Tail probabilities of St. Petersburg sums, trimmed sums, and their limit

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Abstract

We provide exact asymptotics for the tail probabilities $\mathbb{P}\{S_n > x\}$ and $\mathbb{P}\{S_n - X_n^* > x\}$ as $x \to \infty$, for fix n, where S_n and X_n^* is the partial sum and partial maximum of i.i.d. St. Petersburg random variables. We show that while the order of the tail of the sum S_n is x^{-1}, the order of the tail of the trimmed sum $S_n - X_n^*$ is x^{-2}. In particular, we prove that although the St. Petersburg distribution is only O-subexponential, the subexponential property almost holds. We also provide an infinite series representation of the distribution function of the limiting distribution of the trimmed sum, and analyze its tail behavior.

1 Introduction

Peter offers to let Paul toss a fair coin repeatedly until it lands heads and pays him 2^k ducats if this happens on the k^{th} toss, where $k \in \mathbb{N} = \{1, 2, \ldots\}$. This is the so-called classical St. Petersburg game. If X denotes Paul's winning, then $\mathbb{P}\left\{X = 2^k\right\} = 2^{-k}$, $k \in \mathbb{N}$. Put $\lfloor x \rfloor$ for the lower integer part, $\lceil x \rceil$ for the upper integer part and $\{x\}$ for the fractional part of x. Then the distribution function of the gain is

$$F(x) = \mathbb{P}\{X \le x\} = \begin{cases} 0, & x < 2\,, \\ 1 - \frac{1}{2^{\lfloor \log_2 x \rfloor}} = 1 - \frac{2^{\{\log_2 x\}}}{x}, & x \ge 2\,. \end{cases} \tag{1}$$

and its quantile function $F^{-1}(s) = Q(s) = \inf\{x : s \le F(x)\}$ is

$$Q(s) = \begin{cases} 2, & s = 0, \\ 2^{\lceil -\log_2(1-s) \rceil} = \frac{2^{\{\log_2(1-s)\}}}{1-s}, & s \in (0,1). \end{cases} \tag{2}$$

Let $X, X_1, X_2, \ldots$ be i.i.d. St. Petersburg random variables, and let

$$S_n = X_1 + \ldots + X_n \quad \text{and} \quad X_n^* = \max_{1 \le i \le n} X_i$$

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denote their partial sum and their maximum, respectively. In order to state the necessary and sufficient condition for the existence of the limit, we introduce the positional parameter

$$\gamma_n = \frac{n}{2^{\lceil \log_2 n \rceil}} \in (1/2, 1],$$

which shows the position of n between two consecutive powers of 2. Since the function $2^{\{\log_2 x\}}$ in the numerator in (1) is not slowly varying at infinity, the St. Petersburg distribution is not in the domain of attraction of any stable law or max-stable law, so limit distribution neither for the centered and normed sum, nor for the centered and normed maximum, holds true. What holds instead for the sum is the merging theorem

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n}{n} - \log_2 n \le x \right\} - G_{\gamma_n}(x) \right| \to 0, \quad \text{as } n \to \infty, \tag{3}$$

shown by Csörgő [6], where G_γ is the distribution function of the infinitely divisible random variable W_γ, $\gamma \in (1/2, 1]$ with characteristic function

$$\mathbb{E}\left(e^{\mathrm{i}tW_\gamma}\right) = \exp\left(\mathrm{i}t\,[s_\gamma + u_\gamma] + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) \mathrm{d}R_\gamma(x)\right)$$

with $s_\gamma = -\log_2 \gamma$, $u_\gamma = \sum_{k=1}^\infty \frac{\gamma^2}{\gamma^2+4^k} - \sum_{k=0}^\infty \frac{1}{1+\gamma^2 4^k}$, and right-hand-side Lévy function

$$R_\gamma(x) = -\frac{\gamma}{2^{\lfloor \log_2(\gamma x)\rfloor}} = -\frac{2^{\{\log_2(\gamma x)\}}}{x}, \quad x > 0.$$

For the maximum we have

$$\sup_{j\in\mathbb{Z}} \left| \mathbb{P}\left\{ X_n^* = 2^{\lceil \log_2 n\rceil + j} \right\} - p_{j,\gamma_n} \right| = O(n^{-1}), \tag{4}$$

in particular $\mathbb{P}\left\{X_n^* = 2^{\lceil \log_2 n \rceil + j}\right\} \sim e^{-\gamma_n 2^{-j}}\left(1 - e^{-\gamma_n 2^{-j}}\right)$ for any $j \in \mathbb{Z}$, as $n \to \infty$, where

$$p_{j,\gamma} = e^{-\gamma 2^{-j}}\left(1 - e^{-\gamma 2^{-j}}\right), \quad j \in \mathbb{Z},\ \gamma \in [1/2, 1].$$

See formula (4) by Berkes et al. in [2], or Lemma 1 by Fukker et al. in [9] in the general case.

The limit theorems (3) and (4) suggest that the irregular oscillating behavior is due to the maximum, which is also indicated by the following fact. It is well-known (see Chow and Robbins [5] and Adler [1]) that

$$1 = \liminf_{n\to\infty} \frac{S_n}{n\log_2 n} < \limsup_{n\to\infty} \frac{S_n}{n \log_2 n} = \infty \quad \text{a.s.},$$

while the trimmed sum has nicer behavior, concerning at least the almost sure limits, since

$$\lim_{n\to\infty} \frac{S_n - X_n^*}{n\log_2 n} = 1 \quad \text{a.s.}$$

(cf. Csörgő and Simons [7]). For further results and history of St. Petersburg games see Csörgő [6] and the references therein.

As a continuation of our studies of the joint behavior of S_n and X_n^* in [9], we investigate the properties of the trimmed sum $S_n - X_n^*$ both for fix n and for $n \to \infty$. Figure **??** shows the histograms of the St. Petersburg sum and of the trimmed St. Petersburg sum. One can see that the histogram of $\log_2 S_n$ is mixtures of unimodal densities such that the first lobe is a mixture of overlapping densities, while the side-lobes have disjoint support. For the histogram of $\log_2(S_n - X_n^*)$ the side-lobes almost disappear, so the trimmed version has smaller tail. According to Proposition 7 in [9], for large X_n^* one gets $S_n/X_n^* \approx 1$, or equivalently $(S_n - X_n^*)/X_n^* \approx 0$, which explains the disappearance of side-lobes.

In Section 2 we investigate asymptotic behavior of the tail of the distribution function of the sum S_n and of the trimmed sum $S_n - X_n^*$ for fix n. In Theorem 1 we determine the exact tail behavior of $\mathbb{P}\{S_n^{(r)} > x\}$. It turns out that the orders are surprisingly different. In particular, we show that the St. Petersburg distribution is almost subexponential in a well-defined sense.

In Section 3 we let $n \to \infty$. In Theorem 2 we show that $\mathbb{P}\{(S_n - X_n^*)/n - \log_2 n > x\} \leq c_1(\ln x)/x^2$ holds uniformly in n. In Theorem 3 we determine $\{W_\gamma^* : \gamma \in (1/2, 1]\}$, the set of the possible subsequential limit distributions of $(S_n - X_n^*)/n - \log_2 n$. This result was first obtained by Gut and Martin-Löf in their Theorem 6.1 in [12]. They investigate the so-called max-trimmed St. Petersburg game, where from the sum S_n all the maximum values are canceled. In Theorem 5 we prove that $\mathbb{P}\{W_\gamma^* > x\} \leq K(\ln x)/x^2$. The latter result is surprising in view of the recent result by Watanabe and Yamamuro [19], from which follows that for the untrimmed limit

$$1 = \liminf_{x\to\infty} x\mathbb{P}\{W_\gamma > x\} < \limsup_{x\to\infty} x\mathbb{P}\{W_\gamma > x\} = 2.$$

Finally, in Section 4 we mention some of these results without proof in case of generalized St. Petersburg games.

2 Tail behavior of the sum and the trimmed sum

In this section the number of summands n is fix, and we are interested in the tail behavior of S_n and $S_n - X_n^*$.

2.1 Preliminaries

Here we gather some facts about the St. Petersburg sums which we need later.

For the number of maximal terms put

$$N_n = \left|\{k : 1 \leq k \leq n, X_k = X_n^*\}\right|.$$

In Lemma 3 in [9] we obtained that the conditional generating function of N_n given X_n^* is

$$g_{k,n}(s) = \mathbb{E}\left[s^{N_n} | X_n^* = 2^k\right] = \frac{\left(1 - 2^{1-k}(1 - s/2)\right)^n - \left(1 - 2^{1-k}\right)^n}{(1 - 2^{-k})^n - (1 - 2^{1-k})^n},$$

and the corresponding probabilities are

$$\mathbb{P}\left\{N_n = m | X_n^* = 2^k\right\} = \frac{\binom{n}{m} 2^{-km}(1-2^{1-k})^{n-m}}{(1-2^{-k})^n - (1-2^{1-k})^n}. \tag{5}$$

Given that $X \leq 2^k$ for $i \leq k$ we have $\mathbb{P}\left\{X = 2^i | X \leq 2^k\right\} = 2^{-i}/(1-2^{-k})$. Introduce the corresponding distribution function

$$F_k(x) = \mathbb{P}\left\{X \leq x | X \leq 2^k\right\} = \begin{cases} \frac{1}{1-2^{-k}} \left[1 - \frac{2^{\{\log_2 x\}}}{x}\right], & \text{for } x \in [2, 2^k], \\ 1, & \text{for } x \geq 2^k. \end{cases}$$

In the following $X^{(k)}, X_1^{(k)}, X_2^{(k)}, \ldots$, are i.i.d. random variables with distribution function F_k, and $S_n^{(k)}$ stands for their partial sums. For the moments we obtain (see (29) in [9])

$$\mathbb{E}(X^{(k)})^\ell = \frac{1}{1-2^{-k}} \sum_{i=1}^{k} 2^{i\ell} 2^{-i} = \begin{cases} \frac{2^{\ell-1}}{1-2^{-k}} \frac{2^{(\ell-1)k}-1}{2^{\ell-1}-1}, & \text{for } \ell \geq 2, \\ \frac{k}{1-2^{-k}}, & \text{for } \ell = 1. \end{cases} \tag{6}$$

According to Lemma 2 in [9] conditioning on $X_n^* = 2^k$,

$$S_n \stackrel{\mathcal{D}}{=} N_{n,k} 2^k + \sum_{i=1}^{n-N_{n,k}} X_i^{(k-1)} = N_{n,k} 2^k + S_{n-N_{n,k}}^{(k-1)} \leq N_{n,k} 2^k + S_n^{(k-1)}, \tag{7}$$

such that $N_{n,k}$ and $S_n^{(k-1)}$ are independent. Here, $N_{n,k}$ is the number of maximums given that $X_n^* = 2^k$.

Let $X_{1n} \geq X_{2n} \geq \ldots \geq X_{nn}$ be the ordered sample of the variables $X_1, X_2, \ldots, X_n$. Using the well-known quantile representation and that $U \stackrel{\mathcal{D}}{=} 1 - U$, for $U \sim \text{Uniform}(0,1)$, we obtain

$$(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} \left(2^{\lceil \log_2 U_{1n}^{-1} \rceil}, \ldots, 2^{\lceil \log_2 U_{nn}^{-1} \rceil}\right), \tag{8}$$

where $U_{1n} \leq U_{2n} \leq \ldots \leq U_{nn}$ is the ordered sample of n independent Uniform$(0,1)$ random variables. Introducing the function $\Psi(x) = 2^{\{\log_2 x\}}$, i.e. it grows linearly from 1 to 2 on each interval $[2^j, 2^{j+1})$, $j = 1, 2, \ldots$, we have

$$(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{\Psi(U_{1n})}{U_{1n}}, \ldots, \frac{\Psi(U_{nn})}{U_{nn}}\right).$$

Introduce the r-trimmed sum, $r = 0, 1, \ldots, n-1$,

$$S_n^{(r)} = \sum_{i=r+1}^{n} X_{in}. \tag{9}$$

Note that $r = 0$ corresponds to the untrimmed sum S_n.

2.2 The O-subexponentiality of the St. Petersburg distribution

First we summarize some basic facts on subexponential distributions. Let G be a distribution function of a non-negative random variable Y. Put $\overline{G}(x) = 1 - G(x)$. The distribution G is *subexponential*, $G \in \mathcal{S}$, if

$$\lim_{x\to\infty} \frac{\overline{G*G}(x)}{\overline{G}(x)} = 2, \tag{10}$$

where $*$ stands for the usual convolution, and G^{n*} is the n^{th} convolution power, for $n \geq 2$. The characterizing property of the subexponential distributions is that the sum of i.i.d. random variables behaves like the maximum of these variables, that is for any $n \geq 1$

$$\lim_{x\to\infty} \frac{\mathbb{P}\{Y_1 + \ldots + Y_n > x\}}{\mathbb{P}\{\max\{Y_i : i = 1, 2, \ldots, n\} > x\}} = 1,$$

or equivalently

$$\lim_{x\to\infty} \frac{\mathbb{P}\{Y_1 + \ldots + Y_n > x\}}{\mathbb{P}\{Y_1 > x\}} = n. \tag{11}$$

For properties of subexponential distributions and their use in practice we refer to the survey paper by Goldie and Klüppelberg [11].

It is well-known that distributions with regularly varying tails are subexponential. What makes the St. Petersburg game so interesting is that its tail is not regularly varying. In fact it was already noted by Goldie [10] that the St. Petersburg distribution F is not subexponential. What we have instead is that

$$2 = \liminf_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} < \limsup_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 4. \tag{12}$$

This can be proved by showing that for $1 \leq k \leq \ell$

$$\mathbb{P}\{X_1 + X_2 > 2^k + 2^\ell\} = \begin{cases} 2 \cdot 2^{-\ell} + 2 \cdot 2^{-(\ell+k)} - 4 \cdot 2^{-2\ell}, & \text{for } \ell > k, \\ 2 \cdot 2^{-\ell} - 2^{-2\ell}, & \text{for } \ell = k, \end{cases}$$

from which

$$\lim_{\ell\to\infty} \frac{\mathbb{P}\{X_1 + X_2 > 2^\ell\}}{\mathbb{P}\{X_1 > 2^\ell\}} = 4, \quad \text{and} \quad \lim_{\ell\to\infty} \frac{\mathbb{P}\{X_1 + X_2 > 2^\ell - 1\}}{\mathbb{P}\{X_1 > 2^\ell - 1\}} = 2.$$

Moreover, it is simple to see that 4 is in fact the limsup.

This naturally leads to the extension of subexponentiality. A distribution G is *O-subexponential*, $G \in \mathcal{OS}$, if

$$l^*(G) := \limsup_{x\to\infty} \frac{\overline{G*G}(x)}{\overline{G}(x)} < \infty.$$

It is known that the corresponding lim inf is always greater than, or equal to 2, and it was shown recently by Foss and Korshunov [8] that it is exactly 2 for any heavy-tailed distribution. The notion of O-subexponentiality was introduced by Klüppelberg [13]. The properties of the $\mathcal{OS}$ class, in particular when the distribution is also infinitely divisible, were investigated by Shimura and Watanabe [17]. In their Proposition 2.4 they prove that if $G \in \mathcal{OS}$ then for every $\varepsilon > 0$ there is a $c > 0$ such that for all n and $x \geq 0$

$$\frac{\overline{G^{n*}}(x)}{\overline{G}(x)} \leq c(l^*(G) - 1 + \varepsilon)^n.$$

In the St. Petersburg case $l^*(F) = 4$. In Theorem 1 we determine the exact asymptotic behavior of $\overline{F^{n*}}(x)$, which, in particular, implies a linear bound in n instead of the exponential.

Let us examine the case $n = 2$ in detail. Note that $\mathbb{P}\{X_1 > 2^\ell\} = \mathbb{P}\{X_1 > 2^\ell + 2^k\}$ for $k < \ell$, therefore from (12)

$$\frac{\mathbb{P}\{X_1 + X_2 > 2^\ell + 2^k\}}{\mathbb{P}\{X_1 > 2^\ell + 2^k\}} = 2 + 2 \cdot 2^{-k} - 4 \cdot 2^{-\ell}.$$

From this it is clear that when both ℓ and k tends to infinity, then the limit exists and equal to 2; in particular for any $\delta > 0$

$$\lim_{x\to\infty, \{\log_2 x\} \geq \delta} \frac{\mathbb{P}\{X_1 + X_2 > x\}}{\mathbb{P}\{X_1 > x\}} = 2,$$

where $\{x\}$ stands for the fractional part of x. That is, the St. Petersburg distribution is 'almost subexponential'. We prove the corresponding result for general n, i.e. for any $\delta > 0$

$$\lim_{x\to\infty, \{\log_2 x\} \geq \delta} \frac{\mathbb{P}\{S_n > x\}}{\mathbb{P}\{X_1 > x\}} = n.$$

Theorem 1. *For any $0 \leq r < n$ we have as $x \to \infty$*

$$\mathbb{P}\{S_n^{(r)} > x\} \sim \frac{2^{(r+1)\{\log_2 x\}}}{x^{r+1}} \binom{n}{r+1} \left(1 + \mathbb{P}\left\{S_{n-r-1} > x(1 - 2^{-\{\log_2 x\}})\right\} (2^{r+1} - 1)\right). \tag{13}$$

In particular, for any $0 < \delta < 1$,

$$\lim_{x\to\infty, \{\log_2 x\} > \delta} \mathbb{P}\{S_n^{(r)} > x\} \frac{x^{r+1}}{2^{(r+1)\{\log_2 x\}}} = \binom{n}{r+1}. \tag{14}$$

Proof. The density function of $U_{r+1,n}$ is $\binom{n}{r+1}(r+1)x^r(1-x)^{n-r-1}$, therefore

$$\mathbb{P}\left\{\frac{\Psi(U_{r+1,n})}{U_{r+1,n}} > x\right\} = \mathbb{P}\{U_{r+1,n} < 2^{-\lfloor \log_2 x \rfloor}\} \sim \binom{n}{r+1} \frac{2^{\{\log_2 x\}(r+1)}}{x^{r+1}}. \tag{15}$$

Considering the asymptotics, write

$$\begin{aligned}
\mathbb{P}\{S_n^{(r)} > x\} &= \sum_{m=1}^{\infty} \mathbb{P}\left\{S_n^{(r+1)} > x - 2^m, \ \frac{\Psi(U_{r+1,n})}{U_{r+1,n}} = 2^m\right\} \\
&= \sum_{m=1}^{\lfloor \log_2 x \rfloor - 1} \mathbb{P}\left\{S_n^{(r+1)} > x - 2^m, \ \frac{\Psi(U_{r+1,n})}{U_{r+1,n}} = 2^m\right\} \\
&\quad + \mathbb{P}\left\{S_n^{(r+1)} > x - 2^{\lfloor \log_2 x \rfloor}, \ \frac{\Psi(U_{r+1,n})}{U_{r+1,n}} = 2^{\lfloor \log_2 x \rfloor}\right\} + \mathbb{P}\left\{\frac{\Psi(U_{r+1,n})}{U_{r+1,n}} > x\right\} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For $m \le \lfloor \log_2 x \rfloor - 1$ we have $x - 2^m \ge x/2$, thus, by (15) the first sum

$$I_1 \le \lfloor \log_2 x \rfloor \mathbb{P}\{S_n^{(r+1)} > x/2\} = O(x^{-(r+2)} \ln x).$$

For I_2 we have

$$\begin{aligned} I_2 &= \mathbb{P}\left\{ \sum_{i=r+2}^{n} \frac{\Psi(U_{in})}{U_{in}} > x(1 - 2^{-\{\log_2 x\}}) \Big| \frac{\Psi(U_{r+1,n})}{U_{r+1,n}} = 2^{\lfloor \log_2 x \rfloor} \right\} \mathbb{P}\left\{ \frac{\Psi(U_{r+1,n})}{U_{r+1,n}} = 2^{\lfloor \log_2 x \rfloor} \right\} \\ &\sim \mathbb{P}\left\{ S_{n-r-1} > x(1 - 2^{-\{\log_2 x\}}) \right\} \binom{n}{r+1} (2^{r+1} - 1) \frac{2^{(r+1)\{\log_2 x\}}}{x^{r+1}}. \end{aligned} \tag{16}$$

Here we used the simple fact that conditioning on $U_{n,r+1} \to 0$

$$(U_{n,r+2}, \dots, U_{nn}) \xrightarrow{\mathcal{D}} (U_{1,n-r-1}, \dots, U_{n-r-1,n-r-1}).$$

Combining (16) and (15) formula (13) follows. To show (14) notice that $x(1 - 2^{-\{\log_2 x\}}) \to \infty$, as $x \to \infty$, $\{\log_2 x\} > \delta$, thus $\mathbb{P}\left\{S_{n-r-1} > x(1 - 2^{-\{\log_2 x\}})\right\} \to 0$. $\square$

When $r = 0$ the result describes the tail behavior of the untrimmed sum S_n. In Figure **??** the oscillatory behavior of $\mathbb{P}\{S_n > x\}$ is clearly visible. We also see that at each power of 2 there is a large jump, that is where the asymptotic (14) fails.

We mention some important consequences.

Theorem 1 readily implies that for any $n \ge 1$ we have

$$n = \liminf_{x \to \infty} x \mathbb{P}\left\{S_n > x\right\} < \limsup_{x \to \infty} x \mathbb{P}\left\{S_n > x\right\} = 2n. \tag{17}$$

Since $x\mathbb{P}\{X > x\} = 2^{\{\log_2 x\}}$, $x \ge 2$, we have

$$\lim_{x \to \infty, \{\log_2 x\} \ge \delta} \frac{\mathbb{P}\{S_n > x\}}{\mathbb{P}\{X > x\}} = n.$$

This convergence also shows that (14) does not hold without the restriction, since by (11) that would imply the subexponentiality of F.

For $c > 1$ fixed as $m \to \infty$

$$1 - 2^{-\{\log_2(2^m + c)\}} \sim 1 - e^{-c2^{-m}} \sim c2^{-m}.$$

Therefore from (13) we obtain that for any $c > 1$

$$\lim_{m \to \infty} \mathbb{P}\{S_n^{(r)} > 2^m + c\} \sim 2^{-m(r+1)} \left(1 + \left(2^{r+1} - 1\right) \mathbb{P}\{S_{n-r-1} > c\}\right). \tag{18}$$

For the maximum term, for fixed n we have $\mathbb{P}\{X_n^* > x\} \sim n\mathbb{P}\{X > x\}$, and so (18) gives

$$\lim_{m \to \infty} \frac{\mathbb{P}\left\{S_n > 2^m + c\right\}}{\mathbb{P}\{X_n^* > 2^m + c\}} = 1 + \mathbb{P}\left\{S_{n-1} > c\right\}.$$

If $c = c(m)$ tends to infinity arbitrarily slowly, then the limit above is 1, that is the St. Petersburg distribution is very close to having the subexponential property.

Lemma 1. *For $x \geq 0$, put*

$$h(x) = (2+x)\ln\left(1+\frac{x}{2}\right) - x.$$

For any $n \geq 1$, $j \geq 1 - \lceil \log_2 n \rceil$ and $x \geq 0$, we have that

$$\mathbb{P}\left\{S_n^{(\lceil \log_2 n \rceil + j)} - \mathbb{E}S_n^{(\lceil \log_2 n \rceil + j)} > nx\right\} \leq e^{-\frac{h(x)}{\eta_{j,\gamma_n}}},$$

where

$$\eta_{j,\gamma} = 2^j \gamma^{-1}. \tag{19}$$

It is worth to mention that the estimates provide exponential bounds on the tails for x fixed and $n \to \infty$. We also note that Lemma 1 is optimal, see the remark before Theorem 3.

Proof of Lemma 1. For any $\lambda > 0$, we apply the Chernoff bounding technique:

$$\begin{aligned}
&\mathbb{P}\left\{n^{-1}\left(S_n^{(\lceil \log_2 n \rceil + j)} - \mathbb{E}S_n^{(\lceil \log_2 n \rceil + j)}\right) > x\right\} \\
&= \mathbb{P}\left\{n^{-1} S_n^{(\lceil \log_2 n \rceil + j)} > x + \mathbb{E}X^{(\lceil \log_2 n \rceil + j)}\right\} \\
&\leq e^{-\lambda\left(x + \mathbb{E}X^{(\lceil \log_2 n \rceil + j)}\right)} \mathbb{E}\exp\left[\lambda n^{-1} S_n^{(\lceil \log_2 n \rceil + j)}\right] \\
&= e^{-\lambda\left(x + \mathbb{E}X^{(\lceil \log_2 n \rceil + j)}\right)} \left(\mathbb{E}\exp\left[\frac{\lambda}{n} X^{(\lceil \log_2 n \rceil + j)}\right]\right)^n.
\end{aligned}$$

One has that

$$\begin{aligned}
\mathbb{E}\exp\left[\frac{\lambda}{n} X^{(\lceil \log_2 n \rceil + j)}\right] &= 1 + \frac{\lambda}{n}\mathbb{E}X^{(\lceil \log_2 n \rceil + j)} + \sum_{\ell=2}^{\infty} \frac{\lambda^\ell \mathbb{E}\left\{(X^{(\lceil \log_2 n \rceil + j)})^\ell\right\}}{n^\ell \, \ell!} \\
&\leq 1 + \frac{\lambda}{n}\mathbb{E}X^{(\lceil \log_2 n \rceil + j)} + 2\sum_{\ell=2}^{\infty} \frac{\lambda^\ell \left(\frac{2^j}{\gamma_n}\right)^{\ell-1}}{n \ell!} \\
&= 1 + \frac{\lambda}{n}\mathbb{E}X^{(\lceil \log_2 n \rceil + j)} + \frac{2}{n}\frac{e^{\lambda \eta_{j,\gamma_n}} - 1 - \lambda \eta_{j,\gamma_n}}{\eta_{j,\gamma_n}} \\
&\leq \exp\left[\frac{\lambda}{n}\mathbb{E}X^{(\lceil \log_2 n \rceil + j)} + \frac{2}{n}\frac{e^{\lambda \eta_{j,\gamma_n}} - 1 - \lambda \eta_{j,\gamma_n}}{\eta_{j,\gamma_n}}\right],
\end{aligned}$$

where we used that by (6)

$$\mathbb{E}\left(X^{(\lceil \log_2 n \rceil + j)}\right)^\ell = \frac{1}{1 - 2^{-(\lceil \log_2 n \rceil + j)}} \frac{2^{\ell-1}}{2^{\ell-1} - 1}\left[\left(\frac{n2^j}{\gamma_n}\right)^{\ell-1} - 1\right] \leq 2\left(\frac{n2^j}{\gamma_n}\right)^{\ell-1},$$

($\ell \geq 2$). Therefore

$$\mathbb{P}\left\{\frac{S_n^{(\lceil \log_2 n \rceil + j)} - \mathbb{E}S_n^{(\lceil \log_2 n \rceil + j)}}{n} > x\right\} \leq \exp\left[2\frac{e^{\lambda \eta_{j,\gamma_n}} - 1 - \lambda \eta_{j,\gamma_n}}{\eta_{j,\gamma_n}} - \lambda x\right].$$

With the choice $\lambda = \left[\ln\left(1 + \frac{x}{2}\right)\right]/\eta_{j,\gamma_n}$ the lemma is proved. $\square$

Remark 1. Note that $h(x) \sim x \ln x$, as $x \to \infty$, therefore the upper bound for large x is approximately $\exp\left[-\gamma 2^{-j} x \ln x\right]$.

Applying the elementary inequalities

$$\frac{u}{1+u/2} \leq \ln(1+u) \leq u, \quad u \geq 0,$$

one has that

$$\frac{x^2}{4+x} \leq h(x) \leq \frac{x^2}{2},$$

and so for any $x \geq 0$

$$e^{-\frac{x^2}{2\eta_{j,\gamma_n}}} \leq e^{-\frac{h(x)}{\eta_{j,\gamma_n}}} \leq e^{-\frac{x^2}{(4+x)\eta_{j,\gamma_n}}}.$$

Since $h(x) = x^2/4 + o(x^2)$ as $x \to 0$, for small $x \geq 0$, we have $e^{-\frac{h(x)}{\eta_{j,\gamma_n}}} \approx e^{-\frac{x^2}{4\eta_{j,\gamma_n}}}$.

Remark 2. We note that this exponential inequality (and its straightforward extension to generalized St. Petersburg games) allows us to show that arbitrary powers of the random variables $(S_n^{(k_n)} - \mathbb{E}S_n^{(k_n)})/\mathbf{Var}\, S_n^{(k_n)}$ are uniformly integrable, whenever $\log_2 n - k_n \to \infty$. The latter implies that in Propositions 2 and 3 in [9] not only distributional convergence, but also moment convergence holds.

3 Properties of the limit

3.1 Uniform tail bound for the trimmed sums

In this section we further investigate the properties of the trimmed sum as $n \to \infty$. First we obtain a uniform tail bound for the centralized and normalized trimmed sum.

Theorem 2. *For $x \geq e$ and $n \geq 1$ there is a finite constant $C > 0$ such that*

$$\mathbb{P}\left\{\frac{S_n - X_n^*}{n} - \log_2 n > x\right\} \leq C \frac{\ln x}{x^2}.$$

Proof. To ease the notation put $\tilde{q}_{n,j} = \mathbb{P}\left\{X_n^* = 2^{\lceil \log_2 n \rceil + j}\right\}$ and

$$U_{n,j} = \frac{S_n^{(\lceil \log_2 n \rceil + j - 1)}}{n} + (N_{n, \lceil \log_2 n \rceil + j} - 1)\eta_{j,\gamma_n},$$

where $\eta_{j,\gamma}$ is defined in (19). Using (7) for the tail probability we obtain the decomposition

$$\begin{aligned}
\mathbb{P}\left\{\frac{S_n - X_n^*}{n} - \log_2 n > x\right\} &\leq \sum_{j=1-\lceil \log_2 n \rceil}^{\infty} \mathbb{P}\left\{U_{n,j} - \log_2 n > x\right\} \mathbb{P}\left\{X_n^* = 2^{\lceil \log_2 n \rceil + j}\right\} \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$I_1 = \sum_{j=\lfloor 2\log_2 x \rfloor + 1}^{\infty} \mathbb{P}\left\{U_{n,j} - \log_2 n > x\right\} \tilde{q}_{n,j}, \quad I_2 = \sum_{j=1}^{\lfloor 2\log_2 x \rfloor} \mathbb{P}\left\{U_{n,j} - \log_2 n > x\right\} \tilde{q}_{n,j},$$

and

$$I_3 = \sum_{j=1-\lceil \log_2 n \rceil}^{0} \mathbb{P}\left\{U_{n,j} - \log_2 n > x\right\} \tilde{q}_{n,j}.$$

We have that

$$I_1 \leq \mathbb{P}\left\{X_n^* > 2^{\lceil \log_2 n \rceil + \lfloor 2 \log_2 x \rfloor}\right\} = 1 - \left(1 - \frac{\gamma_n}{n} 2^{-\lfloor 2 \log_2 x \rfloor}\right)^n \leq \frac{2}{x^2}.$$

One can check that uniformly in n, x and $j \leq \lfloor 2 \log_2 x \rfloor$

$$x + \log_2 n - \mathbb{E}X^{(\lceil \log_2 n \rceil + j - 1)} - (\mathbb{E}N_{n, \lceil \log_2 n \rceil + j} - 1)\eta_{j, \gamma_n} \geq x - j - 2.$$

Therefore

$$\begin{aligned}
I_2 &= \sum_{j=1}^{\lfloor 2 \log_2 x \rfloor} \mathbb{P}\left\{U_{n,j} - \mathbb{E}U_{n,j} > x + \log_2 n - \mathbb{E}U_{n,j}\right\} \tilde{q}_{n,j} \\
&\leq \sum_{j=1}^{\lfloor 2 \log_2 x \rfloor} \mathbb{P}\left\{U_{n,j} - \mathbb{E}U_{n,j} > x - j - 2\right\} \tilde{q}_{n,j} \\
&\leq \sum_{j=0}^{\lfloor 2 \log_2 x \rfloor} \mathbb{P}\left\{U_{n,j} - \mathbb{E}U_{n,j} > x - 2 \log_2 x - 2\right\} \tilde{q}_{n,j}.
\end{aligned}$$

The Chebyshev–Cantelli inequality implies that

$$\mathbb{P}\left\{U_{n,j} - \mathbb{E}U_{n,j} > x - 2 \log_2 x - 2\right\} \leq \frac{\mathbf{Var}\, U_{n,j}}{\mathbf{Var}\,(U_{n,j}) + (x - 2 \log_2 x - 2)^2}.$$

We can verify that for any n and $j \geq 1 - \log_2 n$

$$\mathbf{Var}\, U_{n,j} = \mathbf{Var}\,(S_n^{(\lceil \log_2 n \rceil + j - 1)}/n) + \mathbf{Var}\,(N_{n, \lceil \log_2 n \rceil + j})\eta_{j,\gamma_n}^2 \leq c_1 \max\{1, 2^j\}$$

and $\tilde{q}_{n,j} \leq c_2 2^{-j}$, with some constants $c_1, c_2 > 0$ independent of n and x. Thus

$$\begin{aligned}
I_2 &\leq \sum_{j=1}^{\lfloor 2 \log_2 x \rfloor} \frac{c_1 2^j}{c_1 2^j + (x - 2 \log_2 x - 2)^2} \tilde{q}_{n,j} \\
&\leq \sum_{j=1}^{\lfloor 2 \log_2 x \rfloor} \frac{c_1 2^j c_2 2^{-j}}{c_1 2^j + (x - 2 \log_2 x - 2)^2} \\
&\leq \frac{c_1 c_2 2 \log_2 x}{2 c_1 + (x - 2 \log_2 x - 2)^2}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
I_3 &\leq \sum_{j=1-\lceil \log_2 n \rceil}^{0} \mathbb{P}\left\{U_{n,j} - \mathbb{E}U_{n,j} > x - j - 2\right\} \tilde{q}_{n,j} \\
&\leq \sum_{j=1-\lceil \log_2 n \rceil}^{0} \mathbb{P}\left\{U_{n,j} - \mathbb{E}U_{n,j} > x - 2\right\} \tilde{q}_{n,j} \\
&\leq \sum_{j=1-\lceil \log_2 n \rceil}^{0} \frac{c_1}{c_1 + (x-2)^2} \tilde{q}_{n,j} \\
&\leq \frac{c_1}{c_2 + (x-2)^2},
\end{aligned}
$$

which proves the theorem. □

3.2 Properties of the 1-trimmed limit

In the following we determine the possible limit distributions of the trimmed sum, and we investigate the limit. Introduce the infinitely divisible random variables $W_{j,\gamma}$, $j \in \mathbb{Z}, \gamma \in [1/2, 1]$ with characteristic function

$$
\varphi_{j,\gamma}(t) = \mathbb{E}e^{\mathrm{i}tW_{j,\gamma}} = \exp\left[\mathrm{i}tu_{j,\gamma} + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \mathrm{i}tx\right) \mathrm{d}L_{j,\gamma}(x)\right], \tag{20}
$$

with

$$
L_{j,\gamma}(x) = \begin{cases} \gamma 2^{-j} - \frac{2^{\{\log_2(\gamma x)\}}}{x}, & \text{for } x < 2^j \gamma^{-1}, \\ 0, & \text{for } x \geq 2^j \gamma^{-1}, \end{cases}
$$

and $u_{j,\gamma} = j - \log_2 \gamma$. Note that each $W_{j,\gamma}$ has finite exponential moment of any order. We pointed out in [9] that the distribution function $G_{j,\gamma}(x) = \mathbb{P}\{W_{j,\gamma} \leq x\}$ is infinitely many times differentiable. Moreover, expanding the exponential in Taylor-series and changing the order of the summation we obtain

$$
\log \varphi_{j,\gamma}(t) = \mathrm{i}t \log_2 \eta_{j,\gamma} + \sum_{k=2}^{\infty} \frac{(\mathrm{i}t)^k}{k!} \eta_{j,\gamma}^{k-1} \frac{2^{k-1}}{2^{k-1}-1} =: \mathrm{i}t \log_2 \eta_{j,\gamma} + f_{\eta_{j,\gamma}}(t), \tag{21}
$$

with $\eta_{j,\gamma} = 2^j/\gamma$ as in (19). The distribution of $W_{j,\gamma}$ depends only on the single parameter $\eta_{j,\gamma} = 2^j/\gamma$. Denote Z_η a random variable with the characteristic function $e^{f_\eta(t)}$. Then, by the definition of f_η

$$
\mathbb{E}e^{\mathrm{i}t\frac{Z_\eta}{\eta}} = e^{f_\eta(t/\eta)} = e^{f_1(t)/\eta},
$$

thus from the properties of Z_1 we can derive the properties of Z_η, for any η. For example, for the density function g_η of Z_η we have

$$
g_\eta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{f_\eta(t)} e^{-\mathrm{i}tx} \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{f_1(t\eta)/\eta} e^{-\mathrm{i}tx} \mathrm{d}t = \frac{1}{2\pi} \frac{1}{\eta} \int_{-\infty}^{\infty} e^{(f_1(t) - \mathrm{i}tx)/\eta} \mathrm{d}t.
$$

Thus, g_η can be derived from the characteristic function e^{f_1} of Z_1 by a simple transformation. It also shows, that the proper scaling is the variance instead of the standard deviation.

From (21) it is apparent that

$$\frac{W_{j,\gamma} - \log_2 \eta_{j,\gamma}}{\sqrt{2\eta_{j,\gamma}}} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1), \quad \text{as } \eta_{j,\gamma} \to 0,$$

while

$$\frac{W_{j,\gamma}}{\eta_{j,\gamma}} \xrightarrow{\mathbb{P}} 0, \quad \text{as } \eta_{j,\gamma} \to \infty.$$

These limit theorems are in complete accordance with Proposition 3 in [9], which states that conditioning on small maximum the limit is normal, and with Proposition 7 [9], which states that conditioning on large maximum the limit is deterministic.

Remark 3. Since the support of the Lévy measure is bounded, according to Theorem 26.1 in [16] for the tail behavior of $W_{j,\gamma}$ we have the following. For any $0 < c < \gamma/2^j$

$$\mathbb{E} \exp\left\{cW_{j,\gamma} |\ln W_{j,\gamma}|\right\} < \infty,$$

and so

$$\mathbb{P}\left\{|W_{j,\gamma}| > x\right\} = o(\exp\{-cx \ln x\}) \quad \text{as } x \to \infty,$$

while for $c > \gamma/2^j$

$$\mathbb{E} \exp\left\{cW_{j,\gamma} |\ln W_{j,\gamma}|\right\} = \infty,$$

and

$$\mathbb{P}\left\{|W_{j,\gamma}| > x\right\} \exp\{cx \ln x\} \to \infty \quad \text{as } x \to \infty.$$

This result combining with Proposition 5 in [9] implies that the tail bound in Lemma 1 is optimal.

According to Corollary 2 in [9] we have that

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n^{(\lceil \log_2 n\rceil + j)}}{n} - \log_2 n \le x \right\} - G_{j,\gamma_n}(x) \right| \to 0. \tag{22}$$

Moreover, by Proposition 6 in [9] for each $j \in \mathbb{Z}$

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n}{n} - \log_2 n \le x \middle| X_n^* = 2^{\lceil \log_2 n\rceil + j} \right\} - \widetilde{G}_{j,\gamma_n}(x) \right| \to 0, \tag{23}$$

where

$$\widetilde{G}_{j,\gamma}(x) = \sum_{m=1}^{\infty} G_{j-1,\gamma}\left(x - m\frac{2^j}{\gamma}\right) r_{j,\gamma}(m) \tag{24}$$

with

$$r_{j,\gamma}(m) = \frac{(2^{-j}\gamma)^m}{m!} \left(e^{2^{-j}\gamma} - 1\right)^{-1}, \quad m \ge 1.$$

Note that the distribution corresponding to $(r_{j,\gamma}(m))_{m\geq 1}$ is a Poisson-distribution conditioned on being nonzero. From Proposition II.2.7 in [18] it follows that this distribution is *not* infinitely divisible. In Theorem 1 in [9] we showed that for any $\gamma \in [1/2, 1]$

$$G_\gamma(x) = \sum_{j=-\infty}^{\infty} \widetilde{G}_{j,\gamma}(x) p_{j,\gamma}.$$

For the trimmed sum we have the following the merging theorem, together with the infinite series representation of the limiting distribution function.

Theorem 3. *We have*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n - X_n^*}{n} - \log_2 n \leq x \right\} - G^*_{\gamma_n}(x) \right| \to 0, \quad \text{as } n \to \infty,$$

where

$$G^*_\gamma(x) = \sum_{j=-\infty}^{\infty} \sum_{m=1}^{\infty} G_{j-1,\gamma}\left(x - (m-1)2^j/\gamma\right) r_{j,\gamma}(m) p_{j,\gamma}, \quad \gamma \in (1/2, 1]. \tag{25}$$

Proof. Since (23) holds uniformly in x, we obtain

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n - X_n^*}{n} - \log_2 n \leq x \middle| X_n^* = 2^{\lceil \log_2 n \rceil + j} \right\} - \widetilde{G}_{j,\gamma_n}(x + 2^j/\gamma_n) \right| \to 0.$$

Using (4) and the same conditioning as in the proof of Theorem 1 in [9] we obtain the statement. □

This result implies that, as usual in this setup, along subsequences there is distributional convergence. For the subsequence $n_k = \gamma 2^k$, $\gamma \in [1/2, 1]$, which in fact covers all the possible limits, this was shown by Gut and Martin-Löf in Theorem 6.1 [12].

The infinite series representation of G_γ in Theorem 1 in [9] is in fact equivalent to the distributional representation

$$W_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma - 1,\gamma} + M_{Y_\gamma,\gamma} 2^{Y_\gamma} \gamma^{-1},$$

where $(W_{j,\gamma})_{j\in\mathbb{Z}}$, $(M_{j,\gamma})_{j\in\mathbb{Z}}$ and Y_γ are independent random variables, Y_γ has probability distribution $(p_{j,\gamma})_{j\in\mathbb{Z}}$, $M_{j,\gamma}$ has Poisson$(\gamma 2^{-j})$ distribution, conditioned on not being 0, and $W_{j,\gamma}$ is an infinitely divisible distribution given in (20). Let W^*_γ be a random variable with distribution function G^*_γ. Then, the same way (25) reads as

$$W^*_\gamma \stackrel{\mathcal{D}}{=} W_{Y_\gamma - 1,\gamma} + (M_{Y_\gamma,\gamma} - 1) 2^{Y_\gamma} \gamma^{-1}. \tag{26}$$

Looking at the infinitely divisible random variable W_γ as a semistable Lévy process at time 1, the meaning of the representation above is the following. The value $2^{Y_\gamma}/\gamma$ corresponds to the maximum jump, $M_{Y_\gamma,\gamma}$ is the number of the maximum jumps, and $W_{Y_\gamma - 1,\gamma}$ has the law of the Lévy process conditioned on that the maximum jump is strictly less than $2^{Y_\gamma}/\gamma$. This kind of distributional representations for general Lévy processes were obtained by Buchmann, Fan and Maller, see Theorem 2.1 in [4].

3.3 Representation of the r-trimmed limit

Let $\eta_k, k=1,2,\dots$ be iid Exp(1) random variables and $Z_k=\eta_1+\ldots+\eta_k$.

Lemma 2. *For any $\gamma>0$, the sum*

$$Y_{r,\gamma}=\sum_{k=r+1}^{\infty}\left(\frac{\Psi(Z_k/\gamma)}{Z_k}-\frac{\Psi(k/\gamma)}{k}\right) \tag{27}$$

converges absolutely with probability 1 and its sum belongs to L_p for any $1\le p<r+1$.

Proof. We have

$$\begin{aligned}\left|\frac{\Psi(Z_k/\gamma)}{Z_k}-\frac{\Psi(k/\gamma)}{k}\right| &\le \Psi(Z_k/\gamma)\left|\frac{1}{Z_k}-\frac{1}{k}\right|+|\Psi(Z_k/\gamma)-\Psi(k/\gamma)|\frac{1}{k}\\ &\le 2|Z_k-k|\frac{1}{kZ_k}+|\Psi(Z_k/\gamma)-\Psi(k/\gamma)|\frac{1}{k}=:I_k+J_k.\end{aligned} \tag{28}$$

By the Hölder inequality we have for any $p\ge 1$ and any $P,Q>1$ with $1/P+1/Q=1$,

$$\mathbb{E}(|Z_k-k|^p(kZ_k)^{-p})\le k^{-p}\left(\mathbb{E}(|Z_k-k|^{pP}\right)^{1/P}\mathbb{E}(Z_k^{-pQ})^{1/Q}. \tag{29}$$

By a classical inequality (reference?? uniform integrability and central limit theorem) we have

$$\mathbb{E}(|Z_k-k|^{pP})\le c_1k^{pP/2}, \tag{30}$$

with some constant $c_1>0$ depending only on pP. In the following $c_2,c_3,\dots$ are universal positive constants, whose value is not important. On the other hand, Z_k is $\Gamma(k,1)$ distributed and thus for any $\beta>0$ we have

$$\mathbb{E}(Z_k^{-\beta})=\int_0^\infty\frac{1}{x^\beta}\frac{x^{k-1}}{\Gamma(k)}e^{-x}\mathrm{d}x=\frac{\Gamma(k-\beta)}{\Gamma(k)}\le c_2k^{-\beta}$$

for $k>\beta$, and thus in (29) we have

$$\mathbb{E}(Z_k^{-pQ})^{1/Q}\le c_3k^{-p}\qquad\text{for } k>pQ. \tag{31}$$

Since $p<r+1$, we can choose $Q>1$ so close to 1 that for $k\ge r+1$ we have $k>pQ$ and thus (31) holds. Choosing Q close to 1 will make $P=Q/(Q-1)$ very large, but (30) is still valid. Therefore, the left hand side of (29) is $\le c_4k^{-3p/2}$, and consequently in (28) we have

$$\|I_k\|_p\le c_5k^{-3/2}\quad\text{for } k\ge r+1. \tag{32}$$

To estimate J_k we first observe that by large deviation theory we have $|Z_k-k|\le k^{2/3}$ except on a set A_k with $\mathbb{P}\{A_k\}\le a\exp(-k^\delta)$ $(k\ge 1)$ for some absolute constants $a>0$, $\delta>0$. To estimate the difference $\Psi(Z_k/\gamma)-\Psi(k/\gamma)$ we have to make sure that Z_k/γ and k/γ falls into the same dyadic interval. Note that when k/γ or Z_k/γ is close to a discontinuity point of Ψ, i.e. to an integer power of 2, then we cannot give a good estimate. Therefore assume that $2^j+2^{2(j+1)/3}\le k/\gamma\le 2^{j+1}-2^{2(j+1)/3}$. Then on the set A_k^c we have $Z_k/\gamma\in[2^j,2^{j+1}]$ and thus the change

$|\Psi(Z_k/\gamma) - \Psi(k/\gamma)| \le 2^{1-j}|Z_k - k| \le 4k^{-1/3}$. Therefore, for such k's in (28) we have $J_k \le 4k^{-4/3}$ except on A_k, and on A_k trivially $J_k \le 2$. Thus we proved

$$\|J_k\|_p \le c_6 k^{-4/3} \qquad \text{for } k \ge r+1, k \in M, \tag{33}$$

with

$$M = \cup_{j=1}^{\infty} \left[\gamma(2^j + 2^{2(j+1)/3}), \gamma(2^{j+1} + 2^{2(j+1)/3})\right].$$

For $k \notin M$ we only have that $J_k \le 2/k$, but since there is not so many such k's it is enough, more precisely

$$\sum_{k \notin M} \|J_k\|_p \le \sum_{j=1}^{\infty} 4\, 2^{2(j+1)/3} 2^{-j} < \infty.$$

Consequently by (28), (32) we proved that

$$\sum_{k=r+1}^{\infty} \left\| \frac{\Psi(Z_k/\gamma)}{Z_k} - \frac{\Psi(k/\gamma)}{k} \right\|_p < \infty \qquad \text{for } p < r+1,$$

completing the proof of the lemma. □

Theorem 4. *Whenever along a subsequence $\gamma_{n_k} \to \gamma$*

$$\frac{1}{n_k} S_{n_k}^{(r)} - a_{n,\gamma}^{(r)} \xrightarrow{\mathcal{D}} Y_{r,\gamma}, \tag{34}$$

where

$$a_{n,\gamma}^{(r)} = \sum_{k=r+1}^{n} \frac{\Psi(k/\gamma)}{k}. \tag{35}$$

Proof. We rewrite the representation (8) in terms of the Poisson process determined by (η_i). Since for n fix

$$(U_{1n}, U_{2n}, \ldots, U_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{Z_1}{Z_{n+1}}, \frac{Z_2}{Z_{n+1}}, \ldots, \frac{Z_n}{Z_{n+1}} \right),$$

we obtain

$$(X_{1n}, \ldots, X_{nn}) \stackrel{\mathcal{D}}{=} \left(\frac{Z_{n+1}}{Z_1} \Psi(Z_1/Z_{n+1}), \ldots, \frac{Z_{n+1}}{Z_n} \Psi(Z_n/Z_{n+1}) \right) =: (X_{1n}^*, \ldots, X_{nn}^*).$$

By the strong law of large numbers $Z_{n+1}/n \to 1$ a.s. whence it follows

$$X_{1,n}^* = \frac{n}{Z_1} \Psi\left(\frac{Z_1}{n}\right)(1 + o(1)) \qquad \text{a.s.} \tag{36}$$

Now if along a subsequence $\gamma_{n_k} \to \gamma \in (1/2, 1]$ we obtain, using (36),

$$\frac{X_{1,n_k}^*}{n_k} \to \frac{1}{Z_1} \Psi\left(\frac{Z_1}{\gamma}\right) \qquad \text{a.s.}$$

Note that although Ψ is not continuous, the probability that Z_1/γ falls in $2^{\mathbb{Z}}$ is zero. Similar formulas apply for $X^*_{j,n_k}/n_k$ for any fixed $K \geq 1$ and thus we get

$$\frac{1}{n_k}(X_{1,n_k}, \ldots, X_{K,n_k}) \xrightarrow{\mathcal{D}} \left(\frac{\Psi\left(Z_1/\gamma\right)}{Z_1}, \ldots, \frac{\Psi\left(Z_K/\gamma\right)}{Z_K} \right). \tag{37}$$

Observe that

$$\frac{1}{n} S_n^{(r)} \stackrel{\mathcal{D}}{=} \sum_{j=r+1}^{n} \frac{\Psi(Z_j/Z_{n+1})}{n Z_j/Z_{n+1}} = \frac{Z_{n+1}}{n} \sum_{j=r+1}^{n} \frac{\Psi(Z_j/Z_{n+1})}{Z_j}. \tag{38}$$

Now by (38)

$$\frac{1}{n_k} S_{n_k}^{(r)} - a_{n_k,\gamma}^{(r)} \stackrel{\mathcal{D}}{=} \left(\frac{Z_{n_k+1}}{n_k} - 1 \right) \sum_{j=r+1}^{n_k} \frac{\Psi(Z_j/Z_{n_k+1})}{Z_j} + \sum_{j=r+1}^{n_k} \left(\frac{\Psi(Z_j/Z_{n_k+1})}{Z_j} - \frac{\Psi(j/\gamma)}{j} \right) := U_{n_k} + V_{n_k}. \tag{39}$$

By the strong law of large numbers, the first sum on the right hand side of (39) is $O(\log n)$ a.s., further Chebyshev's inequality implies $|Z_{n+1}/n - 1| = O_P(n^{-1/2})$, and thus $U_n \to 0$ in probability. On the other hand, for each j

$$\frac{\Psi(Z_j/Z_{n_k+1})}{Z_j} \to \frac{\Psi(Z_j/\gamma)}{Z_j} \quad \text{a.s.}$$

and the a.s. convergence of the series (27) imply that $V_{n_k} \to Y_{r,\gamma}$ a.s. as $n_k \to \infty$, completing the proof of the theorem.

*********** Szerintem annyi hianyzik a bizonyitasbol, hogy

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m+1}^{n} \left[\frac{\Psi(Z_j/Z_{n+1})}{Z_j} - \frac{\Psi(j/c)}{j} \right] = 0 \quad \text{a.s.}$$

*********** □

Introduce the notation

$$A_{r,\gamma} = \sum_{k=1}^{r} \frac{\Psi(k/\gamma)}{k}. \tag{40}$$

Using Lemma 2 we can determine the tail distribution of the trimmed limit. The tail behavior of the semistable limit along the subsequence $2^m + c$ (c fix, $m \to \infty$) was determined by Martin-Löf [15, Theorem 4]. Our proof in the general r-trimmed setup and also the proof of Theorem 1 use the same idea as Martin-Löf: conditioning on the maximum term.

Theorem 5. *For any* $r = 0, 1, \ldots$

$$\mathbb{P}\{Y_{r,\gamma} > x\} \sim \frac{2^{\{\log_2(\gamma x)\}(r+1)}}{(r+1)!\, x^{r+1}} \left[2^{-r-1} + (2^{r+1} - 1) \sum_{\ell=0}^{1} 2^{-\ell(r+1)} \mathbb{P}\left\{ Y_{0,\gamma} + A_{r,\gamma} > x \left(1 - 2^{\ell - \{\log_2(\gamma x)\}} \right) \right\} \right]. \tag{41}$$

Proof. Note that $\Psi(z/\gamma)/z = 2^{-\lfloor \log_2(z/\gamma) \rfloor}/\gamma$. Simple calculation shows that for any $k \geq 1$

$$\mathbb{P}\left\{ \frac{\Psi(Z_k/\gamma)}{Z_k} > x \right\} \sim \frac{1}{k!} \frac{2^{\{\log_2(\gamma x)\} k}}{x^k},$$

and by Lemma 2

$$\mathbb{P}\{Y_{r+1,\gamma} > x\} = o(x^{-(r+3/2)}).$$

We have for x large enough

$$\begin{aligned}
&\mathbb{P}\{Y_{r,\gamma} > x\} = \sum_{m=-\infty}^{\infty} \mathbb{P}\left\{ Y_{r,\gamma} > x,\, 2^{-\lfloor \log_2(Z_{r+1}/\gamma) \rfloor} = 2^m \right\} \\
&= \mathbb{P}\{Y_{r,\gamma} > x,\, -\lfloor \log_2(Z_{r+1}/\gamma) \rfloor \leq \lfloor \log_2(x\gamma) \rfloor - 1\} + \mathbb{P}\{Y_{r,\gamma} > x,\, -\lfloor \log_2(Z_{r+1}/\gamma) \rfloor = \lfloor \log_2(x\gamma) \rfloor\} \\
&\quad + \mathbb{P}\{Y_{r,\gamma} > x,\, -\lfloor \log_2(Z_{r+1}/\gamma) \rfloor = \lfloor \log_2(x\gamma) \rfloor + 1\} + \mathbb{P}\{-\lfloor \log_2(Z_{r+1}/\gamma) \rfloor \geq \lfloor \log_2(x\gamma) \rfloor + 2\} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since $2^{\lfloor \log_2(\gamma x) \rfloor - 1} \leq \gamma x/2$

$$I_1 \leq \mathbb{P}\{Y_{r+1,\gamma} > x/2\} = o(x^{-(r+3/2)}).$$

Conditioning on $Z_{r+1} \to 0$ we have

$$Y_{r+1,\gamma} = \sum_{k=r+2}^{\infty} \left(\frac{\Psi(Z_k/\gamma)}{Z_k} - \frac{\Psi((k-r-1)/\gamma)}{k-r-1} \right) + \sum_{k=1}^{r+1} \frac{\Psi(k/\gamma)}{k} \xrightarrow{\mathcal{D}} Y_{0,\gamma} + A_{r+1,\gamma}.$$

Therefore, for I_2, I_3

$$\begin{aligned}
&\mathbb{P}\{Y_{r,\gamma} > x,\, -\lfloor \log_2(Z_{r+1}/\gamma) \rfloor = \lfloor \log_2(x\gamma) \rfloor + \ell\} \\
&= \mathbb{P}\{-\lfloor \log_2(Z_{r+1}/\gamma) \rfloor = \lfloor \log_2(x\gamma) \rfloor + \ell\} \\
&\quad \times \mathbb{P}\left\{ Y_{r+1,\gamma} - \frac{\Psi((r+1)/\gamma)}{r+1} > x(1 - 2^{\ell - \{\log_2(\gamma x)\}}) \middle| - \lfloor \log_2(Z_{r+1}/\gamma) \rfloor = \lfloor \log_2(x\gamma) \rfloor + \ell \right\} \\
&\sim \frac{2^{\{(\log_2(\gamma x)\} - \ell)(r+1)}}{(r+1)! x^{r+1}} \left(2^{r+1} - 1\right) \mathbb{P}\left\{ Y_{0,\gamma} + A_{r,\gamma} > x(1 - 2^{\ell - \{\log_2(\gamma x)\}}) \right\}.
\end{aligned}$$

Finally,

$$I_4 \sim \frac{2^{(\{\log_2(\gamma x)\} - 1)(r+1)}}{(r+1)!\, x^{r+1}}.$$

Combining the asymptotics the theorem follows. □

Note that if $\{\log_2(\gamma x)\} > \delta$ for some $\delta > 0$, then $x(1 - 2^{-\{\log_2(\gamma x)\}}) \to \infty$, and so the term corresponding to $\ell = 0$ in (41) converges to 0. While if $\{\log_2(\gamma x)\} < 1 - \delta$ for some $\delta > 0$, then $x(1 - 2^{1 - \{\log_2(\gamma x)\}}) \to -\infty$, and so the term corresponding to $\ell = 1$ in (41) converges to 1. Thus the asymptotic has a simple form when γx is not close to a power of 2. In particular for any $\delta \in (0, 1/2)$ we have

$$\lim_{x \to \infty, \delta < \{\log_2(\gamma x)\} < 1 - \delta} \mathbb{P}\{Y_{r,\gamma} > x\} \frac{x^{r+1}}{2^{\{\log_2(\gamma x)\}(r+1)}} = (r+1)!\,.$$

Moreover, for $\gamma x = 2^m + c$

$$\lim_{m\to\infty} (2^m + c)(1 - 2^{-\{\log_2(2^m+c)\}}) \to c,$$

thus (41) reads as

$$\mathbb{P}\{Y_{r,\gamma} > (2^m + c)/\gamma\} \sim \frac{\gamma^{r+1} 2^{-m(r+1)}}{(r+1)!} \left[1 + (2^{r+1} - 1)\mathbb{P}\{Y_{0,\gamma} + A_{r,\gamma} > c/\gamma\}\right], \quad \text{as } m \to \infty.$$

In the untrimmed case ($r = 0$) for $\gamma = 1$ this gives

$$\mathbb{P}\{Y_{0,1} > 2^m + c\} \sim 2^{-m} \left[1 + \mathbb{P}\{Y_{0,1} > c\}\right], \quad \text{as } m \to \infty,$$

which is exactly Martin-Löf's asymptotics [15, Theorem 4, formula (9)].

Remark 4. Let $Y, Y_1, Y_2, \ldots$ be i.i.d. random variables from the domain of attraction of an α-stable law, $\alpha \in (0, 2)$. That is

$$\mathbb{P}\{|Y| > y\} = \ell(y) y^{-\alpha}, \quad \lim_{y\to\infty} \frac{\mathbb{P}\{Y > y\}}{\mathbb{P}\{|Y| > y\}} = p \in [0, 1], \quad \lim_{y\to\infty} \frac{\mathbb{P}\{Y < -y\}}{\mathbb{P}\{|Y| > y\}} = q \in [0, 1],$$

with $p + q = 1$. Let Z_n denote the partial sum, and let $a_n > 0$ and b_n such that $(Z_n - nb_n)/a_n$ converges in distribution to an α-stable law Z. Let $|Y_{1,n}| \geq |Y_{2,n}| \geq \ldots \geq |Y_{n,n}|$ denote the monotone reordering of $|Y_1|, \ldots, |Y_n|$. LePage, Woodroofe and Zinn [14, Theorem 1'] proved that the limit has the representation

$$Z = \sum_{k=1}^{\infty} \left(\delta_k \Gamma_k^{-1/\alpha} - (p - q)\mathbb{E}\Gamma_k^{-1/\alpha} I(\Gamma_k^{-1/\alpha} < 1) \right),$$

where $\delta_1, \delta_2, \ldots$ are i.i.d. ± 1 random variables with $\mathbb{P}\{\delta = 1\} = p$, and independently of δ's $\omega_1, \omega_2, \ldots$ are i.i.d. Exp(1) random variables and $\Gamma_k = \omega_1 + \ldots + \omega_k$. Moreover,

$$\left(\frac{Z_n - nb_n}{a_n}, \frac{1}{a_n} (|Y_{1,n}|, |Y_{2,n}|, \ldots, |Y_{n,n}|) \right) \xrightarrow{\mathcal{D}} \left(Z, (\Gamma_1^{-1/\alpha}, \Gamma_2^{-1/\alpha}, \ldots) \right).$$

The latter convergence allows us to obtain a representation for the limit of the trimmed sums, from which the tail behavior can be deduced.

In case of the two-sided (symmetric) version of the St. Petersburg game similar results were obtained by Berkes, Horváth and Schauer [3, Corollary 1.4].

4 The generalized St. Petersburg game

In this last section we consider the previous results in a more general setup, in the case of the so-called generalized St. Petersburg game. Since the proofs are similar to the proofs in the classical case, we omit them.

In this setup Peter tosses a possibly biased coin, where the probability of heads at each throw is $p = 1 - q$, and Paul's winning is $q^{-k/\alpha}$, if the first heads appears on the k^{th} toss, where $k \in$

$\mathbb{N} = \{1, 2, \ldots\}$, while $\alpha > 0$ is a payoff parameter. The classical St. Petersburg game corresponds to $\alpha = 1$ and $p = 1/2$. If X denotes Paul's winning in this St. Petersburg(α, p) game, then $\mathbb{P}\left\{X = q^{-k/\alpha}\right\} = q^{k-1}p$, $k \in \mathbb{N}$. In this section $X, X_1, \ldots$ are i.i.d. St. Petersburg(α, p) random variables, and S_n and X_n^* stands for the partial sum and partial maximum, respectively.

Note that when n is fix and $x \to \infty$ then the parameter α can be any positive number. While when we consider asymptotics as $n \to \infty$ the parameter is less than 2. The reason is that for $\alpha \geq 2$ the generalized St. Petersburg distribution belongs to the domain of attraction of the normal law.

For general α, p we do not have a closed formula for the probabilities $\mathbb{P}\{S_n > x\}$, nevertheless it turns out that the generalized St. Petersburg distributions are not subexponential for any choice of the parameters.

Lemma 3. *Let $\alpha > 0$. Let X_1, X_2 be independent St. Petersburg(α, p) random variables. Then*

$$2 = \liminf_{x \to \infty} \frac{\mathbb{P}\{X_1 + X_2 > x\}}{\mathbb{P}\{X_1 > x\}} < \limsup_{x \to \infty} \frac{\mathbb{P}\{X_1 + X_2 > x\}}{\mathbb{P}\{X_1 > x\}} = 2q^{-1}.$$

The liminf result is a consequence of a recent result by Foss and Korshunov [8], as they proved that for any heavy-tailed distribution the liminf is 2. The proof is simple, so we omit it.

By the definition of subexponential distributions in (10) the consequence of the lemma is that there is no subexponential generalized St. Petersburg random variable.

The tail behavior of $S_n^{(r)}$ in the general setup is the following. The proof of both theorems are almost identical to the proof in the classical case.

Theorem 6. *Let $\alpha > 0$. For any $n > r$*

$$\mathbb{P}\left\{S_n^{(r)} > x\right\} \sim \binom{n}{r+1} \frac{q^{-(r+1)\{\log_{q^{-1}} x^\alpha\}}}{x^{(r+1)\alpha}} \left(1 + (q^{-r-1} - 1)\mathbb{P}\{S_{n-r-1} > x(1 - q^{\{\log_{q^{-1}} x^\alpha\}/\alpha})\}\right).$$

For general α the analog of Theorem 5 is the following.

Theorem 7. *For $\alpha \in (0, 1)$ there exists a constant $K > 0$ such that for $x > 0$*

$$\mathbb{P}\{W_\gamma^* > x\} \leq K\, x^{-\frac{2\alpha}{2-\alpha}},$$

while for $\alpha \in (1, 2)$ there exists a constant $K > 0$ such that for $x > 0$

$$\mathbb{P}\{W_\gamma^* > x\} \leq K\, x^{-2}.$$

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