Planar 3-dimensional assignment problems with Monge-like cost arrays

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Abstract

Given an $n \times n \times p$ cost array C we consider the problem p-P3AP which consists in finding p pairwise disjoint permutations $\varphi_1, \varphi_2, \ldots, \varphi_p$ of $\{1, \ldots, n\}$ such that $\sum_{k=1}^{p} \sum_{i=1}^{n} c_{i\varphi_k(i)k}$ is minimized. For the case $p = n$ the planar 3-dimensional assignment problem P3AP results.

Our main result concerns the p-P3AP on cost arrays C that are layered Monge arrays. In a layered Monge array all $n \times n$ matrices that result from fixing the third index k are Monge matrices. We prove that the p-P3AP and the P3AP remain NP-hard for layered Monge arrays. Furthermore, we show that in the layered Monge case there always exists an optimal solution of the p-3PAP which can be represented as matrix with bandwidth $\leq 4p - 3$. This structural result allows us to provide a dynamic programming algorithm that solves the p-P3AP in polynomial time on layered Monge arrays when p is fixed.

Keywords. layered Monge arrays, planar 3-dimensional assignment problem, repeated assignment problem, block structure.

1 Introduction

Let S_n denote the set of permutations on $\{1, \ldots, n\}$. Given an $n \times n$ cost matrix M, the classical linear assignment problem LAP can be stated as follows

$$\min_{\varphi \in S_n} \sum_{i=1}^{n} m_{i\varphi(i)}.$$

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The LAP can be generalized to three dimensions in two ways. The resulting two problems are the axial 3-dimensional assignment problem and the planar 3-dimensional assignment problem [4].

Let an $n \times n \times n$ array $C = (c_{ijk})$ be given. For notational convenience let us introduce *planes* and *lines* of C. We refer to the subarray (matrix) that results from C by fixing the third index k as k-planes. Lines arise if a subset of two indices is fixed.

The *axial 3-dimensional assignment problem* A3AP can be stated as follows:

$$\min_{\varphi, \psi \in S_n} \sum_{i=1} c_{i\varphi(i)\psi(i)}.$$

In the *planar 3-dimensional assignment problem* P3AP the goal is to find n pairwise disjoint permutations $\varphi_1, \varphi_2, \ldots, \varphi_n \in S_n$ so as to minimize the cost function

$$\sum_{k=1}^{n} \sum_{i=1}^{n} c_{i\varphi_k(i)k}$$

where two permutations $\varphi, \psi \in S_n$ are said to be disjoint if $\varphi(i) \neq \psi(i)$ for all $i \in \{1, \ldots, n\}$.

Both problems can be formulated as integer linear programs. Below the ILP formulation of the P3AP is given.

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk} x_{ijk} \tag{1}$$

$$\text{s.t.} \ \sum_{k=1}^{n} x_{ijk} = 1 \qquad i, j = 1, 2, \ldots, n \tag{2}$$

$$\sum_{i=1}^{n} x_{ijk} = 1 \qquad j, k = 1, 2, \ldots, n \tag{3}$$

$$\sum_{j=1}^{n} x_{ijk} = 1 \qquad i, k = 1, 2, \ldots, n \tag{4}$$

$$x_{ijk} \in \{0, 1\} \qquad i, j, k = 1, 2, \ldots, n. \tag{5}$$

Let $c(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk} x_{ijk}$ denote the cost of the assignment x.

Both the A3AP and the P3AP are known to be NP-hard already for 0-1 cost arrays. For the axial case this has been proved by Karp [14] and for the planar case by Frieze [10].

The *p-layer planar 3-dimensional assignment problem*, p-P3AP for short, is the following variant/generalization of the P3AP. Given an $n \times n \times p$ cost array C where $p \in \{2, \ldots, n\}$, the goal is to find p pairwise disjoint

permutations $\varphi_1, \ldots, \varphi_p$ so as to minimize the cost function

$$\sum_{k=1}^{p} \sum_{i=1}^{n} c_{i\varphi_k(i)k}.$$

Note that for $p = n$ the P3AP results.

The associated integer linear program arises from the one for the 3PAP by replacing the equality constraints in (2) by $\sum_{k=1}^{p} x_{ijk} \leq 1$ and setting the upper limit for the index k in the objective function (1) and in the constraints (3), (4) and (5) to p.

Frieze [10] proved that the DISJOINT MATCHING problem described below is NP-complete.

DISJOINT MATCHINGS (DM)
Input: Disjoint finite sets Q, S with $|Q| = |S|$ and sets $A_1, A_2 \subseteq P \times Q$.
Question: Do there exist matchings $M_i \subseteq A_i$, for $i = 1, 2$ such that $M_1 \cap M_2 = \emptyset$?

It immediately follows that the p-layer planar 3-dimensional assignment problem is NP-hard for every fixed $p \geq 2$.

For the P3AP both exact and heuristic approaches have been devised, see e.g. [15, 16, 19]. The polyhedral structure of the P3AP has been studied in [3, 7, 8]. Frieze and Sorkin [9] establish high probability lower and upper bounds for the optimal objective function value under the assumption that the costs c_{ijk} are independent, identically distributed exponential random variables with parameter 1. The P3AP has applications in many areas such as timetabling, rostering and satellite launching. For references and more details see the survey by Spieksma [18] and the monograph by Burkard et al. [4].

The p-P3AP has received less attention in the literature than the P3AP. Gimadi and Glazkov [12] suggested an approximation algorithm for the p-P3AP and analysed its performance for the case where the costs are independent, identically distributed uniform random variables. Yokoya et al. [20] present an algorithm for the p-P3AP (called repeated assignment problem in their paper) that outperforms general ILP-solvers for instances with small p.

Polynomially solvable special cases. For the axial 3-dimensional several polynomially solvable special cases are known, see [4]. We are not aware, however, of any result in the literature on non-trivial polynomially solvable cases of the planar 3-dimensional assignment problem. For the p-P3AP the situation is hardly better. The only known special case concerns the case when all k-planes of C are the same, i.e. $c_{ij1} = c_{ij2} = \ldots = c_{ijp}$. In this case the p-P3AP reduces to finding a p-factor with minimum cost on a bipartite graph (the p-factor can then be decomposed into p pairwise disjoint perfect matchings/assignments) and is solvable by network flow methods.

Many hard combinatorial optimization problems become efficiently solvable for Monge-like cost structures. Examples include the traveling salesman problem [11], economic lot-sizing problems [2] and the axial d-dimensional assignment and transportation problems [5].

The most famous Monge structures are *Monge matrices* which arise in many areas of mathematics [5].

Definition 1.1. *An $n \times n$ matrix $M = (m_{ij})$ is called Monge matrix if the following property holds*

$$m_{ij} + m_{kl} \leq m_{il} + m_{kj} \qquad \text{for all } 1 \leq i < k \leq n, 1 \leq j < l \leq n. \qquad (6)$$

Monge arrays are a generalization of the concept of Monge matrices to $d \geq 3$ dimensions. A d-dimensional array is a Monge array if and only if all its 2×2 submatrices are Monge matrices. More details are provided in Section 2.

The following folklore result for the linear assignment problem LAP and the axial 3-dimensional assignment problem A3AP is well known.

Observation 1.1. *Let ε_n denote the identity permutation with $\varepsilon_n(i) := i$.*

(i) *ε_n is an optimal solution for the LAP on $n \times n$ Monge matrices.*

(ii) *$\varphi = \varepsilon_n$ and $\psi = \varepsilon_n$ provides an optimal solution for the A3AP on $n \times n \times n$ Monge arrays.*

This result motivates to ask whether Monge properties are helpful also for the planar 3-dimensional assignment problem 3PAP and its p-layered variant p-3PAP.

The main focus in this paper will be on the class of layered Monge arrays which form a superclass of Monge arrays. The array $C = (c_{ijk})$ is called *layered Monge array* if all its k-planes are Monge matrices.

Organization of the paper. The paper is organized as follows. In Section 3, we provide intractability results for the P3AP and the p-P3AP. First, it is shown that the P3AP remains hard if the cost array C is monotone in all three coordinate directions (along all lines). ¡As main result of Section 3 it is shown that the P3AP and the p-P3AP stay NP-hard on the class of layered Monge arrays.

In Section 4 the structure of an optimal solution of the p-P3AP on Monge arrays and on layered Monge arrays is investigated. For the case of Monge arrays it is shown that there always exists an optimal solution of the 2-P3AP which has a nice block structure. The result breaks down already for $p = 3$. For the case of layered Monge arrays it is shown that there always exists an optimal solution of the P3AP which can be represented as matrix with a bandwidth $\leq 4p - 3$. This result improves and generalizes the result on k-matchings in a Monge matrix from [17]. In Section 5 the

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bandwidth bound is exploited to provide a dynamic programming algorithm that obtains an optimal solution of the p-P3AP on layered Monge arrays and runs in polynomial time if p is fixed.

2 Preliminaries

2.1 Basic facts about planar 3-dimensional assignment problems

A *Latin rectangle* is an $m \times n$ table L filled with integers from the set $\{1, \ldots, n\}$ such that every row and every column contains every integer at most once. An $n \times n$ Latin rectangle is called *Latin square* (of order n). A *partial Latin square* results if entries of the table can stay unfilled.

The following observation is straightforward and well-known.

Observation 2.1. *The set of feasible solutions of the p-3PAP is in one-to-one correspondence with*

 (i) *the set of $p \times n$ Latin rectangles*

 (ii) *the set of partial $n \times n$ Latin squares in which each of the integers $1, 2, \ldots, p$ appears exactly n times and the integers from $\{p+1, \ldots, n\}$ do not appear at all.*

To see this, let x be a feasible solution of the p-P3AP. For (i) place integer i in column j and row k if and only if $x_{ijk} = 1$. One ends up with a $p \times n$ Latin rectangle.

For (ii) place integer k in row i and column j if and only if $x_{ijk} = 1$. One obtains a partial Latin square of the claimed type. For the special case of the P3AP one ends up with a Latin square. See Figure 1 for an illustrating example with $n = 4$ and $p = 3$.

$$
\begin{aligned}
x_{113} = x_{121} = x_{132} = 1 \\
x_{211} = x_{233} = x_{242} = 1 \\
x_{322} = x_{331} = x_{343} = 1 \\
x_{412} = x_{423} = x_{441} = 1
\end{aligned}
\qquad \longleftrightarrow \qquad
\begin{array}{|c|c|c|c|}
\hline
3 & 1 & 2 & \\
\hline
1 & & 3 & 2 \\
\hline
& 2 & 1 & 3 \\
\hline
2 & 3 & & 1 \\
\hline
\end{array}
\qquad \longleftrightarrow \qquad
\begin{array}{|c|c|c|c|}
\hline
2 & 1 & 3 & 4 \\
\hline
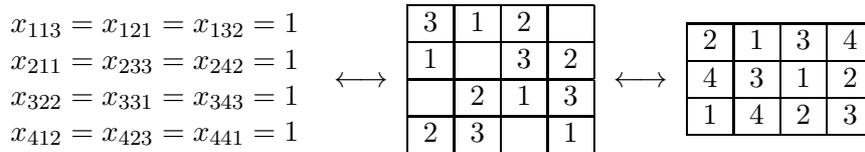4 & 3 & 1 & 2 \\
\hline
1 & 4 & 2 & 3 \\
\hline
\end{array}
$$

Figure 1: Example for partial Latin square/Latin rectangle representation

The partial Latin square representation uses an $n \times n$ table regardless of the value of p. For small values of p the Latin rectangle representation is more compact and will turn out to be useful in Section 4.

The following observation is straightforward, but turns out to be useful, for example for showing that assuming monotonicity of C does not make the P3AP and the p-P3AP any easier (see Section 3).

Observation 2.2. *Let C be an $n \times n \times p$ array and let $A = (a_{ij})$ be an $n \times n$ matrix and $B = (b_{ij})$ and $D = (d_{ij})$ be $n \times p$ matrices.*

- *Let $p = n$. The P3AP instances with cost array C and with cost array $C + C'$ where*

$$c'_{ijk} = a_{ij} + b_{ik} + d_{jk} \qquad \text{for all} \quad i, j, k \in \{1, \ldots, n\} \qquad (7)$$

   are equivalent. The objective function value is shifted by the constant $\alpha = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} + b_{ij} + d_{ij})$.

- *The p-P3AP instances with cost array C and with cost array $C + C''$ where*

$$c''_{ijk} = b_{ik} + d_{jk} \qquad \text{for all} \quad i, j \in \{1, \ldots, n\}, k \in \{1, \ldots, p\} \qquad (8)$$

   are equivalent. The objective function value is shifted by the constant $\beta = \sum_{i=1}^{n} \sum_{k=1}^{p} (b_{ik} + d_{ik})$.

We mention that arrays that fulfill the property (7) have been named sum-decomposable with parameters $d = 3$ and $s = 2$ in [6]. There it was proved that all feasible solutions of the P3AP have the same cost if and only if the cost array C is sum-decomposable with parameters $d = 3$ and $s = 2$. The cost arrays that fulfill (8) play an analogous role for the p-P3AP for the $p < n$ case.

2.2 Cost arrays with Monge properties

The notion of Monge matrices can be generalized to $d > 2$ dimensions.

Definition 2.1. *An $n_1 \times n_2 \times \cdots \times n_d$ d-dimensional array $C = (c_{i_1 i_2 \cdots i_d})$ is called **Monge array** if for all $i_k = 1, \ldots, n$ and $j_k = 1, \ldots, n$, $k = 1 \ldots, d$, we have*

$$c_{s_1 s_2 \cdots s_d} + c_{t_1 t_2 \cdots t_d} \leq c_{i_1 i_2 \cdots i_d} + c_{j_1 j_2 \cdots j_d}, \qquad (9)$$

where $s_k = \min\{i_k, j_k\}$ and $t_k = \max\{i_k, j_k\}$, $k = 1 \ldots, n$.

Note that a two dimensional Monge array is a Monge matrix. It is well known that a d-dimensional array C is a Monge array if and only if every two-dimensional subarray (matrix) of C corresponding to fixed values for $d - 2$ of the d indices in C is a Monge matrix, see [1].

Next we define a special class of Monge arrays.

Definition 2.2. *An $n \times n \times \cdots \times n$ d-dimensional array $C = (c_{i_1 i_2 \cdots i_d})$ is called* distribution array *if*

$$c_{i_1 i_2 \cdots i_d} = -\sum_{j_1=1}^{i_1} \sum_{j_2=1}^{i_2} \cdots \sum_{j_d=1}^{i_d} p_{j_1 j_2 \cdots j_d} \qquad (10)$$

where $p_{j_1 j_2 \cdots j_d} \geq 0$ for all $(j_1, j_2, \ldots, j_d) \in \{1, \ldots, n\}^d$.

It is easy to show that every distribution array is a Monge array. Next we consider a superclass of the class of Monge arrays.

Definition 2.3. *Let $C = (c_{ijk})$ be a three-dimensional $n \times n \times p$ array such that for every k, $1 \leq k \leq p$, the matrix $M^k = (m_{ij}^k)$ where $m_{ij}^k = c_{ijk}$ is a Monge matrix. Then C is called an* array with Monge layers, *or* layered Monge array *for short.*

3 Intractability results

Some hard combinatorial optimization problems become easy if the cost structure fulfills a monotonicity property. Unfortunately, monotonicity does not help for the P3AP.

Definition 3.1. *An $n_1 \times n_2 \times n_3$ cost array C is called triply graded if C is monotone increasing along all lines of C.*

Theorem 3.1. *The P3AP and the p-P3AP stay NP-hard on the class of triply graded cost arrays.*

Proof. Let a $n \times n \times p$ cost array C with $p \in \{2, \ldots, n\}$ be given. We will show that $C = (c_{ijk})$ can be turned into a triply graded cost array $\widetilde{C}$ such that the order of the feasible solutions of the p-3PAP with respect to the objective function value does not change.

Let
$$m \geq |\max_{i,j,k} c_{ijk} - \min_{i,j,k} c_{ijk}|.$$

Define $\widetilde{c}_{ijk} = c_{ijk} + (i + j + k)m$. It follows from Observation 2.2 that the p-P3AP instances with cost arrays C and $\widetilde{C}$ are equivalent. Moreover, it can be seen easily that $\widetilde{C}$ is monotone increasing along all lines and hence triply graded. □

Note that the same approach also works if the direction of monoticity is not the same in all three coordinate directions. Next we turn to the case of layered Monge arrays.

Theorem 3.2. *The P3AP stays NP-hard on the class of layered Monge cost arrays.*

Proof. Consider the $n \times n$ matrix $M = (m_{ij})$ with $m_{ij} = -(i + j)^2$ (or alternatively $m_{ij} = 4n^2 - (i + j)^2$ if one prefers to deal with nonnegative cost arrays). It is easy to check that M is a Monge matrix. Let $F = (f_{ijk})$ be the $n \times n \times n$ cost array obtained from M by setting $f_{ijk} = m_{ij}$ for all $i, j, k \in \{1, \ldots, n\}$. Since M is a Monge matrix, F is a layered Monge array.

Let C be an arbitrary $n \times n \times n$ 0-1 array. Consider the $n \times n \times n$ cost array $C' = (c'_{ijk})$ with $c'_{ijk} = f_{ijk} + c_{ijk}$. It is easy to check that the layered Monge array property is inherited to C' from F.

It follows from Observation 2.2 that the P3AP on cost matrix C is equivalent to the P3AP on cost array C'. Since the P3AP is NP-hard for general 0-1 cost arrays it follows that the P3AP stays hard when restricted to layered Monge arrays. $\square$

Theorem 3.3. *The p-P3AP stays NP-hard on the class of layered Monge cost arrays.*

Proof. The proof builds on the idea used in the proof of Theorem 3.2. We make again use of the $n \times n$ Monge matrix $M = (m_{ij})$ with $m_{ij} = -(i+j)^2$. Let $n' = 2n$ and expand M into an $n' \times n'$ matrix M' as follows

$$M' = \begin{pmatrix} M & Y \\ Y^t & Z \end{pmatrix}$$

where Z is the $n \times n$ zero matrix and $Y = (y_{ij})$ is an $n \times n$ matrix with $y_{ij} = i \cdot n$ for all $i, j \in \{1, \ldots, n\}$. It is easy to check that M' is again a Monge matrix.

Let an instance of the P3AP with an $n \times n \times n$ 0-1 cost array C be given. Expand C into an $n' \times n' \times n$ array $\widehat{C} = (\widehat{c}_{ijk})$ by defining

$$\widehat{c}_{ijk} = \begin{cases} c_{ijk} & \text{for } i,j,k \in \{1, \ldots, n\} \\ 0 & \text{for } i,j \in \{n+1, \ldots, n'\}, k \in \{1, \ldots, n\} \end{cases}$$

Now create the $n' \times n' \times n$ cost arrays $F' = (f'_{ijk})$ and $C' = (c'_{ijk})$ with

$$f'_{ijk} = m'_{ij} \quad \text{for all } i,j \in \{1, \ldots, n'\}, k \in \{1, \ldots, n\}$$

and

$$c'_{ijk} = f_{ijk} + \widehat{c}_{ijk} \quad \text{for all } i,j \in \{1, \ldots, n'\}, k \in \{1, \ldots, n\}.$$

It is easy to check that C' is a layered Monge array. Now consider the instance I of the p-P3AP with cost array C' array and $p = n$.

Note that the cost entries in the Y and the Y^t block of M' (which carry over to the k-planes of F' and C') are positive numbers $\geq n$ while the entries in Z are zero and the entries in M are negative. Thus an optimal solution of the p-P3AP will contain only elements (i, j, k) for which either $i, j \in \{1, \ldots, n\}$ or $i, j \in \{n+1, \ldots, n'\}$ holds when such solutions exist (which is easily seen to be the case).

Hence an optimal solution of the P3AP with cost array C can be obtained from an optimal solution of the p-P3AP instance I by dropping all 3-tuples in the solution that involve indices $> n$. The result now follows from the NP-hardness of the P3AP for 0-1 cost arrays. $\square$

Note that the cost arrays used in the hardness reductions in the proofs for Theorems 3.2 and 3.3 do not belong to the class of Monge arrays. The complexity status of the P3AP and of the p-P3AP on the class of Monge arrays and its subclass distribution arrays remains open.

4 The structure of the optimal solution of the p-P3AP on layered Monge arrays

Note that in the hard instance constructed in the proof of Theorem 3.3 the number of layers p is of the order of n. This leaves the complexity status of the p-P3AP for layered Monge arrays unsettled when p is a constant. In this section we present results on the structure of the optimal solution of the p-P3AP on layered Monge arrays. The bandwidth result shown in Theorem 4.3 will be used in Section 5 to provide an algorithm that solves the p-P3AP on layered Monge arrays in polynomial time if p is fixed.

4.1 Block structure result for the 2-P3AP

In this subsection we will prove that there always exists an optimal solution of the 2-P3AP on layered Monge arrays which fulfills a nice block structure. To formulate this result and to prove it we will need the following definitions. It will be convenient to represent the feasible solutions of the 2-P3AP by Latin rectangles, cf. Section 2.1.

Definition 4.1. *A partition of a Latin rectangle into* blocks *is a partition of the columns into sets of adjacent columns such that such each set of m columns contains only m different integers, and is minimal with respect to that property. A block which spreads from k-th to the l-th column, $k < l$ and contains only integers from the set $\{k, \ldots, l\}$ is called* normalized block.

As an illustrative example consider the 2×12 Latin rectangle below. It is partitioned into 4 blocks. The second block which contains the numbers 3, 4 and 5 is the only normalized block.

6	2	4	3	5	8	11	12	1	7	10	9
2	6	3	5	4	12	1	11	8	10	9	7

Note that if the block B is a normalized block which consists of the m columns $j, \ldots, j + m - 1$ the following property is fulfilled for all $i = 1, \ldots, m - 1$

(*) The first i columns of B contain an integer $x > i + j$ and the last i columns of B contain an integer $x < j + m - i$.

The local operation of a swap that exchanges two integers in a row of an $p \times n$ Latin rectangle will play a fundamental role in what follows.

Definition 4.2. *Let a feasible solution x of the p-P3AP be given and let $r, q \in \{1, \ldots, n\}$, $r < q$. The operation that exchanges the positions of r and q in a row k of the Latin rectangle representation of x is referred to as a* swap *and denoted as* $\mathrm{SWAP}(r, q, k)$. *The swap is called* feasible *if the newly obtained solution is feasible. The swap is called* non-increasing *if the newly obtained obtained solution x' has cost $c(x') \leq c(x)$.*

Observation 4.1. *Let C be an $n \times n \times p$ layered Monge array. Then the swap $\mathrm{SWAP}(r, q, k)$ is non-increasing if r is placed to the right of q.*

Proof. Let q be placed in column j of row k and $r < q$ be placed in column $s > j$, i.e. $x_{qjk} = 1$ and $x_{rsk} = 1$. Since C is a layered Monge array, the following property is fulfilled $c_{rjk} + c_{qsk} \leq c_{rsk} + c_{qjk}$. It follows that exchanging the position of r and q in row k will lead to a new (not necessarily feasible) solution x' with $x'_{rjk} = 1$ and $x'_{qsk} = 1$ such that $c(x') \leq c(x)$. $\quad\square$

Proposition 4.2. *For the 2-P3AP with an $n \times n \times 2$ cost array C which is a layered Monge array, there always exists an optimal solution such that its corresponding Latin rectangle decomposes into normalized blocks with 2 or 3 columns.*

Our proof is based on an extensive case distinction and is unfortunately not elegant. To improve the readability of the paper, we have decided to move the proof to an appendix (which will be included in the second version of the paper).

For the special case of distribution arrays we have a more elegant proof which is based on the fact that for distribution arrays there exists an explicit formula for the cost of a feasible solution which in terms of the entries of the density matrix. As the result above is more general, we decided to omit the proof for the special case.

4.2   Example with a single large block for p = 3

It would be nice if the block structure result for the 2-P3AP from Proposition 4.2 carried over to the 3-P3AP. Unfortunately, this is not the case, not even for the subclass of distribution arrays as is demonstrated by the following counterexample in which the unique optimal solution consists of a single large block.

Let $n = 10$ and $p = 3$. We consider the distribution matrix C which is generated by the matrix $P = (p_{ijk})$ with

$$
p_{ij1} = \begin{cases} 100 & \text{if } i = j = 7 \\ 1 & \text{otherwise} \end{cases} \qquad p_{ij2} = \begin{cases} 10a & \text{if } i = 4 \text{ and } j = 5 \\ 10a^3 & \text{if } i = 9 \text{ and } j = 10 \\ a_j & \text{otherwise} \end{cases}
$$

and $p_{ij3} = a^a$ for all $i, j \in \{1, \ldots, 10\}$ where $a_n = (1, 1, a, a, a, a^2, a^2, a^3, a^3, a^3)$ and a is some large number. For notational convenience let us refer to the 10×10 matrix that corresponds to the k-plane of P as Q_k.

We claim that the following 3×10 Latin rectangle L

$$
\begin{array}{|cccccccccc|}
\hline
3 & 4 & 1 & 2 & 6 & 5 & 8 & 10 & 7 & 9 \\
2 & 1 & 4 & 5 & 3 & 7 & 6 & 9 & 10 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\end{array}
\tag{11}
$$

is the unique optimal solution for the 3-P3AP instance with cost array C. This can be checked by computational means. Below we provide the rationale behind why L shows up as unique optimal solution.

Consider the $10 \times 10 \times 2$ subarray C_2 of C where the the k-plane of C with $k = 1$ is dropped. We claim that the following Latin rectangle provides an optimal solution of the 2-P3AP on C_2

$$
\begin{array}{|cc|ccc|cc|ccc|}
\hline
2 & 1 & 4 & 5 & 3 & 7 & 6 & 9 & 10 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
\end{array}
\tag{12}
$$

This is easy to check. Note that the entries of Q_3 are much larger than the entries in Q_2 and Q_1. It can easily be checked that it thus pays off to choose the overall best permutation for any Monge matrix, namely the identity permutation, as solution for the k-plane with $k = 3$. Choosing any other permutation more will be lost than can be won for the k-planes with $k = 1$ and $k = 2$. Hence the second row of (12) is the identity permutation.

As a consequence of the structure of the vector a_n and its role in Q_3 (12) splits into blocks of sizes 2,3,2 and 3 respectively. The two entries of Q_2 which involve the multiplicative factor 10 determine the type of the two blocks of size 3 in (12).

Since Q_1 has much smaller entries than Q_2 and Q_3, the optimal solution of the $10 \times 10 \times 3$ P3AP with cost array C will have (12) as last two rows. This is true since every set of Latin rows can be completed to Latin square [13].

Because $p_{1,7,7} = 100$, the rightmost four columns of the first row in the optimal Latin rectangle L will be filled with integers greater or equal than 7, if possible. There indeed exist three ways to achieve this goal and we take the one with the lowest cost, i.e., the block (8,10,7,9), and fill the rest of the first row of the Latin rectangle in an obviously unique optimal manner. The Latin rectangle L results.

As the Latin rectangle L consists of a single block of size $n = 10$, this destroys any hope for a result along the lines of Proposition 4.2 for $p \geq 3$.

The same approach can be used to construct examples where an arbitrarily large block arises in the optimal solution. Just add an arbitrary number of 2×2 blocks in the middle of the candidate solution (12).

The construction above can be carried over $p > 3$. Add more k-planes on top of P with much smaller values than in the first three k-planes. Then the optimal solution will have a Latin rectangle representation such that the first three rows are as in the example provided for $p = 3$. (Use Hall's theorem [13].)

4.3 Bandwidth result for the p-P3AP

Recall that for the linear assignment problem on a Monge matrix the identity permutation provides an optimal solution, cf. Observation 1.1.

A feasible solution of the p-P3AP is a set of p pairwise disjoint permutations (assignments). Hence, one might expect that for the p-P3AP on a layered Monge array the filled cells of the partial Latin square representing the optimal solution tend to group around the main diagonal. It is shown below that this is indeed the case.

Theorem 4.3. *Let an instance I of the p-P3AP with the $n \times n \times p$ layered Monge cost array C be given. There exists an optimal solution x for I such that the partial Latin square L that represents x has the following property:*

(P) Whenever the cell $L(i,j)$ is filled, we have $|i - j| \leq 2p - 2$.

Proof. Consider an optimal solution x and its representation as partial Latin square L. We start at the upper right corner and shift a line parallel to the main diagonal towards the center of the partial Latin square until one hits for the first time a filled cell. Choose such a cell and call it *pivotal cell*. Suppose that the pivotal cell is $L(i,j)$ and contains the integer k, see Figure 2 for an illustration. We define four sub-rectangles of the partial Latin square L

Figure 2: Pushing elements closer to the diagonal

as follows depending in which of the four quadrants around $L(i,j)$ the cells are located.

- Q_{la}: Contains all cells (r,q) with $r < i$ and $q < j$.

- Q_{lb}: Contains all cells (r,q) with $r > i$ and $q < j$.

- Q_{ra}: Contains all cells (r,q) with $r < i$ and $q > j$.

- Q_{rb}: Contains all cells (r, q) with $r > i$ and $q > j$.

Let us refer to Q_{lb} as *candidate area*. If there exists a cell $L(q, r)$ in the candidate area that is filled with the integer k while the cells $L(i, r)$ and $L(q, j)$ are empty, we can fill the cells $L(i, r)$ and $L(q, j)$ with the integer k and restore the cells $L(i, j)$ and $L(q, r)$ to the empty state by deleting k from them. The new solution x' is feasible again and since each 3-plane of the cost array C is a Monge matrix, it follows that

$$c_{irk} + c_{qjk} \leq c_{ijk} + c_{qrk},$$

and hence the cost does not increase if we move to the new solution.

Next we examine for which i and j it is always possible to perform a move of the type described above. First we count how many cells in the candidate area are filled with the integer k. Since there are no filled cells in Q_{ra} by assumption, it follows that k occurs in Q_{la} exactly $i - 1$ times. Analogously, it follows that k occurs in Q_{rb} exactly $n - j$ times. Consequently, k occurs in the the candidate area exactly $n - (i - 1) - (n - j) - 1 = j - i$ times.

Note that filled cells in row i left of the pivotal cell and filled cells in column j below of the pivotal element (see the gray area in Figure 2) can lead to the in the situation that a candidate integer k cannot be used to perform a move of the type described above. Note that there are exactly $p - 1$ filled cells in row i left of $L(i, j)$ and exactly $p - 1$ filled cells in column j below $L(i, j)$. Therefore, if

$$j - i > 2p - 2,$$

holds, we always can choose an integer such the performed move is feasible.

Similarly, if we consider the elements below the diagonal, we get that a feasible move can be performed if $i - j > 2p - 2$. Therefore there always exists an optimal solution such that for all filled cells $L(i, j)$ we have

$$|i - j| \leq 2p - 2.$$

$\square$

Note that property (P) implies that the partial Latin square L is a matrix with bandwidth $\leq 4p - 3$. It is natural to ask whether this bound is tight. For $p = 2$ and odd n it is easy to see that this is the case. It suffices to choose the two 3-planes of the cost array C such that in the optimal partial Latin square all diagonal entries will be filled by 1-entries which is easy to achieve.

We conjecture that the bound $4p - 3$ is tight for infinitely many p. More specifically, we have constructed a class of instances for which we conjecture that the bound is achieved when p is prime. We have checked the correctness for values of p up to 31 by computational means. Work on the proof is in progress.

5 A dynamic programming algorithm for the p-PLAP on layered Monge arrays

Based on Theorem 4.3 we can devise a dynamic programming algorithm that finds an optimal solution of the p-P3AP on layered Monge arrays. The algorithm's running time will be linear in n and exponential in p.

Our dynamic programming algorithm will start with the empty partial Latin square and will then fill the rows of the partial Latin square with integers iteratively, going from the top to the bottom. Integers will be placed in the band around the diagonal according to Theorem 4.3 while at the same time taking the feasibility aspect into account.

The state space of our dynamic program is defined as follows. Each state corresponds to a partial Latin square. Let $\mathcal{S}_i$ denote the set of states built in the i-th step, that is, the set of partial Latin squares with the first i rows filled with integers 1 to p while the remaining rows are empty. Furthermore, $\mathcal{S}_i$ will fulfill the following additional properties:

i) The partial Latin squares that correspond to a state in $\mathcal{S}_i$ contain exactly p integers in the columns 1 to $i + 2p - 2$ and no integers in the columns $i + 2p - 1$ to n.

ii) To every state S from $\mathcal{S}_i$ we associate a $(4p - 4)$-tuple of subsets of $\{1, \ldots, p\}$ that correspond to integers that occur in columns $i - 2p + 3$ to $i + 2p - 2$ of S, respectively. No two states in $\mathcal{S}_i$ have the same corresponding $(4p - 4)$-tuples.

We start the algorithm with $\mathcal{S}_0 = \{S_0\}$, where S_0 is an empty $n \times n$ table. In the i-th step, for each state S_{i-1} from $\mathcal{S}_{i-1}$ do the following. Place integers 1 to p into the columns $i - 2p + 2$ to $i + 2p - 2$ of the row i, in every possible way. Let S_i be one such table. If S_i is not a partial Latin square or does not satisfy the condition i), discard it. Otherwise, check the $(4p - 4)$-tuple defined in ii). If there is no state in $\mathcal{S}_i$ with the same corresponding $(4p-4)$-tuple, put S_i into $\mathcal{S}_i$. If there already exists such state, then compare the corresponding cost values of the two states, and put in $\mathcal{S}_i$ only the state with the smaller cost. In doing so, condition ii) will be ensured. At the end of the step n, $\mathcal{S}_n$ will consist of only one state, which will correspond to an optimal solution.

The size of $\mathcal{S}_i$ does not depend on n. Also, the number of ways one can fill a new row, and the time needed to check whether there already exists a state in $\mathcal{S}_i$ with the same $(4p - 4)$-tuple than RyRythe $(4p - 4)$-tuple of the currently considered partial solution. does not depend on n. Hence the time complexity of the algorithm is $O(f_p n)$, where f_p is an exponential function in p. Note that to build $\mathcal{S}_i$ we only need $\mathcal{S}_{i-1}$, hence space complexity does does not depend on n either.

Corollary 5.1. *The p-P3AP for layered Monge cost arrays is fixed parameter tractable with respect to the parameter p.*

Acknowledgement

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A   Appendix: Proof for Proposition 4.2

The proof will be part of version 2.