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An energy space finite element approach for elliptic Dirichlet boundary control problems

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Abstract

In this paper we present a finite element analysis for a Dirichlet boundary control problem where the Dirichlet control is considered in a convex closed subspace of the energy space $H^{1/2}(\Gamma)$. As an equivalent norm in $H^{1/2}(\Gamma)$ we use the energy norm induced by the so–called Steklov–Poincaré operator which realizes the Dirichlet to Neumann map, and which can be implemented by using standard finite element methods. The presented stability and error analysis of the discretization of the resulting variational inequality is based on the mapping properties of the solution operators related to the primal and adjoint boundary value problems, and their finite element approximations. Some numerical results are given, which confirm on one hand the theoretical estimates, but on the other hand indicate the differences when modelling the control in $L_2(\Gamma)$.

1 Introduction

In this paper, the focus is on the a priori error analysis of the finite element approximation to minimise the cost functional

$$\mathcal{J}(u,z) \,=\, \mathcal{F}(u) + \frac{1}{2}\varrho\|z\|^2_{\mathcal{V}}\,, \tag{1.1}$$

where the state u is the unique solution of a second order elliptic partial differential equation in a polygonal bounded domain Ω, satisfying a Dirichlet boundary condition $u = z$ on

$\Gamma = \partial\Omega$. The non–negative cost functional $\mathcal{F}(u)$ describes, e.g., some physical quantity to be minimised, while $\|z\|^2_{\mathcal{V}}$ reflects either the costs of the control $z \in \mathcal{U} \subset \mathcal{V}$, or represents some regularization when considering, for example, an inverse source problem. Note that $\mathcal{V}$ is an appropriate Hilbert space to be specified, and $\mathcal{U} \subset \mathcal{V}$ is a convex closed subset which describes additional constraints on the control.

For a general analysis on optimal control problems governed by partial differential equations we refer to, e.g., [28, 41], for numerical methods to solve related variational inequalities, see, e.g., [13], and for a more recent overview on the theory and numerics of optimal control problems, see, for example, [16].

Optimal control problems (1.1) with a Dirichlet boundary control play an important role, for example, in the context of computational fluid mechanics., see, e.g., [12, 15], and the references given therein. In [15], the cost functional $\mathcal{J}(u, z) = \mathcal{F}(u)$ is the domain integral over the strain tensor of the velocity field u satisfying the steady state Navier–Stokes equations with the Dirichlet boundary condition $u = z \in \mathcal{U} \subset H^{1/2}(\Gamma)$. A similar minimisation problem is considered in [12], where the cost functional $\mathcal{J}(u, z)$ is either written as boundary integral to describe the work needed to overcome the drag exerted on a given body, or equivalently as a domain integral over the strain of the velocity, and some additional boundary integral. In both cases, the cost functional $\mathcal{J}(u, z)$ describes an energy in $H^1(\Omega)$, or equivalently, in the Sobolev trace space $H^{1/2}(\Gamma)$. Hence, when considering the minimisation problem (1.1) subject to a Dirichlet boundary value problem, $\mathcal{V} = H^{1/2}(\Gamma)$ appears as a natural choice [3]. To obtain smoother optimal solutions one may even consider more regular cost functionals: In [21], $\mathcal{V} = H^2(\Gamma)$ is considered where the norm $\| \cdot \|_{H^2(\Gamma)}$ is realized by using the Laplace–Beltrami operator on the boundary Γ. Note that such an approach requires sufficient regularity of the domain Ω which is assumed to be of the class $C^{2,1}$. In [28], several Neumann and Dirichlet boundary control problems with observations in the domain Ω and on the boundary Γ are considered from an analytic point of view. To describe the involved Sobolev norms on the boundary Γ, in particular $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, certain fractional powers of the Laplace–Beltrami operator are used which seems to be complicated from a numerical point of view. The situation simplifies when $\mathcal{U} = L_2(\Gamma)$ is used as control space. But in this case, the associated partial differential equation has to be considered within an ultra–weak variational formulation, see, for example, [28], and [4] for an appropriate finite element approximation using standard piecewise linear basis functions. The use of the ultra–weak variational formulation of the primal Dirichlet boundary value problem in the context of an optimal control problem requires the adjoint variable p to be sufficiently regular, i.e., $p \in H^2(\Omega) \cap H^1_0(\Omega)$. Since the adjoint variable p itself is the unique solution of the adjoint partial differential equation with homogeneous Dirichlet boundary conditions, either a smooth boundary Γ, or a polygonal or polyhedral but convex domain Ω has to be assumed. For related finite element approximations, see, e.g., [8, 10, 14, 26, 30], or [44] in the case of a finite dimensional Dirichlet control. Several variational formulations of Dirichlet control problems are discussed in [25]. To include a Dirichlet boundary condition $u = z \in L_2(\Gamma)$ in a standard variational formulation, alternatively one may consider a penalty approximation of the Dirichlet boundary condition by using a Robin boundary condition, see, e.g., [3, 17, 19, 20]. Again, sufficient smoothness of the boundary Γ has to be assumed.

The aim of this paper is to present a numerical analysis of an energy space finite element approach when the control z is considered as an element of the boundary trace space $H^{1/2}(\Gamma)$, where an equivalent norm is induced by the so–called Steklov–Poincaré operator which realizes the Dirichlet to Neumann map. Note that in this case the costs represent the energy of the harmonic extension of the Dirichlet control z. The main difference to the more common approach when using $L_2(\Gamma)$ as control space appears in the optimality condition. In particular, the Steklov–Poincaré operator links the Dirichlet control to the normal derivative of the adjoint variable. The related optimality condition results then in a higher regularity of the control, and requires less assumptions on the smoothness of the adjoint variable. Indeed, the use of $H^{1/2}(\Gamma)$ as control space reflects the proper mapping properties of the Dirichlet to Neumann map which appears within the optimality condition. As a consequence, we also obtain higher order convergence results for the approximate finite element solution. Instead, when using $L_2(\Gamma)$ as control space, Dirichlet and Neumann boundary data are identified with each other. In particular for polygonal and polyhedral domains Ω the control z will then be zero in all corner points which seems to be not motivated from the application in mind.

The paper is organised as follows: In Sect. 2, we describe the considered Dirichlet boundary control problem, the primal boundary value problem, and the reduced cost functional as well as the related adjoint boundary value problem. The minimiser of the reduced cost functional is characterised as the unique solution of a variational inequality of the first kind. The finite element discretization of the variational inequality is described in Sect. 3, where also finite element approximations of both the primal and adjoint boundary value problems and related error estimates are given. These approximations are used to describe and to analyse the finite element discretization of the perturbed variational inequality. The main results of this paper are the error estimates as given in Corollary 3.8. Some numerical results are finally given in Sect. 4, where we also give a comparison with the more common approach when considering the control in $L_2(\Gamma)$.

For the ease of presentation we restrict our considerations to the case of a convex and polygonal bounded domain. However, the presented approach applies to general two– and three–dimensional Lipschitz domains when taking into account related results on the regularity of the solutions and on appropriate finite element approximations.

For an overview on the used Sobolev spaces in the domain and on the boundary, see, for example, [1, 31, 37, 40].

2 Dirichlet control problems

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal bounded domain with boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Dirichlet boundary control problem to minimise the cost functional

$$J(u,z) \,=\, \frac{1}{2}\int_\Omega [u(x)-\overline{u}(x)]^2\,dx + \frac{1}{2}\varrho|z|^2_{H^{1/2}(\Gamma)} \tag{2.1}$$

subject to the constraint

$$-\Delta u = f \quad \text{in } \Omega, \quad u = z \quad \text{on } \Gamma, \tag{2.2}$$

and where the control satisfies the box constraints

$$z \in \mathcal{U} := \Big\{ w \in H^{1/2}(\Gamma) \,:\, g_a \leq w \leq g_b \quad \text{on } \Gamma \Big\}. \tag{2.3}$$

We assume $f \in L_2(\Omega)$, $\overline{u} \in L_2(\Omega)$, $\varrho \in \mathbb{R}_+$, $g_a, g_b \in H^{1/2}(\Gamma)$, and $g_a < g_b$ on Γ.

The choice $\mathcal{V} = H^{1/2}(\Gamma)$ ensures the well–posedness of the standard variational formulation of the Dirichlet boundary value problem (2.2). There are several possibilities to describe a (semi–)norm in $H^{1/2}(\Gamma)$, e.g., by using the Sobolev–Slobodeckii semi–norm

$$|z|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{[z(x) - z(y)]^2}{|x-y|^n} \, ds_x ds_y,$$

other realizations rely on the use of Fourier or multilevel representations which imply equivalent (semi–)norms in $H^{1/2}(\Gamma)$. However, since $H^{1/2}(\Gamma)$ is the boundary trace of $H^1(\Omega)$, we may also consider the equivalent Dirichlet trace norm

$$\|z\|_{H^{1/2}(\Gamma)} = \min_{Z \in H^1(\Omega): Z_{|\Gamma} = z} \|Z\|_{H^1(\Omega)}.$$

In particular, for any $z \in H^{1/2}(\Gamma)$ there exists the harmonic extension $u_z \in H^1(\Omega)$ as the unique solution of the homogeneous Dirichlet boundary value problem

$$-\Delta u_z = 0 \quad \text{in } \Omega, \quad u_z = z \quad \text{on } \Gamma. \tag{2.4}$$

Using Green's first formula, this motivates the use of the semi–norm

$$|z|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \frac{\partial}{\partial n_x} u_z(x) u_z(x) \, ds_x = \int_\Omega |\nabla u_z(x)|^2 \, dx,$$

which describes the energy of the harmonic extension $u_z \in H^1(\Omega)$ of the Dirichlet control $z \in H^{1/2}(\Gamma)$. By introducing the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$,

$$(Sz)(x) := \frac{\partial}{\partial n_x} u_z(x) \quad \text{for almost all } x \in \Gamma,$$

which realizes the Dirichlet to Neumann map related to the Dirichlet boundary value problem (2.4), we can rewrite the semi–norm as

$$|z|^2_{H^{1/2}(\Gamma)} = \langle Sz, z \rangle_\Gamma \tag{2.5}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

2.1 Primal boundary value problem

To rewrite the Dirichlet boundary control problem (2.1)–(2.3) by using a reduced cost functional, we introduce a linear solution operator describing the application of the constraint (2.2). Therefore we consider the homogeneous partial differential equation (2.4) with the control as Dirichlet boundary condition, and an inhomogeneous partial differential equation with zero Dirichlet boundary conditions to describe a particular solution. Let u_f be the weak solution of the Dirichlet boundary value problem

$$-\Delta u_f \,=\, f \quad \text{in } \Omega, \quad u_f \,=\, 0 \quad \text{on } \Gamma, \tag{2.6}$$

i.e., $u_f \in H_0^1(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla u_f(x) \cdot \nabla v(x)\, dx \,=\, \int_\Omega f(x) v(x)\, dx \quad \text{for all } v \in H_0^1(\Omega). \tag{2.7}$$

The solution of the Dirichlet boundary value problem (2.2) is given by $u = u_z + u_f$, where $u_z \in H^1(\Omega)$ is the weak solution of the Dirichlet boundary value problem (2.4). By applying the inverse trace theorem, see, e.g., [22, 31, 40] or [37, Theorem 2.22], there exists a bounded extension $\mathcal{E}z \in H^1(\Omega)$ for $z \in H^{1/2}(\Gamma)$. With $u_z = u_0 + \mathcal{E}z$, the variational formulation of the Dirichlet boundary value problem (2.4) is to find $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_0(x) \cdot \nabla v(x)\, dx \,=\, -\int_\Omega \nabla \mathcal{E}z(x) \cdot \nabla v(x)\, dx \quad \text{for all } v \in H_0^1(\Omega). \tag{2.8}$$

By using standard arguments we can ensure the unique solvability of the variational formulations (2.7) and (2.8), respectively. Due to the compact imbedding $H^1(\Omega) \subset L_2(\Omega)$ we may introduce the bounded solution operator $\mathcal{H} : H^{1/2}(\Gamma) \to L_2(\Omega)$, $u_z = \mathcal{H}z$. Hence we can write the solution of the primal boundary value problem (2.2) as $u = \mathcal{H}z + u_f$.

2.2 Reduced cost functional and adjoint boundary value problem

By using $u = \mathcal{H}z + u_f$, we can write the cost functional (2.1) as the reduced cost functional

$$\widetilde{J}(z) \,=\, \frac{1}{2}\langle \mathcal{H}^*\mathcal{H}z, z\rangle_\Gamma + \langle \mathcal{H}^*(u_f - \overline{u}), z\rangle_\Gamma + \frac{1}{2}\,\|u_f - \overline{u}\|^2_{L_2(\Omega)} + \frac{1}{2}\varrho\langle Sz, z\rangle_\Gamma, \tag{2.9}$$

where $\mathcal{H}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is the adjoint operator of $\mathcal{H} : H^{1/2}(\Gamma) \to L_2(\Omega)$. For the application $\tau = \mathcal{H}^*\psi \in H^{-1/2}(\Gamma)$, $\psi \in L_2(\Omega)$, we have

$$\langle \tau, \varphi\rangle_\Gamma = \langle \mathcal{H}^*\psi, \varphi\rangle_\Gamma = -\langle \frac{\partial}{\partial n} p, \varphi\rangle_\Gamma \quad \text{for all } \varphi \in H^{1/2}(\Gamma), \tag{2.10}$$

where the adjoint variable p is the weak solution of the Dirichlet boundary value problem

$$-\Delta p \,=\, \psi \quad \text{in } \Omega, \quad p \,=\, 0 \quad \text{on } \Gamma, \tag{2.11}$$

i.e., $p \in H^1_0(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla p(x) \cdot \nabla q(x)\, dx \;=\; \int_\Omega \psi(x) q(x)\, dx \quad \text{for all } q \in H^1_0(\Omega). \tag{2.12}$$

Finally, by using Green's first formula we can rewrite (2.10) for $\varphi \in H^{1/2}(\Gamma)$ as

$$\langle \tau, \varphi \rangle_\Gamma = -\langle \frac{\partial}{\partial n} p, \varphi \rangle_\Gamma = -\int_\Omega \nabla p(x) \cdot \nabla \mathcal{E}\varphi(x)\, dx + \int_\Omega \psi(x) \mathcal{E}\varphi(x)\, dx, \tag{2.13}$$

where $\mathcal{E} : H^{1/2}(\Gamma) \to H^1(\Omega)$ is again the bounded extension operator.

2.3 Optimality condition

To characterise the minimiser of the reduced cost functional (2.9) we introduce the self–adjoint and bounded operator

$$T_\varrho \;:=\; \mathcal{H}^* \mathcal{H} + \varrho S \;:\; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \tag{2.14}$$

satisfying

$$\|T_\varrho z\|_{H^{-1/2}(\Gamma)} \;\le\; c_2^{T_\varrho}\, \|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma),$$

and define

$$g \;:=\; \mathcal{H}^*(\overline{u} - u_f) \in H^{-1/2}(\Gamma).$$

Hence we can rewrite (2.9) as

$$\widetilde{J}(z) \;=\; \frac{1}{2}\, \langle T_\varrho z, z \rangle_\Gamma - \langle g, z \rangle_\Gamma + \frac{1}{2}\, \|u_f - \overline{u}\|^2_{L_2(\Omega)}. \tag{2.15}$$

Lemma 2.1 *The operator T_ϱ as defined in* (2.14) *is $H^{1/2}(\Gamma)$–elliptic, i.e., there exists a positive constant $c_1^{T_\varrho}$ such that*

$$\langle T_\varrho z, z \rangle_\Gamma \;\ge\; c_1^{T_\varrho}\, \|z\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. For $z \in H^{1/2}(\Gamma)$, let $u_z = \mathcal{H} z \in H^1(\Omega)$ be the harmonic extension satisfying the Dirichlet boundary value problem (2.4). Then, by (2.5)

$$\langle T_\varrho z, z \rangle_\Gamma = \|\mathcal{H} z\|^2_{L_2(\Omega)} + \varrho \langle S z, z \rangle_\Gamma = \|u_z\|^2_{L_2(\Omega)} + \varrho \|\nabla u_z\|^2_{L_2(\Omega)} \ge \min\{1, \varrho\} \|u_z\|^2_{H^1(\Omega)},$$

and the assertion follows from the trace theorem. ∎

Since $\mathcal{U} \subset H^{1/2}(\Gamma)$ is convex and closed, and since T_ϱ is self–adjoint and $H^{1/2}(\Gamma)$–elliptic, the minimisation of (2.15) is equivalent to solving a variational inequality to find $z \in \mathcal{U}$ such that

$$\langle T_\varrho z, w - z \rangle_\Gamma \;\ge\; \langle g, w - z \rangle_\Gamma \quad \text{for all } w \in \mathcal{U}. \tag{2.16}$$

Since (2.16) is an elliptic variational inequality of the first kind, we can use standard arguments as given, for example in [6, 13, 28, 29], to establish unique solvability of the variational inequality (2.16).

In what follows we will give another characterization of the solution $z \in \mathcal{U}$ of the variational inequality (2.16) as a Dirichlet trace of a solution of a bilateral Signorini boundary value problem. We introduce

$$\lambda := T_\varrho z - g = \mathcal{H}^*[\mathcal{H}z - (\overline{u} - u_f)] + \varrho S z = \mathcal{H}^*(u - \overline{u}) + \varrho S z = -\frac{\partial}{\partial n} p + \varrho \frac{\partial}{\partial n} u_z,$$

where p is the unique solution of the adjoint Dirichlet boundary value problem

$$-\Delta p = u - \overline{u} \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma, \tag{2.17}$$

and where u_z is the harmonic extension of z. Then we can rewrite the variational inequality (2.16) as

$$\langle \lambda, w - z \rangle_\Gamma \ \geq \ 0 \quad \text{for all } w \in \mathcal{U}.$$

Since the solution $z \in \mathcal{U}$ is uniquely determined, we introduce the active zones

$$\Gamma_a := \{x \in \Gamma : z(x) = g_a(x)\}, \quad \Gamma_b := \{x \in \Gamma : z(x) = g_b(x)\},$$

and the inactive zone

$$\Gamma_n := \Gamma \backslash (\Gamma_a \cup \Gamma_b).$$

Let us first investigate the behavior of the solution z in the vicinity of the upper bound g_b. For an arbitrary test function $w \in \mathcal{U}$ with $w \leq z$ we define a non–negative function $\varphi := z - w \in \widetilde{H}^{1/2}(\Gamma \backslash \Gamma_a)$ satisfying

$$\varphi(x) = 0 \quad \text{for } x \in \Gamma_a, \quad 0 < \varphi(x) \leq z(x) - g_a(x) \quad \text{for } x \in \Gamma \backslash \Gamma_a.$$

Hence we have

$$\langle \lambda, \varphi \rangle_\Gamma \leq 0$$

for all appropriate test functions φ. By using a scaling argument we conclude $\lambda \leq 0$ in the sense of $H^{-1/2}(\Gamma \backslash \Gamma_a)$. In the same way we find $\lambda \geq 0$ in the sense of $H^{-1/2}(\Gamma \backslash \Gamma_b)$ when considering the lower bound, and we conclude

$$\langle \lambda, \psi \rangle_\Gamma = 0 \quad \text{for all } \psi \in \widetilde{H}^{1/2}(\Gamma_n) = \widetilde{H}^{1/2}(\Gamma \backslash \Gamma_a) \cap \widetilde{H}^{1/2}(\Gamma \backslash \Gamma_b),$$

i.e., $\lambda = 0$ in the sense of $H^{-1/2}(\Gamma_n)$. Hence we have obtained the complementary conditions

$$\begin{array}{lcll} \lambda & \leq & 0 & \text{for } z = g_b, \\ \lambda & = & 0 & \text{for } g_a < z < g_b, \\ \lambda & \geq & 0 & \text{for } z = g_a \end{array}$$

which have to be understood as above.

Finally we have that $u_z \in H^1(\Omega)$ is the unique solution of the Signorini boundary value problem

$$-\Delta u_z = 0 \quad \text{in } \Omega \tag{2.18}$$

with the bilateral constraints on Γ

$$u_z \le g_b, \quad \varrho \frac{\partial}{\partial n} u_z \le \frac{\partial}{\partial n} p \quad \text{for } u_z = g_b, \quad u_z \ge g_a, \quad \varrho \frac{\partial}{\partial n} u_z \ge \frac{\partial}{\partial n} p \quad \text{for } u_z = g_a, \tag{2.19}$$

and

$$\varrho \frac{\partial}{\partial n} u_z = \frac{\partial}{\partial n} p \quad \text{for } g_a < u_z < g_b. \tag{2.20}$$

Note that the complementary conditions (2.19) and (2.20) are nothing than the Karush–Kuhn–Tucker conditions which are related to the variational inequality (2.16).

Related to the regularity of solutions of the Dirichlet boundary value problem of the Laplace equation in a convex polygonal bounded domain Ω we introduce the Sobolev trace space $H^{3/2}_{\text{pw}}(\Gamma) = H^2(\Omega)_{|\Gamma}$ of piecewise smooth functions, i.e. for $g \in H^{3/2}_{\text{pw}}(\Gamma)$ there exists the unique harmonic extension $u_g \in H^2(\Omega)$ satisfying

$$-\Delta u_g = 0 \quad \text{in } \Omega, \quad u_g = g \quad \text{on } \Gamma = \partial\Omega. \tag{2.21}$$

Now we are in a position to formulate a regularity result for solutions of the bilateral Signorini problem (2.18)–(2.20).

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal bounded domain. Let $\overline{u} \in L_2(\Omega)$ and $f \in L_2(\Omega)$ be given. Moreover, we assume $g_a, g_b \in H^{3/2}_{pw}(\Gamma)$, $g_a < g_b$. For the solution of the bilateral Signorini problem* (2.18)–(2.20) *we then have $u_z \in H^2(\Omega)$, and therefore $z \in H^{3/2}_{pw}(\Gamma)$.*

Proof. For the solution of the adjoint boundary value problem (2.17) we first have $p \in H^2(\Omega)$. For $g_a, g_b \in H^{3/2}_{\text{pw}}(\Gamma)$ there exist harmonic extensions $u_a, u_b \in H^2(\Omega)$ as defined in (2.21).

Let $z \in H^{1/2}(\Gamma)$ be the unique solution of the variational inequality (2.16), and let $u_z \in H^1(\Omega)$ be the harmonic extension, which satisfies the bilateral Signorini boundary value problem (2.18)–(2.20), i.e.

$$-\Delta u_z = 0 \quad \text{in } \Omega, \quad u_z = g_b \quad \text{on } \Gamma_b, \quad u_z = g_a \quad \text{on } \Gamma_a, \quad \varrho \frac{\partial}{\partial n} u_z = \frac{\partial}{\partial n} p \quad \text{on } \Gamma_n, \tag{2.22}$$

where in addition we have the constraints

$$\varrho \frac{\partial}{\partial n} u_z \le \frac{\partial}{\partial n} p \quad \text{on } \Gamma_b, \quad \varrho \frac{\partial}{\partial n} u_z \ge \frac{\partial}{\partial n} p \quad \text{on } \Gamma_a, \quad g_a < u_z < g_b \quad \text{on } \Gamma_n.$$

Since the given boundary data of the mixed boundary value problem (2.22) are sufficient regular, it remains to consider boundary points where the boundary condition changes from an inactive zone to an active one. In particular we investigate the existence of singularity functions in the vicinity of those points. In contrast to boundary value problems with mixed boundary conditions of Dirichlet and Neumann type, no singularity functions appear in the case of Signorini boundary conditions.

Without loss of generality let $x_0 \in \overline{\Gamma}_a \cap \overline{\Gamma}_n$. By introducing local polar coordinates, see Fig. 1

$$x = x_0 + r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \quad \text{for } r > 0, \quad \phi \in [0, \phi_0], \quad \phi_0 \in (0, \pi],$$

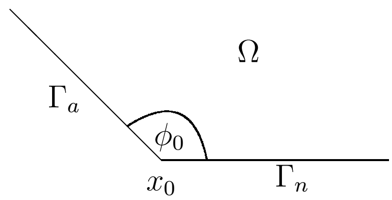


Figure 1: Local polar coordinates in the vicinity of x_0.

the solution u_z can be written in the vicinity of x_0 as

$$u_z(x) = \widetilde{u}_z(r, \phi) = r^\alpha[A\cos\phi + B\sin\phi] + u_z^{\text{reg}}(x), \quad \alpha \in (0, 1),$$

where $u_z^{\text{reg}} \in H^2(\Omega)$ is the regular part. To determine the coefficients A and B of the singular part we consider the Signorini type boundary conditions on Γ_a and Γ_n, respectively.

For $x \in \Gamma_n$, i.e. $\phi = 0$, and $n_x = (0, -1)^\top$, the normal derivative of u_z reads

$$\frac{\partial}{\partial n_x} u_z(x) = \alpha r^{\alpha-1}[A\sin(\alpha-1)\phi - B\cos(\alpha-1)\phi]_{|\phi=0} + \frac{\partial}{\partial n_x} u_z^{\text{reg}}(x),$$

and for the Neumann boundary condition on Γ_n we obtain

$$\varrho \frac{\partial}{\partial n_x} u_z(x) = -\varrho\alpha r^{\alpha-1} B + \varrho \frac{\partial}{\partial n_x} u_z^{\text{reg}}(x) = \frac{\partial}{\partial n_x} p(x).$$

Since the regular functions $p, u_z^{\text{reg}} \in H^2(\Omega)$ do not contain any singularity, we conclude $B = 0$. On Γ_n we also have the complementary condition $g_a = u_a < u_z$, which gives for $\phi = 0$

$$F(r) := \widetilde{u}_z(r, 0) - \widetilde{u}_a(r, 0) = r^\alpha A + \widetilde{u}_z^{\text{reg}}(r, 0) - \widetilde{u}_a(r, 0) > 0 \quad \text{for } r > 0.$$

Recall that $\widetilde{u}_z(0, 0) = u_z(x_0) = u_a(x_0) = \widetilde{u}_a(0, 0)$. A first order Taylor expansion of the regular part then gives

$$0 < F(r) = r^\alpha A + r\,\frac{d}{dr}\left[\widetilde{u}_z^{\text{reg}}(r, 0) - \widetilde{u}_a(r, 0)\right]_{|r=\widetilde{r}} \quad \text{for some } \widetilde{r} \in (0, r),$$

and therefore,

$$r^{\alpha-1}A + \frac{d}{dr}\left[\widetilde{u}_z^{\text{reg}}(r,0) - \widetilde{u}_a(r,0)\right]_{|r=\widetilde{r}} > 0, \quad \text{for some } \widetilde{r} \in (0,r), r > 0.$$

Since the first order derivative of the regular part is bounded for all $\widetilde{r} \in (0,r)$, we conclude $A \geq 0$ when considering $r \to 0$.

For $x \in \Gamma_a$, i.e. $\phi = \phi_0$, we have

$$u_z(x) = \widetilde{u}_z(r,\phi_0) = r^\alpha A \cos\alpha\phi_0 + u_z^{\text{reg}}(x) = u_a(x),$$

from which we conclude

$$\alpha_0 = \frac{\pi}{2\phi_0} \geq \frac{1}{2} \quad \text{for } \phi_0 \in (0,\pi].$$

For $x \in \Gamma_a$ the normal vector is $n_x = (-\sin\phi_0, \cos\phi_0)^\top$. Therefore, the normal derivative of u_z on Γ_a is given as

$$\frac{\partial}{\partial n_x}u_z(x) = -A\alpha_0 r^{\alpha_0-1}\sin[(\alpha_0-1)\phi + \phi_0]_{|\phi=\phi_0} + \frac{\partial}{\partial n_x}u_z^{\text{reg}}(x).$$

The complementary Neumann condition on Γ_a then reads

$$\varrho\frac{\partial}{\partial n_x}u_z(x) = -\varrho A\alpha_0 r^{\alpha_0-1} + \varrho\frac{\partial}{\partial n_x}u_z^{\text{reg}}(x) \geq \frac{\partial}{\partial n_x}p(x),$$

from which we conclude $A \leq 0$. Hence, together with $A \geq 0$ this gives $A = 0$, i.e. no singularity function occurs, in particuluar we have $u_z \in H^2(\Omega)$. ■

Remark 2.1 *Although we have given a proof of Theorem* 2.2 *for the particular case of a convex and polygonal bounded domain only, we may comment on more general situations as follows:*

1. *As explicitely shown in the proof of Theorem* 2.2 *the solutions of Signorini boundary value problem do not involve any singularity function. This remains true for smooth boundaries. For a more general discussion of bilateral Signorini boundary conditions, see, e.g.,* [2, 43].

2. *In the case of additional Dirichlet or Neumann boundary conditions one may have only a reduced regularity, see, e.g., the example as given in* [24, p. 617f.].

3. *Depending on the regularity of the given data one may even have higher regularity results than proven in Theorem* 2.2, *but even for smooth data one cannot expect more than* $u_z \in H^{5/2-\varepsilon}(\Omega)$, *i.e.* $z \in H^{2-\varepsilon}(\Gamma)$.

4. *In the case of a non–convex polygonal bounded domain we can not ensure* $p \in H^2(\Omega)$ *which would also reduce the regularity of the solution of the bilateral Signorini problem.*

3 Finite element approximations

For the ease of presentation we consider the case where Ω is a convex two–dimensional polygonal bounded domain. Let

$$S_h^1(\Omega) := \text{span}\{\varphi_k\}_{k=1}^{M} \subset H^1(\Omega), \quad S_{h,0}^1(\Omega) := \text{span}\{\varphi_k\}_{k=1}^{M_\Omega} \subset H_0^1(\Omega) \tag{3.1}$$

be some conforming finite element spaces of, e.g., piecewise linear and continuous basis functions φ_k, which are defined with respect to some admissible domain triangulation Ω_h of mesh size h. Let

$$S_h^1(\Gamma) := S_h^1(\Omega)_{|\Gamma} = \text{span}\{\phi_i\}_{i=1}^{M_\Gamma} \subset H^{1/2}(\Gamma) \tag{3.2}$$

be the boundary finite element space of, e.g., piecewise linear and continuous basis functions ϕ_i which are the boundary traces of the domain basis functions $\varphi_{M_\Omega+i}$ as given in (3.1). For continuous functions g_a and g_b we define the discrete convex set

$$\mathcal{U}_h := \left\{ w_h \in S_h^1(\Gamma) \, : \, g_a(x_i) \leq w_h(x_i) \leq g_b(x_i) \quad \text{for all nodes } x_i \in \Gamma \right\}.$$

Then the Galerkin discretization of the variational inequality (2.16) is to find $z_h \in \mathcal{U}_h$ such that

$$\langle T_\varrho z_h, w_h - z_h \rangle_\Gamma \geq \langle g, w_h - z_h \rangle_\Gamma \quad \text{for all } w_h \in \mathcal{U}_h. \tag{3.3}$$

Theorem 3.1 *Let $z \in \mathcal{U}$ and $z_h \in \mathcal{U}_h$ be the unique solutions of the variational inequalities* (2.16) *and* (3.3)*, respectively. We assume $g_a, g_b \in H^{3/2}_{pw}(\Gamma)$ implying $z \in H^{3/2}_{pw}(\Gamma)$. Then there hold the error estimates*

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \leq c\, h \, \|z\|_{H^{3/2}_{\text{pw}}(\Gamma)} \tag{3.4}$$

and

$$\|z - z_h\|_{L_2(\Gamma)} \leq c\, h^{3/2} \, \|z\|_{H^{3/2}_{\text{pw}}(\Gamma)} . \tag{3.5}$$

Proof. The error estimate (3.4) follows from an abstract theory for the boundary element approximation of variational inequalities [38, Theorem 3.4]. The error estimate (3.5) follows from the Aubin–Nitsche trick for variational inequalities in $H^{1/2}(\Gamma)$, see [38, Corollary 4.6]. Note that $T_\varrho : H^1(\Gamma) \to L_2(\Gamma)$ which follows from the mapping properties of all operators which are involved in the definition of T_ϱ. ■

Remark 3.1 *For a more regular solution $z \in H^{2-\varepsilon}_{pw}(\Gamma)$ we obtain an almost quadratic order of convergence when considering the error in $L_2(\Gamma)$, see* [38]*.*

Although the error estimates (3.4) and (3.5) seem to be optimal, the operator T_ϱ as considered in the variational inequality (3.3) does not allow a practical implementation, since this would require the discretizations of the composed solution operator $\mathcal{H}^*\mathcal{H}$ and of the

Steklov–Poincaré operator S, which are not possible in general. Hence, instead of (3.3) we need to consider a perturbed variational inequality to find $\widetilde{z}_h \in \mathcal{U}_h$ such that

$$\langle \widetilde{T}_\varrho \widetilde{z}_h, w_h - \widetilde{z}_h \rangle_\Gamma \;\geq\; \langle \widetilde{g}, w_h - \widetilde{z}_h \rangle_\Gamma \quad \text{for all } w_h \in \mathcal{U}_h, \tag{3.6}$$

where $\widetilde{T}_\varrho$ and $\widetilde{g}$ are appropriate approximations of T_ϱ and g, respectively. The following theorem presents an abstract consistency result, which will be used to analyse the finite element approximations of the Steklov–Poincaré operator S, and of the primal and adjoint boundary value problems.

Theorem 3.2 *Let $\widetilde{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ be a bounded and $S_h^1(\Gamma)$–elliptic approximation of T_ϱ satisfying*

$$\langle \widetilde{T}_\varrho z_h, z_h \rangle_\Gamma \;\geq\; c_1^{\widetilde{T}_\varrho} \, \|z_h\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z_h \in S_h^1(\Gamma)$$

and

$$\|\widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_2^{\widetilde{T}_\varrho} \, \|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Let $\widetilde{g} \in H^{-1/2}(\Gamma)$ be some approximation of g. For the unique solution $\widetilde{z}_h \in \mathcal{U}_h$ of the perturbed variational inequality (3.6) *there holds the error estimate*

$$\begin{aligned} \|z - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} \;\leq\; & \left(1 + \frac{1}{c_1^{\widetilde{T}_\varrho}}[c_2^{T_\varrho} + c_2^{\widetilde{T}_\varrho}]\right) \|z - z_h\|_{H^{1/2}(\Gamma)} \\ & + \frac{1}{c_1^{\widetilde{T}_\varrho}} \left[\|(T_\varrho - \widetilde{T}_\varrho) z\|_{H^{-1/2}(\Gamma)} + \|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} \right], \end{aligned} \tag{3.7}$$

where $z_h \in \mathcal{U}_h$ is the unique solution of the discrete variational inequality (3.3)*.*

Proof. The unique solvability of the discrete variational inequality (3.6) follows from the $S_h^1(\Gamma)$–ellipticity and boundedness of $\widetilde{T}_\varrho$. We further obtain

$$\begin{aligned} c_1^{\widetilde{T}_\varrho} \, \|z_h - \widetilde{z}_h\|^2_{H^{1/2}(\Gamma)} \;\leq\; & \langle \widetilde{T}_\varrho (z_h - \widetilde{z}_h), z_h - \widetilde{z}_h \rangle_\Gamma \\ \leq\; & \langle \widetilde{T}_\varrho z_h, z_h - \widetilde{z}_h \rangle_\Gamma + \langle \widetilde{g} - g, \widetilde{z}_h - z_h \rangle_\Gamma + \langle T_\varrho z_h, \widetilde{z}_h - z_h \rangle_\Gamma \\ \leq\; & \left[\|(\widetilde{T}_\varrho - T_\varrho) z_h\|_{H^{-1/2}(\Gamma)} + \|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} \right] \|z_h - \widetilde{z}_h\|_{H^{1/2}(\Gamma)} \, . \end{aligned}$$

The assertion now follows from the triangle inequality, and by using the boundedness of T_ϱ and $\widetilde{T}_\varrho$, respectively. ■

It remains to define a suitable approximation $\widetilde{T}_\varrho := \widetilde{\mathcal{H}}^* \widetilde{\mathcal{H}} + \varrho \widetilde{S}$ which is based on finite element approximations $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}^*$ of the primal and adjoint boundary value problem as well as on the approximation $\widetilde{S}$ of the Steklov–Poincaré operator.

3.1 Primal boundary value problem

The application of the solution operator $u_z = \mathcal{H}z = u_0 + \mathcal{E}z$, $u_0 \in H_0^1(\Omega)$, is given as the unique solution of the variational problem (2.8). The finite element approximation of (2.8) is to find $u_{0,h} \in S_{h,0}^1(\Omega)$ such that

$$\int_\Omega \nabla u_{0,h}(x) \cdot \nabla v_h(x)\, dx \;=\; -\int_\Omega \nabla \mathcal{E}z(x) \cdot \nabla v_h(x)\, dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega). \tag{3.8}$$

By using Cea's lemma, see, e.g., [5] or [37, Theorem 8.1], and by using the approximation property of $S_{h,0}^1(\Omega)$, see, e.g., [37, Theorem 9.10], we conclude the unique solvability of the Galerkin formulation (3.8) as well as the error estimate

$$\|u_0 - u_{0,h}\|_{H^1(\Omega)} \;\leq\; c\, h\, |u_0|_{H^2(\Omega)}. \tag{3.9}$$

By applying the Aubin–Nitsche trick, see, e.g., [5] or [37, Theorem 11.1], we also obtain an error estimate in $L_2(\Omega)$,

$$\|u_0 - u_{0,h}\|_{L_2(\Omega)} \;\leq\; c\, h^2\, |u_0|_{H^2(\Omega)}. \tag{3.10}$$

However, instead of the variational problem (3.8), we will consider a perturbed variational problem to find an approximate solution $\widetilde{u}_{0,h} \in S_{h,0}^1(\Omega)$ such that

$$\int_\Omega \nabla \widetilde{u}_{0,h}(x) \cdot \nabla v_h(x)\, dx \;=\; -\int_\Omega \nabla Q_h \mathcal{E}z(x) \cdot \nabla v_h(x)\, dx \quad \text{for all } v_h \in S_{h,0}^1(\Omega), \tag{3.11}$$

where $Q_h : H^1(\Omega) \to S_h^1(\Omega)$ denotes a quasi–interpolation operator [35]. Since the perturbed variational problem (3.11) admits a unique solution $\widetilde{u}_{0,h} \in S_{h,0}^1(\Omega)$, a finite element approximation $\widetilde{\mathcal{H}}z$ of the solution operator $\mathcal{H}z = u_0 + \mathcal{E}z$ can be defined by

$$\widetilde{\mathcal{H}}z \;:=\; \widetilde{u}_{0,h} + Q_h \mathcal{E}z. \tag{3.12}$$

Lemma 3.3 *The approximate solution operator $\widetilde{\mathcal{H}} : H^{1/2}(\Gamma) \to L_2(\Omega)$ as defined in* (3.12) *is bounded. Moreover, for $z \in H_{\text{pw}}^{3/2}(\Gamma)$ there holds the approximation error estimate*

$$\|\mathcal{H}z - \widetilde{\mathcal{H}}z\|_{L_2(\Omega)} \;\leq\; c\, h^2 \left[|u_0|_{H^2(\Omega)} + \|z\|_{H_{\text{pw}}^{3/2}(\Gamma)} \right]. \tag{3.13}$$

Proof. The boundedness of $\widetilde{\mathcal{H}} : H^{1/2}(\Gamma) \to L_2(\Omega)$ follows from the stability of the finite element approximation scheme, and from the boundedness of the quasi–interpolation operator $Q_h : H^1(\Omega) \to S_h^1(\Omega) \subset H^1(\Omega)$, see [35]. The error estimate (3.13) follows from the application of the Strang lemma, see, e.g., [5] or [37, Theorem 8.2], the Aubin–Nitsche trick, see, e.g., [37, Lemma 11.3], and by using the L_2 error estimate of the linear quasi–interpolation operator Q_h, see [35]. ■

In the same way as above we may define a finite element approximation of the particular solution u_f, i.e., $u_{f,h} \in S^1_{h,0}(\Omega)$ is the unique solution of the Galerkin variational problem

$$\int_\Omega \nabla u_{f,h}(x) \cdot \nabla v_h(x)\, dx \;=\; \int_\Omega f(x) v_h(x)\, dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega)$$

satisfying the error estimate

$$\|u_f - u_{f,h}\|_{L_2(\Omega)} \;\leq\; c\, h^2\, \|u_f\|_{H^2(\Omega)} \tag{3.14}$$

in the case of a convex polygonal bounded domain Ω.

3.2 Steklov–Poincaré operator

The application of the Steklov–Poincaré operator S for $z \in H^{1/2}(\Gamma)$ is defined by

$$\langle Sz, w\rangle_\Gamma = \int_\Omega \nabla \mathcal{H}z(x) \cdot \nabla \mathcal{E}w(x)\, dx \quad \text{for all } w \in H^{1/2}(\Gamma).$$

Based on the approximation $\widetilde{\mathcal{H}}$ of the solution operator $\mathcal{H}$, we define the approximation $\widetilde{S}$ by

$$\langle \widetilde{S}z, w\rangle_\Gamma = \int_\Omega \nabla \widetilde{\mathcal{H}}z(x) \cdot \nabla \mathcal{E}w(x)\, dx \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.15}$$

Lemma 3.4 [36, Theorem 3.5] *The approximate Steklov–Poincaré operator $\widetilde{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded and $H^{1/2}(\Gamma)$ semi–elliptic. For $z \in H^{3/2}_{pw}(\Gamma)$ there holds the approximation error estimate*

$$\|Sz - \widetilde{S}z\|_{H^{-1/2}(\Gamma)} \;\leq\; c\, h\, \left[|u_0|_{H^2(\Omega)} + \|z\|_{H^{3/2}_{\text{pw}}(\Gamma)} \right]. \tag{3.16}$$

3.3 Adjoint boundary value problem

Next we consider a finite element approximation of the adjoint solution operator $\tau = \mathcal{H}^*\psi$ as defined in (2.10), i.e., of

$$\tau(x) = -\frac{\partial}{\partial n_x} p(x) \quad \text{for } x \in \Gamma,$$

where $p \in H^1_0(\Omega)$ is the unique solution of the variational problem (2.12). The finite element approximation of (2.12) is to find $p_h \in S^1_{h,0}(\Omega)$ such that

$$\int_\Omega \nabla p_h(x) \cdot \nabla v_h(x)\, dx \;=\; \int_\Omega \psi(x) v_h(x)\, dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega). \tag{3.17}$$

Again, we conclude the unique solvability of the Galerkin formulation (3.17) by means of Cea's lemma, as well as the quasi–optimal error estimate, in the case of a convex polygonal bounded domain,

$$\|p - p_h\|_{H^1(\Omega)} \;\leq\; c\, h\, |p|_{H^2(\Omega)} \;\leq\; c\, h\, \|\psi\|_{L_2(\Omega)}. \tag{3.18}$$

Now we are able to define an approximation $\widetilde{\tau} = \widetilde{\mathcal{H}}^* \psi$ of $\tau = \mathcal{H}^* \psi$ for $\psi \in L_2(\Omega)$ by

$$\langle \widetilde{\tau}, w \rangle_\Gamma = -\int_\Omega \nabla p_h(x) \cdot \nabla \mathcal{E} w(x)\, dx + \int_\Omega \psi(x) \mathcal{E} w(x)\, dx \quad \text{for all } w \in H^{1/2}(\Gamma). \tag{3.19}$$

Lemma 3.5 *For $\psi \in L_2(\Omega)$ let $\widetilde{\tau} = \widetilde{\mathcal{H}}^* \psi$ be the approximation as defined in* (3.19)*. Then, $\widetilde{\mathcal{H}}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is bounded, and there holds the error estimate*

$$\|(\mathcal{H}^* - \widetilde{\mathcal{H}}^*)\psi\|_{H^{-1/2}(\Gamma)} \;=\; \|\tau - \widetilde{\tau}\|_{H^{-1/2}(\Gamma)} \;\leq\; c\, h\, \|\psi\|_{L_2(\Omega)}. \tag{3.20}$$

Proof. Both the boundedness of $\widetilde{\mathcal{H}}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ and the error estimate (3.20) follow from the finite element error estimate (3.18) when using duality arguments and the inverse trace theorem. ■

3.4 Approximation error estimates

By using the finite element approximations $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{H}}^*$, and $\widetilde{S}$ as defined in (3.12), (3.19) and (3.15), respectively, we can introduce the finite element approximations

$$\widetilde{T}_\varrho := \varrho \widetilde{S} + \widetilde{\mathcal{H}}^* \widetilde{\mathcal{H}}, \quad \widetilde{g} := \widetilde{\mathcal{H}}^*(\overline{u} - u_{f,h}) \tag{3.21}$$

to be used within the perturbed variational inequality (3.6). For the application of Theorem 3.2 we need to estimate the related approximation error.

Lemma 3.6 *For $z \in H^{3/2}_{\text{pw}}(\Gamma)$, let $\mathcal{H}z = u_0 + \mathcal{E}z \in H^2(\Omega)$ be the solution of the Dirichlet boundary value problem* (2.4)*. Let $p \in H^2(\Omega)$ be the weak solution of the adjoint boundary value problem* (2.11) *with $\psi = \mathcal{H}z$. Then there holds the error estimate*

$$\|(T_\varrho - \widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1\, h\, \|\mathcal{H}z\|_{L_2(\Omega)} + c_2\, h \left[\|u_0\|_{H^2(\Omega)} + \|z\|_{H^{3/2}_{\text{pw}}(\Gamma)}\right]. \tag{3.22}$$

Proof. By the triangle inequality and by using the boundedness of $\widetilde{\mathcal{H}}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ we have

$$\begin{aligned}
\|(T_\varrho - \widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} \;&=\; \|(\mathcal{H}^*\mathcal{H} - \widetilde{\mathcal{H}}^*\widetilde{\mathcal{H}})z + (S - \widetilde{S})z\|_{H^{-1/2}(\Gamma)} \\
&\leq \|(\mathcal{H}^* - \widetilde{\mathcal{H}}^*)\mathcal{H}z\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathcal{H}}^*(\mathcal{H} - \widetilde{\mathcal{H}})z\|_{H^{-1/2}(\Gamma)} + \|(S - \widetilde{S})z\|_{H^{-1/2}(\Gamma)} \\
&\leq \|(\mathcal{H}^* - \widetilde{\mathcal{H}}^*)\mathcal{H}z\|_{H^{-1/2}(\Gamma)} + c_2^{\widetilde{\mathcal{H}}^*} \|(\mathcal{H} - \widetilde{\mathcal{H}})z\|_{L_2(\Omega)} + \|(S - \widetilde{S})z\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

The assertion now follows from the error estimates (3.13), (3.16), and (3.20). ■

Lemma 3.7 *Let $u_{f,h} \in S^1_{h,0}(\Omega)$ be the finite element approximation of the particular solution $u_f \in H^2(\Omega)$. Let $p_{\overline{u}}, p_{u_f} \in H^2(\Omega)$ be the weak solutions of the adjoint boundary value problem* (2.11) *with $\psi = \overline{u}$ and $\psi = u_f$, respectively. Then there holds the error estimate*

$$\|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1\, h^2\, \|u_f\|_{H^2(\Omega)} + c_2\, h\, \left[\|\overline{u}\|_{L_2(\Omega)} + \|u_f\|_{L_2(\Omega)}\right]. \tag{3.23}$$

Proof. From the triangle inequality we first have

$$\begin{aligned}\|g-\widetilde{g}\|_{H^{-1/2}(\Gamma)} &= \|\mathcal{H}^*(\overline{u}-u_f)-\widetilde{\mathcal{H}}^*(\overline{u}-u_{f,h})\|_{H^{-1/2}(\Gamma)} \\ &\leq \|(\mathcal{H}^*-\widetilde{\mathcal{H}}^*)\overline{u}\|_{H^{-1/2}(\Gamma)}+\|\widetilde{\mathcal{H}}^*(u_{f,h}-u_f)\|_{H^{-1/2}(\Gamma)}+\|(\widetilde{\mathcal{H}}^*-\mathcal{H}^*)u_f\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

The assertion now follows from the boundedness of $\widetilde{\mathcal{H}}^*: L_2(\Omega)\to H^{-1/2}(\Gamma)$, and the finite element error estimates (3.14) and (3.20). For the application of Lemma 3.5 we need to assume $p_{\overline{u}}, p_{u_f}\in H^2(\Omega)$ where $p_{\overline{u}}$ and p_{u_f} are the weak solutions of the adjoint boundary value problem (2.11) with $\psi=\overline{u}$ and $\psi=u_f$, respectively. But this assumption is satisfied in the case of a convex polygonal domain. ■

It remains to prove the $S_h^1(\Gamma)$–ellipticity of the finite element approximation $\widetilde{T}_\varrho$. This will be a consequence of the matrix representation of $\widetilde{T}_\varrho$ as discussed in the next subsection.

Now we are in the position to present the final error estimate for the approximate solution $\widetilde{z}_h$ of the variational inequality (3.6).

Corollary 3.8 *Let $\Omega\subset\mathbb{R}^2$ be a convex polygonal bounded domain. For $g_a, g_b\in H^{3/2}_{\rm pw}(\Gamma)$ and $f,\overline{u}\in L_2(\Omega)$ let $z\in H^{3/2}_{\rm pw}(\Gamma)$ be the unique solution of the variational inequality* (2.16)*. Let $\widetilde{z}_h\in\mathcal{U}_h$ be the unique solution of the perturbed variational inequality* (3.6)*. Then there holds the error estimate*

$$\|z-\widetilde{z}_h\|_{H^{1/2}(\Gamma)}\leq c(z,\overline{u},f)\,h\,. \tag{3.24}$$

Moreover, we have

$$\|z-\widetilde{z}_h\|_{L_2(\Gamma)}\leq c(z,\overline{u},f)\,h^{3/2}. \tag{3.25}$$

Proof. The error estimate (3.24) follows from Theorem 3.2 by using the approximation error estimates (3.22) and (3.23). By using the Aubin–Nitsche trick for the perturbed variational inequality (3.6) we obtain the error estimate (3.25). ■

From the error estimate (3.25) we also conclude, by applying the Aubin–Nitsche trick for the primal boundary value problem (2.2), the error estimate

$$\|u-u_h\|_{L_2(\Omega)}\leq c(z,\overline{u},f)\,h^2\,. \tag{3.26}$$

Moreover, by using the Aubin–Nitsche trick for the adjoint boundary value problem (2.11) with $\psi=u-\overline{u}$ we also obtain the error estimate

$$\|p-p_h\|_{L_2(\Omega)}\leq c(z,\overline{u},f)\,h^2\,. \tag{3.27}$$

3.5 Approximate variational inequality

Now we are in a position to describe the finite element approximation of the perturbed variational inequality (3.6) to find $\widetilde{z}_h\in\mathcal{U}_h$ such that

$$\varrho\langle\widetilde{S}\widetilde{z}_h, w_h-\widetilde{z}_h\rangle_\Gamma+\langle\widetilde{\mathcal{H}}^*(\widetilde{\mathcal{H}}\widetilde{z}_h+u_{f,h}-\overline{u}), w_h-\widetilde{z}_h\rangle_\Gamma\geq 0\quad\text{for all } w_h\in\mathcal{U}_h\,.$$

Note that $u_h = \widetilde{\mathcal{H}}\widetilde{z}_h + u_{f,h} \in S_h^1(\Omega)$ is the unique solution of the Galerkin variational formulation

$$\int_\Omega \nabla u_h(x) \cdot \nabla v_h(x)\, dx \;=\; \int_\Omega f(x) v_h(x)\, dx \quad \text{for all } v_h \in S^1_{h,0}(\Omega). \tag{3.28}$$

By using

$$u_h(x) \;=\; \sum_{k=1}^{M_\Omega} u_{I,k}\varphi_k(x) + \sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \varphi_{M_\Omega+\ell}(x), \quad \widetilde{z}_h(x) = \sum_{\ell=1}^{M_\Gamma} \widetilde{z}_\ell \phi_\ell(x), \quad \phi_\ell = \varphi_{M_\Omega+\ell|\Gamma},$$

the variational formulation (3.28) is equivalent to the linear system

$$K_{II}\underline{u}_I + K_{CI}\underline{\widetilde{z}} \;=\; \underline{f}_I, \tag{3.29}$$

where

$$K_{II}[j,k] = \int_\Omega \nabla\varphi_k(x)\cdot\nabla\varphi_j(x)\, dx, \quad K_{CI}[j,\ell] = \int_\Omega \nabla\varphi_{M_\Omega+\ell}(x)\cdot\nabla\varphi_j(x)\, dx,$$

for $k, j = 1, \ldots, M_\Omega$, $\ell = 1, \ldots, M_\Gamma$, describe the standard finite element stiffness matrices, and

$$f_{I,j} = \int_\Omega f(x)\varphi_j(x)\, dx \quad \text{for } j = 1, \ldots, M_\Omega.$$

Next we consider, by using (3.19),

$$\begin{aligned}\langle \widetilde{\mathcal{H}}^*(\widetilde{\mathcal{H}}\widetilde{z}_h + u_{f,h} - \overline{u}), w_h - \widetilde{z}_h\rangle_\Gamma \;&=\; \langle \widetilde{\mathcal{H}}^*(u_h - \overline{u}), w_h - \widetilde{z}_h\rangle_\Gamma \\ &= -\int_\Omega \nabla p_h(x)\cdot\nabla\mathcal{E}(w_h - \widetilde{z}_h)(x)\, dx + \int_\Omega [u_h(x) - \overline{u}(x)]\mathcal{E}(w_h - \widetilde{z}_h)(x)\, dx,\end{aligned}$$

where $p_h \in S^1_{h,0}(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla p_h(x)\cdot\nabla q_h(x)\, dx \;=\; \int_\Omega [u_h(x) - \overline{u}(x)] q_h(x)\, dx \quad \text{for all } q_h \in S^1_{h,0}(\Omega).$$

Hence we conclude the matrix representation

$$\langle \widetilde{\mathcal{H}}^*(\widetilde{\mathcal{H}}\widetilde{z}_h + u_{f,h} - \overline{u}), w_h - \widetilde{z}_h\rangle_\Gamma \;=\; (-K_{IC}\underline{p} + M_{IC}\underline{u}_I + M_{CC}\underline{\widetilde{z}} - \underline{g}_C, \underline{w} - \underline{\widetilde{z}}),$$

where

$$K_{II}\underline{p} \;=\; M_{II}\underline{u}_I + M_{CI}\underline{\widetilde{z}} - \underline{g}_I\,.$$

Note that the matrices defined by

$$M_{II}[j,k] = \int_\Omega \varphi_k(x)\varphi_j(x)\, dx, \quad M_{CI}[j,\ell] = M_{IC}[\ell,j] = \int_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_j(x)\, dx$$

for $k, j = 1, \ldots, M_\Omega$, $\ell = 1, \ldots, M_\Gamma$ and

$$M_{CC}[j,\ell] = \int\limits_\Omega \varphi_{M_\Omega+\ell}(x)\varphi_{M_\Omega+j}(x)\,dx \quad \text{for } j,\ell = 1,\ldots,M_\Gamma$$

are the standard finite element mass matrices. Moreover,

$$g_{I,j} = \int\limits_\Omega \overline{u}(x)\varphi_j(x)\,dx \text{ for } j = 1,\ldots,M_\Omega, \quad g_{C,\ell} = \int\limits_\Omega \overline{u}(x)\varphi_{M_\Omega+\ell}(x)\,dx \text{ for } \ell = 1,\ldots,M_\Gamma.$$

Let $\widetilde{S}_h$ be the Galerkin matrix of the approximate Steklov–Poincaré operator $\widetilde{S}$, i.e.,

$$\widetilde{S}_h \;=\; K_{CC} - K_{IC}K_{II}^{-1}K_{CI}$$

where

$$K_{CC}[j,\ell] = \int_\Omega \nabla\varphi_{M_\Omega+\ell}(x)\cdot\nabla\varphi_{M_\Omega+j}(x)\,dx,$$

for $j,\ell = 1,\ldots,M_\Gamma$. The matrix representation of the variational inequality (3.6) is then given by the discrete variational inequality

$$\varrho(\widetilde{S}_h\underline{\widetilde{z}}, \underline{w}-\underline{\widetilde{z}}) + (-K_{IC}\underline{p} + M_{IC}\underline{u}_I + M_{CC}\underline{\widetilde{z}} - \underline{g}_C, \underline{w}-\underline{\widetilde{z}}) \;\geq\; 0 \quad \text{for all } \underline{w} \in \mathbb{R}^{M_\Gamma} \leftrightarrow w_h \in \mathcal{U}_h,$$

or

$$(\widetilde{T}_{\varrho,h}\underline{\widetilde{z}} - \underline{\widetilde{g}}, \underline{w}-\underline{\widetilde{z}}) \;\geq\; 0 \quad \text{for all } \underline{w} \in \mathbb{R}^{M_\Gamma} \leftrightarrow w_h \in \mathcal{U}_h, \tag{3.30}$$

where

$$\widetilde{T}_{\varrho,h} := \varrho\widetilde{S}_h + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} \tag{3.31}$$

and

$$\underline{\widetilde{g}} := \underline{g}_C - K_{IC}K_{II}^{-1}\underline{g}_I + K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}\underline{f}_I - M_{IC}K_{II}^{-1}\underline{f}_I.$$

Lemma 3.9 *The approximate operator $\widetilde{T}_\varrho = \varrho\widetilde{S} + \widetilde{\mathcal{H}}^*\widetilde{\mathcal{H}}$ is $S_h^1(\Gamma)$–elliptic. In particular, the matrix $\widetilde{T}_{\varrho,h}$ as defined in* (3.31) *is positive definite, i.e.,*

$$\langle\widetilde{T}_\varrho w_h, w_h\rangle_\Gamma = (\widetilde{T}_{\varrho,h}\underline{w},\underline{w}) \;\geq\; c_1^{\widetilde{T}_\varrho}\,\|w_h\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } w_h \in S_h^1(\Gamma) \leftrightarrow \underline{w} \in \mathbb{R}^{M_\Gamma}.$$

Proof. For $\underline{w} \in \mathbb{R}^{M_\Gamma}$ and by defining $\underline{v} = -K_{II}^{-1}K_{CI}\underline{w}$ we have

$$\begin{aligned}
(\widetilde{T}_{\varrho,h}\underline{w},\underline{w}) &= ([K_{IC}K_{II}^{-1}M_{II}K_{II}^{-1}K_{CI} - K_{IC}K_{II}^{-1}M_{CI} - M_{IC}K_{II}^{-1}K_{CI} + M_{CC} + \varrho\widetilde{S}_h]\underline{w},\underline{w}) \\
&= (M_{II}K_{II}^{-1}K_{CI}\underline{w}, K_{II}^{-1}K_{CI}\underline{w}) - (M_{CI}\underline{w}, K_{II}^{-1}K_{CI}\underline{w}) - (M_{IC}K_{II}^{-1}K_{CI}\underline{w},\underline{w}) \\
&\qquad + (M_{CC}\underline{w},\underline{w}) + \varrho(\widetilde{S}_h\underline{w},\underline{w}) \\
&= (M_{II}\underline{v},\underline{v}) + (M_{CI}\underline{w},\underline{v}) + (M_{IC}\underline{v},\underline{w}) + (M_{CC}\underline{w},\underline{w}) + \varrho(\widetilde{S}_h\underline{w},\underline{w}) \\
&= \left(\begin{pmatrix} M_{II} & M_{CI} \\ M_{IC} & M_{CC}\end{pmatrix}\begin{pmatrix}\underline{v}\\ \underline{w}\end{pmatrix}, \begin{pmatrix}\underline{v}\\ \underline{w}\end{pmatrix}\right) + \varrho(\widetilde{S}_h\underline{w},\underline{w}).
\end{aligned}$$

Since the mass matrix

$$M_h = \begin{pmatrix} M_{II} & M_{CI} \\ M_{IC} & M_{CC} \end{pmatrix}$$

is positive definite, the assertion follows. ■

In the particular case of a non–constrained minimisation problem, instead of the discrete variational inequality (3.30) we have to solve the linear system

$$\widetilde{T}_{\varrho,h}\underline{\widetilde{z}} = \underline{\widetilde{g}},$$

which is equivalent to the system

$$\begin{pmatrix} -M_{II} & K_{II} & -M_{CI} \\ K_{II} & & K_{CI} \\ M_{IC} & -K_{IC} & M_{CC} + \varrho\widetilde{S}_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{p} \\ \underline{\widetilde{z}} \end{pmatrix} = \begin{pmatrix} -\underline{g}_I \\ \underline{f}_I \\ \underline{g}_C \end{pmatrix}. \tag{3.32}$$

Note that the coupled linear system (3.32) also results from a direct finite element approximation of the adjoint boundary value problem, of the primal boundary value problem, and of the optimality condition.

4 Numerical results and concluding remarks

As numerical example we consider as in [8], see also [30], the Dirichlet boundary control problem (2.1)–(2.3) for the domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$ where

$$\overline{u}(x) = (x_1^2 + x_2^2)^{-1/3}, \quad f(x) = 0, \quad \varrho = 1,$$

and without box constraints. For the finite element discretization we introduce a uniform triangulation of Ω on several refinement levels L with the mesh size $h_L = 2^{-(L+2)}$, where for $L = 0$ the coarsest level is given by 4 uniform triangles. Since the minimiser of (2.1) is not known for this example, we use the finite element solutions (z_{h_9}, u_{h_9}) on the 9th level as reference solutions.

In Table 1 we give the finite element errors for the control variable z in the $L_2(\Gamma)$–norm, for the primal variable u and for the adjoint state p in the $L_2(\Omega)$–norm. We observe an estimated order of convergence (eoc) close to 2 as predicted by the error estimates (3.26) and (3.27) for u and p. For the control z the observed order of convergence is larger than 3/2 which is predicted by the error estimate (3.25). So far we are not able to prove this higher order of convergence for a finite element discretization but for the boundary element discretization presented in [32] when assuming $z \in H^2_{\text{pw}}(\Gamma)$. In this case the second order of convergence reflects the optimal approximation property when using piecewise linear basis functions on the boundary.

For comparison we present in Table 2 the numerical results when considering the Dirichlet control problem in $L_2(\Gamma)$, see also [8, 30]. While we obtain a quadratic convergence

L	$\|z_h - z_{h_9}\|_{L_2(\Gamma)}$	eoc	$\|u_h - u_{h_9}\|_{L_2(\Omega)}$	eoc	$\|p_h - p_{h_9}\|_{L_2(\Omega)}$	eoc
2	1.23e-03		2.25e-04		1.47e-04	
3	3.47e-04	1.83	5.51e-05	2.03	4.77e-05	1.63
4	9.48e-05	1.87	1.34e-05	2.04	1.40e-05	1.76
5	2.57e-05	1.88	3.44e-06	1.96	3.88e-06	1.85
6	6.96e-06	1.88	9.72e-07	1.82	1.03e-06	1.92
7	1.87e-06	1.89	2.91e-07	1.74	2.57e-07	2.00
8	4.54e-07	2.04	7.70e-08	1.92	5.52e-08	2.22

Table 1: Finite element errors for the Dirichlet control problem in $H^{1/2}(\Gamma)$.

behaviour for the adjoint state p with slightly larger errors than for $H^{1/2}(\Gamma)$ control, the convergence rate for the control z and for the primal variable u changes significantly. The total errors are worse by orders of magnitude than the ones in Table 1. The main reason for this different convergence behaviour lies in the optimality condition which reads for the $L_2(\Gamma)$ control

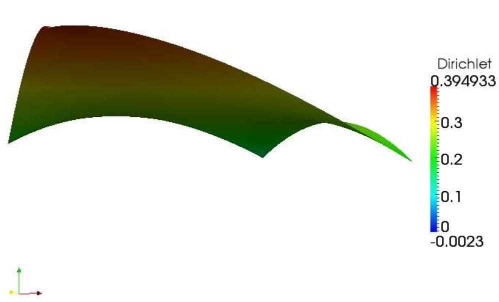
$$z = P_{\mathcal{U}}\xi, \quad \varrho\xi + \tau = 0, \quad \tau = -\frac{\partial}{\partial n}p \tag{4.1}$$

where $P_{\mathcal{U}}$ denotes the pointwise projection onto $\mathcal{U}$.

L	$\|z_h - z_{h_9}\|_{L_2(\Gamma)}$	eoc	$\|u_h - u_{h_9}\|_{L_2(\Omega)}$	eoc	$\|p_h - p_{h_9}\|_{L_2(\Omega)}$	eoc
2	3.20e-02		6.76e-03		4.92e-04	
3	1.84e-02	0.80	2.66e-03	1.35	1.38e-04	1.84
4	1.05e-02	0.80	1.05e-03	1.34	3.63e-05	1.92
5	6.07e-03	0.80	4.20e-04	1.32	9.34e-06	1.96
6	3.53e-03	0.78	1.69e-04	1.31	2.35e-06	1.99
7	2.07e-03	0.77	6.85e-05	1.30	5.71e-07	2.04
8	1.14e-03	0.85	2.57e-05	1.41	1.20e-07	2.25

Table 2: Finite element errors for the Dirichlet control problem in $L_2(\Gamma)$.

Indeed, for the given $\overline{u}$ we obtain $z \in H^{2/3}(\Gamma)$ only, instead of $z \in H^{3/2}_{\text{pw}}(\Gamma)$ when considering the control in $H^{1/2}(\Gamma)$. Moreover, from the optimality conditions (4.1) we conclude, that z is zero in all corner points due to the zero Dirichlet boundary condition of the adjoint state p. Hence we obtain a zero control z in all corner points independent of the target function $\overline{u}$. Instead, when considering the control in $H^{1/2}(\Gamma)$, the application of $\widetilde{S}$ reflects the proper mapping properties of the Dirichlet to Neumann map. This results in a more feasible Dirichlet control in corner points. For illustration we give the plots of the states u in Fig. 1 for the $H^{1/2}(\Gamma)$ setting, and in Fig. 2 for the $L_2(\Gamma)$ setting. Note that for this example we used a $H^{1/2}$ norm, which is defined by the semi–norm induced by the Steklov–Poincaré operator S plus a $L_2(\Gamma)$ equivalent term, and different values of ϱ to ensure comparable values for the tracking functional $\|u - \overline{u}\|_{L_2(\Omega)}$.



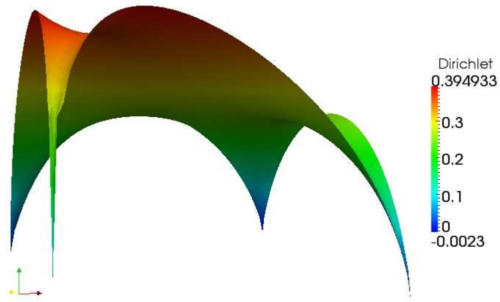


Fig. 1: State u for the control in $H^{1/2}(\Gamma)$. Fig. 2: State u for the control in $L_2(\Gamma)$.

In Table 3, we present the computational times in seconds related to Table 1 and Table 2. N denotes the number of triangles of the triangulation. The slightly higher effort for the approach of the $H^{1/2}(\Gamma)$ semi–norm is due to a few more iteration steps needed to solve the Schur complement system of (3.32) iteratively. Note that advanced preconditioners are not applied here. Taking into account the higher order of convergence, the additional effort is justifiable.

L	N	$H^{1/2}(\Gamma)$	$L_2(\Gamma)$
4	1024	0.04 sec	0.03 sec
5	4096	0.18 sec	0.16 sec
6	16384	0.79 sec	0.70 sec
7	65536	3.70 sec	3.05 sec
8	262144	15.92 sec	12.81 sec

Table 3: Comparison of the computational times for $H^{1/2}(\Gamma)$ and $L_2(\Gamma)$ costs.

Recall that there are no box constraints considered in this example, i.e., we have to solve the coupled linear system (3.32), or the related Schur complement system. Since the Schur complement $\widetilde{T}_{\varrho,h}$ defines an equivalent discrete norm in $H^{1/2}(\Gamma)$, appropriate preconditioners can be constructed, e.g. by using multilevel methods. For the solution of the discrete variational inequality (3.30) one may consider either multigrid methods, see, e.g., [18, 34], or semi–smooth Newton methods, see, e.g., [23, 42].

As an example with active constraints and a $H^{1/2}(\Gamma)$ control, we consider the first example where $\varrho = 1$ and choose $g_b = 2.2$ and g_a inactive. Again we use the semi–norm induced by the Steklov–Poincaré operator. Fig. 3 and Fig. 4 clearly show the difference of the solutions of the unconstrained and the constrained case. The constrained problem is solved by a semi–smooth Newton method, see e.g. [23]. Fig. 5 shows the control of the constrained and unconstrained problem along the line $x_2 = 0$.

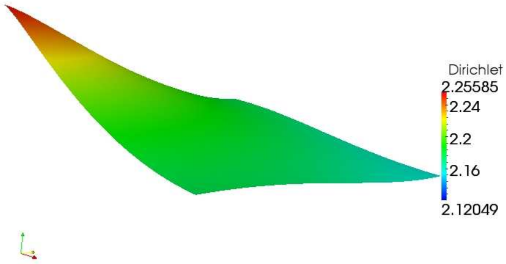


Fig. 3: State u for the unconstrained case.

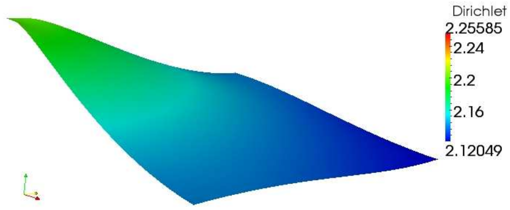


Fig. 4: State u for the constrained case.

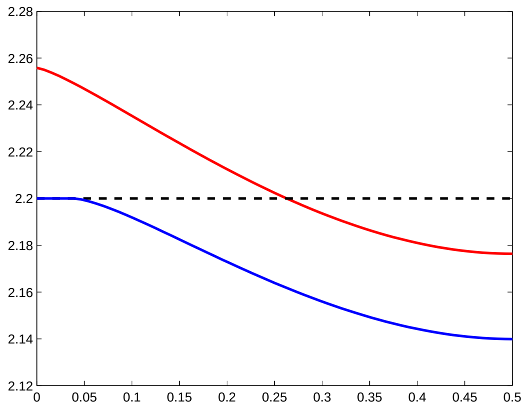


Fig. 5: Comparison of the constrained and unconstrained control z along the line $x_2 = 0$.

In this paper we have presented a rigorous stability and error analysis for a finite element approximation of a Dirichlet boundary control problem where the control is considered in the energy space $H^{1/2}(\Gamma)$. For the ease of presentation we have restricted our considerations to the case of a convex two–dimensional polygonal bounded domain. However, using standard finite element approximations for partial differential equations which are formulated in domains with smooth curved boundaries, our approach can be extended to the case of two– and three–dimensional domains with smooth boundary. Moreover, the finite element analysis can be transfered also to general polygonal or polyhedral bounded domains, but then related regularity results for Signorini boundary value problems are required.

Note that for the approximation of the operator $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as defined in (2.14) we may also consider boundary element methods which require an appropriate handling of the inhomogeneous adjoint partial differential equation in (2.11). The stability and error analysis of the resulting boundary element approach for the solution of Dirichlet control problems is discussed in [32], see also [33].

Instead of the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ one may use any other bounded and semi–elliptic operator to realize an equivalent norm in $H^{1/2}(\Gamma)$. A possible choice is to consider as in [32] the so–called hypersingular boundary integral operator D which does not require an inversion as in the definition of $\widetilde{S}_h$. But the use of the hypersingular integral operator may result in a lower regularity of the control z, and therefore in a lower order of convergence of the finite element approximation, see [32, 33].

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