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Abstract

In this paper we discuss the application of boundary element methods for the solution of Dirichlet boundary control problems subject to the Poisson equation with box constraints on the control. The primal and adjoint boundary value problems are rewritten as systems of boundary integral equations involving the standard boundary integral operators of the Laplace equation and of the Bi–Laplace equation. While the first approach is based on the use of the standard boundary integral equation based on the Bi–Laplace fundamental solution, the additional use of the normal derivative of the related representation formula results in a symmetric formulation, which is also symmetric in the discrete case. We prove the unique solvability of both boundary integral approaches and discuss related boundary element discretisations. In particular, we prove stability and related error estimates which are confirmed by a numerical example.

1 Introduction

Optimal control problems of elliptic or parabolic partial differential equations with a Dirichlet boundary control play an important role, for example, in the context of computational fluid mechanics, see, e.g., [9, 11, 12], and the references given therein. A difficulty in the handling of Dirichlet control problems is the choice of the control space, where the Sobolev trace space $H^{1/2}(\Gamma)$ appears as a natural choice [1]. To obtain smoother optimal solutions one may even consider $H^2(\Gamma)$ as control space, where the Sobolev norm for sufficient regular boundaries can be realised by using the Laplace–Beltrami operator [16]. The most popular choice is to consider $L_2(\Gamma)$ as control space. Although this choice allows the use of a piecewise constant control function, the associated partial differential equation has to be considered within an ultra–weak variational formulation, see, for example, [20], and [2] for an appropriate finite element approximation using standard piecewise linear basis functions. The use of the ultra–weak variational formulation of the primal Dirichlet boundary value problem in the context of an optimal control problem requires the adjoint variable p to be sufficiently regular, i.e., $p \in H^2(\Omega) \cap H^1_0(\Omega)$. Since the adjoint variable p itself is the unique solution of the adjoint partial differential equation with homogeneous Dirichlet boundary conditions, either a smooth boundary Γ, or a polygonal or polyhedral but convex domain Ω has to be assumed. For related finite element approximations, see, e.g., [5, 6, 11, 19, 22], or [29] in the case of a finite dimensional Dirichlet control. To include a Dirichlet boundary condition $u = z \in L_2(\Gamma)$ in a standard variational formulation, alternatively one may consider a penalty approximation of the Dirichlet boundary condition by using a Robin boundary condition, see, e.g., [1, 13, 14, 15]. Again, sufficient smoothness of the boundary Γ has to be assumed.

In [25], a finite element approach was considered, where the energy norm was realized by using some hypersingular boundary integral operator which links the Dirichlet control with the normal derivative of the adjoint variable. The related optimality condition results then in a higher regularity of the control and requires less assumptions on the smoothness of the adjoint variable, in fact, one may even consider general Lipschitz domains Ω. Moreover, for polygonal or polyhedral bounded domains Ω one also obtains higher order convergence results for the approximate finite element solution.

Since the unknown function in Dirichlet boundary control problems is to be found on the boundary $\Gamma = \partial\Omega$ of the computational domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, the use of boundary integral equations seems to be a natural choice. But to our knowledge, there are only a few results known on the use of boundary integral equations to solve optimal boundary control problems, see, e.g., [7, 30] for problems with point observations. In this paper, we consider the Poisson equation as a model problem, however, this approach can be applied to any elliptic partial differential equation, if a fundamental solution is known. In this case, solutions of partial differential equations can be described by the means of surface and volume potentials. To find the complete Cauchy data, boundary integral equations have to be solved. For an overview on boundary integral equations, see, e.g., [17, 23] and the references given therein. The numerical solution of boundary integral equations results in boundary element methods, see, e.g., [26, 27].

The model problem is described in Section 2, where we also discuss the adjoint problem which characterises the solution of the reduced minimisation problem. In Section 3, we present the representation formulae to describe the solutions of both the primal and adjoint Dirichlet boundary value problems. To find the unknown normal derivatives of the state variable and of the adjoint variable, weakly singular boundary integral equations are formulated. Since the state enters the adjoint boundary value problem as a volume density, an additional volume integral has to be considered. By applying integration by parts, this Newton potential can be reformulated by using boundary potentials of the Bi–Laplace operator. Hence we recall some properties of boundary integral operators for the Bi–Laplace operator in Section 4. In Section 5, we analyse a first boundary integral formulation to solve the Dirichlet boundary control problem, and we discuss stability and error estimates of the related Galerkin boundary element method. Since this boundary element approximation leads to a non–symmetric matrix representation of a self–adjoint operator, we introduce and analyse a symmetric boundary element approach, which includes a second, the so–called hypersingular boundary integral equation in Section 6. Again we discuss the related stability and error analysis. Finally, we present a numerical example in Section 7.

2 Dirichlet control problems

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with boundary $\Gamma = \partial\Omega$. As a model problem, we consider the Dirichlet boundary control problem to minimise the cost functional

$$J(u,z) \;=\; \frac{1}{2}\int_\Omega [u(x)-\overline{u}(x)]^2 dx + \frac{1}{2}\varrho\,\langle Dz, z\rangle_\Gamma \tag{2.1}$$

subject to the constraint

$$-\Delta u(x) \;=\; f(x) \quad \text{for } x \in \Omega, \quad u(x) \;=\; z(x) \quad \text{for } x \in \Gamma, \tag{2.2}$$

and where the control z satisfies the box constraints

$$z \in \mathcal{U} \;:=\; \left\{ w \in H^{1/2}(\Gamma) \;:\; g_a(x) \le w(x) \le g_b(x) \quad \text{for } x \in \Gamma \right\}. \tag{2.3}$$

We assume $f, \overline{u} \in L_2(\Omega)$, $\varrho \in \mathbb{R}_+$, and $g_a, g_b \in H^{1/2}(\Gamma)$. Moreover, we use the hypersingular boundary integral operator $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ to describe the cost, or some regularisation term, via a semi–norm in $H^{1/2}(\Gamma)$. In particular, for $z \in H^{1/2}(\Gamma)$ we have

$$(Dz)(x) = -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) z(y) ds_y \quad \text{for } x \in \Gamma,$$

where

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } n = 2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } n = 3 \end{cases} \tag{2.4}$$

is the fundamental solution of the Laplace operator.

Let $u_f \in H^1_0(\Omega)$ be the weak solution of the homogeneous Dirichlet boundary value problem

$$-\Delta u_f(x) = f(x) \quad \text{for } x \in \Omega, \quad u_f(x) = 0 \quad \text{for } x \in \Gamma.$$

The solution of the Dirichlet boundary value problem (2.2) is then given by $u = u_z + u_f$, where $u_z \in H^1(\Omega)$ is the unique solution of the Dirichlet boundary value problem

$$-\Delta u_z(x) = 0 \quad \text{for } x \in \Omega, \quad u_z(x) = z(x) \quad \text{for } x \in \Gamma. \tag{2.5}$$

Note that the solution of the Dirichlet boundary value problem (2.5) defines a linear map $u_z = \mathcal{S}z$ with $\mathcal{S} : H^{1/2}(\Gamma) \to H^1(\Omega) \subset L_2(\Omega)$. Then, by using $u = \mathcal{S}z + u_f$, we consider the problem to find the minimiser $z \in \mathcal{U} \subset H^{1/2}(\Gamma)$ of the reduced cost functional

$$\widetilde{J}(z) \;=\; \frac{1}{2}\int_\Omega [(\mathcal{S}z)(x) + u_f(x) - \overline{u}(x)]^2 dx + \frac{1}{2}\varrho\,\langle Dz, z\rangle_\Gamma. \tag{2.6}$$

To characterise the minimiser $z \in \mathcal{U}$ of the reduced cost functional (2.6) we introduce the self–adjoint, bounded and $H^{1/2}(\Gamma)$–elliptic operator

$$T_\varrho \,:=\, \varrho D + \mathcal{S}^*\mathcal{S} \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \tag{2.7}$$

satisfying, see, e.g., [25],

$$\langle T_\varrho z, z\rangle_\Gamma \geq c_1^{T_\varrho}\|z\|^2_{H^{1/2}(\Gamma)}, \quad \|T_\varrho z\|_{H^{-1/2}(\Gamma)} \leq c_2^{T_\varrho}\|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma),$$

where $\mathcal{S}^* : L_2(\Omega) \to H^{-1/2}(\Gamma)$ is the adjoint operator of $\mathcal{S} : H^{1/2}(\Gamma) \to L_2(\Omega)$, i.e.,

$$\langle \mathcal{S}^*\psi, \varphi\rangle_\Gamma = \langle \psi, \mathcal{S}\varphi\rangle_\Omega = \int_\Omega \psi(x)(\mathcal{S}\varphi)(x)dx \quad \text{for all } \varphi \in H^{1/2}(\Gamma), \psi \in L_2(\Omega).$$

Moreover, we define

$$g := \mathcal{S}^*(\overline{u} - u_f) \in H^{-1/2}(\Gamma). \tag{2.8}$$

Hence we can rewrite the reduced cost functional (2.6) as

$$\widetilde{J}(z) \,=\, \frac{1}{2}\,\langle T_\varrho z, z\rangle_\Gamma - \langle g, z\rangle_\Gamma + \frac{1}{2}\,\|u_f - \overline{u}\|^2_{L_2(\Omega)}.$$

Since $\mathcal{U} \subset H^{1/2}(\Gamma)$ is convex and closed, and since T_ϱ is self–adjoint and $H^{1/2}(\Gamma)$–elliptic, the minimisation of the reduced cost functional $\widetilde{J}(z)$ is equivalent to solving a variational inequality to find $z \in \mathcal{U}$ such that

$$\langle T_\varrho z, w - z\rangle_\Gamma \,\geq\, \langle g, w - z\rangle_\Gamma \quad \text{for all } w \in \mathcal{U}. \tag{2.9}$$

Since (2.9) is an elliptic variational inequality of the first kind, we can use standard arguments as given, for example in [3, 10, 20, 21], to establish unique solvability of the variational inequality (2.9).

By using the primal variable $u = \mathcal{S}z + u_f$, and by introducing the adjoint variable $\tau = \mathcal{S}^*(u - \overline{u}) \in H^{-1/2}(\Gamma)$, we can rewrite the variational inequality (2.9) as

$$\langle \varrho D z + \tau, w - z\rangle_\Gamma \,\geq\, 0 \quad \text{for all } w \in \mathcal{U}. \tag{2.10}$$

Note that for given $z \in H^{1/2}(\Gamma)$ and $f \in L_2(\Omega)$ the application of $u = Sz + u_f$ corresponds to the solution of the Dirichlet boundary value problem (2.2). The application of the adjoint operator $\tau = \mathcal{S}^*(u - \overline{u})$ is characterised by the Neumann datum

$$\tau \,=\, -\frac{\partial}{\partial n}p \quad \text{in the sense of } H^{-1/2}(\Gamma), \tag{2.11}$$

where $p \in H^1_0(\Omega)$ is the unique solution of the adjoint Dirichlet boundary value problem

$$-\Delta p(x) \,=\, u(x) - \overline{u}(x) \quad \text{for } x \in \Omega, \quad p(x) \,=\, 0 \quad \text{for } x \in \Gamma. \tag{2.12}$$

Since the unknown control $z \in \mathcal{U} \subset H^{1/2}(\Gamma)$ is considered on the boundary $\Gamma = \partial\Omega$, the use of boundary integral equations to solve both the primal boundary value problem (2.2) and the adjoint boundary value problem (2.12) seems to be a natural choice. In what follows we will describe and analyse two different boundary element methods to solve the variational inequality (2.9) numerically. This will be based on the use of appropriate boundary integral operator representations of T_ϱ and g as introduced in (2.7) and (2.8), respectively.

3 Laplace boundary integral equations

3.1 Primal boundary value problem

The solution of the Dirichlet boundary value problem (2.2),

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = z(x) \quad \text{for } x \in \Gamma,$$

is given by the representation formula for $\widetilde{x} \in \Omega$,

$$u(\widetilde{x}) \;=\; \int_\Gamma U^*(\widetilde{x}, y)t(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(\widetilde{x}, y)z(y)ds_y + \int_\Omega U^*(\widetilde{x}, y)f(y)dy, \tag{3.1}$$

where $t = \dfrac{\partial}{\partial n} u \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(Vt)(x) \;=\; (\frac{1}{2}I + K)z(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma. \tag{3.2}$$

Note that

$$(Vt)(x) \;=\; \int_\Gamma U^*(x, y)t(y)ds_y \quad \text{for } x \in \Gamma$$

is the Laplace single layer integral operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, and

$$(Kz)(x) \;=\; \int_\Gamma \frac{\partial}{\partial n_y} U^*(x, y)z(y)ds_y \quad \text{for } x \in \Gamma$$

is the Laplace double layer integral operator $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$. Moreover,

$$(N_0 f)(x) \;=\; \int_\Omega U^*(x, y)f(y)dy \quad \text{for } x \in \Gamma$$

is the related Newton potential. The single layer integral operator V is $H^{-1/2}(\Gamma)$–elliptic; for $n = 2$ we assume the scaling condition diam $\Omega < 1$ to ensure this. For the solution of the boundary integral equation (3.2) we therefore obtain

$$t = V^{-1}(\frac{1}{2}I + K)z - V^{-1}N_0 f\,. \tag{3.3}$$

3.2 Adjoint boundary value problem

The solution of the adjoint Dirichlet boundary value problem (2.12),

$$-\Delta p(x) = u(x) - \overline{u}(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma,$$

is given correspondingly by the representation formula for $\widetilde{x} \in \Omega$,

$$p(\widetilde{x}) \;=\; \int_\Gamma U^*(\widetilde{x}, y)q(y)ds_y + \int_\Omega U^*(\widetilde{x}, y)[u(y) - \overline{u}(y)]dy, \tag{3.4}$$

where $q = \dfrac{\partial}{\partial n} p \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(Vq)(x) = (N_0 \overline{u})(x) - (N_0 u)(x) \quad \text{for } x \in \Gamma. \tag{3.5}$$

Remark 3.1 *While the boundary integral equation* (3.2) *can be used to determine the unknown Neumann datum* $t \in H^{-1/2}(\Gamma)$ *of the primal Dirichlet boundary value problem* (2.2)*, the unknown Neumann datum* $q \in H^{-1/2}(\Gamma)$ *of the adjoint Dirichlet boundary value problem* (2.12) *is given as the solution of the boundary integral equation* (3.5)*. Then, by using* $\tau = -q$*, the control* $z \in H^{1/2}(\Gamma)$ *is determined by the variational inequality* (2.10)*. However, since the solution* u *of the primal Dirichlet boundary value problem* (2.2) *enters the volume potential* $N_0 u$ *in the boundary integral equation* (3.5)*, it seems to be necessary to include the representation formula* (3.1)*. In this case we would have to solve a coupled system of boundary and domain integral equations, which still would require some domain mesh. Instead, we will now describe a system of only boundary integral equations to solve the adjoint boundary value problem* (2.12)*.*

To end up with a system of boundary integral equations only, instead of (3.4), we will introduce a modified representation formula for the adjoint state p as follows. First we note that

$$V^*(x,y) \;=\; \begin{cases} -\dfrac{1}{8\pi}|x-y|^2\big(\log|x-y|-1\big) & \text{for } n=2, \\ \dfrac{1}{8\pi}\,|x-y| & \text{for } n=3 \end{cases} \tag{3.6}$$

is a solution of the Poisson equation

$$\Delta_y V^*(x,y) \;=\; U^*(x,y) \quad \text{for } x \neq y, \tag{3.7}$$

i.e., $V^*(x,y)$ is the fundamental solution of the Bi–Laplacian. Hence we can rewrite the volume integral for u in (3.4), by using Green's second formula, as follows:

$$\begin{aligned} \int_\Omega U^*(\widetilde{x},y)u(y)dy \;&=\; \int_\Omega [\Delta_y V^*(\widetilde{x},y)]u(y)dy \\ &= \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x},y)u(y)ds_y - \int_\Gamma V^*(\widetilde{x},y)\frac{\partial}{\partial n_y}u(y)ds_y + \int_\Omega V^*(\widetilde{x},y)[\Delta u(y)]dy \\ &= \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x},y)z(y)ds_y - \int_\Gamma V^*(\widetilde{x},y)t(y)ds_y - \int_\Omega V^*(\widetilde{x},y)f(y)dy\,. \end{aligned}$$

Therefore, we now obtain from (3.4) the modified representation formula for $\widetilde{x} \in \Omega$,

$$\begin{aligned} p(\widetilde{x}) \;&=\; \int_\Gamma U^*(\widetilde{x},y)q(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x},y)z(y)ds_y - \int_\Gamma V^*(\widetilde{x},y)t(y)ds_y \\ &\qquad - \int_\Omega U^*(\widetilde{x},y)\overline{u}(y)dy - \int_\Omega V^*(\widetilde{x},y)f(y)dy, \end{aligned} \tag{3.8}$$

where the volume potentials involve given data only.

The representation formula (3.8) results, when taking the limit $\Omega \ni \widetilde{x} \to x \in \Gamma$, in the boundary integral equation for $x \in \Gamma$,

$$\begin{aligned} 0 = p(x) \;&=\; \int_\Gamma U^*(x,y)q(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y)z(y)ds_y - \int_\Gamma V^*(x,y)t(y)ds_y \\ &\qquad - \int_\Omega U^*(x,y)\overline{u}(y)dy - \int_\Omega V^*(x,y)f(y)dy, \end{aligned}$$

which can be written as

$$(Vq)(x) \;=\; (V_1 t)(x) - (K_1 z)(x) + (N_0 \overline{u})(x) + (M_0 f)(x) \quad \text{for } x \in \Gamma. \tag{3.9}$$

Note that

$$(V_1 t)(x) \;=\; \int_\Gamma V^*(x,y) t(y) ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Laplace single layer integral operator $V_1 : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$, see, for example, [17, Theorem 5.7.3]. Moreover,

$$(K_1 z)(x) \;=\; \int_\Gamma \frac{\partial}{\partial n_y} V^*(x,y) z(y) ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Laplace double layer potential $K_1 : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$. In addition, we have introduced a second Newton potential, which is related to the fundamental solution of the Bi–Laplace operator,

$$(M_0 f)(x) \;=\; \int_\Omega V^*(x,y) f(y) dy \quad \text{for } x \in \Gamma.$$

With (3.3), we conclude from (3.9) the boundary integral equation

$$Vq = V_1 V^{-1}(\frac{1}{2}I + K)z - K_1 z + N_0 \overline{u} + M_0 f - V_1 V^{-1} N_0 f,$$

and therefore

$$q = V^{-1} V_1 V^{-1}(\frac{1}{2}I + K)z - V^{-1} K_1 z + V^{-1} N_0 \overline{u} + V^{-1} M_0 f - V^{-1} V_1 V^{-1} N_0 f\,. \tag{3.10}$$

By replacing $\tau = -q$ in (2.10) we therefore obtain a boundary integral representation of the operator T_ϱ as defined in (2.7),

$$T_\varrho \;:=\; \varrho D + V^{-1} K_1 - V^{-1} V_1 V^{-1}(\frac{1}{2}I + K), \tag{3.11}$$

and a related representation for g as defined in (2.8),

$$g \;:=\; V^{-1} N_0 \overline{u} + V^{-1} M_0 f - V^{-1} V_1 V^{-1} N_0 f. \tag{3.12}$$

To investigate the unique solvability of the variational inequality (2.9) based on the boundary integral representations (3.11) and (3.12), we first will recall some mapping properties of boundary integral operators which are related to the Bi–Laplace partial differential equation, see also [17, 18].

4 Bi–Laplace boundary integral equations and properties of T_ϱ

In this section, we consider a representation formula and related boundary integral equations for the Bi–Laplace equation

$$\Delta^2 u(x) \;=\; 0 \quad \text{for } x \in \Omega, \tag{4.1}$$

which can be written as a system,

$$\Delta u_\Delta(x) \;=\; 0, \quad \Delta u(x) \;=\; u_\Delta(x) \quad \text{for } x \in \Omega. \tag{4.2}$$

As for the Laplace equation we can first write the boundary integral equations

$$u_\Delta(x) \;=\; (V t_\Delta)(x) + \frac{1}{2} u_\Delta(x) - (K u_\Delta)(x) \quad \text{for } x \in \Gamma \tag{4.3}$$

and

$$t_\Delta(x) \;=\; \frac{1}{2} t_\Delta(x) + (K' t_\Delta)(x) + (D u_\Delta)(x) \quad \text{for } x \in \Gamma, \tag{4.4}$$

where

$$(K' t_\Delta)(x) \;=\; \int_\Gamma \frac{\partial}{\partial n_x} U^*(x,y) t_\Delta(y) ds_y \quad \text{for } x \in \Gamma$$

is the adjoint Laplace double layer integral operator $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. Note that

$$u_\Delta = \Delta u \quad \text{and} \quad t_\Delta = \frac{\partial}{\partial n} u_\Delta = n \cdot \nabla u_\Delta = n \cdot \nabla \Delta u$$

are the associated Cauchy data on Γ.

To obtain a representation formula for the solution u of the Bi–Laplace equation (4.1), we first consider the related Green's first formula

$$\int_\Omega \Delta u(y) \Delta v(y) dy = \int_\Gamma \frac{\partial}{\partial n_y} u(y) \Delta v(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} \Delta v(y) u(y) ds_y + \int_\Omega [\Delta^2 v(y)] u(y) dy, \tag{4.5}$$

and in the sequel Green's second formula,

$$\begin{aligned} &\int_\Gamma \frac{\partial}{\partial n_y} u(y) \Delta v(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} \Delta v(y) u(y) ds_y + \int_\Omega [\Delta^2 v(y)] u(y) dy \\ &\quad = \int_\Gamma \frac{\partial}{\partial n_y} v(y) \Delta u(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} \Delta u(y) v(y) ds_y + \int_\Omega [\Delta^2 u(y)] v(y) dy \,. \end{aligned}$$

When choosing $v(y) = V^*(\widetilde{x}, y)$ for $\widetilde{x} \in \Omega$, i.e., the fundamental solution (3.6) of the Bi–Laplace operator, the solution of the Bi–Laplace partial differential equation (4.1) is given by the representation formula for $\widetilde{x} \in \Omega$ by

$$\begin{aligned} u(\widetilde{x}) \;=\; &\int_\Gamma \frac{\partial}{\partial n_y} u(y) \Delta_y V^*(\widetilde{x}, y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} \Delta_y V^*(\widetilde{x}, y) u(y) ds_y \\ &- \int_\Gamma \frac{\partial}{\partial n_y} V^*(\widetilde{x}, y) \Delta u(y) ds_y + \int_\Gamma \frac{\partial}{\partial n_y} \Delta u(y) V^*(\widetilde{x}, y) ds_y \,. \end{aligned}$$

By using (3.7), this can be written as

$$\begin{aligned} u(\widetilde{x}) \;=\; & \int_\Gamma U^*(\widetilde{x},y)t(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y}U^*(\widetilde{x},y)u(y)ds_y \\ & - \int_\Gamma \frac{\partial}{\partial n_y}V^*(\widetilde{x},y)u_\Delta(y)ds_y + \int_\Gamma V^*(\widetilde{x},y)t_\Delta(y)ds_y\,. \end{aligned} \tag{4.6}$$

Hence we obtain the boundary integral equation

$$u(x) \;=\; (Vt)(x) + \frac{1}{2}u(x) - (Ku)(x) - (K_1u_\Delta)(x) + (V_1t_\Delta)(x) \quad \text{for } x\in\Gamma. \tag{4.7}$$

Moreover, when taking the normal derivative of the representation formula (4.6), this gives another boundary integral equation for $x \in \Gamma$,

$$t(x) \;=\; \frac{1}{2}t(x) + (K't)(x) + (Du)(x) + (D_1u_\Delta)(x) + (K_1't_\Delta)(x), \tag{4.8}$$

where

$$(K_1't_\Delta)(x) \;=\; \int_\Gamma \frac{\partial}{\partial n_x}V^*(x,y)t_\Delta(y)ds_y \quad \text{for } x\in\Gamma$$

is the adjoint Bi–Laplace double layer integral operator $K_1' : H^{-3/2}(\Gamma) \to H^{1/2}(\Gamma)$, and

$$(D_1u_\Delta)(x) \;=\; -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}V^*(x,y)u_\Delta(y)ds_y \quad \text{for } x\in\Gamma$$

with $D_1 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

The boundary integral equations (4.3), (4.4), (4.7), and (4.8) can now be written as a system, including the so–called Calderon projection $\mathcal{C}$,

$$\begin{pmatrix} u \\ t \\ u_\Delta \\ t_\Delta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ & & \frac{1}{2}I - K & V \\ & & D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \\ u_\Delta \\ t_\Delta \end{pmatrix}. \tag{4.9}$$

Lemma 4.1 *The Calderon projection* $\mathcal{C}$ *as defined in* (4.9) *is a projection, i.e.,* $\mathcal{C}^2 = \mathcal{C}$.

Proof. The proof follows as in the case of the Laplace equation [23, 27], for the Bi–Laplace equation see also [18]. ■

From the projection property as stated in Lemma 4.1 we obtain some well–known relations of all boundary integral operators which were introduced for both the Laplace and the Bi–Laplace equation.

Lemma 4.2 *For the boundary integral operators as introduced above there hold the relations*

$$KV = VK', \quad DK = K'D, \quad VD = \frac{1}{4}I - K^2, \quad DV = \frac{1}{4}I - K'^2 \tag{4.10}$$

and

$$\begin{aligned} K_1V - VK_1' &= V_1K' - KV_1, &(4.11)\\ K_1'D - DK_1 &= D_1K - K'D_1, &(4.12)\\ VD_1 + V_1D + KK_1 + K_1K &= 0, &(4.13)\\ DV_1 + D_1V + K'K_1' + K_1'K' &= 0. &(4.14) \end{aligned}$$

Proof. The relations of (4.10) for the Laplace operator are well–known, see, e.g., [27], for the Bi–Laplace operator see also [18]. ∎

To prove the ellipticity of the boundary integral operator T_ϱ as defined in (3.11), we use the following result:

Lemma 4.3 *For any $t \in H^{-1/2}(\Gamma)$ there holds the equality*

$$\|\widetilde{V}t\|^2_{L_2(\Omega)} = \langle K_1Vt, t\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')t, t\rangle_\Gamma \tag{4.15}$$

where

$$(\widetilde{V}t)(x) = \int_\Gamma U^*(x,y)t(y)ds_y \quad \textit{for } x \in \Omega\,.$$

Proof. For $x \in \Omega$ and $t \in H^{-1/2}(\Gamma)$ we define the Bi–Laplace single layer potential

$$u_t(x) = (\widetilde{V}_1t)(x) = \int_\Gamma V^*(x,y)t(y)ds_y,$$

which is a solution of the Bi–Laplace differential equation (4.1). Then, the related Cauchy data are given by

$$u_t(x) = (V_1t)(x), \quad \frac{\partial}{\partial n_x}u_t(x) = (K_1't)(x) \quad \text{for } x \in \Gamma.$$

On the other hand, for $x \in \Omega$

$$w_t(x) = \Delta_x u_t(x) = \Delta_x \int_\Gamma V^*(x,y)t(y)ds_y = \int_\Gamma U^*(x,y)t(y)ds_y = (\widetilde{V}t)(x)$$

is a solution of the Laplace equation. Hence, the related Cauchy data are given by

$$w_t(x) = (Vt)(x), \quad \frac{\partial}{\partial n_x}w_t(x) = \frac{1}{2}t(x) + (K't)(x) \quad \text{for } x \in \Gamma.$$

Now, for $u = v = u_t$, Green's first formula (4.5) reads

$$\int_\Omega [\Delta u_t(x)]^2 dx = \int_\Gamma \frac{\partial}{\partial n_x} u_t(x) \Delta u_t(x) ds_x - \int_\Gamma \frac{\partial}{\partial n_x} \Delta u_t(x) u_t(x) ds_x,$$

and therefore we conclude

$$\begin{aligned}
\int_\Omega [w_t(x)]^2 dx &= \int_\Gamma \frac{\partial}{\partial n_x} u_t(x) w_t(x) ds_x - \int_\Gamma \frac{\partial}{\partial n_x} w_t(x) u_t(x) ds_x \\
&= \int_\Gamma (K_1' t)(x)(Vt)(x) ds_x - \int_\Gamma [\frac{1}{2} t(x) + (K't)(x)](V_1 t)(x) ds_x \\
&= \langle K_1' t, Vt \rangle_\Gamma - \langle \frac{1}{2} t + K't, V_1 t \rangle_\Gamma \\
&= \langle t, K_1 V t \rangle_\Gamma - \langle V_1 (\frac{1}{2} I + K') t, t \rangle_\Gamma .
\end{aligned}$$

The assertion now follows with $w_t = \widetilde{V} t$. ■

Now we are able to state the mapping properties of the boundary integral operator T_ϱ as defined in (3.11), see also the properties of T_ϱ as introduced in (2.7).

Theorem 4.4 *The composed boundary integral operator*

$$T_\varrho := \varrho D + V^{-1} K_1 - V^{-1} V_1 V^{-1} (\frac{1}{2} I + K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is self–adjoint, bounded and $H^{1/2}(\Gamma)$–elliptic, i.e.,

$$\langle T_\varrho z, z \rangle_\Gamma \geq c_1^{T_\varrho} \|z\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. The mapping properties of $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ follow from the boundedness of all used boundary integral operators [23, 26, 27]. In addition, we use the compact embedding of $H^{3/2}(\Gamma)$ in $H^{1/2}(\Gamma)$.

Next we will show the self–adjointness of T_ϱ. For $z, w \in H^{1/2}(\Gamma)$ we have

$$\begin{aligned}
\langle T_\varrho z, w \rangle_\Gamma &= \langle \varrho D z, w \rangle_\Gamma + \langle V^{-1} K_1 z, w \rangle_\Gamma - \frac{1}{2} \langle V^{-1} V_1 V^{-1} z, w \rangle_\Gamma - \langle V^{-1} V_1 V^{-1} K z, w \rangle_\Gamma \\
&= \langle z, \varrho D w \rangle_\Gamma + \langle z, K_1' V^{-1} w \rangle_\Gamma - \frac{1}{2} \langle z, V^{-1} V_1 V^{-1} w \rangle_\Gamma - \langle z, K' V^{-1} V_1 V^{-1} w \rangle_\Gamma \\
&= \langle z, \varrho D w \rangle_\Gamma - \frac{1}{2} \langle z, V^{-1} V_1 V^{-1} w \rangle_\Gamma + \langle z, [K_1' V^{-1} - K' V^{-1} V_1 V^{-1}] w \rangle_\Gamma .
\end{aligned}$$

Now, we conclude, by using the relations (4.10) and (4.11),

$$\begin{aligned}
K_1' V^{-1} - K' V^{-1} V_1 V^{-1} &= K_1' V^{-1} - V^{-1} K V_1 V^{-1} = V^{-1} [V K_1' - K V_1] V^{-1} \\
&= V^{-1} [K_1 V - V_1 K'] V^{-1} = V^{-1} K_1 - V^{-1} V_1 K' V^{-1} \\
&= V^{-1} K_1 - V^{-1} V_1 V^{-1} K \, .
\end{aligned}$$

Hence we have

$$\begin{aligned}\langle T_\varrho z, w\rangle_\Gamma &= \langle z, \varrho D w\rangle_\Gamma - \frac{1}{2}\langle z, V^{-1}V_1V^{-1}w\rangle_\Gamma + \langle z, [V^{-1}K_1 - V^{-1}V_1V^{-1}K]w\rangle_\Gamma \\ &= \langle z, [\varrho D + V^{-1}K_1 - V^{-1}V_1V^{-1}(\frac{1}{2}I + K)]w\rangle_\Gamma \;=\; \langle z, T_\varrho w\rangle_\Gamma,\end{aligned}$$

i.e., T_ϱ is self–adjoint.

Moreover, for $z \in H^{1/2}(\Gamma)$ we have, by using (4.10), $t = V^{-1}z$, and by Lemma 4.3,

$$\begin{aligned}\langle T_\varrho z, z\rangle_\Gamma &= \varrho\langle Dz, z\rangle_\Gamma + \langle V^{-1}K_1 z, z\rangle_\Gamma - \langle V^{-1}V_1V^{-1}(\frac{1}{2}I + K)z, z\rangle_\Gamma \\ &= \varrho\langle Dz, z\rangle_\Gamma + \langle K_1 V V^{-1}z, V^{-1}z\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')V^{-1}z, V^{-1}z\rangle_\Gamma \\ &= \varrho\langle Dz, z\rangle_\Gamma + \langle K_1 V t, t\rangle_\Gamma - \langle V_1(\frac{1}{2}I + K')t, t\rangle_\Gamma \\ &= \varrho\langle Dz, z\rangle_\Gamma + \|\widetilde{V}t\|^2_{L_2(\Omega)}\,.\end{aligned}$$

Since the last expression defines an equivalent norm in $H^{1/2}(\Gamma)$, the $H^{1/2}(\Gamma)$–ellipticity of T_ϱ follows. ∎

5 A non–symmetric boundary element method

Let

$$S^1_H(\Gamma) = \text{span}\{\varphi_i\}_{i=1}^M \subset H^{1/2}(\Gamma) \tag{5.1}$$

be a boundary element space of, e.g., piecewise linear and continuous basis functions φ_i, which are defined with respect to a globally quasi–uniform and shape regular boundary mesh Γ_H of mesh size H. Define the discrete convex set

$$\mathcal{U}_H \;:=\; \left\{ w_H \in S^1_H(\Gamma) \;:\; g_a(x_i) \le w_H(x_i) \le g_b(x_i) \quad \text{for all nodes } x_i \in \Gamma \right\}.$$

Then the Galerkin discretisation of the variational inequality (2.9) is to find $z_H \in \mathcal{U}_H$ such that

$$\langle T_\varrho z_H, w_H - z_H\rangle_\Gamma \;\ge\; \langle g, w_H - z_H\rangle_\Gamma \quad \text{for all } w_H \in \mathcal{U}_H. \tag{5.2}$$

Theorem 5.1 *Let $z \in \mathcal{U}$ and $z_H \in \mathcal{U}_H$ be the unique solutions of the variational inequalities* (2.9) *and* (5.2)*, respectively. If we assume $z, g_a, g_b \in H^s(\Gamma)$ for some $s \in [\frac{1}{2}, 2]$, then there hold the error estimates*

$$\|z - z_H\|_{H^{1/2}(\Gamma)} \;\le\; c\, H^{s-\frac{1}{2}}\, \|z\|_{H^s(\Gamma)} \tag{5.3}$$

and

$$\|z - z_H\|_{L_2(\Gamma)} \;\le\; c\, H^s\, \|z\|_{H^s(\Gamma)}\,. \tag{5.4}$$

Proof. The error estimate (5.3) in the energy norm follows from the general abstract theory as presented, e.g., in [4, 8], see also [10]. The error estimate (5.4) follows from the Aubin–Nitsche trick for variational inequalities, see [24] for the case $\mathcal{U}_H \subset \mathcal{U}$, and [28] for the more general case $\mathcal{U}_H \not\subset \mathcal{U}$. ∎

Although the error estimates (5.3) and (5.4) seem to be optimal, the operator T_ϱ as considered in the variational inequality (5.2) does not allow a practical implementation, since this would require the discretisation of the operator T_ϱ as defined in (3.11), which is not possible in general. Hence, instead of (5.2) we need to consider a perturbed variational inequality to find $\widetilde{z}_H \in \mathcal{U}_H$ such that

$$\langle \widetilde{T}_\varrho \widetilde{z}_H, w_H - \widetilde{z}_H \rangle_\Gamma \;\geq\; \langle \widetilde{g}, w_H - \widetilde{z}_H \rangle_\Gamma \quad \text{for all } w_H \in \mathcal{U}_H, \tag{5.5}$$

where $\widetilde{T}_\varrho$ and $\widetilde{g}$ are appropriate approximations of T_ϱ and g, respectively. The following theorem, see, e.g., [25], presents an abstract consistency result, which will later be used to analyse the boundary element approximation of both the primal and adjoint boundary value problems.

Theorem 5.2 *Let $\widetilde{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ be a bounded and $S_H^1(\Gamma)$–elliptic approximation of T_ϱ satisfying*

$$\langle \widetilde{T}_\varrho z_H, z_H \rangle_\Gamma \;\geq\; c_1^{\widetilde{T}_\varrho} \, \|z_H\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } z_H \in S_H^1(\Gamma)$$

and

$$\|\widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_2^{\widetilde{T}_\varrho} \, \|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Let $\widetilde{g} \in H^{-1/2}(\Gamma)$ be some approximation of g. For the unique solution $\widetilde{z}_H \in \mathcal{U}_H$ of the perturbed variational inequality (5.5) *there holds the error estimate*

$$\|z - \widetilde{z}_H\|_{H^{1/2}(\Gamma)} \leq c_1 \, \|z - z_H\|_{H^{1/2}(\Gamma)} + c_2 \, \left[\|(T_\varrho - \widetilde{T}_\varrho) z\|_{H^{-1/2}(\Gamma)} + \|g - \widetilde{g}\|_{H^{-1/2}(\Gamma)} \right], \tag{5.6}$$

where $z_H \in \mathcal{U}_H$ is the unique solution of the discrete variational inequality (5.2).

5.1 Boundary element approximation of T_ϱ

For an arbitrary but fixed $z \in H^{1/2}(\Gamma)$, the application of $T_\varrho z$ reads

$$T_\varrho z = \varrho D z + V^{-1} K_1 z - V^{-1} V_1 V^{-1} (\frac{1}{2} I + K) z = \varrho D z - q_z,$$

where $q_z \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(V q_z)(x) = (V_1 t_z)(x) - (K_1 z)(x) \quad \text{for } x \in \Gamma,$$

and $t_z \in H^{-1/2}(\Gamma)$ solves

$$(V t_z)(x) \;=\; (\frac{1}{2} I + K) z(x) \quad \text{for } x \in \Gamma.$$

For a Galerkin approximation of the above boundary integral equations, let

$$S_h^0(\Gamma) = \operatorname{span}\{\psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma)$$

be another boundary element space of, e.g., piecewise constant basis functions ψ_k, which are defined with respect to a second globally quasi–uniform and shape regular boundary element mesh of mesh size h. Now, $\widetilde{q}_{z,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle V\widetilde{q}_{z,h}, \tau_h\rangle_\Gamma = \langle V_1 t_{z,h} - K_1 z, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma),$$

and $t_{z,h} \in S_h^0(\Gamma)$ solves

$$\langle V t_{z,h}, \tau_h\rangle_\Gamma = \langle(\frac{1}{2}I + K)z, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

Hence we can define an approximation $\widetilde{T}_\varrho$ of the operator T_ϱ by

$$\widetilde{T}_\varrho z \;:=\; \varrho D z - \widetilde{q}_{z,h}\,. \tag{5.7}$$

Lemma 5.3 *The approximate operator* $\widetilde{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *as defined in* (5.7) *is bounded, i.e.,*

$$\|\widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \le c_2^{\widetilde{T}_\varrho}\, \|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Proof. The assertion is a direct consequence of the mapping properties of all boundary integral operators involved, we skip the details. ■

Lemma 5.4 *Let* $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *be given by* (3.11), *and let* $\widetilde{T}_\varrho$ *be defined as in* (5.7). *Then there holds the error estimate*

$$\|T_\varrho z - \widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\le\; c_1 \inf_{\tau_h \in S_h^0(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)} + c_2\,\|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)}. \tag{5.8}$$

Proof. For an arbitrary chosen but fixed $z \in H^{1/2}(\Gamma)$ we have, by definition,

$$T_\varrho z = \varrho D z - q_z, \quad q_z = V^{-1}[V_1 t_z - K_1 z], \quad t_z = V^{-1}(\frac{1}{2}I + K)z.$$

By using (5.7), we also have

$$\widetilde{T}_\varrho z \;=\; \varrho D z - \widetilde{q}_{z,h},$$

and therefore,

$$T_\varrho z - \widetilde{T}_\varrho z \;=\; \widetilde{q}_{z,h} - q_z\,.$$

Let us further define $q_{z,h} \in S_h^0(\Gamma)$ as the unique solution of the variational problem

$$\langle V q_{z,h}, \tau_h\rangle_\Gamma \;=\; \langle V_1 t_z - K_1 z, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.9}$$

Then, the perturbed Galerkin orthogonality

$$\langle V(q_{z,h} - \widetilde{q}_{z,h}), \tau_h \rangle_\Gamma \;=\; \langle V_1(t_z - t_{z,h}), \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma)$$

follows. From this we further conclude

$$\|q_{z,h} - \widetilde{q}_{z,h}\|_{H^{-1/2}(\Gamma)} \;\leq\; c\, \|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)}.$$

The assertion now follows from the triangle inequality, and by applying Cea's lemma. ■

By using the approximation property of the trial space $S_h^0(\Gamma)$ and the Aubin–Nitsche trick, we conclude an error estimate from (5.8) when assuming some regularity of q_z and t_z, respectively.

Corollary 5.5 *Assume $q_z, t_z \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Then there holds the error estimate*

$$\|T_\varrho z - \widetilde{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1\, h^{s+\frac{1}{2}}\, \|q_z\|_{H^s_{\text{pw}}(\Gamma)} + c_2\, h^{s+\frac{3}{2}}\, \|t_z\|_{H^s_{\text{pw}}(\Gamma)}. \tag{5.10}$$

5.2 Boundary element approximation of g

As in (5.7), we may also define a boundary element approximation of the right hand side g as defined in (3.12),

$$g = V^{-1}N_0\overline{u} + V^{-1}M_0 f - V^{-1}V_1V^{-1}N_0 f.$$

In particular, $g \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle Vg, \tau \rangle_\Gamma \;=\; \langle N_0\overline{u} + M_0 f, \tau \rangle_\Gamma - \langle V_1 t_f, \tau \rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma),$$

where $t_f = V^{-1}N_0 f \in H^{-1/2}(\Gamma)$ solves the variational problem

$$\langle V t_f, \tau \rangle_\Gamma \;=\; \langle N_0 f, \tau \rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Hence we can define a boundary element approximation $\widetilde{g}_h \in S_h^0(\Gamma)$ as the unique solution of the Galerkin variational problem

$$\langle V\widetilde{g}_h, \tau_h \rangle_\Gamma \;=\; \langle N_0\overline{u} + M_0 f, \tau_h \rangle_\Gamma - \langle V_1 t_{f,h}, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma), \tag{5.11}$$

where $t_{f,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin problem

$$\langle V t_{f,h}, \tau_h \rangle_\Gamma \;=\; \langle N_0 f, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.12}$$

Lemma 5.6 *Let g be the right hand side as defined in* (3.12)*, and let $\widetilde{g}_h$ be the boundary element approximation as defined in* (5.11)*. Then there holds the error estimate*

$$\|g - \widetilde{g}_h\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1 \inf_{\tau_h \in S_h^0(\Gamma)} \|g - \tau_h\|_{H^{-1/2}(\Gamma)} + c_2\, \|t_f - t_{f,h}\|_{H^{-3/2}(\Gamma)}. \tag{5.13}$$

Proof. The assertion follows as in the proof of Lemma 5.4, we skip the details. ∎

By using the approximation property of the trial space $S_h^0(\Gamma)$ and the Aubin–Nitsche trick, we conclude an error estimate from (5.13) when assuming some regularity of g and t_f, respectively.

Corollary 5.7 *Assume $g, t_f \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Then there holds the error estimate*

$$\|g - \widetilde{g}_h\|_{H^{-1/2}(\Gamma)} \le c_1\, h^{s+\frac{1}{2}}\, \|g\|_{H^s_{\text{pw}}(\Gamma)} + c_2\, h^{s+\frac{3}{2}}\, \|t_f\|_{H^s_{\text{pw}}(\Gamma)}. \tag{5.14}$$

5.3 Approximate variational inequality

We consider the variational inequality (2.10) with $\tau = -q$ to find $z \in \mathcal{U}$ such that

$$\langle \varrho D z - q, w - z\rangle_\Gamma \ge 0 \quad \text{for all } w \in \mathcal{U}, \tag{5.15}$$

where $q \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(Vq)(x) = (V_1 t)(x) - (K_1 z)(x) + (N_0 \overline{u})(x) + (M_0 f)(x) \quad \text{for } x \in \Gamma, \tag{5.16}$$

and $t \in H^{-1/2}(\Gamma)$ is the unique solution of

$$(Vt)(x) = (\frac{1}{2} I + K) z(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma. \tag{5.17}$$

The Galerkin boundary element approximation of the variational inequality (5.15), and therefore the boundary element discretisation of the perturbed variational inequality (5.5), is to find $\widetilde{z}_H \in \mathcal{U}_H$ such that

$$\langle \varrho D \widetilde{z}_H - q_h, w_H - \widetilde{z}_H\rangle_\Gamma \ge 0 \quad \text{for all } w_H \in \mathcal{U}_H, \tag{5.18}$$

where $q_h \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle V q_h, \tau_h\rangle_\Gamma = \langle V_1 t_h - K_1 \widetilde{z}_H + N_0 \overline{u} + M_0 f, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma), \tag{5.19}$$

and $t_h \in S_h^0(\Gamma)$ solves

$$\langle V t_h, \tau_h\rangle_\Gamma = \langle (\frac{1}{2} I + K)\widetilde{z}_H - N_0 f, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.20}$$

The Galerkin formulation (5.19) is equivalent to the linear system

$$V_h \underline{q} = V_{1,h} \underline{t} - K_{1,h} \underline{\widetilde{z}} + \underline{f}_1, \tag{5.21}$$

and (5.20) is equivalent to

$$V_h \underline{t} = (\frac{1}{2} M_h + K_h) \underline{\widetilde{z}} - \underline{f}_2, \tag{5.22}$$

where

$$\begin{array}{rclrcl} V_h[\ell,k] &=& \langle V\psi_k,\psi_\ell\rangle_\Gamma, & K_h[\ell,i] &=& \langle K\varphi_i,\psi_\ell\rangle_\Gamma, \\ V_{1,h}[\ell,k] &=& \langle V_1\psi_k,\psi_\ell\rangle_\Gamma, & K_{1,h}[\ell,i] &=& \langle K_1\varphi_i,\psi_\ell\rangle_\Gamma, \\ & & & M_h[\ell,i] &=& \langle \varphi_i,\psi_\ell\rangle_\Gamma, \end{array}$$

and

$$f_{1,\ell} = \langle N_0\overline{u} + M_0 f, \psi_\ell\rangle_\Gamma, \quad f_{2,\ell} = \langle N_0 f, \psi_\ell\rangle_\Gamma$$

for $k,\ell = 1,\ldots,N$ and $i = 1,\ldots,M$. Recall that we use piecewise linear basis functions φ_i, and piecewise constant basis functions ψ_k. Moreover, let D_H be the Galerkin matrix of the hypersingular boundary integral operator D, i.e.

$$D_H[j,i] \,=\, \langle D\varphi_i,\varphi_j\rangle_\Gamma \quad \text{for } i,j = 1,\ldots,M.$$

The matrix representation of the variational inequality (5.18) is then given by the discrete variational inequality

$$(\varrho D_H \underline{\widetilde{z}} - M_h^\top \underline{q}, \underline{w} - \underline{\widetilde{z}}) \,\geq\, 0 \quad \text{for all } \underline{w} \in \mathbb{R}^M \leftrightarrow w_H \in \mathcal{U}_H,$$

or

$$(\widetilde{T}_{\varrho,H}\underline{\widetilde{z}} - \underline{\widetilde{g}}, \underline{w} - \underline{\widetilde{z}}) \,\geq\, 0 \quad \text{for all } \underline{w} \in \mathbb{R}^M \leftrightarrow w_H \in \mathcal{U}_H, \tag{5.23}$$

where

$$\widetilde{T}_{\varrho,H} = \varrho D_H + M_h^\top V_h^{-1} K_{1,h} - M_h^\top V_h^{-1} V_{1,h} V_h^{-1}(\frac{1}{2}M_h + K_h) \tag{5.24}$$

defines a non–symmetric Galerkin boundary element approximation of the self–adjoint boundary integral operator T_ϱ as defined in (3.11). Moreover,

$$\underline{\widetilde{g}} \,=\, M_h^\top V_h^{-1}\Big[\underline{f}_1 - V_{1,h}V_h^{-1}\underline{f}_2\Big]$$

is the boundary element approximation of g as defined in (3.12).

Theorem 5.8 *The approximate Schur complement $\widetilde{T}_{\varrho,H}$ as defined in* (5.24) *is positive definite, i.e.,*

$$(\widetilde{T}_{\varrho,H}\underline{z}, \underline{z}) \,\geq\, \frac{1}{2}c_1^{T_\varrho}\,\|z_H\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } \underline{z} \in \mathbb{R}^M \leftrightarrow z_H \in S_H^1(\Gamma),$$

if $h \leq c_0 H$ is sufficiently small.

Proof. For an arbitrary chosen but fixed $\underline{z} \in \mathbb{R}^M$ let $z_H \in S_H^1(\Gamma)$ be the associated boundary element function. Then we have

$$\begin{array}{rcl} (\widetilde{T}_{\varrho,H}\underline{z}, \underline{z}) &=& \langle \widetilde{T}_\varrho z_H, z_H\rangle_\Gamma \,=\, \langle T_\varrho z_H, z_H\rangle_\Gamma - \langle (T_\varrho - \widetilde{T}_\varrho) z_H, z_H\rangle_\Gamma \\ &\geq& c_1^{T_\varrho}\,\|z_H\|^2_{H^{1/2}(\Gamma)} - \|(T_\varrho - \widetilde{T}_\varrho)z_H\|_{H^{-1/2}(\Gamma)}\|z_H\|_{H^{1/2}(\Gamma)}. \end{array}$$

Since $z_H \in S^1_H(\Gamma)$ is a continuous function, we have $z_H \in H^1(\Gamma)$. Hence we find

$$t_{z_H} = V^{-1}(\frac{1}{2}I + K)z_H \in L_2(\Gamma), \quad q_{z_H} = V^{-1}[V_1 t_{z_H} - K_1 z_H] \in L_2(\Gamma).$$

Therefore we can apply the error estimate (5.10) for $s = 0$ to obtain

$$\|T_\varrho z_H - \widetilde{T}_\varrho z_H\|_{H^{-1/2}(\Gamma)} \le c_1\, h^{1/2}\, \|q_{z_H}\|_{L_2(\Gamma)} + c_2\, h^{3/2}\, \|t_{z_H}\|_{L_2(\Gamma)} \le c_3\, h^{1/2}\, \|z_H\|_{H^1(\Gamma)}.$$

Now, by applying the inverse inequality for $S^1_H(\Gamma)$,

$$\|z_H\|_{H^1(\Gamma)} \le c_I\, H^{-1/2}\, \|z_H\|_{H^{1/2}(\Gamma)},$$

we obtain

$$\|T_\varrho z_H - \widetilde{T}_\varrho z_H\|_{H^{-1/2}(\Gamma)} \le c_3 c_I \left(\frac{h}{H}\right)^{1/2} \|z_H\|_{H^{1/2}(\Gamma)}.$$

Hence we finally obtain

$$(\widetilde{T}_{\varrho,H}\underline{z}, \underline{z}) \ge \left[c_1^{T_\varrho} - c_3 c_I \left(\frac{h}{H}\right)^{1/2}\right] \|z_H\|^2_{H^{1/2}(\Gamma)} \ge \frac{1}{2} c_1^{T_\varrho}\, \|z_H\|^2_{H^{1/2}(\Gamma)},$$

if

$$c_3 c_I \left(\frac{h}{H}\right)^{1/2} \le \frac{1}{2} c_1^{T_\varrho}$$

is satisfied. ∎

Now we are in a position to apply Theorem 5.2 to ensure unique solvability of the perturbed Galerkin variational inequality (5.5), and to derive related error estimates.

Corollary 5.9 *When combining the error estimate* (5.6) *with the approximation property of the ansatz space* $S^1_H(\Gamma)$*, and with the error estimates* (5.10) *and* (5.14)*, we finally obtain the error estimate*

$$\begin{aligned}\|z - \widetilde{z}_H\|_{H^{1/2}(\Gamma)} \le\;& c_1\, H^{s+1/2}\, |z|_{H^{1+s}(\Gamma)} + c_2\, h^{s+1/2}\, \|q_z\|_{H^s_{pw}(\Gamma)} + c_3\, h^{s+3/2}\, \|t_z\|_{H^s_{pw}(\Gamma)} \\ &+ c_4\, h^{s+1/2}\, \|g\|_{H^s_{pw}(\Gamma)} + c_5\, h^{s+3/2}\, \|t_f\|_{H^s_{pw}(\Gamma)}\end{aligned}$$

when assuming $z \in H^{1+s}(\Gamma)$*, and* $q_z, t_z, g, t_f \in H^s_{pw}(\Gamma)$ *for some* $s \in [0, 1]$*. For* $h \le c_0 H$ *we therefore obtain the error estimate*

$$\|z - \widetilde{z}_H\|_{H^{1/2}(\Gamma)} \le c(z, \overline{u}, f)\, H^{s+\frac{1}{2}}\,. \tag{5.25}$$

Moreover, we are also able to derive an error estimate in $L_2(\Gamma)$*, i.e.,*

$$\|z - \widetilde{z}_H\|_{L_2(\Gamma)} \le c(z, \overline{u}, f)\, H^{s+1}\,, \tag{5.26}$$

when applying the Aubin–Nitsche trick.

In the particular case of a non–constrained minimisation problem, instead of the discrete variational inequality (5.23) we have to solve the linear system

$$\widetilde{T}_{\varrho,H}\underline{\widetilde{z}} = \underline{\widetilde{g}},$$

which is equivalent to the system

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -(\frac{1}{2}M_h + K_h) \\ & -M_h^\top & \varrho D_H \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{q} \\ \underline{\widetilde{z}} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ -\underline{f}_2 \\ \underline{0} \end{pmatrix}. \tag{5.27}$$

Remark 5.1 *The error estimates* (5.25) *and* (5.26) *provide optimal convergence rates when approximating the control z by using piecewise linear basis functions. However, we have to assume $h \leq c_0 H$ to ensure the unique solvability of the perturbed Galerkin variational inequality* (5.5)*, where the constant c_0 is in general unknown. Moreover, the matrix $\widetilde{T}_{\varrho,H}$ as given in* (5.24) *defines a non–symmetric approximation of the self–adjoint operator T_ϱ. Hence we are interested in deriving a symmetric boundary element method which is stable without any additional constraints in the choice of the boundary element trial spaces.*

6 A symmetric boundary element method

The boundary integral formulation of the primal boundary value problem (2.2) is given by (3.2), while the adjoint boundary value problem (2.12) corresponds to the modified boundary integral equation (3.9). In what follows, we will use a second boundary integral equation of the adjoint boundary value problem to obtain an alternative representation for q and therefore of the adjoint operator $\mathcal{S}^*$. In particular, when computing the normal derivative of the representation formula, (3.8), this gives

$$q(x) = (\frac{1}{2}I + K')q(x) - (D_1 z)(x) - (K_1' t)(x) - (N_1 \overline{u})(x) - (M_1 f)(x) \quad \text{for } x \in \Gamma, \tag{6.1}$$

where

$$(N_1\overline{u})(x) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} \int_\Omega U^*(\widetilde{x}, y)\overline{u}(y) dy \quad \text{for } x \in \Gamma$$

and

$$(M_1 f)(x) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} \int_\Omega V^*(\widetilde{x}, y) f(y) dy \quad \text{for } x \in \Gamma.$$

Hence, from (2.11) we obtain

$$\tau = \mathcal{S}^*(u - \overline{u}) = -q = -(\frac{1}{2}I + K')q + D_1 z + K_1' t + N_1 \overline{u} + M_1 f,$$

and, by using (3.3) and (3.10), we conclude the alternative representations

$$T_\varrho = \varrho D + D_1 - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K) + K_1'V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 \quad (6.2)$$

and

$$g = K_1'V^{-1}N_0f - N_1\overline{u} - M_1f + (\frac{1}{2}I + K')V^{-1}\left[N_0\overline{u} + M_0f - V_1V^{-1}N_0f\right] . \quad (6.3)$$

Theorem 6.1 *The boundary integral operator* $T_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *as defined in* (6.2) *is self–adjoint, bounded, and* $H^{1/2}(\Gamma)$*–elliptic.*

Proof. While the self–adjointness of T_ϱ in the symmetric representation (6.2) is obvious, the boundedness and ellipticity estimates follow as in the proof of Theorem 4.4. In particular, the operators T_ϱ in the symmetric representation (6.2) and in the non–symmetric representation (3.11) coincide. Indeed, by using (4.10) and (4.11) we obtain

$$\begin{aligned}
T_\varrho &= \varrho D + D_1 - (\frac{1}{2}I + K')V^{-1}V_1V^{-1}(\frac{1}{2}I + K) + K_1'V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 \\
&= \varrho D + D_1 + \left[K_1' - (\frac{1}{2}I + K')V^{-1}V_1\right]V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 \\
&= \varrho D + D_1 + V^{-1}\left[VK_1' - KV_1 - \frac{1}{2}V_1\right]V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 \\
&= \varrho D + D_1 + V^{-1}\left[K_1V - V_1K' - \frac{1}{2}V_1\right]V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1 \\
&= \varrho D + D_1 + V^{-1}K_1(\frac{1}{2}I + K) - V^{-1}V_1(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K')V^{-1}K_1.
\end{aligned}$$

Due to the representation of the Laplace Steklov–Poincaré operator, see, e.g., [27],

$$S = V^{-1}(\frac{1}{2}I + K) = D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K),$$

we further conclude

$$(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) = V^{-1}(\frac{1}{2}I + K) - D.$$

Therefore, by using (4.10) and (4.13) we have

$$\begin{aligned}
T_\varrho &= \varrho D + D_1 + V^{-1}K_1(\frac{1}{2}I + K) - V^{-1}V_1\left[V^{-1}(\frac{1}{2}I + K) - D\right] + V^{-1}(\frac{1}{2}I + K)K_1 \\
&= \varrho D + V^{-1}\left[VD_1 + V_1D + K_1(\frac{1}{2}I + K) - V_1V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K)K_1\right] \\
&= \varrho D + V^{-1}\left[-KK_1 - K_1K + K_1(\frac{1}{2}I + K) - V_1V^{-1}(\frac{1}{2}I + K) + (\frac{1}{2}I + K)K_1\right] \\
&= \varrho D + V^{-1}\left[K_1 - V_1V^{-1}(\frac{1}{2}I + K)\right],
\end{aligned}$$

and we finally obtain the non–symmetric representation (3.11). Therefore, the ellipticity of T_ϱ follows as in Theorem 4.4. ■

6.1 Symmetric boundary element approximation of T_ϱ

For an arbitrary but fixed given $z \in H^{1/2}(\Gamma)$, the application of $T_\varrho z$ reads, by using the symmetric representation (6.2),

$$T_\varrho z \;=\; \varrho D z + D_1 z + K_1' t_z - (\frac{1}{2} I + K') q_z,$$

where $q_z \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(V q_z)(x) \;=\; (V_1 t_z)(x) - (K_1 z)(x) \quad \text{for } x \in \Gamma,$$

and $t_z \in H^{-1/2}(\Gamma)$ solves

$$(V t_z)(x) \;=\; (\frac{1}{2} I + K) z(x) \quad \text{for } x \in \Gamma.$$

As for the non–symmetric representation of T_ϱ we can define approximate Galerkin solutions $t_{z,h}, \widetilde{q}_{z,h} \in S_h^0(\Gamma)$, and therefore we can introduce the approximation

$$\widehat{T}_\varrho z \;:=\; \varrho D z + D_1 z + K_1' t_{z,h} - (\frac{1}{2} I + K') \widetilde{q}_{z,h}. \tag{6.4}$$

Lemma 6.2 *The approximate operator $\widehat{T}_\varrho : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as defined in* (6.4) *is bounded, i.e.,*

$$\|\widehat{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_2^{\widehat{T}_\varrho} \, \|z\|_{H^{1/2}(\Gamma)} \quad \textit{for all } z \in H^{1/2}(\Gamma).$$

Moreover, there holds the error estimate

$$\|T_\varrho z - \widehat{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \leq c_1 \inf_{\tau_h \in S_h^0(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)} + c_2 \, \|t_z - t_{z,h}\|_{H^{-3/2}(\Gamma)}. \tag{6.5}$$

Proof. The proof follows as for the boundary element approximation of the non–symmetric formulation, see Lemma 5.3 and Lemma 5.4. ■

By using the approximation property of the trial space $S_h^0(\Gamma)$ and the Aubin–Nitsche trick, we then conclude an error estimate from (6.5) when assuming some regularity of q_z and t_z, respectively.

Corollary 6.3 *Assume $q_z, t_z \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$. Then there holds the error estimate*

$$\|T_\varrho z - \widehat{T}_\varrho z\|_{H^{-1/2}(\Gamma)} \;\leq\; c_1 \, h^{s+\frac{1}{2}} \, \|q_z\|_{H^s_{\text{pw}}(\Gamma)} + c_2 \, h^{s+\frac{3}{2}} \, \|t_z\|_{H^s_{\text{pw}}(\Gamma)}. \tag{6.6}$$

6.2 Boundary element approximation of g

As in the approximation (6.4), we can define a boundary element approximation of g as defined in (6.3),

$$g \,=\, K_1' t_f - N_1 \overline{u} - M_1 f + (\frac{1}{2} I + K') q_f,$$

where $q_f \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(V q_f)(x) \,=\, (N_0 \overline{u})(x) + (M_0 f)(x) - (V_1 t_f)(x) \quad \text{for } x \in \Gamma,$$

and $t_f \in H^{-1/2}(\Gamma)$ solves

$$(V t_f)(x) \,=\, (N_0 f)(x) \quad \text{for } x \in \Gamma.$$

Hence we can define approximate Galerkin solutions $\widehat{q}_{f,h}, t_{f,h} \in S_h^0(\Gamma)$, and therefore, we can introduce the approximation

$$\widehat{g} \,:=\, K_1' t_{f,h} - N_1 \overline{u} - M_1 f + (\frac{1}{2} I + K') \widehat{q}_{f,h}. \tag{6.7}$$

As in (5.14) we conclude the error estimate

$$\| g - \widehat{g} \|_{H^{-1/2}(\Gamma)} \,\leq\, c_1 \, h^{s+\frac{1}{2}} \, \| q_f \|_{H^s_{\text{pw}}(\Gamma)} + c_2 \, h^{s+\frac{3}{2}} \, \| t_f \|_{H^s_{\text{pw}}(\Gamma)} \tag{6.8}$$

when assuming $q_f, t_f \in H^s_{\text{pw}}(\Gamma)$ for some $s \in [0,1]$.

6.3 Approximate variational inequality

The use of the symmetric approximations (6.4) and (6.7) results in the approximate variational inequality

$$(\widehat{T}_{\varrho,H} \underline{\widehat{z}} - \underline{\widehat{g}}, \underline{w} - \underline{\widehat{z}}) \,\geq\, 0 \quad \text{for all } \underline{w} \in \mathbb{R}^M \leftrightarrow w_H \in \mathcal{U}_H, \tag{6.9}$$

where

$$\begin{aligned} \widehat{T}_{\varrho,H} \;&=\; \varrho D_H + D_{1,H} - (\frac{1}{2} M_h^\top + K_h^\top) V_h^{-1} V_{1,h} V_h^{-1} (\frac{1}{2} M_h + K_h) \\ &\qquad + K_{1,h}^\top V_h^{-1} (\frac{1}{2} M_h + K_h) + (\frac{1}{2} M_h^\top + K_h^\top) V_h^{-1} K_{1,h} \end{aligned} \tag{6.10}$$

defines a symmetric Galerkin boundary element approximation of the self–adjoint operator T_ϱ, and

$$\underline{\widehat{g}} = K_{1,h}^\top V_h^{-1} \underline{f}_2 - \underline{f}_3 + (\frac{1}{2} M_h^\top + K_h^\top) V_h^{-1} \left[\underline{f}_1 - V_{1,h} V_h^{-1} \underline{f}_2 \right]$$

is the related boundary element approximation of g as defined in (6.3). Note, that in addition to those entries of the non–symmetric approximation, we use

$$D_{1,H}[j,i] = \langle D_1 \varphi_i, \varphi_j \rangle_\Gamma, \quad f_{3,j} = \langle N_1 \overline{u} + M_1 f, \varphi_j \rangle_\Gamma \quad \text{for } i,j = 1, \ldots, M.$$

Lemma 6.4 *The symmetric matrix*

$$\widehat{T}_H \;:=\; \widehat{T}_{\varrho,H} - \varrho D_H - D_{1,H} \;=\; K_{1,h}^\top V_h^{-1}(\frac{1}{2}M_h + K_h) + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}K_{1,h}$$
$$-(\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}V_{1,h}V_h^{-1}(\frac{1}{2}M_h + K_h)$$

is positive semi–definite, i.e.,

$$(\widehat{T}_H\underline{z},\underline{z}) \;\geq\; 0 \quad \textit{for all } \underline{z} \in \mathbb{R}^M.$$

Proof. We consider the generalized eigenvalue problem

$$\widehat{T}_H\underline{z} = \mu\left[\widetilde{S}_H + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h)\right]\underline{z}, \tag{6.11}$$

where the stabilised discrete Steklov–Poincaré operator

$$\widetilde{S}_H = D_H + \underline{a}\,\underline{a}^\top + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h)$$

is symmetric and positive definite. Note that the vector $\underline{a}$ is given by

$$a_i = \int_\Gamma \varphi_i(x)ds_x \quad \text{for } i = 1,\dots,M.$$

Since the eigenvalue problem (6.11) can be written as

$$\begin{pmatrix} (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1} & I \end{pmatrix}\begin{pmatrix} -V_{1,h} & K_{1,h} \\ K_{1,h}^\top & \end{pmatrix}\begin{pmatrix} V_h^{-1}(\frac{1}{2}M_h + K_h) \\ I \end{pmatrix}\underline{z}$$
$$= \mu\begin{pmatrix} (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1} & I \end{pmatrix}\begin{pmatrix} V_h & \\ & \widetilde{S}_H \end{pmatrix}\begin{pmatrix} V_h^{-1}(\frac{1}{2}M_h + K_h) \\ I \end{pmatrix}\underline{z},$$

it is sufficient to consider the generalised eigenvalue problem

$$\begin{pmatrix} -V_{1,h} & K_{1,h} \\ K_{1,h}^\top & \end{pmatrix}\begin{pmatrix} \underline{w} \\ \underline{z} \end{pmatrix} = \mu\begin{pmatrix} V_h\underline{w} \\ \widetilde{S}_H\underline{z} \end{pmatrix}, \tag{6.12}$$

where

$$\underline{w} = V_h^{-1}(\frac{1}{2}M_h + K_h)\underline{z}\,.$$

From (6.12) we conclude

$$\begin{aligned} (K_{1,h}\underline{z},\underline{w}) - (V_{1,h}\underline{w},\underline{w}) &= \mu(V_h\underline{w},\underline{w}), \\ (K_{1,h}^\top\underline{w},\underline{z}) &= \mu(\widetilde{S}_H\underline{z},\underline{z}), \end{aligned}$$

and by taking the difference we obtain

$$(V_{1,h}\underline{w},\underline{w}) = \mu[(\widetilde{S}_H\underline{z},\underline{z}) - (V_h\underline{w},\underline{w})] = \mu((D_H + \underline{a}\,\underline{a}^\top)\underline{z},\underline{z}).$$

Hence, $\mu \geq 0$ follows, which implies the assertion. ■

As a corollary of Lemma 6.4, we find the positive definiteness of the symmetric Schur complement matrix $\widehat{T}_{\varrho,H}$ as defined in (6.10).

Corollary 6.5 *The approximate Schur complement $\widehat{T}_{\varrho,H}$ as defined in* (6.10) *is positive definite, i.e.,*

$$(\widehat{T}_{\varrho,H}\underline{z},\underline{z}) \;\geq\; \varrho(D_H\underline{z},\underline{z}) + (D_{1,H}\underline{z},\underline{z}) \;=\; \langle(\varrho D + D_1)z_H, z_H\rangle_\Gamma \geq c\,\|z_H\|^2_{H^{1/2}(\Gamma)}$$

for all $\underline{z} \in \mathbb{R}^M \leftrightarrow z_H \in S^1_H(\Gamma)$, since $\varrho D + D_1$ implies an equivalent norm in $H^{1/2}(\Gamma)$.

Hence we can apply Theorem 5.2 to ensure unique solvability of the perturbed variational inequality to find $\widehat{\underline{z}} \in \mathbb{R}^M \leftrightarrow \widehat{z}_H \in \mathcal{U}_H$ such that

$$(\widehat{T}_{\varrho,H}\widehat{\underline{z}} - \widehat{\underline{g}}, \underline{w} - \widehat{\underline{z}}) \;\geq\; 0 \quad \text{for all } \underline{w} \in \mathbb{R}^M \leftrightarrow w_H \in \mathcal{U}_H. \tag{6.13}$$

Corollary 6.6 *When combining the general error estimate* (5.6) *with the approximation property of the ansatz space $S^1_H(\Gamma)$, and with the error estimates* (6.6) *and* (6.8)*, we finally obtain the error estimate*

$$\begin{aligned}\|z-\widehat{z}_H\|_{H^{1/2}(\Gamma)} \;\leq\;& c_1\,H^{s+1/2}\,|z|_{H^{1+s}(\Gamma)} + c_2\,h^{s+1/2}\,\|q_z\|_{H^s_{pw}(\Gamma)} + c_3\,h^{s+3/2}\,\|t_z\|_{H^s_{pw}(\Gamma)} \\ &+ c_4\,h^{s+1/2}\,\|g\|_{H^s_{pw}(\Gamma)} + c_5\,h^{s+3/2}\,\|t_f\|_{H^s_{pw}(\Gamma)}\end{aligned}$$

when assuming $z \in H^{1+s}(\Gamma)$, and $q_z, t_z, g, t_f \in H^s_{pw}(\Gamma)$ for some $s \in [0,1]$. In particular for $h = H$ we therefore obtain the error estimate

$$\|z-\widehat{z}_H\|_{H^{1/2}(\Gamma)} \;\leq\; c(z,\overline{u},f)\,H^{s+\frac{1}{2}}\,. \tag{6.14}$$

Moreover, we are also able to derive an error estimate in $L_2(\Gamma)$, i.e.,

$$\|z-\widehat{z}_H\|_{L_2(\Gamma)} \;\leq\; c(z,\overline{u},f)\,H^{s+1}\,, \tag{6.15}$$

when applying the Aubin–Nitsche trick.

In the particular case of a non–constrained minimisation problem, instead of the discrete variational inequality (6.13) we have to solve the linear system

$$\widehat{T}_{\varrho,H}\widehat{\underline{z}} \;=\; \widehat{\underline{g}},$$

which is equivalent to a system of linear equations,

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & & -(\frac{1}{2}M_h + K_h) \\ K_{1,h}^\top & -(\frac{1}{2}M_h^\top + K_h^\top) & \varrho D_H + D_{1,H}\end{pmatrix}\begin{pmatrix}\underline{t}\\ \underline{q}\\ \widehat{\underline{z}}\end{pmatrix} = \begin{pmatrix}\underline{f}_1\\ -\underline{f}_2\\ -\underline{f}_3\end{pmatrix}. \tag{6.16}$$

Remark 6.1 *The symmetric boundary element approximation $\widehat{T}_{\varrho,H}$ is positive definite for any choice of conformal boundary element spaces $S^1_H(\Gamma) \subset H^{1/2}(\Gamma)$ and $S^0_h(\Gamma) \subset H^{-1/2}(\Gamma)$. In particular we may use the same boundary element mesh with mesh size $h = H$ to define the basis functions φ_i and ψ_k, respectively. From a theoretical point of view, this is not possible when using the non–symmetric approximation $\widetilde{T}_{\varrho,H}$.*

7 Numerical results

As numerical example we consider as in [5, 22], see also [25], the Dirichlet boundary control problem (2.1)–(2.3) for the domain $\Omega = (0, \frac{1}{2})^2 \subset \mathbb{R}^2$ where

$$\overline{u}(x) = (x_1^2 + x_2^2)^{-1/3}, \quad f(x) = 0, \quad \varrho = 1, \quad [g_a, g_b] = [-1, 2].$$

Note that the box constraints $[g_a, b_b] = [-1, 2]$ as considered in this example are not active, i.e., we have to solve the coupled linear system (5.27)) in the case of the non–symmetric boundary element approach, and (6.16) for the symmetric approach.

For the boundary element discretisation, we introduce a uniform triangulation of the boundary $\Gamma = \partial\Omega$ on several levels where the mesh size is $h_L = 2^{-(L+1)}$. Since the minimiser of (2.1) is not known in this case, we use the boundary element solution z_{h_9} of the 9th refinement level as reference solution. The boundary element discretisation is done by using the trial space $S_h^0(\Gamma)$ of piecewise constant basis functions, and $S_h^1(\Gamma)$ of piecewise linear and continuous functions. In particular, we use the same boundary element mesh to approximate the control z by a piecewise linear approximation, and piecewise constant approximations for the fluxes t and q. Note that we have $h = H$ in this case, and therefore we can not ensure the $S_h^1(\Gamma)$–ellipticity of the non–symmetric boundary element approximation, see Theorem 5.8. However, the numerical example shows stability in this case.

	Non–symmetric BEM (5.27)		Symmetric BEM (6.16)		FEM [25]	
L	$\\|\widetilde{z}_{h_L} - \widetilde{z}_{h_9}\\|_{L_2(\Gamma)}$	eoc	$\\|\widehat{z}_{h_L} - \widehat{z}_{h_9}\\|_{L_2(\Gamma)}$	eoc	$\\|z_{h_L}^{\mathrm{FEM}} - z_{h_9}^{\mathrm{FEM}}\\|_{L_2(\Gamma)}$	eoc
2	2.19 –3		7.77 –3		1.34 –3	
3	5.06 –4	2.11	1.95 –3	1.99	5.33 –4	1.33
4	1.92 –4	1.40	4.84 –4	2.01	2.08 –4	1.36
5	7.84 –5	1.29	1.28 –4	1.92	8.00 –5	1.38
6	3.13 –5	1.32	3.85 –5	1.73	3.14 –5	1.35
7	1.25 –5	1.33	1.34 –5	1.53	1.25 –5	1.33
8	4.58 –6	1.45	4.66 –6	1.52	4.55 –6	1.45

Table 1: Comparison of BEM/FEM errors of the Dirichlet control.

In Table 1, we present the errors for the control z in the $L_2(\Gamma)$ norm and the estimated order of convergence (eoc). These results correspond to the error estimate (5.26) of the non–symmetric boundary element approximation, and to the error estimate (6.15) of the symmetric boundary element approximation. Note that we have $z \in H^{7/6}(\Gamma)$, and therefore $s = 1/6$. Hence we can expect 7/6 as order of convergence. For comparison, we also give the error of the related finite element solution, see [25]. From the numerical results we conclude, that all three different approaches behave almost similar, as predicted by the theory.

8 Concluding remarks

In this paper, we have shown that we can use boundary element methods to solve Dirichlet boundary control problems. The numerical results coincide with those of a comparable finite element approach. The advantage of using boundary element methods lies in the fact, that only a discretisation of the boundary is required. In the case of smooth data we can prove, with respect to the used lowest order trial spaces, the best possible order of convergence for the boundary element approximation of the control z, while for a finite element approximation we are only able to prove some reduced order, see [25]. Moreover, optimal control problems subject to partial differential equations in unbounded exterior domains can be handled analoguesly.

While this paper is on the stability and error analysis of boundary element methods for optimal control problems only, further research will be done for an efficient solution of the resulting discrete systems. Hereby, special focus will be on appropriate solution methods to solve the discrete variational inequalities. This also involves the construction of efficient preconditioners, as well as the use of fast boundary element methods.

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