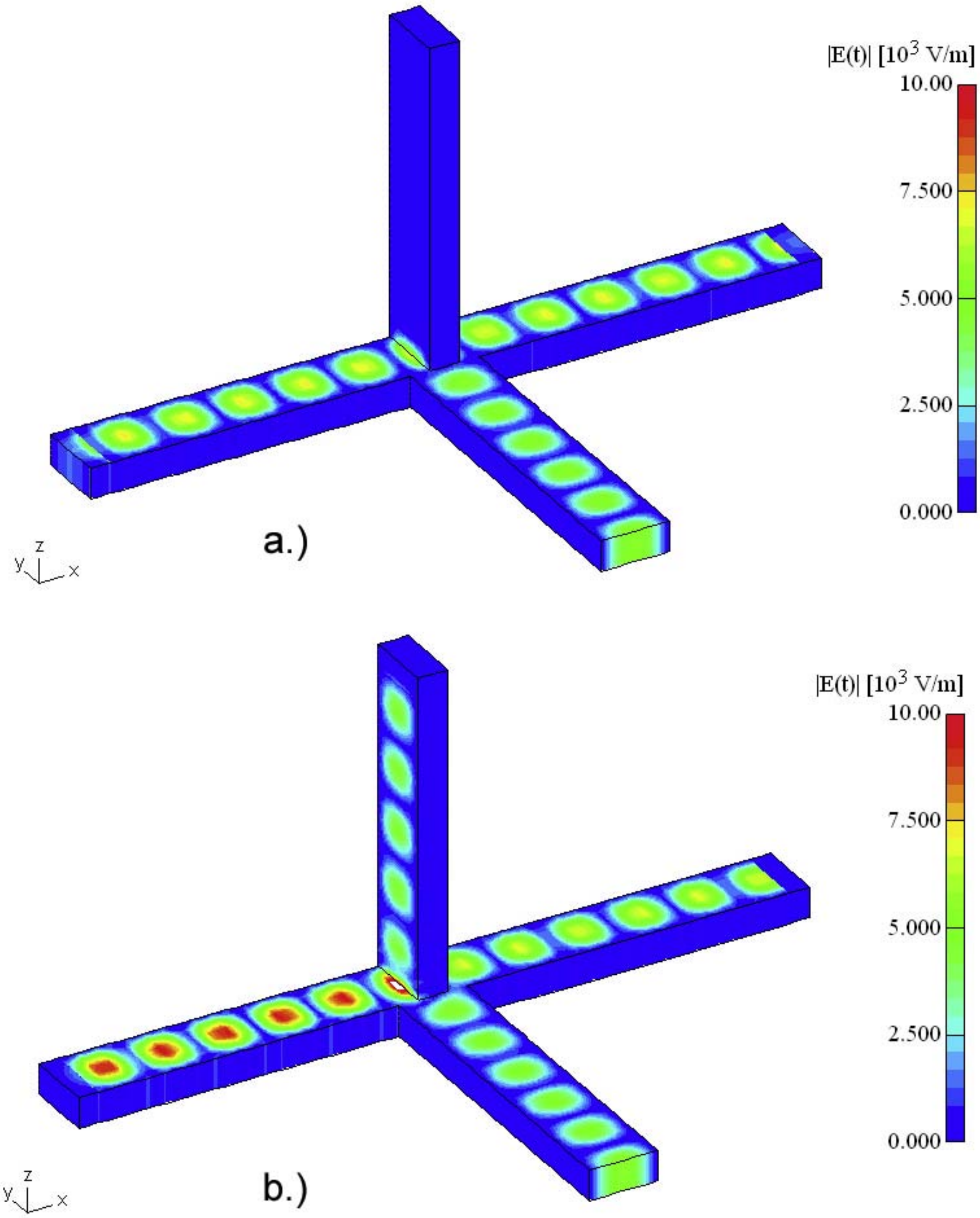

Figure 2: Electric field in the waveguide calculated with one PML layer.
a: $\mathbf{T}, \Phi$ formulation
b: $\mathbf{A}, V$ formulation

This example shows that it is adequate to use only one PML layer with the **T**, Φ formulation. In addition, the reflection of the PML layers can be improved if the thickness of the layers is tuned.

Dipole antenna

For the second example, a λ/2-dipole antenna has been treated. Making use of symmetry, one eighth of the arrangement was modeled only. The edge size of the cube was 1.5λ and its material constants were $\varepsilon_r=\mu_r=1$ and $\sigma=0.2$ mS/m. An excitation frequency of 2.5 GHz has been chosen. The antenna was modeled as a long cylindrical wire with a diameter of 1mm. The walls of the antenna were modeled as perfect electric conductors and the finite elements inside the wire were deleted.

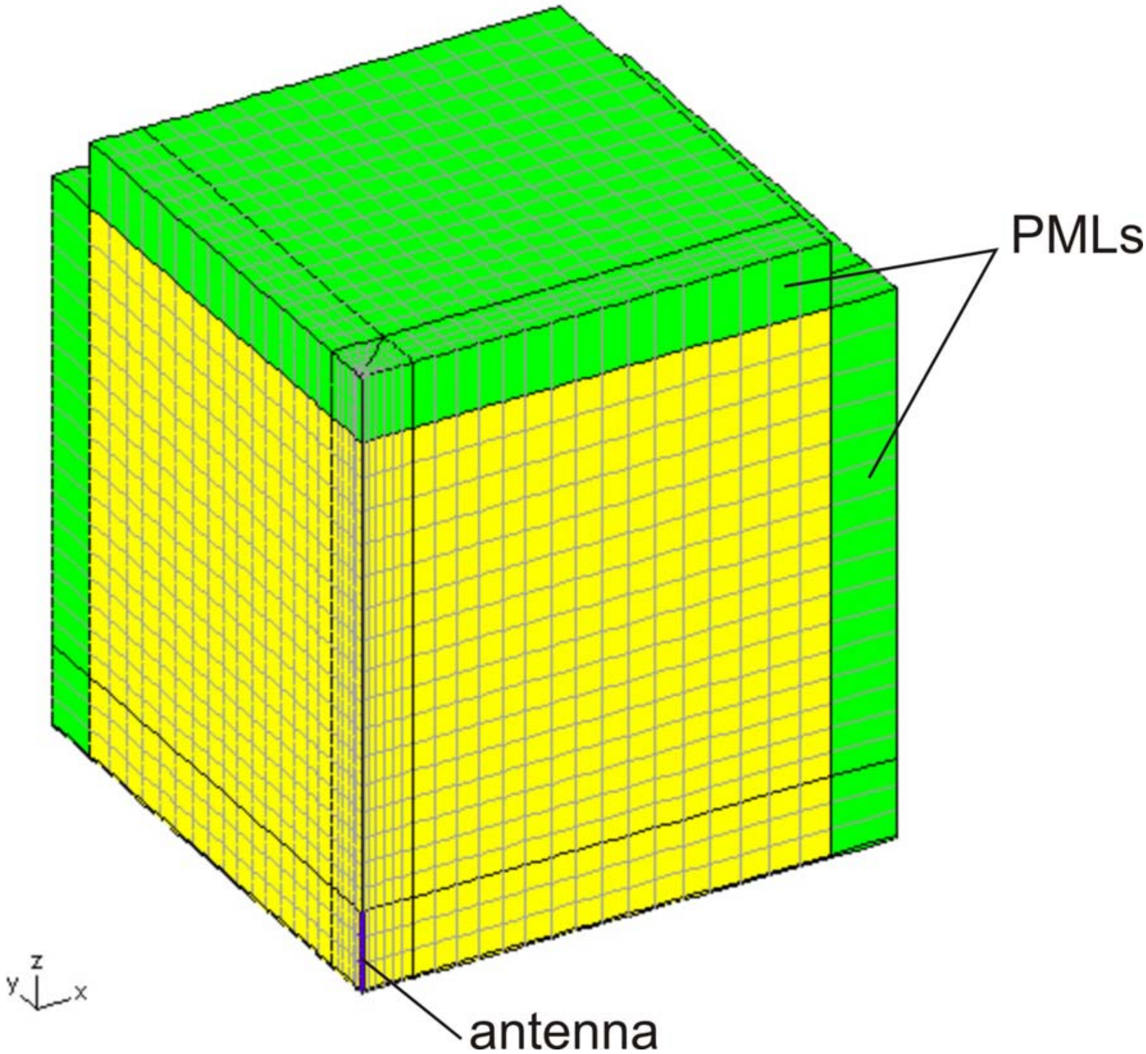

Figure 3: finite element model of λ/2-dipole antenna

For the **T**, Φ formulation one PML layer was used and for the ***A***, *V* formulation three PML layers were employed.

The PML parameters were set to the following values: $a=b=e=1.5$, $r_{min}=0.16m$, $r_{max}=0.28m$, $\beta=52.39$, $\rho_{\text{ref}}=10^{-4}$ and $d=3*10^{-3}$. The thickness of one PML layer was calculated for the **A**,V formulation with equation (3) and for the **T**,Φ formulation with equation (7). Fig. 3 shows the finite element model of the λ/2-dipole antenna. The PML layer is plotted in green and the λ/2-dipole antenna is drawn in blue.

The solution of the second example is presented as field plots of the electric field and the magnetic field.

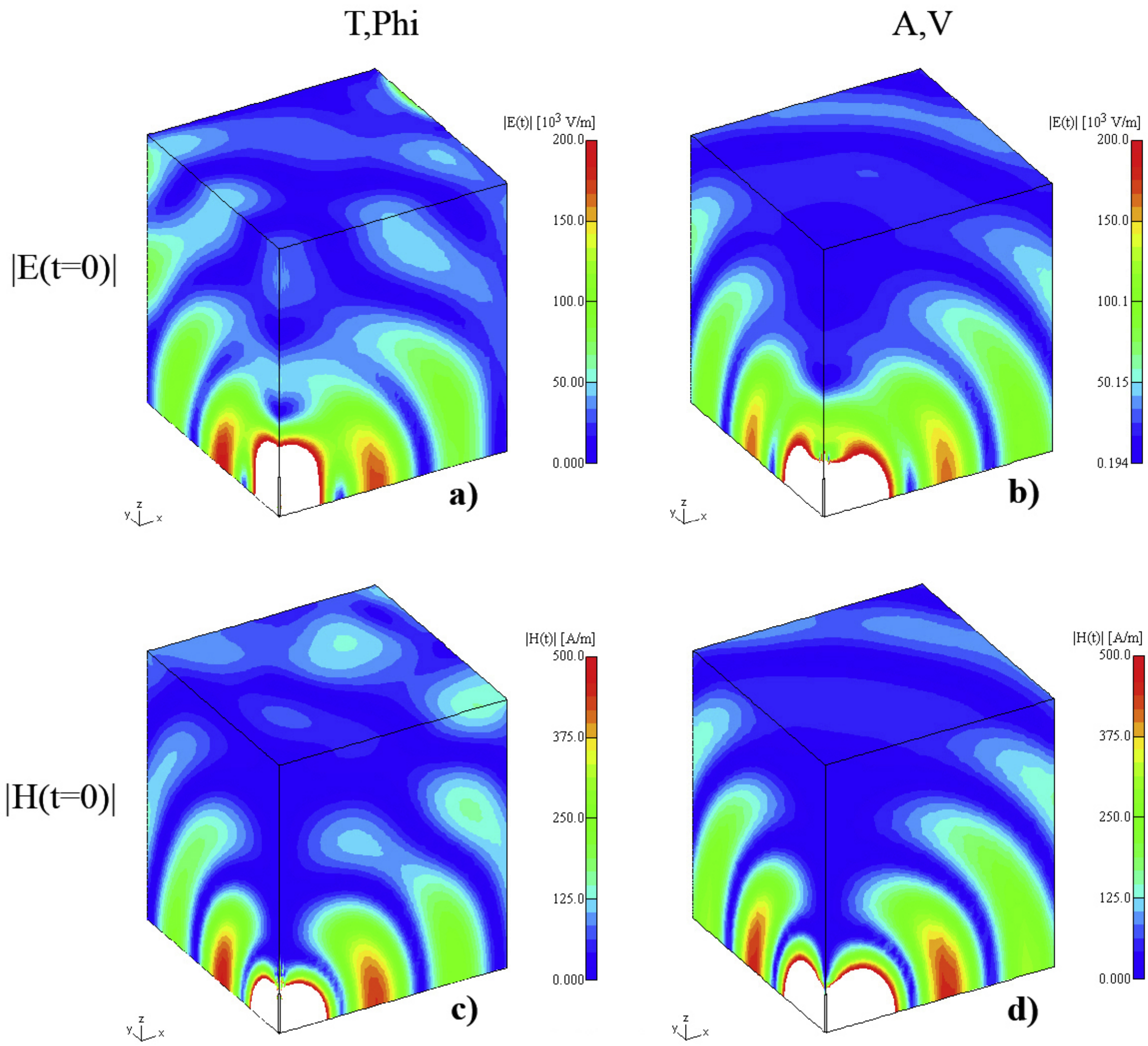

Figure 4: Field quantities for the electric and the magnetic field for the **A**,V formulation and the **T**,Φ formulation.

The solution obtained from the **T**,Φ formulation is shown in Figs 4a and 4c and the solution obtained by the **A**,V formulation is plotted in Figs 4b and 4d. The solutions for the electric and magnetic fields look quite the same for both formulations. The PML layers are not plotted in the figures. The excitations for the **T**,Φ formulation and for the **A**,V formulation have not been scaled equally. The deviation between the two solutions is very small indicating that the PML also works with free space wave propagation for a **T**, Φ formulation.

Conclusion

It has been shown that the reflectionless truncation of the problem boundaries using FEM also works with the **T**, Φ formulation. Furthermore, for the **T**, Φ formulation it is possible to use only one PML layer. The reflection at the PML layer and the convergence rate are better at the lower bound of (7) for the complex PML constant.
In addition the **T**, Φ PML implementation can be improved by reducing the layer thickness *h* by a factor k_n as shown in (8).

References:

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