# CUMULANTS, SPREADABILITY AND THE CAMPBELL-BAKER-HAUSDORFF SERIES

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Abstract. We define spreadability systems as a generalization of exchangeability systems in order to unify various notions of independence and cumulants known in noncommutative probability. In particular, our theory covers monotone independence and monotone cumulants which do not satisfy exchangeability. To this end we study generalized zeta and Möbius functions in the context of the incidence algebra of the semilattice of ordered set partitions and prove an appropriate variant of Faa di Bruno's theorem. With the aid of this machinery we show that our cumulants cover most of the previously known cumulants. Due to noncommutativity of independence the behaviour of these cumulants with respect to independent random variables is more complicated than in the exchangeable case and the appearance of Goldberg coefficients exhibits the role of the Campbell-Baker-Hausdorff series in this context. In a final section we exhibit an interpretation of the Campbell-Baker-Hausdorff series as a sum of cumulants in a particular spreadability system, thus providing a new derivation of the Goldberg coefficients.

Contents



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## 1. Introduction

Cumulants were introduced by Thiele in the late 19th century as a combinatorial means to describe independence of classical random variables. In free probability existence of cumulants was indicated by Voiculescu [Voi85] and described explicitly by Speicher [Spe94]. Free cumulants are one of the cornerstones in free probability, complementing the analytic machinery of Cauchy transforms, see [NS06] for many applications. Later on other kinds of cumulants were introduced in noncommutative probability, e.g., various kinds of $q$-deformed cumulants were considered in [Nic95, Ans01] in order to interpolate between classical and free cumulants; however no $q$-convolution has been found so far; Boolean cumulants were defined in [SW97] in the context of Boolean independence (see also [vW73, vW75]); conditionally free cumulants were defined in [BLS96] which generalize both free and Boolean cumulants. The second-named author gave a unified theory of the cumulants mentioned above in the framework of so-called *exchangeability systems* as a general notion of independence [Leh04]. In the present paper we develop a yet more general framework which comprises also Muraki's monotone independence [Mur01], which is not covered by the approach of [Leh04] because it does not satisfy exchangeability, the obstruction being that monotone independence is sensitive to the order on random variables:

(O) Independence of $X$ and $Y$ is not equivalent to independence of $Y$ and $X$.

Hence in order to avoid misinterpretations we say "the (ordered) pair $(X,Y)$ is monotone independent" rather than "$X$ and $Y$ are monotone independent". This property sharply distinguishes monotone independence from classical, free, and Boolean independences. Despite lack of exchangeability, the first-named author together with H. Saigo managed to define monotone cumulants [HS11b, HS11a] relying only on the property called *extensivity* defined below. If we denote the monotone cumulants by $K_n^{\mathrm{M}}(X_1,X_2,\dots,X_n)$ with respect to a noncommutative probability space $(\mathcal{A},\varphi)$, then we have the moment-cumulant formula:

$$\varphi(X_1X_2\cdots X_n)=\sum_{\pi\in\mathcal{M}_n}\frac{1}{|\pi|!}K_{(\pi)}^{\mathrm{M}}(X_1,X_2,\dots,X_n). \tag{MC}$$

The notations are introduced in Sections 2, 3, and 4. The set $\mathcal{M}_n$ of monotone partitions is not a subclass of set partitions, but a subclass of ordered set partitions. The factor $\frac{1}{|\pi|!}$ is new in the sense that it is hidden in the classical, free or Boolean cases, the reason being that $|\pi|!$ is the number of possible orderings of the blocks of the underlying set partition of $\pi$.

Cumulants carry essential information on independence, in particular the *vanishing* of mixed cumulants, that is, of cumulants with independent entries, characterize independence [Leh04, Prop. 3.5], which is the major reason for their usefulness in free probability [NS06]. More precisely, if a finite family of random variables $X_1,X_2,\dots,X_n$ can be partitioned into two mutually independent subfamilies (in the general sense of Definition 3.6 below) then

$$K_n(X_1,X_2,\dots,X_n)=0. \tag{V}$$

As a consequence, cumulants are additive, that is, the cumulant of the sum of two independent tuples $(X_1,X_2,\dots,X_n)$ and $(Y_1,Y_2,\dots,Y_n)$ decomposes as

$$K_n(X_1+Y_1,X_2+Y_2,\dots,X_n+Y_n)=K_n(X_1,X_2,\dots,X_n)+K_n(Y_1,Y_2,\dots,Y_n). \tag{1.1}$$

By contrast, because of property (O), monotone cumulants do not satisfy additivity and thus mixed cumulants do not necessarily vanish. Instead, they satisfy the weaker notion of *extensivity*: if $\{(X_1^{(j)},X_2^{(j)},\dots,X_n^{(j)})\}_{j=1}^{\infty}$ is a sequence of monotone independent random vectors such that $(X_1,X_2,\dots,X_n)\overset{d}{=}(X_1^{(j)},X_2^{(j)},\dots,X_n^{(j)})$ for any $j\geq 1$, then

$$K_n^{\mathrm{M}}(N.X_1,N.X_2,\dots,N.X_n)=NK_n^{\mathrm{M}}(X_1,X_2,\dots,X_n), \tag{E}$$

where $N.X_i=X_i^{(1)}+X_i^{(2)}+\cdots+X_i^{(N)}$ is the sum of i.i.d. copies.

Extensivity is strictly weaker than the property of vanishing of mixed cumulants, but extensivity (together with some other properties) still suffices to prove uniqueness of cumulants even in the case of exchangeability. Therefore extensivity is a natural generalization of the property of vanishing of mixed cumulants.

The first goal of this paper is the unification of second-named author's approach based on exchangeability systems and first-named author's monotone cumulants based on natural products of states. The second-named author's definition of cumulants includes $q$-deformed cumulants as well as tensor (or classical), free and Boolean cumulants. On the other hand, the approach of the first-named author and Saigo comprises monotone cumulants as well as tensor, free and Boolean cumulants, but not $q$-deformed cumulants. In the present paper we establish a unified theory based on the concept of spreadability which has been considered recently by Köstler [Kös10] in the noncommutative context. Similar to the transition from symmetric to quasisymmetric functions, the concept of *spreadability systems* naturally arises as a generalization of exchangeability systems and allows to unify various kinds of independence and cumulants, including conditionally monotone independence [Has11] and two other generalized notions of independence [Has10], with a generalization of the moment-cumulant formula (MC) and the property of extensivity (E).

Our approach is combinatorial on the basis of *ordered set partitions*. An ordered set partition is defined as an ordered *sequence* of disjoint subsets whose union is the entire set (say $\{1, 2, \dots, n\}$). We first investigate the structure of the semilattices $\mathcal{OP}_n$ of ordered set partitions (with respect to dominance order) and show that they locally look like the lattices $\mathcal{I}_n$ of interval partitions, in the sense that any interval in $\mathcal{OP}_n$ is isomorphic to an interval in $\mathcal{I}_k$ for an appropriate $k$. This property is crucial for the study of multiplicative functions in the incidence algebra of ordered set partitions and we establish an isomorphism with the composition algebra of generating functions analogous to the well known formula of Faa di Bruno. In particular, we obtain the fundamental convolution identity for the generalized zeta- and Möbius functions.

Our interest in ordered set partitions was stipulated by the appearance of monotone partitions in the classification of independence [Mur03] and their role for monotone cumulants [HS11b], but presently it turns out that the structure of ordered set partitions is actually easier to describe than that of monotone partitions. For example we do not have a good understanding of the structure of intervals $[\sigma, \pi] \subseteq \mathcal{M}_n$ for $\sigma, \pi \in \mathcal{M}_n$ and in particular the values of the Möbius function remain mysterious. This and some other open problems on monotone partitions are listed at the end of the paper and left for future investigations.

The second goal is the application of our results to *free Lie algebras*. Connections of free probability to formal groups have been pointed out early by Voiculescu [Voi85] and more recently by Friedrich and McKay [FM15] and in the context of monotone probability by Manzel and Schürmann [MS17]; see also an approach via shuffle algebras [EFP18].

Here we obtain new formulations of some well known identities in terms of ordered set partitions and cumulants. In particular our cumulants turn out to coincide with the homogenous components of the *Campbell-Baker-Hausdorff formula*, if an appropriate spreadability system is chosen. Thus a new derivation of the coefficients of the Baker-Campbell-Hausdorff formula ("Goldberg coefficients") drops out as a by-product of our results on spreadability systems.

We hope that our results will stimulate more connections to combinatorics, in particular Hopf algebras and noncommutative quasi-symmetric functions [BZ09]. Not being familiar with the abstract theory behind these concepts we proceed in a pedestrian way. Our proofs only use elementary and at times tedious calculations, yet we suspect that many of our results have been obtained in different contexts before and that those with the right knowledge will find easier and more conceptual proofs.

The paper is organized as follows. In Section 2 we collect basic definitions and results concerning ordered set partitions, including the description of intervals in the poset of ordered set partitions, incidence algebras, multiplicative functions, generalized zeta functions and the corresponding Möbius functions.

In Section 3, we define spreadability systems and give examples coming from the four (or five) natural products of linear maps. Then we define a generalized notion of independence and show that it indeed extends the existing theories.

In Section 4, we define cumulants associated to a spreadability system and express cumulants in terms of moments and vice versa. We prove extensivity and uniqueness of cumulants and list the previous moment-cumulant formulas in the literature as special cases.

In Section 5 we discuss the central limit theorem associated to spreadability systems satisfying the singleton condition.

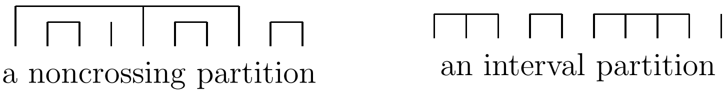


FIGURE 1. Examples of partitions

In Section 6, we obtain recursive differential equations for the time evolution of moments. This generalizes for example the complex Burger's equation in free probability.

In Section 7, we compute *mixed cumulants*, i.e., we express cumulants of random variables, which split into "independent subsets" in terms of lower order moments or cumulants. These formulas shed light on the gap between extensivity (E) and the vanishing property (V) and uncover the role of the Campbell-Baker-Hausdorff formula.

In Section 8 we define a spreadability system whose cumulants are the homogeneous components of the Baker-Campbell-Hausdorff formula; in particular, the specialization of our cumulants to free algebras yields Lie projectors. Finally we give a new derivation of the coefficients of the Baker-Campbell-Hausdorff formula (also known as "Goldberg coefficients").

We conclude the paper with several open problems.

## 2. Ordered set partitions

2.1. **Set partitions.** Let $\mathbb{N}$ denote the set of natural numbers $\{1,2,3,\dots\}$, and let $[n]$ denote the finite set $\{1,2,\dots,n\}$.

A *set partition*, or simply partition, of a finite set $A$ is a set of mutually disjoint subsets $\pi = \{P_1, P_2, \dots, P_k\}$ such that $\cup_{i=1}^k P_i = A$. The number $k$ is the *size* of the partition and denoted by $|\pi|$. The elements $P \in \pi$ are called *blocks* of $\pi$. The set of partitions of $A$ is denoted by $\mathcal{P}_A$. We are mostly concerned with the case $A = [n]$ and in this case $\mathcal{P}_{[n]}$ is abbreviated to $\mathcal{P}_n$ and called (set) *partitions of order n*. As is well known there is a one-to-one correspondence between set partitions $\pi$ of $[n]$ and equivalence relations on $[n]$ by defining for $\pi \in \mathcal{P}_n$ and $i,j \in [n]$ the relation $i \sim_\pi j$ to hold if and only if there is a block $P \in \pi$ such that both $i,j \in P$. The set partitions of fixed order $n$ form a lattice under *refinement order*: For partitions $\pi$ and $\sigma$ we write $\pi \le \sigma$ if for any block $P \in \pi$, there exists a block $S \in \sigma$ such that $P \subseteq S$. In other words, every block of $\sigma$ is a union of blocks of $\pi$. The minimal element of this lattice is $\hat{0}_n = \{\{1\},\{2\},\dots,\{n\}\}$ and the maximal element $\hat{1}_n = \{\{1,2,\dots,n\}\}$. We proceed with the description of several classes of set partitions.

**Definition 2.1.** Let $\pi \in \mathcal{P}_n$ be a set partition.

(i) Two (distinct) blocks $B$ and $B' \in \pi$ are said to be *crossing* if there are elements $i < i' < j < j'$ such that $i,j \in B$ and $i',j' \in B'$. $\pi$ is called *noncrossing* if there are no crossing blocks, i.e., if there is no quadruple of elements $i < j < k < l$ s.t. $i \sim_\pi k$, $j \sim_\pi l$ and $i \not\sim_\pi j$. The noncrossing partitions of order $n$ form a sublattice which we denote by $\mathcal{NC}_n$.

(ii) Two blocks $B, B'$ of a noncrossing partition $\pi$ are said to form a *nesting* if there are $i,j \in B$ such that $i < k < j$ for any $k \in B'$. In this case $B$ is called the *outer block* of the nesting and $B'$ is called the *inner block* of the nesting.

(iii) A block $B$ of a noncrossing partition $\pi$ is *inner* if $B$ is the inner block of a nesting of $\pi$. If this is not the case $B$ is called an *outer block* of $\pi$. The set of inner blocks of $\pi$ is denoted by $\operatorname{Inner}(\pi)$ and the set of outer blocks of $\pi$ by $\operatorname{Outer}(\pi)$.

(iv) An *interval partition* is a partition $\pi$ for which every block is an interval. Equivalently, this means that $\pi$ is noncrossing and has no nestings. The set of interval partitions of $[n]$ is denoted by $\mathcal{I}_n$. The blocks of an interval partition $\{I_1, I_2, \dots, I_p\}$ can be uniquely ordered so that $i < j$ whenever $i \in I_s$, $j \in I_t$, $s < t$. This ordering provides a natural embedding $\mathcal{I}_n \subseteq \mathcal{OI}_n$ (see Definition 2.6(ii)) and we may write $(I_1, I_2, \dots, I_p) \in \mathcal{I}_n$ rather than $\{I_1, I_2, \dots, I_p\} \in \mathcal{I}_n$.

Analogous definitions apply to any finite linearly ordered set $A$ and the corresponding sets of partitions are denoted $\mathcal{P}_A$, $\mathcal{NC}_A$, $\mathcal{I}_A$ etc.

Examples of partitions are shown in Fig. 1. The lattice of interval partitions plays a central role in this paper and has a particularly simple structure.

**Proposition 2.2.** *The lattice of interval partitions $\mathcal{I}_n$ is anti-isomorphic to the Boolean lattice $\mathcal{B}_{n-1}$ via the lattice antiisomorphism*

$$(I_1, I_2, \dots, I_p) \mapsto \{r_1, r_2, \dots, r_{p-1}\} \subseteq [n-1],$$

*where the blocks $I_i$ are uniquely determined by their maximal elements $r_i$; note that always $r_p = n$.*

The following construction inverts the previous bijection in a certain sense.

**Definition 2.3.** Fix a number $n \in \mathbb{N}$ and a subset $A \subseteq [n]$.

(i) Among all noncrossing partitions containing $A^c$ as an outer block there is a maximal one, which we denote by $\nu_{\max}(A)$. Removing $A^c$ we obtain an interval partition of $A$ which we denote by $\iota_{\max}(A)$; in other words, the blocks of $\iota_{\max}(A)$ consist of the maximal contiguous subintervals of $A$. Yet in another interpretation, the blocks of $\iota_{\max}(A)$ are the connected components of the graph induced on $A$ from the integer line.

(ii) Among all noncrossing partitions containing $A^c$ as a block there is a maximal one, which we denote by $\tilde{\nu}_{\max}(A)$. Removing $A^c$ we obtain a noncrossing partition of $A$ which we denote by $\tilde{\iota}_{\max}(A)$; in other words, the blocks of $\tilde{\iota}_{\max}(A)$ consist of the maximal contiguous subintervals of $A$ when we consider it on the circle, i.e., the blocks of $\tilde{\iota}_{\max}(A)$ are the connected components of the graph induced on $A$ from the Cayley graph of $\mathbb{Z}_n$.

**Remark 2.4.** 1. Construction (i) will occur in some examples below (see, e.g., Examples 6.6 and 6.7).
2. Construction (ii) occurs in the recursion (6.1) for free cumulants and Example 6.8 below.

The lattices considered so far have the following structural property. It is easy to see for both $\mathcal{P}_n$ and $\mathcal{I}_n$ while for $\mathcal{NC}_n$ it is proved in [Spe94].

**Proposition 2.5.** *Let $P_n$ be one of $\mathcal{P}_n$, $\mathcal{NC}_n$ and $\mathcal{I}_n$. Then for any pair of elements $\sigma, \pi \in P_n$ such that $\sigma \leq \pi$ there are uniquely determined numbers $k_j$ such that the interval $[\sigma, \pi]$ is isomorphic (as a lattice) to the direct product*

$$P_1^{k_1} \times P_2^{k_2} \times \cdots \times P_n^{k_n}.$$

2.2. **Ordered set partitions.** An *ordered set partition* of a set $A$ is a sequence $(P_1, P_2, \dots, P_p)$ of distinct blocks such that $\{P_1, P_2, \dots, P_p\}$ is a set partition of $A$. In other words, it is a set partition with a total ordering of its blocks. Ordered set partitions are also known under the name of *set compositions*, see, e.g., [BZ09], *pseudopermutations* [KLN+00], *packed words* [NT06], etc. The set of ordered set partitions of $A$ is denote by $\mathcal{OP}_A$ and $\mathcal{OP}_{[n]}$ is abbreviated to $\mathcal{OP}_n$. Let $\pi \mapsto \bar{\pi}$ be the map from $\mathcal{OP}_n$ onto $\mathcal{P}_n$ which drops the order on blocks, that is, $(P_1, P_2, \dots) \mapsto \{P_1, P_2, \dots\}$. We say that an ordered set partition is *in canonical order* if the blocks are sorted in ascending order according to their minimal elements. It will be convenient to transfer as much structure as possible from ordinary set partitions to ordered set partitions when no confusion can arise. For example, the notation $P \in \pi$ indicates that $P$ is a block of $\bar{\pi}$. There is a natural partial order relation on $\mathcal{OP}_n$, namely if $\pi = (P_1, P_2, \dots, P_p), \sigma = (S_1, S_2, \dots, S_s) \in \mathcal{OP}_n$, we define $\pi \leq \sigma$ if the following conditions hold.

- $\bar{\pi} \leq \bar{\sigma}$ as set partitions.
- If $P_i \subseteq S_k, P_j \subseteq S_l$ for $i < j$, then $k \leq l$.

In other words, $\pi \leq \sigma$ if every block of $\sigma$ is a union of a contiguous sequence of blocks of $\pi$. This order makes $(\mathcal{OP}_A, \leq)$ a poset (but not a lattice as will be seen shortly). Note that $\pi \leq \sigma$ implies $\bar{\pi} \leq \bar{\sigma}$, but not vice versa.

We consider the following subclasses of ordered set partitions on $[n]$.

**Definition 2.6.** (i) An ordered set partition $\pi \in \mathcal{OP}_n$ is called *noncrossing* if the underlying set partition $\bar{\pi}$ has this property. The set of ordered noncrossing partitions of $[n]$ is denoted by $\mathcal{ONC}_n$. Outer blocks and inner blocks of a ordered noncrossing partition are defined according to the case of noncrossing partitions.

(ii) An *interval ordered set partition* of $[n]$ is an ordered set partition $\pi$ such that $\bar{\pi} \in \mathcal{I}_n$. The set of interval ordered set partitions of $[n]$ is denoted by $\mathcal{OI}_n$.

(iii) A *monotone partition* is a noncrossing ordered set partition $\pi = (P_1, P_2, \dots, P_{|\pi|})$ such that for every nesting the outer block precedes the inner block; in other words, the order of the blocks implements a linearization of the partial order given by the nesting relation. The set of monotone partitions is denoted by $\mathcal{M}_n$.

**Example 2.7.** Figure 2 shows some examples of monotone and non-monotone partitions whose underlying noncrossing partition is $\pi =$ .

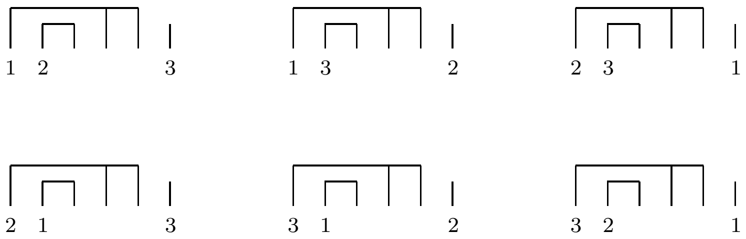


FIGURE 2. Monotone partitions (upper row) and non-monotone partitions (lower row). The labeled numbers denote the order of blocks.

In section 8, we need the following extension of ordered set partitions.

**Definition 2.8.** An *ordered pseudopartition* of $[n]$ is a sequence $(P_1, P_2, \dots, P_p)$ of disjoint subsets of $[n]$ such that $\cup_{i=1}^p P_i = [n]$ with empty blocks allowed. We keep the notation $|\pi| = p$ for the length, now including empty blocks. The set of ordered pseudopartitions of $[n]$ is denoted by $\mathcal{OPP}_n$.

**Lemma 2.9** (see, e.g., [BS96])**.** *The poset of ordered set partitions $\mathcal{OP}_n$ is isomorphic to the poset of nonempty chains in the boolean lattice $2^n$ with the reverse refinement order, i.e., a chain $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_k = [n]$ is smaller than a chain $\varnothing = B_0 \subset B_1 \subset \cdots \subset B_l = [n]$ if it is finer, i.e., as sets $\{B_0, B_1, \dots, B_l\} \subseteq \{A_0, A_1, \dots, A_k\}$.*

*Proof.* The bijection is given by the map

$$\Phi_n : (A_1 \subset A_2 \subset \cdots \subset A_k) \mapsto (A_1, A_2 \setminus A_1, A_3 \setminus A_2, \dots, A_k \setminus A_{k-1}).$$

$\square$

Moreover, when the empty chain is added, $\mathcal{OP}_n$ becomes a lattice isomorphic to the face lattice of the *permutohedron* [BS96, Ton97]. With this alternative picture it is now easy to see a join semilattice structure, namely the join operation corresponds to the intersection of chains in the chain poset.

The maximal ordered set partition is unique and is $\hat{1}_n := ([n])$, while there are several minimal elements, namely all permutations of the minimal set partition $\hat{0}_n := (\{1\}, \{2\}, \dots, \{n\})$. Consequently there is no meet operation, but we define the following associative but noncommutative replacement. It turns $\mathcal{OP}_n$ into a *band*, i.e., a semigroup in which every element is an idempotent; however it is not a skew lattice.

**Definition 2.10.** (i) For an ordered set partition $\sigma = (S_1, S_2, \dots, S_s) \in \mathcal{OP}_n$ and a nonempty subset $P \subseteq [n]$, let the *restriction* $\sigma\rfloor_P \in \mathcal{OP}_P$ be the ordered set partition $(P \cap S_1, P \cap S_2, \dots, P \cap S_s) \in \mathcal{OP}_P$, where empty sets are dropped; see [PS08], where this operation arises in the context of Hopf algebras.

(ii) For $\pi = (P_1, P_2, \dots, P_p), \sigma = (S_1, S_2, \dots, S_s) \in \mathcal{OP}_n$, we define the *quasi-meet operation* $\pi \curlywedge \sigma \in \mathcal{OP}_n$ to be $(P_1 \cap S_1, P_1 \cap S_2, \dots, P_1 \cap S_s, P_2 \cap S_1, P_2 \cap S_2, \dots, P_2 \cap S_s, \dots, P_p \cap S_s)$, where empty sets are skipped. In other words, the quasi-meet is the concatenation of the restrictions $\pi \curlywedge \sigma = \sigma\rfloor_{P_1} \sigma\rfloor_{P_2} \cdots \sigma\rfloor_{P_p}$.

**Remark 2.11.** The quasi-meet operation is associative and coincides with the multiplication operation in the Solomon-Tits algebra of the symmetric group [Sol76] which has received a lot of attention recently from the point of view of Markov chains [Bro00, Bro04] and Hopf algebras of noncommutative quasi-symmetric functions [BZ09].

**Proposition 2.12.** *(i) The quasi-meet operation on $\mathcal{OP}_n$ is compatible with the meet operation on $\mathcal{P}_n$, in the sense that $\overline{\pi \curlywedge \sigma} = \bar{\pi} \wedge \bar{\sigma}$.*

*(ii)* $\pi \wedge \sigma \leq \pi$ *for any* $\pi, \sigma \in \mathcal{OP}_n$.
*(iii) If* $\sigma \leq \rho \in \mathcal{OP}_n$, *then* $\sigma \wedge \pi \leq \rho \wedge \pi$ *for any* $\pi \in \mathcal{OP}_n$.
*(iv)* $\pi \wedge \sigma = \pi \iff \bar{\pi} \leq \bar{\sigma}$.
*(v)* $\pi \wedge \sigma = \sigma \iff \sigma \leq \pi$.

*Proof.* The first three items are immediate from the definition. To see (iv), assume first $\pi \wedge \sigma = \pi$. Then by (i) also $\bar{\pi} = \bar{\pi} \wedge \bar{\sigma} \leq \bar{\sigma}$. Conversely, if $\bar{\pi} \leq \bar{\sigma}$, then $\bar{\pi} \wedge \bar{\sigma} = \bar{\pi}$ and by (ii) $\pi \wedge \sigma \leq \pi$. Since the number of blocks of $\bar{\pi} \wedge \bar{\sigma}$ and $\pi \wedge \sigma$ are equal, we must have $\pi \wedge \sigma = \pi$.

As for (v), if $\pi \wedge \sigma = \sigma$ then it follows from (ii) that $\sigma \leq \pi$. On the other hand, if $\sigma \leq \pi$ and $\pi = (B_1, B_2, \ldots, B_k)$, then $\pi \wedge \sigma = \sigma\rfloor_{B_1} \sigma\rfloor_{B_2} \cdots \sigma\rfloor_{B_k}$ and the order of the blocks of $\sigma$ remains unchanged, so $\pi \wedge \sigma = \sigma$. □

The interval structure of the poset $\mathcal{OP}_n$ can be described as follows. Recall that a *down-set* or *order ideal* in a poset $P$ is a subset $I$ such that $x \in I$ and $y \leq x$ implies $y \in I$. The *principal ideal generated by* $x$, denoted by $\downarrow x$, is the smallest down-set containing $x$, i.e.,

$$\downarrow x = \{y \in P : y \leq x\}.$$

The following proposition is immediate.

**Proposition 2.13.** *The principal ideal generated by an element* $\pi = (P_1, P_2, \ldots, P_p) \in \mathcal{OP}_n$ *is canonically isomorphic to*

$$\mathcal{OP}_{P_1} \times \mathcal{OP}_{P_2} \times \cdots \times \mathcal{OP}_{P_p}$$

*via the map*

$$\sigma \mapsto (\sigma\rfloor_{P_1}, \sigma\rfloor_{P_2}, \ldots, \sigma\rfloor_{P_p})$$

There is no direct analogue of Proposition 2.5 for ordered set partitions; instead the next proposition shows that the interval structure can be expressed in terms of lattices of interval partitions.

**Proposition 2.14.** *Let* $\sigma, \pi \in \mathcal{OP}_n$ *be ordered set partitions of size* $|\sigma| = s$ *and* $|\pi| = p$ *respectively such that* $\sigma \leq \pi$. *If* $\pi = (P_1, P_2, \ldots P_p)$, *let* $k_j$ *be the number of blocks of* $\sigma$ *contained in* $P_j$, $j \in \{1, 2, \ldots, p\}$. *Then* $k_1 + k_2 + \cdots + k_p = s$ *and as a poset the interval* $[\sigma, \pi]$ *is canonically isomorphic to* $\mathcal{I}_{k_1} \times \mathcal{I}_{k_2} \times \cdots \times \mathcal{I}_{k_p}$. *More precisely, if* $\sigma = (S_1, S_2, \ldots, S_s) \leq \pi \in \mathcal{OP}_n$, *then there exists a unique* $\tau = (T_1, T_2, \ldots, T_p) \in \mathcal{I}_s$ *such that*

$$\pi = (\bigcup_{i \in T_1} S_i, \bigcup_{i \in T_2} S_i, \ldots, \bigcup_{i \in T_p} S_i).$$

*The map* $\Phi : \mathcal{I}_{T_1} \times \mathcal{I}_{T_2} \times \cdots \times \mathcal{I}_{T_p} \to [\sigma, \pi]$ *by*

$$(\tau_1, \tau_2, \ldots, \tau_p) \mapsto (\bigcup_{i \in T_{1,1}} S_i, \bigcup_{i \in T_{1,2}} S_i, \ldots, \bigcup_{i \in T_{1,k_1}} S_i, \ldots, \bigcup_{i \in T_{p,1}} S_i, \ldots, \bigcup_{i \in T_{p,k_p}} S_i),$$

*where* $\tau_i = (T_{i,1}, T_{i,2}, \ldots, T_{i,k_i})$, *is a bijection, and so its inverse establishes a bijection*

$$\Psi := \Phi^{-1} : [\sigma, \pi] \to \mathcal{I}_{k_1} \times \mathcal{I}_{k_2} \times \cdots \times \mathcal{I}_{k_p}$$

*with* $k_i := |T_i|$. *The composition* $(k_1, k_2, \ldots, k_p)$ *is called the* type *of the interval* $[\sigma, \pi]$.

*Proof.* Let $\pi = (P_1, P_2, \ldots, P_p)$. Each block of $\pi$ is the union of blocks of $\sigma$, and so we can find $A \subseteq [s]$ such that $P_1 = \cup_{i \in A} S_i$. We show that there exists $k$ such that $A = [k]$. Suppose that there are $1 \leq u < v \leq p$ such that $u \notin A$ and $v \in A$. Then there is $j \geq 2$ such that $S_u \subseteq P_j$. This contradicts the fact that $u < v$, $S_v \subseteq P_1$ and $\pi \leq \sigma$. Hence $A = [k]$ for some $k$. Removing the first block of $\pi$ and the first $k$ blocks of $\sigma$ we can repeat the argument with the ordered set partitions $(S_{k+1}, S_{k+2}, \ldots, S_s) \leq (P_2, P_3, \ldots, P_p)$ and find $P_2 = \cup_{k+1 \leq i \leq k+l} S_i$ for some $l$. After a finite number of iterations we thus construct a unique interval partition $\tau = (T_1, T_2, \ldots, T_p) \in \mathcal{I}_s$ such that

$$\pi = (\bigcup_{i \in T_1} S_i, \bigcup_{i \in T_2} S_i, \ldots, \bigcup_{i \in T_p} S_i).$$

Clearly the image of $\Phi$ is contained in $[\sigma, \pi]$ and $\Phi$ is injective and it remains to show surjectivity. To this end pick an arbitrary $\rho \in [\sigma, \pi]$. From the first part of the proposition we infer that $\rho \geq \sigma$ is of the form

$$\rho = (\bigcup_{i \in G_1} S_i, \bigcup_{i \in G_2} S_i, \ldots, \bigcup_{i \in G_t} S_i)$$

for some $\gamma = (G_1, G_2, \dots, G_t) \in \mathcal{I}_s$. Since $\rho \le \pi$, the partition $\gamma$ must be finer than $\tau$. Hence, each restriction $\gamma\rfloor_{T_i} \in \mathcal{I}_{T_i}, i \in [p]$, consists of sequence of consecutive blocks of $\gamma$ without splitting any original block $G_1, G_2, \dots, G_t \in \gamma$. Hence we obtain $\Phi((\gamma_1, \gamma_2, \dots, \gamma_p)) = \rho$ and the map $\Phi$ is indeed bijective. □

2.3. **Incidence algebras and multiplicative functions.** Let $(P, \le)$ be a (finite) partially ordered set. The incidence algebra $\mathfrak{I}(P) = \mathfrak{I}(P, \mathbb{C})$ is the algebra of functions supported on the set of pairs $\{(x, y) \in P \times P : x, y \in P; x \le y\}$ with convolution

$$f * g(x, y) = \sum_{x \le z \le y} f(x, z)\, g(z, y)$$

For example, if $P$ is the $n$-set $\{1, 2, \dots, n\}$ with the natural order, then $\mathfrak{I}(P)$ is the algebra of $n \times n$ upper triangular matrices. The algebra $\mathfrak{I}(P)$ is unital with the Kronecker function $\delta(x, y)$ serving as the unit element and a function $f \in \mathfrak{I}(P)$ is invertible if and only if $f(x, x)$ is nonzero for every $x \in P$. An example of an invertible function is the *Zeta function*, which is defined as $\zeta(x, y) \equiv 1$. Its inverse is called the *Möbius function* of $P$, denoted $\mu(x, y)$. For functions $F, G : P \to \mathbb{C}$ we have the fundamental equivalence ("Möbius inversion formula")

$$\left(\forall x \in P : F(x) = \sum_{y \le x} G(y)\right) \quad \Longleftrightarrow \quad \left(\forall x \in P : G(x) = \sum_{y \le x} F(y)\, \mu(y, x)\right)$$

A function $f \in \mathfrak{I}(\mathcal{P}_n)$ (actually a family of functions) is called *multiplicative* if there is a *characteristic sequence* $(f_n)_{n \ge 1}$ such that for any pair $\sigma, \pi \in \mathcal{P}$ we have

$$f(\sigma, \pi) = \prod f_i^{k_i}$$

where $k_i$ are the structural constants of the interval $[\sigma, \pi]$ from Proposition 2.5. It can be shown [DRS72] that the multiplicative functions form a subalgebra of the incidence algebra $\mathfrak{I}(\mathcal{P}_n)$. For example, the Zeta function is multiplicative with characteristic sequence $(1, 1, \dots)$ and the Möbius function is multiplicative as well with characteristic sequence $\mu_n = (-1)^{n-1}(n-1)!$, cf. [Sch54, Rot64]; more precisely, if $\pi = \{P_1, P_2, \dots, P_p\}$ then

$$\mu_{\mathcal{P}}(\sigma, \pi) = \prod_{i=1}^{p} (-1)^{k_i - 1}(k_i - 1)! \tag{2.1}$$

where $k_i = \#(\sigma\rfloor_{P_i})$. Multiplicative functions on the lattice of set partitions provide a combinatorial model for Faa di Bruno's formula which expresses the Taylor coefficients of a composition of exponential formal power series in terms of the coefficients of the original functions, see [DRS72, Sta99]; in the case of noncrossing partitions the convolution is commutative and can be modeled as multiplication of certain power series ("$S$-transforms"), see [NS06]. The lattice of interval partitions combinatorially models the composition of ordinary formal power series, see [Joy81]. From Proposition 2.14 one might guess that convolution on the poset of ordered set partitions is also related to some kind of function composition, and Proposition 2.16 below shows that this is indeed the case for a certain class of functions to be defined next.

**Definition 2.15.** Denote by $\mathbb{N}_{\text{fin}}^{\infty}$ the set of finite sequences of positive integers,

$$\mathbb{N}_{\text{fin}}^{\infty} = \bigcup_{p=1}^{\infty} \mathbb{N}^p$$

For $m \in \mathbb{N}$ let $F_{\mathcal{OP}^m}$ be the set of $\mathbb{C}$-valued functions $f$ on the set of $m$-chains

$$\{(\sigma_1, \dots, \sigma_m) \in \cup_{n=1}^{\infty} \underbrace{(\mathcal{OP}_n \times \dots \times \mathcal{OP}_n)}_{m \text{ fold}} \mid \sigma_1 \le \sigma_2 \le \dots \le \sigma_m\}.$$

We are concerned with different levels of reduced incidence algebras.

(i) A function $f \in F_{\mathcal{OP}^2}$ is said to be *adapted* if there is a family $(f_{\underline{k}})_{\underline{k} \in \mathbb{N}_{\text{fin}}^{\infty}} \subseteq \mathbb{C}$ such that

$$f(\sigma, \pi) = f_{k_1, \dots, k_p},$$

where $(k_i)_{i=1}^{p}$ is the type of the interval $[\sigma, \pi]$ as defined in Proposition 2.14. The family $(f_{\underline{k}})_{\underline{k} \in \mathbb{N}_{\text{fin}}^{\infty}}$ is called the *defining family.*

(ii) To any adapted function $f \in F_{\mathcal{OP}^2}$ we associate its multivariate generating function

$$\underline{Z}_f(\underline{z}) = \sum_{\underline{k} \in \mathbb{N}_{\text{fin}}^\infty} f_{\underline{k}} \underline{z}^{\underline{k}}, \qquad \underline{z} = (z_1, z_2, z_3, \dots)$$

with the usual multiindex convention $\underline{z}^{\underline{k}} = z_1^{k_1} z_2^{k_2} \cdots$, where $z_1, z_2, \dots$ are commuting indeterminates.

(iii) A function $f \in F_{\mathcal{OP}^2}$ is said to be *multiplicative* if it is adapted and moreover the defining family satisfies

$$f_{k_1,\dots,k_p} = \prod_{i=1}^p f_{k_i}.$$

If $f$ is multiplicative, then the sequence $f_n = f(\hat{0}_n, \hat{1}_n)$ is called the *defining sequence* of $f$.

(iv) To a multiplicative function $f \in F_{\mathcal{OP}^2}$ we associate the (univariate) generating function

$$Z_f(z) = \sum_{n=1}^\infty f_n z^n.$$

(v) A function $f \in F_{\mathcal{OP}^3}$ is said to be *quasi-multiplicative* if there is an array $(f_{jk})_{j,k=1}^\infty \subseteq \mathbb{C}$ such that

$$f(\sigma, \rho, \pi) = \prod_{i=1}^p \prod_{G \in \gamma_i} f_{i,|G|},$$

where $(\gamma_1, \dots, \gamma_p)$ is the image of $\rho$ under the map $\Psi$ in Proposition 2.14. The array $(f_{jk})_{j,k=1}^\infty$ is called the *defining array* of $f$.

(vi) To a quasi-multiplicative function $f \in F_{\mathcal{OP}^3}$ we associate a sequence of (univariate) generating functions

$$Z_f^{(j)}(z) = \sum_{k=1}^\infty f_{jk} z^k, \qquad j \in \mathbb{N}.$$

(vii) For $f \in F_{\mathcal{OP}^3}$ and $g \in F_{\mathcal{OP}^2}$ we define the convolution

$$(f \circledast g)(\sigma, \pi) := \sum_{\rho \in [\sigma, \pi]} f(\sigma, \rho, \pi) g(\rho, \pi), \qquad \sigma \leq \pi. \tag{2.2}$$

This latter provides a combinatorial model for the composition of multivariate functions.

**Proposition 2.16.** *If $f \in F_{\mathcal{OP}^3}$ is quasi-multiplicative and $g \in F_{\mathcal{OP}^2}$ is adapted then $f \circledast g \in F_{\mathcal{OP}^2}$ is adapted and*

$$\underline{Z}_{f \circledast g}(\underline{z}) = \underline{Z}_g(Z_f^{(1)}(z_1), Z_f^{(2)}(z_2), \dots), \qquad \underline{z} = (z_1, z_2, \dots).$$

*Proof.* We use the notations in the statement of Proposition 2.14. Pick any $\rho \in [\sigma, \pi]$ and let $(\gamma_1, \gamma_2, \dots, \gamma_p) := \Psi(\rho)$ be its image under $\Psi$. Thus $\gamma_i = (G_{i,1}, G_{i,2}, \dots, G_{i,m_i}) \in \mathcal{I}_{k_i}$ is an interval partition and we have the bijective images

$$\begin{aligned}
\Psi([\sigma, \rho]) &= \mathcal{I}_{|G_{1,1}|} \times \mathcal{I}_{|G_{1,2}|} \times \cdots \times \mathcal{I}_{|G_{1,m_1}|} \times \mathcal{I}_{|G_{2,1}|} \times \cdots \times \mathcal{I}_{|G_{p,m_p}|}, \\
\Psi([\rho, \pi]) &= \mathcal{I}_{|\gamma_1|} \times \mathcal{I}_{|\gamma_2|} \times \cdots \times \mathcal{I}_{|\gamma_p|}.
\end{aligned}$$

Hence

$$\begin{aligned}
(f \circledast g)(\sigma, \pi) &= \sum_{\rho \in [\sigma, \pi]} f(\sigma, \rho, \pi) g(\rho, \pi) \\
&= \sum_{(\gamma_1, \gamma_2, \dots, \gamma_p) \in \mathcal{I}_{k_1} \times \mathcal{I}_{k_2} \times \cdots \times \mathcal{I}_{k_p}} \prod_{i=1}^p \left( \prod_{G \in \gamma_i} f_{i,|G|} \right) g_{|\gamma_1|, |\gamma_2|, \dots, |\gamma_p|} \\
&= \sum_{r_i \in [k_i], 1 \leq i \leq p} \sum_{\substack{(n_{ik})_{i \in [p], k \in [r_i]} \\ n_{i1} + n_{i2} + \cdots + n_{ir_i} = k_i, 1 \leq i \leq p}} \prod_{i=1}^p \left( \prod_{k=1}^{r_i} f_{i,n_{ik}} \right) g_{|\gamma_1|, |\gamma_2|, \dots, |\gamma_p|} \\
&=: (f \circledast g)_{k_1, k_2, \dots, k_p}.
\end{aligned} \tag{2.3}$$

This shows that for every pair $(\sigma,\pi)$ the value $f \circledast g(\sigma,\pi)$ is determined by the structural sequence $(k_1,\dots,k_p)$ and thus $f \circledast g$ is adapted. Now multiplying the terms with $z^{\underline{k}}$ and summing over $\underline{k}$ we obtain

$$\begin{aligned}\underline{Z}_{f\circledast g}(\underline{z}) &= \sum_{\underline{k}=(k_1,k_2,\dots)\in\mathbb{N}^\infty_{\text{fin}}} (f\circledast g)_{\underline{k}} z^{\underline{k}} \\ &= \sum_{p\geq 1}\sum_{\underline{r}=(r_1,r_2,\dots,r_p)\in\mathbb{N}^p}\sum_{\substack{(n_{i1},\dots,n_{ir_i})\in\mathbb{N}^{r_i}\\ 1\leq i\leq p}}\prod_{i=1}^p\left(\prod_{k=1}^{r_i} f_{i,n_{ik}} z_i^{n_{ik}}\right) g_{r_1,r_2,\dots,r_p} \\ &= \sum_{p\geq 1}\sum_{\underline{r}=(r_1,r_2,\dots,r_p)\in\mathbb{N}^p} g_{r_1,r_2,\dots,r_p}\prod_{i=1}^p Z_f^{(i)}(z_i)^{r_i} \\ &= \underline{Z}_g(Z_f^{(1)}(z_1), Z_f^{(2)}(z_2),\dots).\end{aligned} \tag{2.4}$$

□

**Corollary 2.17.** *If $f,g\in F_{\mathcal{OP}^2}$ are multiplicative, then so is $f * g$ and*

$$Z_{f*g}(z) = Z_g(Z_f(z)).$$

*Proof.* Given a multiplicative function $f\in F_{\mathcal{OP}^2}$, we lift it to a quasi-multiplicative function $\tilde{f}\in F_{\mathcal{OP}^3}$ via its defining family $\tilde{f}_{jk} = f_k$, i.e., $\tilde{f}(\sigma,\rho,\pi) \coloneqq f(\sigma,\rho)$ and the generating functions are $Z^{(j)}_{\tilde{f}}(z) = Z_{\tilde{f}}(z)$ for all $j\geq 1$. On the other hand, $g$ is multiplicative, therefore adapted with $g_{k_1,\dots,k_p} = g_{k_1}\cdots g_{k_p}$ and has generating function

$$\underline{Z}_g(z_1,z_2,z_3,\dots) = \sum_{p=1}^\infty\sum_{(k_1,\dots,k_p)\in\mathbb{N}^p} g_{k_1}\cdots g_{k_p} z_1^{k_1}\cdots z_p^{k_p} = \sum_{p=1}^\infty Z_g(z_1)\cdots Z_g(z_p). \tag{2.5}$$

By Proposition 2.16

$$\underline{Z}_{\tilde{f}\circledast g}(z_1,z_2,\dots) = \sum_{p=1}^\infty Z_g(Z_f(z_1))\cdots Z_g(Z_f(z_p)).$$

This shows that $(\tilde{f}\circledast g)_{k_1,\dots,k_p} = \prod_{i=1}^p h_{k_i}$, where $h_k \coloneqq \frac{1}{k!}\frac{d^k}{dz^k}\big|_{z=0} Z_g(Z_f(z))$. Therefore $\tilde{f}\circledast g = f * g$ is multiplicative and $Z_{f*g} = Z_g(Z_f(z))$. □

2.4. **Special functions in the poset of ordered set partitions.** We define several special functions in the case of ordered set partitions, and compute their generating functions.

**Definition 2.18.** Given ordered set partitions $\sigma\leq\rho\leq\pi = (P_1,P_2,\dots,P_p)\in\mathcal{OP}_n$, a sequence $\underline{t} = (t_1,t_2,t_3,\dots)\in\mathbb{R}^{\mathbb{N}}$ and a number $t\in\mathbb{R}$ we define

$$\beta_{\underline{t}}(\sigma,\pi) = \prod_{i=1}^p \binom{t_i}{\sharp(\sigma|_{P_i})}, \tag{2.6}$$

$$\beta_t(\sigma,\pi) = \beta_{(t,t,t,\dots)}(\sigma,\pi), \tag{2.7}$$

$$\gamma_{\underline{t}}(\sigma,\rho,\pi) = \prod_{i=1}^p\prod_{G\in\gamma_i}\binom{t_i}{|G|}, \tag{2.8}$$

where $\binom{t}{n}$ is the generalized binomial coefficient and $(\gamma_1,\dots,\gamma_p)$ is the image of $\rho$ by the map $\Psi$ in Proposition 2.14. Moreover, for $\sigma\leq\pi$ we define

$$[\sigma:\pi] = \prod_{P\in\bar{\pi}}\sharp(\sigma|_P), \tag{2.9}$$

$$[\sigma:\pi]! = \prod_{P\in\pi}\sharp(\sigma|_P)!, \tag{2.10}$$

$$\widetilde{\zeta}(\sigma,\pi) = \frac{1}{[\sigma:\pi]!}, \tag{2.11}$$

$$\widetilde{\mu}(\sigma,\pi) = \frac{(-1)^{|\sigma|-|\pi|}}{[\sigma:\pi]}. \tag{2.12}$$

**Remark 2.19.** The values $\beta_{\underline{t}}(\sigma,\pi)$ and $\gamma_{\underline{t}}(\sigma,\rho,\pi)$ depend only on the first $|\pi|$ elements of $\underline{t}$.

Since every interval in $\mathcal{OP}_n$ is isomorphic to a product of lattices of interval partitions (Proposition 2.14) and thus is a Boolean lattice, it follows that the semilattice of ordered set partitions is eulerian, i.e., its Möbius function only depends on the rank and $\mu_{\mathcal{OP}}(\sigma,\pi) = (-1)^{|\sigma|-|\pi|}$, see [Rot64, Proposition 3 and its Corollary]. Hence if $\sigma \le \pi$ we may write,

$$\widetilde{\zeta}(\sigma,\pi) = \frac{\zeta_{\mathcal{P}}(\bar\sigma,\bar\pi)}{[\sigma:\pi]!}, \tag{2.13}$$

$$\widetilde{\mu}(\sigma,\pi) = \frac{\mu_{\mathcal{OP}}(\sigma,\pi)}{[\sigma:\pi]} = \frac{\mu_{\mathcal{P}}(\bar\sigma,\bar\pi)}{[\sigma:\pi]!} = \mu_{\mathcal{P}}(\bar\sigma,\bar\pi)\,\tilde\zeta(\sigma,\pi), \tag{2.14}$$

where the Möbius function $\mu_{\mathcal{P}}$ was defined in formula (2.1).

**Proposition 2.20.** *(i) The function $\beta_{\underline{t}} \in F_{\mathcal{OP}^2}$ is adapted with defining family*

$$(\beta_{\underline{t}})_{\underline{k}} = \prod_{i=1}^{p} \binom{t_i}{k_i}. \tag{2.15}$$

*(ii) The function $\gamma_{\underline{t}} \in F_{\mathcal{OP}^3}$ is quasi-multiplicative with defining array*

$$(\gamma_{\underline{t}})_{jk} = \binom{t_j}{k}. \tag{2.16}$$

*(iii) The functions $\beta_t, \widetilde{\mu}, \widetilde{\zeta} \in F_{\mathcal{OP}^2}$ are multiplicative with defining sequences*

$$\beta_t(\hat 0_n, \hat 1_n) = \binom{t}{n}, \tag{2.17}$$

$$\widetilde{\zeta}(\hat 0_n, \hat 1_n) = \frac{1}{n!}, \tag{2.18}$$

$$\widetilde{\mu}(\hat 0_n, \hat 1_n) = \frac{(-1)^{n-1}}{n}. \tag{2.19}$$

*Proof.* The claims follow by definition and by Proposition 2.14. □

**Corollary 2.21.** *(i) The inverse function (with respect to the convolution $*$) of $\widetilde{\mu}$ is $\widetilde{\zeta}$.*
*(ii) For $\underline{s}, \underline{t} \in \mathbb{R}^\infty$ we have*

$$\gamma_{\underline{s}} \circledast \beta_{\underline{t}} = \beta_{\underline{s}\circ\underline{t}},$$

*where $\underline{s} \circ \underline{t} = (s_1 t_1, s_2 t_2, \dots)$.*
*(iii) $\beta_t$ satisfies the semigroup property $\beta_s * \beta_t = \beta_{st}$ for $s, t \in \mathbb{R}$.*

*Proof.* (i) Since $Z_{\widetilde{\mu}}(z) = \log(1+z)$ and $Z_{\widetilde{\zeta}}(z) = e^z - 1$, we have $Z_{\widetilde{\mu}}(Z_{\widetilde{\zeta}}(z)) = z$.
(ii) We have

$$Z^{(j)}_{\gamma_{\underline{s}}}(z) = \sum_{k=1}^{\infty} \binom{s_j}{k} = (1+z)^{s_j} - 1,$$

$$\underline{Z}_{\beta_{\underline{t}}}(\underline{z}) = \sum_{\underline{k}} \prod_{i=1}^{p} \binom{t_i}{k_i} z_1^{k_1} \cdots z_p^{k_p} = \sum_{p\ge1} \prod_{i=1}^{p} \left((1+z_i)^{t_i} - 1\right).$$

By Proposition 2.16, $\underline{Z}_{\gamma_{\underline{s}}\circledast\beta_{\underline{t}}}(\underline{z}) = \underline{Z}_{\beta_{\underline{t}}}(Z^{(1)}_{\gamma_{\underline{s}}}(z_1), Z^{(2)}_{\gamma_{\underline{s}}}(z_2), \dots)$, which equals $\underline{Z}_{\beta_{\underline{s}\circ\underline{t}}}(\underline{z})$.
(iii) We can use $Z_{\beta_t}(z) = \sum_{n=1}^{\infty} \binom{t}{n} z^n = (1+z)^t - 1$ and Corollary 2.17. □

## 3. A generalized notion of independence related to spreadability systems

3.1. **Notation and terminology.** From now on we denote by $\mathcal{A}$ and $\mathcal{B}$ unital algebras over $\mathbb{C}$ and by 1 their unit elements. Elements of $\mathcal{A}$ are called *random variables*, and elements of $\mathcal{A}^n, n \in \mathbb{N}$ are called *random vectors.* An (algebraic) $\mathcal{B}$-valued *expectation* is a unital linear map

$$\varphi : \mathcal{A} \to \mathcal{B}$$

and we call the pair $(\mathcal{A}, \varphi)$ an *(algebraic) $\mathcal{B}$-valued noncommutative probability space* ($\mathcal{B}$-ncps). In the case where $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\varphi$ is a $\mathcal{B}$-module map in the sense that $\varphi(bab') = b\,\varphi(a)\,b'$ for all $a \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, the map $\varphi$ is called *conditional expectation.* This property will however not be crucial in the context of the present paper.

Usually the involved algebras are $*$-algebras and the linear maps are positive, in particular in the case where $\mathcal{B} = \mathbb{C}$ and $\varphi$ is a state; then the pair is called a *noncommutative probability space* (ncps). We include the algebraic $\mathcal{B}$-valued setting here because the proofs remain essentially the same and it will be essential in Section 8.

We say that two sequences $(X_i)_{i=1}^\infty, (Y_i)_{i=1}^\infty \subseteq \mathcal{A}$ have *the same distribution* if

$$\varphi(X_{i_1} X_{i_2} \cdots X_{i_n}) = \varphi(Y_{i_1} Y_{i_2} \cdots Y_{i_n})$$

for any tuple $(i_1, i_2, \dots, i_n) \in \mathbb{N}^n, n \in \mathbb{N}$, and in this case we write $(X_i)_{i=1}^\infty \stackrel{d}{=} (Y_i)_{i=1}^\infty$. Alternatively, a unital homomorphism $\iota : \mathcal{D} \to \mathcal{A}$ from some unital algebra $\mathcal{D}$ into $\mathcal{A}$ is also called a *random variable*. This definition extends random variables (elements of $\mathcal{A}$) and more generally random vectors. Indeed, given a random vector $(X_1, X_2, \dots, X_n)$, we get a homomorphism $\iota : \mathcal{D} \to \mathcal{A}$ defined by $I(x_i) = X_i$, where $\mathcal{D} = \mathbb{C}\langle x_1, x_2, \dots, x_n\rangle$ is the unital algebra freely generated by noncommuting indeterminates $x_1, x_2, \dots, x_n$. Let $(\mathcal{A}_1, \varphi_1)$, $(\mathcal{A}_2, \varphi_2)$ be two $\mathcal{B}$-ncps such that $\varphi_1, \varphi_2$ take values in a common algebra $\mathcal{B}$. Sequences of random variables $(\iota_1^{(i)})_{i=1}^\infty \subseteq \operatorname{Hom}(\mathcal{D}, \mathcal{A}_1), (\iota_2^{(j)})_{j=1}^\infty \subseteq \operatorname{Hom}(\mathcal{D}, \mathcal{A}_2)$ have *the same distribution* if

$$\varphi_1(\iota_1^{(i_1)}(x_1)\iota_1^{(i_1)}(x_2)\cdots\iota_1^{(i_n)}(x_n)) = \varphi_2(\iota_2^{(i_1)}(x_1)\iota_2^{(i_2)}(x_2)\cdots\iota_2^{(i_n)}(x_n))$$

for any $(i_1, i_2, \dots, i_n) \in \mathbb{N}^n$ and any $x_1, x_2, \dots, x_n \in \mathcal{D}$. In this case we write

$$(\iota_1^{(i)})_{i=1}^\infty \stackrel{d}{=} (\iota_2^{(j)})_{j=1}^\infty.$$

Given $X_1, X_2, \dots, X_n \in \mathcal{A}$ and $P = \{p_1 < p_2 < \dots < p_k\} \subseteq [n]$, it will be convenient to introduce the notation

$$X_P := \prod_{j=1}^k X_{p_j}, \tag{3.1}$$

which means the ordered product $X_{p_1} X_{p_2} \cdots X_{p_k}$.

For a $k$-linear functional $M : \mathcal{A}^k \to \mathcal{B}$, we denote

$$M(X_P) := M(X_{p_1}, X_{p_2}, \dots, X_{p_k}). \tag{3.2}$$

The following tensor notations will be used in Theorem 6.3. With the setting above,

$$X_{\otimes P} := 1 \otimes \cdots \otimes X_{p_1} \otimes 1 \otimes \cdots \otimes X_{p_2} \otimes 1 \otimes \cdots \otimes X_{p_k} \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{A}^{\otimes n},$$

where $X_{p_j}$ appears in the $p_j$-th position of the total of $n$ factors. When $P = \emptyset \subseteq [n]$, we understand that

$$X_{\otimes\emptyset} = 1 \otimes 1 \otimes \cdots \otimes 1.$$

This allows us to write expansions like

$$(X_1 + Y_1) \otimes (X_2 + Y_2) \otimes \cdots \otimes (X_n + Y_n) = \sum_{I \subseteq [n]} X_{\otimes I} Y_{\otimes I^c} \tag{3.3}$$

in a compact way.

Similarly for a linear map $L : \mathcal{A} \to \mathcal{B}$ we denote by $L^{\otimes P} : \mathcal{A}^{\otimes n} \to \mathcal{B}$ the linear map

$$L^{\otimes P} = I \otimes I \otimes \cdots \otimes L \otimes I \otimes \cdots \otimes L \otimes I \otimes \cdots \otimes L \otimes \cdots \otimes I \tag{3.4}$$

with $L$ appearing exactly at position $p_j$ for every $p_j \in P$.

Recall that the tensor product has the universal property that any multilinear map

$$T : \mathcal{A}^n \to \mathcal{B}$$

has a unique lifting to a linear map

$$\tilde{T} : \mathcal{A}^{\otimes n} \to \mathcal{B}$$

such that on rank 1 tensors we have $\tilde{T}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = T(a_1, a_2, \dots, a_n)$. We will tacitly identify $T$ with $\tilde{T}$ in order to simplify notation.

3.2. **Spreadability systems.** In this subsection we introduce the notation necessary to generalize the notions of partitioned moment and cumulant functionals of [Leh04] from the exchangeable setting to the spreadable setting.

**Definition 3.1.** Let $(\mathcal{A},\varphi)$ be a $\mathcal{B}$-valued ncps.

1. A *spreadability system* for $(\mathcal{A},\varphi)$ is a triplet $\mathcal{S} = (\mathcal{U},\widetilde{\varphi},(\iota^{(i)})_{i=1}^{\infty})$ satisfying the following properties:
   (i) $(\mathcal{U},\widetilde{\varphi})$ is a $\mathcal{B}$-valued ncps.
   (ii) $\iota^{(j)} : \mathcal{A} \to \mathcal{U}$ is a homomorphism such that $\varphi = \widetilde{\varphi} \circ \iota^{(j)}$ for each $j \geq 1$. For simplicity, $\iota^{(j)}(X)$ is denoted by $X^{(j)}$, $X \in \mathcal{A}$, and we denote by $\mathcal{A}^{(j)}$ the image of $\mathcal{A}$ under $\iota^{(j)}$.
   (iii) The identity
$$\widetilde{\varphi}(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}) = \widetilde{\varphi}(X_1^{(h(i_1))} X_2^{(h(i_2))} \cdots X_n^{(h(i_n))}) \tag{3.5}$$
   holds for any $X_1, X_2, \ldots, X_n \in \mathcal{A}$, any $i_1, i_2, \ldots, i_n \in \mathbb{N}$ and any *order preserving* map $h : \{i_1, i_2, \ldots, i_n\} \to \mathbb{N}$, that is, $i_p < i_q$ implies $h(i_p) < h(i_q)$.
2. A triplet $\mathcal{E} = (\mathcal{U},(\iota^{(i)})_{i=1}^{\infty},\widetilde{\varphi})$ is called an *exchangeability system* if, in addition to (i), (ii) above, eq. (3.5) holds for any $X_1, X_2, \ldots, X_n \in \mathcal{A}$, any $i_1, i_2, \ldots, i_n \in \mathbb{N}$ and any *permutation* $h \in \mathfrak{S}_\infty := \bigcup_{n\geq 1} \mathfrak{S}_n$.

If no confusion arises, we write $\varphi$ instead of $\widetilde{\varphi}$.

**Remark 3.2.** 1. In order to be able to include Boolean and monotone products and some other examples (Section 3.3) we do not require the homomorphisms $\iota^{(i)}$ to be unit-preserving.
2. Condition (3.5) can be rephrased as follows:
$$(\iota^{(1)}, \iota^{(2)}, \iota^{(3)}, \ldots) \overset{d}{=} (\iota^{(n_1)}, \iota^{(n_2)}, \iota^{(n_3)}, \ldots) \tag{3.6}$$
   for any strictly increasing sequence $(n_i)_{i=1}^{\infty} \subseteq \mathbb{N}$. This is the definition given in [Kös10].
   If (3.5) holds, then (3.6) is easy to show. If (3.6) holds, then take any $i_1, i_2, \ldots, i_n$ and order-preserving function $h : \{i_1, i_2, \ldots, i_n\} \to \mathbb{N}$. Let $\{j_1, j_2, \ldots, j_m\} = \{i_1, i_2, \ldots, i_n\}$, $j_1 < j_2 < \cdots < j_m$. By (3.6), we have
$$(\iota^{(j_1)}, \iota^{(j_2)}, \ldots, \iota^{(j_m)}) \overset{d}{=} (\iota^{(1)}, \iota^{(2)}, \ldots, \iota^{(m)}) \overset{d}{=} (\iota^{(h(j_1))}, \iota^{(h(j_2))}, \ldots, \iota^{(h(j_m))}),$$
   which implies (3.5).
3. It is easy to extend the definition of exchangeability systems (resp., spreadability systems) from $\mathbb{N}$ to an arbitrary set (resp., arbitrary totally ordered set).

**Definition 3.3.** (i) The *ordered kernel set partition* $\kappa(i_1, i_2, \ldots, i_n)$ of a multiindex $(i_1, i_2, \ldots, i_n)$ is defined as follows. First, pick the smallest value, say $p_1$, from $i_1, i_2, \ldots, i_n$ and define the block $P_1 = \{k \in [n] \mid i_k = p_1\}$. Next, pick the second smallest value $p_2$ from $i_1, i_2, \ldots, i_n$ and define the block $P_2 = \{k \in [n] \mid i_k = p_2\}$. By repeating this procedure, we obtain an ordered set partition $(P_1, P_2, \ldots)$, which we denote by $\kappa(i_1, i_2, \ldots, i_n)$.
(ii) The *kernel set partition* $\bar{\kappa}(i_1, i_2, \ldots, i_n)$ of a sequence of indices is defined as the underlying set partition $\overline{\kappa(i_1, i_2, \ldots, i_n)}$ of the corresponding ordered kernel set partition. In other words, it is the equivalence relation such that by $p \sim q$ if and only if $i_p = i_q$.

Using this notation the condition of spreadability (3.5) is equivalent to the requirement that
$$\varphi(X^{(i_1)} X^{(i_2)} \cdots X^{(i_n)}) = \varphi(X^{(j_1)} X^{(j_2)} \cdots X^{(j_n)}) \tag{3.7}$$
holds whenever $\kappa(i_1, i_2, \ldots, i_n) = \kappa(j_1, j_2, \ldots, j_n)$. That is, the expectation (3.5) only depends on the ordered kernel set partition $\kappa(i_1, i_2, \ldots, i_n)$. Thus for every ordered set partition $\pi \in \mathcal{OP}_n$ we can define a multilinear functional $\varphi_\pi : \mathcal{A}^n \to \mathbb{C}$ by choosing any representative sequence $(i_1, i_2, \ldots, i_n)$ with $\kappa(i_1, i_2, \ldots, i_n) = \pi$ and setting
$$\varphi_\pi(X_1, X_2, \ldots, X_n) = \tilde{\varphi}(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}). \tag{3.8}$$
Invariance (3.7) ensures that this definition is consistent and does not depend on the choice of the representative.

This generalizes the corresponding notions from exchangeability systems [Leh04]: given an exchangeability system $\mathcal{E} = (\mathcal{U},\widetilde{\varphi},(\iota^{(i)})_{i=1}^{\infty})$, we can define a multilinear functional $\varphi_\pi$, this time for any set partition $\pi \in \mathcal{P}_n$,
$$\varphi_\pi(X_1, X_2, \ldots, X_n) = \tilde{\varphi}(X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)}), \tag{3.9}$$

where $(i_1, i_2, \dots, i_n)$ is a representative such that $\overline{\kappa}(i_1, i_2, \dots, i_n) = \pi$.

3.3. **Examples from natural products of linear maps.** Spreadability systems typically arise as universal products of linear maps defined on a free product of $*$-algebras. Universal products were classified by Speicher [Spe97] and Muraki [Mur03] into five types: tensor, free, Boolean, monotone and anti-monotone products. The latter is essentially the reversion of the monotone product and therefore omitted from the discussion below. Usually natural products are defined in the setting of noncommutative probability spaces, i.e. $\mathcal{A}$ is a $*$-algebra and $\varphi$ is a state. In the present paper our interests are in combinatorial aspects and we consider $(\mathcal{A}, \varphi)$ to be an arbitrary pair of an algebra and a unital linear map.

3.3.1. *Tensor exchangeability system.* Let $\mathcal{U} := \otimes_{i=1}^{\infty} \mathcal{A}$ be the algebraic infinite tensor product of the same $\mathcal{A}$'s, and $\widetilde{\varphi} := \otimes_{i=1}^{\infty} \varphi$ be the tensor product. Let $\iota^{(j)}$ be the embedding of $\mathcal{A}$ into the $j$th tensor component:

$$\iota^{(j)}(X) := 1^{\otimes(j-1)} \otimes X \otimes 1 \otimes 1 \otimes \cdots.$$

$\mathcal{A}$ of $\mathcal{U}$. Then $\mathcal{E}_{\mathrm{T}} = (\mathcal{U}, \widetilde{\varphi}, (\iota^{(i)})_{i=1}^{\infty})$ is an exchangeability system for $(\mathcal{A}, \varphi)$, called the *tensor exchangeability system.* In order to stress that $\mathcal{E}_{\mathrm{T}}$ is a spreadability system, we may write $\mathcal{S}_{\mathrm{T}}$ instead of $\mathcal{E}_{\mathrm{T}}$ and call $\mathcal{S}_{\mathrm{T}}$ the tensor spreadability system.

3.3.2. *Free exchangeability system.* Let $\mathcal{U} := *_{i=1}^{\infty} \mathcal{A}$ be the algebraic free product of infinitely many copies of $\mathcal{A}$ identifying the units of $\mathcal{A}$'s, and let $\widetilde{\varphi} := *_{i=1}^{\infty} \varphi$ be the free product of linear maps [Avi82, Voi85]. Let $\iota^{(j)}$ be the embedding of $\mathcal{A}$ into the $j$th component $\mathcal{A}$ of $\mathcal{U}$. Then $\mathcal{E}_{\mathrm{F}} = (\mathcal{U}, \widetilde{\varphi}, (\iota^{(i)})_{i=1}^{\infty})$ (or we may write $\mathcal{S}_{\mathrm{F}}$ to stress the spreadability) is an exchangeability system for $(\mathcal{A}, \varphi)$, called the *free exchangeability (or spreadability) system.*

3.3.3. *Boolean exchangeability system.* Let $\mathcal{U}_0 := \star_{i=1}^{\infty} \mathcal{A}$ be the algebraic free product of infinitely many copies of $\mathcal{A}$, this time without identifying the units of $\mathcal{A}$'s, and let $\mathcal{U} := \mathbb{C}1_{\mathcal{U}} \oplus \mathcal{U}_0$ be its unitization. As before, $\iota^{(j)}$ is the embedding of $\mathcal{A}$ into the $j$th component $\mathcal{A}$ of $\mathcal{U}_0$. The Boolean product $\widetilde{\varphi} = \diamond_{i=1}^{\infty} \varphi$ is defined on $\mathcal{U}$ by the following rule [Boż86]: if $X_k \in \iota^{(i_k)}(\mathcal{A})$ and $i_k \neq i_{k+1}$ for any $1 \le k \le n-1$, then

$$\widetilde{\varphi}(X_1 X_2 \cdots X_n) = \varphi(X_1)\, \varphi(X_2) \cdots \varphi(X_n).$$

The triplet $\mathcal{E}_{\mathrm{B}} = (\mathcal{U}, \widetilde{\varphi}, (\iota^{(i)})_{i=1}^{\infty})$ (or we may write $\mathcal{S}_{\mathrm{B}}$) is an exchangeability system, called the *Boolean exchangeability (or spreadability) system.*

3.3.4. *Monotone spreadability system.* Let $(\mathcal{U}, (\iota^{(i)})_{i=1}^{\infty})$ be the same one as in example 3.3.3. The monotone product $\widetilde{\varphi} = \triangleright_{i=1}^{\infty} \varphi$ is defined on $\mathcal{U}$ by the following recursive rules [Mur00].

(i) $\widetilde{\varphi}|_{\mathcal{A}} = \varphi$, $\widetilde{\varphi}(1_{\mathcal{U}}) = 1$.
(ii) $\widetilde{\varphi}(X_1 X_2 \cdots X_n) = \varphi(X_1)\, \widetilde{\varphi}(X_2 X_3 \cdots X_n)$ if $X_k \in \iota^{(i_k)}(\mathcal{A})$, $1 \le k \le n$ and $i_1 > i_2$.
(iii) $\widetilde{\varphi}(X_1 X_2 \cdots X_n) = \widetilde{\varphi}(X_1 X_2 \cdots X_{n-1})\, \varphi(X_n)$ if $X_k \in \iota^{(i_k)}(\mathcal{A})$, $1 \le k \le n$ and $i_n > i_{n-1}$.
(iv) $\widetilde{\varphi}(X_1 X_2 \cdots X_n) = \widetilde{\varphi}(X_1 X_2 \cdots X_{j-1} X_{j+1} \cdots X_n)\, \varphi(X_j)$ if $2 \le j \le n-1$, $X_k \in \iota^{(i_k)}(\mathcal{A})$ for $1 \le k \le n$ and $i_{j-1} < i_j > i_{j+1}$.

Then $\mathcal{S}_{\mathrm{M}} = (\mathcal{U}, \widetilde{\varphi}, (\iota^{(i)})_{i=1}^{\infty})$ is a spreadability system for $(\mathcal{A}, \varphi)$, called the *monotone spreadability system.* It is a proper spreadability system, i.e., it does not satisfy exchangeability.

3.3.5. *Conditional monotone spreadability system.* This example requires two linear maps $\varphi, \psi : \mathcal{A} \to \mathbb{C}$ and hence is of a different spirit from the previous settings. The conditionally monotone product $(\widetilde{\varphi}, \widetilde{\psi}) = \triangleright_{i=1}^{\infty} (\varphi, \psi)$ is defined on on the full free product $(\mathcal{U}, (\iota^{(i)})_{i=1}^{\infty})$ from example 3.3.4 according to the following rules [Has11].

(i) $\widetilde{\psi}$ is the monotone product of $\psi$ according to Example 3.3.4.
(ii) $\widetilde{\varphi}|_{\mathcal{A}} = \varphi$, $\widetilde{\varphi}(1_{\mathcal{U}}) = 1$.
(iii) $\widetilde{\varphi}(X_1 X_2 \cdots X_n) = \varphi(X_1)\, \widetilde{\varphi}(X_2 X_3 \cdots X_n)$ if $X_k \in \iota^{(i_k)}(\mathcal{A})$, $i_1, i_2, \dots, i_n \in \mathbb{N}$, $i_1 > i_2$.
(iv) $\widetilde{\varphi}(X_1 X_2 \cdots X_n) = \widetilde{\varphi}(X_1 X_2 \cdots X_{n-1})\, \varphi(X_n)$ if $X_k \in \iota^{(i_k)}(\mathcal{A})$, $i_1, i_2, \dots, i_n \in \mathbb{N}$, $i_n > i_{n-1}$.
(v)
$$\begin{aligned}\widetilde{\varphi}(X_1 X_2 \cdots X_n) = {} & \widetilde{\varphi}(X_1 X_2 \cdots X_{j-1})\, (\varphi(X_j) - \psi(X_j))\, \widetilde{\varphi}(X_{j+1} \cdots X_n) \\ & + \psi(X_j)\, \widetilde{\varphi}(X_1 X_2 \cdots X_{j-1} X_{j+1} \cdots X_n)\end{aligned}$$
if $2 \le j \le n-1$, $X_k \in \iota^{(i_k)}(\mathcal{A})$, $i_1, i_2, \dots, i_n \in \mathbb{N}$ and $i_{j-1} < i_j > i_{j+1}$.

Then $\mathcal{S}_{\mathrm{CM}} = (\mathcal{U}, \widetilde{\varphi}, (\iota^{(i)})_{i=1}^{\infty})$ is a spreadability system for $(\mathcal{A}, \varphi)$ which does not satisfy exchangeability. It is called the *c-monotone spreadability system.*

**Remark 3.4.** More examples may be extracted from [BLS96, Has10], but we omit them here.

The definition of the c-monotone spreadability system does not require the second linear map $\widetilde{\psi}$, and so one may skip to define it. However, $\widetilde{\psi}$ is important when we want to formulate the concept of associativity, as below.

**Remark 3.5.** The above examples satisfy associativity in the sense of [Mur03]: Take for instance the monotone spreadability system. For $I \subseteq \mathbb{N}$, we can define $\mathcal{U}_I := \mathbb{C}1_{\mathcal{U}_I} \oplus (\star_{i\in I}\mathcal{A})$ and $\widetilde{\varphi}_I := \rhd_{i\in I}\varphi$ on $\mathcal{U}_I$, similarly to the case $I = \mathbb{N}$. Thus a map $I \mapsto (\mathcal{U}_I, \widetilde{\varphi}_I)$ is obtained. Now take three subsets $I, J, K \subseteq \mathbb{N}$ such that $I < J < K$, i.e. $i < j < k$ for any $i \in I, j \in J, k \in K$. We can naturally identify $\mathcal{U}_I * \mathcal{U}_J$ with $\mathcal{U}_{I\cup J}$. Using this identification it is easy to see that the monotone product is associative in the sense that that $(\widetilde{\varphi}_I \rhd \widetilde{\varphi}_J) \rhd \widetilde{\varphi}_K = \widetilde{\varphi}_I \rhd (\widetilde{\varphi}_J \rhd \widetilde{\varphi}_K)$ on $\mathcal{U}_{I\cup J\cup K}$. Similarly one can show that the tensor and free and exchangeability systems satisfy associativity as well.

For the c-monotone spreadability system associativity is proved in [Has11, Theorem 3.7]. Here the map $I \mapsto (\mathcal{U}_I, \widetilde{\varphi}_I, \widetilde{\psi}_I)$ can be defined similarly, and associativity means that for $I, J, K \subseteq \mathbb{N}$ such that $I < J < K$ one has $\big((\widetilde{\varphi}_I, \widetilde{\psi}_I) \rhd (\widetilde{\varphi}_J, \widetilde{\psi}_J)\big) \rhd (\widetilde{\varphi}_K, \widetilde{\psi}_K) = (\widetilde{\varphi}_I, \widetilde{\psi}_I) \rhd \big((\widetilde{\varphi}_J, \widetilde{\psi}_J) \rhd (\widetilde{\varphi}_K, \widetilde{\psi}_K)\big)$ on $\mathcal{U}_{I\cup J\cup K}$.

3.3.6. *$V$-monotone spreadability system.* Recently Dacko introduced the concept of $V$-monotone independence and constructed a corresponding the $V$-monotone product of probability spaces [Dac19]. These notions are based on the notion of $V$-shaped sequences and partitions. A sequence of numbers $i_1, i_2, \dots, i_n$ is called *$V$-shaped* if there exists an index $1 \le r \le n$ such that $i_1 > i_2 > \cdots > i_r < i_{r+1} < \cdots < i_n$. Given a family of unital algebras $\mathcal{A}_i$ with units $I_i$ and states $\varphi_i$, the $V$-monotone product $\varphi = \circledvee \varphi_i$ is defined on their nonunital free product $\mathcal{U}$ and characterized by the following factorization properties. Let $n \in \mathbb{N}$ and $X_j \in \mathcal{A}_{i_j}$, $j = 1, 2, \dots, n$ be arbitrary elements.

(i) $\varphi(X_1X_2\cdots X_n) = 0$ whenever $i_j \ne i_{j+1}$ for all $j$ and $\varphi_{i_j}(X_j) = 0$.

(ii) In addition,

$$\varphi(X_1X_2\cdots X_{j-1}I_{i_j}X_{j+1}\cdots X_n) = \begin{cases}\varphi(X_1X_2\cdots X_{j-1}X_{j+1}\cdots X_n) & \text{if } (i_1, i_2, \dots, i_j) \text{ is } V\text{-shaped}\\ 0 & \text{otherwise.}\end{cases}$$

whenever $\varphi_{i_1}(X_1) = \varphi_{i_2}(X_2) = \cdots = \varphi_{i_{j-1}}(X_{j-1}) = 0$.

This notion gives rise to a spreadability system, however it is shown in [Dac19] that $(\varphi_1 \circledvee \varphi_2) \circledvee \varphi_3 = \varphi_1 \circledvee \varphi_2 \circledvee \varphi_3 \ne \varphi_1 \circledvee (\varphi_2 \circledvee \varphi_3)$ and therefore associativity does not hold.

3.4. **$\mathcal{E}$-independence.** Recall the notion of independence associated to exchangeability systems [Leh04, Definition 1.8]. Roughly speaking independence of a pair $(X, Y)$ means that the joint distribution of $(X, Y)$ coincides with the joint distribution of $(X^{(1)}, Y^{(2)})$, where the couples $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ are exchangeable copies of the couple $(X, Y)$. This property can be reformulated in a lattice theoretical way as follows.

**Definition 3.6.** (i) Let $\mathcal{E} = (\mathcal{U}, \tilde{\varphi})$ be an exchangeability system for a given $\mathcal{B}$-ncps $\mathcal{A}$. Subalgebras $\{\mathcal{A}_i\}_{i\in I} \subseteq \mathcal{A}$, where $I$ is a subset of $\mathbb{N}$, are said to be *$\mathcal{E}$-independent* if for any tuple of indices $(i_1, i_2, \dots, i_n) \in I^n$, any tuple of random variables $(X_1, X_2, \dots, X_n)$ with $X_j \in \mathcal{A}_{i_j}$ and any set partition $\pi \in \mathcal{P}_n$, we have

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \varphi_{\pi\wedge\bar{\kappa}(i_1,i_2,\dots,i_n)}(X_1, X_2, \dots, X_n).$$

(ii) For a subset $R \subseteq \mathcal{A}$ denote by $\mathbb{C}\langle R\rangle$ the subalgebra of $\mathcal{A}$ generated by $R$ over $\mathbb{C}$ (possibly without unit) and by $\mathbb{C}_1\langle R\rangle$ the unital subalgebra generated by $R$. Random variables $\{X_i\}_{i\in I}$, where $I \subseteq \mathbb{N}$, are said to be *$\mathcal{E}$-independent* if the algebras $\{\mathbb{C}\langle X_i\rangle\}_{i\in I}$ are $\mathcal{E}$-independent. In particular, random vectors $\big\{(X_1(i), X_2(i), \dots, X_n(i))\big\}_{i\in I}$ are said to be *$\mathcal{E}$-independent* if the subalgebras $\{\mathbb{C}\langle X_1(i), X_2(i), \dots, X_n(i)\rangle\}_{i\in I}$ are $\mathcal{E}$-independent.

**Remark 3.7.** We exclude the unit of $\mathcal{A}$ from the polynomial algebras because the homomorphisms $\iota^{(i)}$ may not be unit-preserving, e.g., in the case of the Boolean exchangeability system $\mathcal{E}_{\mathrm{B}}$. By considering the non-unital polynomials, we have the equivalence

$$(X_i)_{i\in\mathbb{N}} \text{ is Boolean independent} \Leftrightarrow (X_i)_{i\in\mathbb{N}} \text{ is } \mathcal{E}_{\mathrm{B}}\text{-independent},$$

where $\mathcal{E}_{\mathrm{B}}$ is the Boolean exchangeability system (see Section 3.3).

For the free and tensor cases we may use the polynomial algebras $\mathbb{C}_1\langle X_1(i), X_2(i) \dots, X_n(i)\rangle$ containing the unit of $\mathcal{A}$ instead of $\mathbb{C}\langle X_1(i), X_2(i), \dots, X_n(i)\rangle$ because the homomorphisms $\iota^{(i)}$ for $\mathcal{E}_{\mathrm{T}}, \mathcal{E}_{\mathrm{F}}$ are unit-preserving.

The reader is referred to [Spe97] for the details of this subtle unital/non-unital problem.

**Example 3.8.** (i) To illustrate the concept of $\mathcal{E}$-independence, two algebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are independent, if for any sequence $X_1, X_2, \dots, X_n \in \mathcal{A}_1 \cup \mathcal{A}_2$ and disjoint subsets $B_1, B_2 \subseteq [n]$ such that $X_i \in \mathcal{A}_1$ for $i \in B_1$ and $X_i \in \mathcal{A}_2$ for $i \in B_2$ we have

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \varphi_{\pi\rfloor_{B_1}\pi\rfloor_{B_2}}(X_1, X_2, \dots, X_n) \tag{3.10}$$

where $\rho = \{B_1, B_2\}$ and $\pi\rfloor_{B_1}\pi\rfloor_{B_2}$ is the set partition of $[n]$ consisting of the restrictions of $\pi$ to $B_1$ and $B_2$.

To be specific, consider the boolean exchangeability system $\mathcal{E}_{\mathrm{B}}$ from Section 3.3. Here for an arbitrary tuple of random variables $X_1, X_2, \dots, X_n$ we have

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \prod_{V \in \tilde{\pi}} \varphi(X_V),$$

where $\tilde{\pi}$ is the maximal interval partition which is dominated by $\pi$. Moreover, suppose that subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are Boolean independent. Pick $X_1, X_2, \dots, X_n \in \mathcal{A}_1 \cup \mathcal{A}_2$ and any partition $\rho = \{B_1, B_2\}$ which splits the $X_i$ into independent subsets as above. Then according to Boolean independence for each block $V \in \tilde{\pi}$ we have

$$\varphi(X_V) = \varphi(X_{V\cap B_1})\,\varphi(X_{V\cap B_2}).$$

In total this yields

$$\begin{aligned}\varphi_\pi(X_1, X_2, \dots, X_n) &= \prod_{V \in \tilde{\pi}} \varphi(X_{V\cap B_1})\,\varphi(X_{V\cap B_2}) = \prod_{V \in \tilde{\pi}} \varphi(X_{V\cap B_1}) \prod_{V \in \tilde{\pi}} \varphi(X_{V\cap B_2}) \\ &= \varphi_{\pi\rfloor_{B_1}}(X_{B_1})\,\varphi_{\pi\rfloor_{B_2}}(X_{B_2}) = \varphi_{\pi\rfloor_{B_1}\pi\rfloor_{B_2}}(X_1, X_2, \dots, X_n)\end{aligned}$$

and condition (3.10) is satisfied. Thus we have shown that Boolean independence of subalgebras $\mathcal{A}_1, \mathcal{A}_2$ implies $\mathcal{E}_{\mathrm{B}}$-independence of $\mathcal{A}_1, \mathcal{A}_2$. Conversely, by supposing $\pi = \hat{1}_n$, we can more easily show that $\mathcal{E}_{\mathrm{B}}$-independence of $\mathcal{A}_1, \mathcal{A}_2$ implies Boolean independence of $\mathcal{A}_1, \mathcal{A}_2$. This argument extends to any number of subalgebras, and hence $\mathcal{E}_{\mathrm{B}}$-independence is equivalent to Boolean independence. Similar reasonings show the analogous equivalence in the case of tensor and free exchangeability systems.

3.5. **$\mathcal{S}$-independence.** In order to generalize independence from exchangeability systems to spreadability systems we replace set partitions by ordered set partitions. This time independence of an ordered pair $(X, Y)$ means that the joint distribution of $(X, Y)$ coincides with the joint distribution of $(X^{(1)}, Y^{(2)})$, where the couples $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ are spread copies of the couple $(X, Y)$.

**Definition 3.9.** Let $\mathcal{S} = (\mathcal{U}, \tilde{\varphi})$ be a spreadability system for a given $\mathcal{B}$-ncps $\mathcal{A}$. A sequence of subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$, where $I \subseteq \mathbb{N}$, is said to be *$\mathcal{S}$-independent* if for any tuple of indices $(i_1, i_2, \dots, i_n) \in I^n$, any tuple of random variables $(X_1, X_2, \dots, X_n)$ with $X_j \in \mathcal{A}_{i_j}$ and any ordered set partition $\pi \in \mathcal{OP}_n$, we have

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \varphi_{\pi\wedge\kappa(i_1, i_2, \dots, i_n)}(X_1, X_2, \dots, X_n). \tag{3.11}$$

A sequence of random variables $(Y_i)_{i\in I}$ is said to be *$\mathcal{S}$-independent* if the sequence $(\mathbb{C}\langle Y_i\rangle)_{i\in I}$ of algebras they generate is $\mathcal{S}$-independent. A sequence of random vectors $((X_1(i), X_2(i), \dots, X_n(i)))_{i\in I}$ is said to be *$\mathcal{S}$-independent* if the sequence of algebras $(\mathbb{C}\langle X_1(i), X_2(i), \dots, X_n(i)\rangle)_{i\in I}$ they generate is $\mathcal{S}$-independent.

**Remark 3.10.** With the notation introduced in Definition 3.1, equation (3.11) may be rewritten as

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \varphi_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}).$$

This condition means that the random vectors $(X_k)_{k\in[n]}$ and $(X_k^{(i_k)})_{k\in[n]}$ have the same "distribution" with respect to $(\varphi_\pi)_\pi$. This is compatible with the concept of spreadability, which is to regard the sequence $(\iota^{(j)}(\mathcal{A}))_{j\in I}$ as independent subalgebras of $(\mathcal{U},\widetilde{\varphi})$. Note that we need the distribution not only with respect to $\widetilde{\varphi}$, but with respect to all $(\varphi_\pi)_{\pi\in\mathcal{OP}_n,n\in\mathbb{N}}$ because a spreadability system may not have a (quasi-) universal calculation rule of mixed moments in the sense of Speicher [Spe97] and Muraki [Mur02]. For instance, we have the exchangeability system associated to the $q$-Fock space (see [Leh05]) and in this case the distribution with respect to $\widetilde{\varphi}$ alone is not sufficient to determine the distribution with respect to $(\varphi_\pi)_\pi$, which is a consequence of the non-existence of "$q$-convolution" [vLM96].

**Example 3.11.** For two subalgebras $\mathcal{S}$-independence reads as follows. A pair of subalgebras $(\mathcal{A}_1,\mathcal{A}_2)$ of $\mathcal{A}$ is $\mathcal{S}$-independent if the following condition holds. Given elements $X_1,X_2,\dots,X_n\in\mathcal{A}_1\cup\mathcal{A}_2$, let $\rho=(B_1,B_2)$ be an ordered set partition of the index set $[n]$ such that $X_i\in\mathcal{A}_1$ for $i\in B_1$ and $X_i\in\mathcal{A}_2$ for $i\in B_2$. Then for any ordered set partition $\pi\in\mathcal{OP}_n$ we have

$$\varphi_\pi(X_1,X_2,\dots,X_n)=\varphi_{\pi\rfloor_{B_1}\pi\rfloor_{B_2}}(X_1,X_2,\dots,X_n).$$

More specifically, the monotone spreadability system $\mathcal{S}_{\mathrm{M}}$ from Section 3.3 induces Muraki's monotone independence.

Moreover, we can show that Boolean independence, $\mathcal{E}_{\mathrm{B}}$-independence and $\mathcal{S}_{\mathrm{B}}$-independence are all equivalent (note that the equivalence of the first two was shown in Example 3.8). A similar equivalence holds for the tensor and free cases.

## 4. Spreadability systems and cumulants

4.1. **Cumulants and factorial Möbius and zeta functions.** Cumulants provide a powerful tool to describe independence of random variables. In [Leh04] the second author defined cumulants by a kind of finite Fourier transform, known as Good's formula in the mathematics literature [Goo75] and Cartier's formula for Ursell functions in the physics literature [Per75, Sim93]. Note that in physics literature cumulants are called *Ursell functions.* This approach apparently fails in the present, non-exchangeable setting; however in their study of monotone cumulants [HS11b, HS11a] the first author and Saigo found a good replacement of the dot operation from umbral calculus [RT94]. We take it as a starting point for the definition of cumulants in full generality.

**Definition 4.1.** Let $(\mathcal{U},(\iota^{(j)})_{j\ge1},\widetilde{\varphi})$ be a spreadability system for a ncps $(\mathcal{A},\varphi)$.

(1) Given a noncommutative random variable $X\in\mathcal{A}$ and a finite subset $A\subseteq\mathbb{N}$ we define

$$\delta_A(X)=\sum_{i\in A}X^{(i)}$$

i.e., the sum of i.i.d. copies of $X$. In the case $A=[N]$ we will also write $\delta_N(X)$ and frequently abbreviate it using Rota's *dot operation*

$$N.X:=X^{(1)}+X^{(2)}+\dots+X^{(N)}$$

whenever it is convenient.

(2) Slightly abusing notation we define

$$\varphi_\pi(X_1^{(i_1)},X_2^{(i_2)},\dots,X_n^{(i_n)}):=\varphi_{\pi\wedge\kappa(i_1,i_2,\dots,i_n)}(X_1,X_2,\dots,X_n), \tag{4.1}$$

and

$$\varphi_\pi(N_1.X_1,N_2.X_2,\dots,N_n.X_n):=\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\cdots\sum_{i_n=1}^{N_n}\varphi_\pi(X_1^{(i_1)},X_2^{(i_2)},\dots,X_n^{(i_n)}).$$

(3) Given an ordered set partition $\pi=(P_1,P_2,\dots,P_p)\in\mathcal{OP}_n$ and $i\in[n]$, the unique number $k$ such that $i\in P_k$ is denoted by $\pi(i)$. When $\pi$ consists of singletons only, i.e., $\#P_i=1$ for all $i\in[p]$ then $\pi$ can be identified with a permutation of $[n]$, and then the notation $\pi(i)$ is consistent with the familiar notation for permutations. Similarly, in the general case $\pi$ can be represented by a multiset permutation. This point of view will be crucial in Section 7 below.

**Remark 4.2.** Note that (4.1) is actually an abuse of notation, because $\varphi_\pi$ is defined for elements of $\mathcal{A}$ only (see (3.8)); we pretend that $\mathcal{S}=(\mathcal{U},(\iota^{(i)})_{i=1}^\infty,\widetilde{\varphi})$ can be interpreted as a spreadability system for the algebra $\mathcal{A}^{(1,2,\dots,N)}$ generated by the images $\mathcal{A}^{(1)},\mathcal{A}^{(2)},\dots,\mathcal{A}^{(N)}$. This is true in the case of free product constructions, but needs justification otherwise; yet (4.1) is well-defined and convenient to keep notation manageable.

Spreadability implies that the value (4.1) is invariant under order preserving changes of the indices; however, for partitioned expectations invariance holds under the weaker assumption that the order is preserved on every individual block. We will only be concerned with the following particular case and therefore refrain from formulating this fact in full generality.

**Lemma 4.3.** *Let $\pi \in \mathcal{OP}_n$ be an ordered set partition. Fix a block $P$ of $\pi$ and a number $m \in \mathbb{N}$ and let $(i_1, i_2, \dots, i_n)$, $(i'_1, i'_2, \dots, i'_n)$ be $n$-tuples such that*

$$i'_k = \begin{cases} i_k + m & \text{if } k \in P \\ i_k & \text{if } k \notin P \end{cases}$$

*Then $\pi \curlywedge \kappa(i_1, i_2, \dots, i_n) = \pi \curlywedge \kappa(i'_1, i'_2, \dots, i'_n)$.*

**Remark 4.4.** Note that without performing the quasi-meet operation in the lemma above the kernel partitions $\kappa(i_1, i_2, \dots, i_n)$ and $\kappa(i'_1, i'_2, \dots, i'_n)$ may well be nontrivial permutations of each other or even $\bar{\kappa}(i'_1, i'_2, \dots, i'_n) \neq \bar{\kappa}(i_1, i_2, \dots, i_n)$.

**Theorem 4.5.** *For any ordered set partition $\pi \in \mathcal{OP}_n$ and numbers $N_1, N_2, \dots \in \mathbb{N}$ we have*

$$\varphi_\pi(N_{\pi(1)}.X_1, N_{\pi(2)}.X_2, \dots, N_{\pi(n)}.X_n) = \sum_{\sigma \le \pi} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \beta_{\underline{N}}(\sigma, \pi),$$

*with $\beta_{\underline{N}}(\sigma, \pi)$ as in Definition 2.18. It follows that $\varphi_\pi(N_{\pi(1)}.X_1, N_{\pi(2)}.X_2, \dots, N_{\pi(n)}.X_n)$ is a polynomial in $N_1, \dots, N_{|\pi|}$ without constant term. In particular for $N \in \mathbb{N}$ and $\pi \in \mathcal{OP}_n$*

$$\varphi_\pi(N.X_1, N.X_2, \dots, N.X_n) = \sum_{\sigma \le \pi} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \beta_N(\sigma, \pi),$$

*is a polynomial in $N$ consisting of monomials of degree at least $|\pi|$.*

*Proof.* The proof boils down to the enumeration of the set

$$S_{\underline{N}}(\sigma, \pi) := \{(i_1, i_2, \dots, i_n) \in [N_{\pi(1)}] \times [N_{\pi(2)}] \times \dots \times [N_{\pi(n)}] : \pi \curlywedge \kappa(i_1, i_2, \dots, i_n) = \sigma\}.$$

Pick an arbitrary block $P_i = \{p_1, p_2, \dots, p_k\}$ of $\pi = (P_1, P_2, \dots)$. There are $\binom{N_i}{\sharp(\sigma\rfloor_{P_i})}$ possible ways to choose a tuple $(i_{p_1}, i_{p_2}, \dots, i_{p_k})$ such that $\kappa(i_{p_1}, i_{p_2}, \dots, i_{p_k})$ defines the partition $\sigma\rfloor_{P_i}$: for each block of $\sigma\rfloor_{P_i}$ we have to choose a distinct label from $[N_i]$ respecting the order prescribed by the labels of the blocks of $\sigma\rfloor_{P_i}$. That is, we have to choose a subset of $[N_i]$ of cardinality $\#(\sigma\rfloor_{P_i})$. This can be done for every block of $\pi$ independently and thus

$$S_{\underline{N}}(\sigma, \pi) = \prod_{i=1}^{|\pi|} \binom{N_i}{\sharp(\sigma\rfloor_{P_i})} = \beta_{\underline{N}}(\sigma, \pi). \tag{4.2}$$

If $N_i < \sharp(\sigma\rfloor_{P_i})$ for some $i$, then $\pi \curlywedge \kappa(i_1, i_2, \dots, i_n)$ can never be equal to $\sigma$, and so the cardinality of $S_{\underline{N}}(\sigma, \pi)$ is 0, in accordance with the generally adopted convention that the generalized binomial coefficient $\binom{N}{k}$ is zero when $N < k$. □

**Definition 4.6.** Given an ordered set partition $\pi = (P_1, P_2, \dots) \in \mathcal{OP}_n$, we define the *(partitioned) cumulant* $K_\pi(X_1, X_2, \dots, X_n)$ to be the coefficient of $N^{|\pi|}$ in $\varphi_\pi(N.X_1, N.X_2, \dots, N.X_n)$. Alternatively, inspecting the proof of Theorem 4.7, this is the same as the coefficient of the multilinear monomial $N_1 N_2 \cdots N_{|\pi|}$ in the polynomial $\varphi_\pi(N_{\pi(1)}.X_1, N_{\pi(2)}.X_2, \dots, N_{\pi(n)}.X_n)$

The next theorem shows that this definition coincides with a natural generalization of [Leh04, Definition 2.6].

**Theorem 4.7** (Cumulants in terms of moments)**.** *For any $\pi \in \mathcal{OP}_n$, we have*

$$K_\pi(X_1, X_2, \dots, X_n) = \sum_{\sigma \le \pi} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi).$$

*Proof.* For $0 < k < N$, there is no constant term in $\binom{N}{k} = \frac{N(N-1)\cdots(N-k+1)}{k!}$ (regarded as a polynomial in $N$), and the coefficient of its linear term is $\frac{(-1)(-2)\cdots(-k+1)}{k!} = \frac{(-1)^{k-1}}{k}$. Therefore the coefficient of $N_1 \cdots N_{|\pi|}$ in $\beta_{\underline{N}}(\sigma, \pi) = \prod_{i=1}^{|\pi|} \binom{N_i}{\sharp(\sigma\rfloor_{P_i})}$ is equal to $\frac{(-1)^{|\pi|-|\sigma|}}{[\sigma:\pi]}$. Note that $\sum_{P \in \pi} \sharp(\sigma\rfloor_P) = |\sigma|$. Comparing with (2.12) the desired formula follows. The same argument holds true if we look at the monomial $N^{|\pi|}$ when $N_1 = N_2 = \dots = N$. □

Using Corollary 2.21 we can immediately express moments in terms of cumulants.

**Theorem 4.8** (Moments in terms of cumulants)**.** *For any $\pi \in \mathcal{OP}_n$, we have*

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \sum_{\sigma \leq \pi} K_\sigma(X_1, X_2, \dots, X_n)\, \widetilde{\zeta}(\sigma, \pi).$$

The next proposition shows that the definition of cumulants for spreadability systems is consistent with the previous definition for exchangeability systems from [Leh04].

**Proposition 4.9.** *If a spreadability system satisfies exchangeability, then $K_\pi = K_{h(\pi)}$ for $\pi = (P_1, P_2, \dots) \in \mathcal{OP}_n$ and $h \in \mathfrak{S}_{|\pi|}$, where $h(\pi) = (P_{h(1)}, P_{h(2)}, \dots)$; i.e., the value is invariant under reordering of the blocks of $\pi$ and thus is determined by the underlying unordered partition $\bar{\pi}$.*

*Proof.* Exchangeability entails invariance under permutation of blocks, that is, $\varphi_\sigma = \varphi_{g(\sigma)}$ for any $\sigma \in \mathcal{OP}_n, g \in \mathfrak{S}_{|\sigma|}$. For any $h \in \mathfrak{S}_{|\pi|}$ and $i_1, i_2, \dots, i_n \in \mathbb{N}$, there exists $g \in \mathfrak{S}_\infty$ such that $h(\pi) \wedge \kappa(i_1, i_2, \dots, i_n) = g(\pi \wedge \kappa(i_1, i_2, \dots, i_n))$. Therefore,

$$\begin{aligned}
\varphi_{h(\pi)}(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}) &= \varphi_{h(\pi)\wedge\kappa(i_1,i_2,\dots,i_n)}(X_1, X_2, \dots, X_n) \\
&= \varphi_{g(\pi\wedge\kappa(i_1,i_2,\dots,i_n))}(X_1, X_2, \dots, X_n) \\
&= \varphi_{\pi\wedge\kappa(i_1,i_2,\dots,i_n)}(X_1, X_2, \dots, X_n) \\
&= \varphi_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}).
\end{aligned}$$

Thus we have equality $\varphi_\pi(N.X_1, N.X_2, \dots, N.X_n) = \varphi_{h(\pi)}(N.X_1, N.X_2, \dots, N.X_n)$ and hence, by definition, $K_\pi(X_1, X_2, \dots, X_n) = K_{h(\pi)}(X_1, X_2, \dots, X_n)$. ☐

**Example 4.10** (Cumulants in terms of moments)**.** We will write the ordered kernel set partition $\kappa(i_1, \dots, i_n)$ simply as the multiset permutation $i_1 i_2 \dots i_n$. For example, $211 = (\{2,3\}, \{1\})$. Examples of Theorem 4.7 are given by

$$K_1(X) = \varphi_1(X), \tag{4.3}$$

$$K_{11}(X,Y) = \varphi_{11}(X,Y) - \frac{1}{2}(\varphi_{12}(X,Y) + \varphi_{21}(X,Y)), \tag{4.4}$$

$$K_{12}(X,Y) = \varphi_{12}(X,Y), \tag{4.5}$$

$$K_{21}(X,Y) = \varphi_{21}(X,Y), \tag{4.6}$$

$$\begin{aligned}
K_{111} = \varphi_{111} &- \frac{1}{2}(\varphi_{112} + \varphi_{121} + \varphi_{122} + \varphi_{211} + \varphi_{212} + \varphi_{221}) \\
&+ \frac{1}{3}(\varphi_{123} + \varphi_{132} + \varphi_{213} + \varphi_{231} + \varphi_{312} + \varphi_{321}),
\end{aligned} \tag{4.7}$$

$$K_{112} = \varphi_{112} - \frac{1}{2}(\varphi_{123} + \varphi_{213}), \tag{4.8}$$

where for the sake of compactness $(X, Y, Z)$ are omitted in the last two formulas.

**Example 4.11** (Moments in terms of cumulants)**.** Examples of Theorem 4.8 are given by

$$\varphi_1(X) = K_1(X), \tag{4.9}$$

$$\varphi_{11}(X,Y) = K_{11}(X,Y) + \frac{1}{2!}(K_{12}(X,Y) + K_{21}(X,Y)), \tag{4.10}$$

$$\varphi_{12}(X,Y) = K_{12}(X,Y), \tag{4.11}$$

$$\varphi_{21}(X,Y) = K_{21}(X,Y), \tag{4.12}$$

$$\begin{aligned}
\varphi_{111} = K_{111} &+ \frac{1}{2!}(K_{112} + K_{121} + K_{122} + K_{211} + K_{212} + K_{221}) \\
&+ \frac{1}{3!}(K_{123} + K_{132} + K_{213} + K_{231} + K_{312} + K_{321}),
\end{aligned} \tag{4.13}$$

$$\varphi_{112} = K_{112} + \frac{1}{2!}(K_{123} + K_{213}), \tag{4.14}$$

where $(X, Y, Z)$ are omitted in the last two formulas for simplicity.

4.2. **Extensivity and uniqueness of cumulants.** In order to show uniqueness of cumulants we first show extensivity.

**Definition 4.12.** (i) For $\pi \in \mathcal{OP}_n$ and $M_i, N_i \in \mathbb{N}, i = 1, 2, \dots, n$, we define

$$\begin{aligned}&\varphi_\pi(M_1.(N_1.X_1), M_2.(N_2.X_2), \dots, M_n.(N_n.X_n))\\ &\qquad := \sum_{(i_1,\dots,i_n)\in[M_1]\times\cdots\times[M_n]} \varphi_{\pi\wedge\kappa(i_1,i_2,\dots,i_n)}(N_1.X_1, N_2.X_2, \dots, N_n.X_n).\end{aligned}$$

(ii) For $\pi \in \mathcal{OP}_n$ and $i_1, i_2, \dots, i_n \in \mathbb{N}$, we define

$$\begin{aligned}K_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}) &:= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_\sigma(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)})\, \widetilde{\mu}(\sigma,\pi)\\ &= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_{\sigma\wedge\kappa(i_1,i_2,\dots,i_n)}(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma,\pi)\end{aligned}$$

and extend it by linearity to

$$K_\pi(N_1.X_1, N_2.X_2, \dots, N_n.X_n) := \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_\sigma(N_1.X_1, N_2.X_2, \dots, N_n.X_n)\, \widetilde{\mu}(\sigma,\pi).$$

**Remark 4.13.** We would like to alert the reader that the formal definition of the expectation $\varphi_\pi(M_1.(N_1.X_1), M_2.(N_2.X_2), \dots, M_n.(N_n.X_n))$ is not necessarily related to $M.(N.X)$ as an element of an enlarged space, obtained, e.g., by iterating a product construction. Yet we show below that the random vectors $(M_{\pi(1)}N_{\pi(1)}.X_1, \dots, M_{\pi(n)}N_{\pi(n)}.X_n)$ and $(M_{\pi(1)}.(N_{\pi(1)}.X_1), \dots, M_{\pi(n)}.(N_{\pi(n)}.X_n))$ (which formally expresses $(N.X)^{(1)} + (N.X)^{(2)} + \cdots + (N.X)^{(M)}$, "the sum of i.i.d. copies of $N.X$") have the same distribution. This property implies extensivity of cumulants, see Theorem 4.15 below. It is a consequence of the associativity of the corresponding product of states in the case of classical, free, monotone and Boolean independences [HS11a]. For general spreadability systems it holds on a formal level (Proposition 4.14), even when it comes from a nonassociative product of states, as the example of V-monotone independence from Section 3.3.6.

**Proposition 4.14.** *Let $\pi \in \mathcal{OP}_n$, $X_i \in \mathcal{A}$ and $M_i, N_i \in \mathbb{N}, i = 1, 2, \dots, |\pi|$.*

*(i) The value $\varphi_\pi(M_{\pi(1)}.(N_{\pi(1)}.X_1), \dots, M_{\pi(n)}.(N_{\pi(n)}.X_n))$ is a polynomial in $M_i, N_i, i = 1, 2, \dots, |\pi|$, and $K_\pi(N_{\pi(1)}.X_1, \dots, N_{\pi(n)}.X_n)$ is the coefficient of $M_1 \cdots M_{|\pi|}$.*

*(ii) The dot operation gives rise to an action of $\mathbb{N}^\infty$ which leaves the distribution invariant:*

$$\varphi_\pi(M_{\pi(1)}N_{\pi(1)}.X_1, \dots, M_{\pi(n)}N_{\pi(n)}.X_n) = \varphi_\pi(M_{\pi(1)}.(N_{\pi(1)}.X_1), \dots, M_{\pi(n)}.(N_{\pi(n)}.X_n)).$$

*Proof.* (i) Let $\underline{M} := (M_1, \dots, M_{|\pi|}, 0, 0, \dots) \in \mathbb{N}^\infty$ and similarly $\underline{N} \in \mathbb{N}^\infty$. Then the following expansion holds:

$$\begin{aligned}&\varphi_\pi(M_{\pi(1)}.(N_{\pi(1)}.X_1), \dots, M_{\pi(n)}.(N_{\pi(n)}.X_n))\\ &\quad= \sum_{(i_1,\dots,i_n)\in[M_{\pi(1)}]\times\cdots\times[M_{\pi(n)}]} \varphi_{\pi\wedge\kappa(i_1,i_2,\dots,i_n)}(N_{\pi(1)}.X_1, \dots, N_{\pi(n)}.X_n)\\ &\quad= \sum_{\sigma\le\pi} \sum_{\substack{(i_1,\dots,i_n)\in[M_{\pi(1)}]\times\cdots\times[M_{\pi(n)}],\\ \pi\wedge\kappa(i_1,i_2,\dots,i_n)=\sigma}} \varphi_\sigma(N_{\pi(1)}.X_1, \dots, N_{\pi(n)}.X_n)\\ &\quad= \sum_{\sigma\le\pi} \varphi_\sigma(N_{\pi(1)}.X_1, \dots, N_{\pi(n)}.X_n)\, \beta_{\underline{M}}(\sigma,\pi),\end{aligned} \tag{4.15}$$

where we used the identity (4.2). Therefore $\varphi_\pi(M_{\pi(1)}.(N_{\pi(1)}.X_1), \dots, M_{\pi(n)}.(N_{\pi(n)}.X_n))$ is a polynomial in $M_i, N_i, i = 1, 2, \dots, |\pi|$, and from the proof of Theorem 4.7 we infer that the coefficient of $M_1 \cdots M_{|\pi|}$ is equal to $K_\pi(N_{\pi(1)}.X_1, \dots, N_{\pi(n)}.X_n)$.

(ii) We proceed with the computation of (4.15).

$$
\begin{aligned}
(4.16)\quad & \varphi_\pi(M_{\pi(1)}.(N_{\pi(1)}.X_1),\dots,M_{\pi(n)}.(N_{\pi(n)}.X_n)) \\
&= \sum_{\sigma\le\pi} \varphi_\sigma(N_{\pi(1)}.X_1,\dots,N_{\pi(n)}.X_n)\,\beta_{\underline{M}}(\sigma,\pi) \\
&= \sum_{\substack{\sigma\in\mathcal{OP}_n,\\ \sigma\le\pi}} \sum_{\substack{\rho\in\mathcal{OP}_n,\\ \rho\le\sigma}} \varphi_\rho(X_1,X_2,\dots,X_n)\,\gamma_{\underline{N}}(\rho,\sigma,\pi)\,\beta_{\underline{M}}(\sigma,\pi) \\
&= \sum_{\substack{\rho\in\mathcal{OP}_n,\\ \rho\le\pi}} \varphi_\rho(X_1,X_2,\dots,X_n)\left(\sum_{\substack{\sigma\in\mathcal{OP}_n,\\ \rho\le\sigma\le\pi}} \gamma_{\underline{N}}(\rho,\sigma,\pi)\,\beta_{\underline{M}}(\sigma,\pi)\right) \\
&= \sum_{\substack{\rho\in\mathcal{OP}_n,\\ \rho\le\pi}} \varphi_\rho(X_1,X_2,\dots,X_n)(\gamma_{\underline{N}} \circledast \beta_{\underline{M}})(\rho,\pi) \\
&= \sum_{\substack{\rho\in\mathcal{OP}_n,\\ \rho\le\pi}} \varphi_\rho(X_1,X_2,\dots,X_n)\,\beta_{\underline{M}\circ\underline{N}}(\rho,\pi) \\
&= \varphi_\pi(M_{\pi(1)}N_{\pi(1)}.X_1,\dots,M_{\pi(n)}N_{\pi(n)}.X_n),
\end{aligned}
$$

where Corollary 2.21 was used in the next to last line. □

**Theorem 4.15.** *Cumulants satisfy extensivity*

$$
\text{(E)}\qquad K_\pi(N.X_1,N.X_2,\dots,N.X_n) = N^{|\pi|}K_\pi(X_1,X_2,\dots,X_n),
$$

*and more generally,*

$$
K_\pi(N_{\pi(1)}.X_1,\dots,N_{\pi(n)}.X_n) = N_1\cdots N_{|\pi|}K_\pi(X_1,X_2,\dots,X_n).
$$

*Proof.* By definition, $\varphi_\pi(M_{\pi(1)}N_{\pi(1)}.X_1,\dots,M_{\pi(n)}N_{\pi(n)}.X_n)$ is of the form

$$
(\prod_{i=1}^{|\pi|} M_iN_i)K_\pi(X_1,X_2,\dots,X_n) + (\text{the sum of monomials on } M_iN_i \text{ with higher degrees}).
$$

On the other hand, from Proposition 4.14(i),

$$
\begin{aligned}
&\varphi_\pi(M_{\pi(1)}.(N_{\pi(1)}.X_1),\dots,M_{\pi(n)}.(N_{\pi(n)}.X_n)) \\
&\quad = M_1M_2\cdots M_{|\pi|}K_\pi(N.X_1,N.X_2,\dots,N.X_n) \\
&\qquad + (\text{the sum of monomials on } M_i \text{ with higher degrees}).
\end{aligned}
$$

We have thus computed the expectation in two ways and it follows from Proposition 4.14(ii) that the coefficients of $M_1\cdots M_{|\pi|}$ coincide. □

Finally, we prove uniqueness of cumulants.

**Proposition 4.16.** *The cumulant functionals $K_\pi$ satisfy the following properties.*

*(i) There exist constants $c(\sigma,\pi)\in\mathbb{C}$ for $\sigma<\pi$, $\sigma,\pi\in\mathcal{OP}_n$ such that*

$$
K_\pi = \varphi_\pi + \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma<\pi}} c(\sigma,\pi)\,\varphi_\sigma.
$$

*(ii) Extensivity holds: $K_\pi(N.X_1,N.X_2,\dots,N.X_n) = N^{|\pi|}K_\pi(X_1,X_2,\dots,X_n)$.*

*Moreover conditions* (i) *and* (ii) *uniquely determine cumulants: If family of functionals $\widetilde{K}_\pi$ satisfies the same conditions with some constants $\widetilde{c}(\sigma,\pi)$, then $K_\pi = \widetilde{K}_\pi$ for all $\pi$.*

*Proof.* We know from Theorems 4.8 and 4.15 that $K_\pi$ satisfy (i) and (ii), and the value of the coefficient $c(\sigma,\pi)$ is explicitly given by $\frac{1}{[\sigma:\pi]!}$, which in fact does not depend on order of blocks of $\pi$ or $\sigma$, i.e., it is uniquely determined by the underlying unordered partitions $\bar{\pi}$ and $\bar{\sigma}$.

To prove uniqueness, we recursively write the first condition in the form

$$
\varphi_\pi = K_\pi + \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma<\pi}} d(\sigma,\pi)K_\sigma
$$

and similarly for $\widetilde{K}_\pi$. Then we can write $\varphi_\pi(N.X_1, N.X_2, \dots, N.X_n)$ in two different ways:

$$\begin{aligned}\varphi_\pi&(N.X_1, N.X_2, \dots, N.X_n)\\ &= N^{|\pi|} K_\pi(X_1, X_2, \dots, X_n) + \sum_{\sigma<\pi} N^{|\sigma|} K_\sigma(X_1, X_2, \dots, X_n)\, d(\sigma, \pi)\\ &= N^{|\pi|} \widetilde{K}_\pi(X_1, X_2, \dots, X_n) + \sum_{\sigma<\pi} N^{|\sigma|} \widetilde{K}_\sigma(X_1, X_2, \dots, X_n)\, \widetilde{d}(\sigma, \pi).\end{aligned} \tag{4.17}$$

The coefficients of $N^{|\pi|}$ must be the same and therefore $K_\pi = \widetilde{K}_\pi$. □

4.3. **Examples.** Let us now briefly review instances of moment-cumulant formulas coming from noncommutative independences under the light of Theorem 4.8. The central objects are the $n$-linear maps

$$K_n := K_{\hat{1}_n}, \qquad \varphi_n := \varphi_{\hat{1}_n}$$

for the maximal ordered set partition $\hat{1}_n$. We call $K_n$ the $n^{\text{th}}$ cumulant and in the next definition we extend the subscript $n$ of $\varphi_n$ and $K_n$ multiplicatively to set partitions and ordered set partitions.

In the remainder of this section, we assume that $\mathcal{B}$ is commutative, but it will not be difficult for readers familiar with operator-valued independence [Spe98, Pop08, HS14, Ske04, Mło02] to generalize the results to general $\mathcal{B}$ in the cases of free, Boolean and monotone products and spreadability systems.

**Definition 4.17.** The multiplicative extensions $(\varphi_{(\pi)})_{\pi\in\mathcal{P}_n}, (K_{(\pi)})_{\pi\in\mathcal{P}_n}$ of $(\varphi_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}}$ are respectively defined by

$$\varphi_{(\pi)}(Y_1, Y_2, \dots, Y_n) := \prod_{P\in\pi} \varphi_{|P|}(Y_P), \qquad Y_j \in \bigcup_{i=1}^\infty \iota^{(i)}(\mathcal{A}), \quad j \in [n],$$

$$K_{(\pi)}(Y_1, Y_2, \dots, Y_n) := \prod_{P\in\pi} K_{|P|}(Y_P), \qquad Y_j \in \bigcup_{i=1}^\infty \iota^{(i)}(\mathcal{A}), \quad j \in [n],$$

where the notation (3.2) is used for multilinear functionals. We also define $\varphi_{(\pi)} := \varphi_{(\bar\pi)}$ and $K_{(\pi)} := K_{(\bar\pi)}$ for ordered set partitions $\pi \in \mathcal{OP}_n$.

These multiplicative extensions are important to understand the cumulants associated to the four natural products. It turns out that for specific spreadability systems certain cumulant functionals vanish identically. This is related to a certain factorization phenomenon which is subsumed in the following lemma.

**Lemma 4.18.** *Let $\mathcal{S}$ be a spreadability system for some noncommutative probability space $(\mathcal{A}, \varphi)$ and let $\pi \in \mathcal{OP}_n$ be an ordered set partition.*

*(i) Assume that*

$$\varphi^{\mathcal{S}}_\pi(X_1, X_2, \dots, X_n) = \varphi_{(\pi)}(X_1, X_2, \dots, X_n) \tag{4.18}$$

*holds for any tuple of random variables $X_i \in \mathcal{A}$. Then*

$$K^{\mathcal{S}}_\pi(X_1, X_2, \dots, X_n) = K^{\mathcal{S}}_{(\pi)}(X_1, X_2, \dots, X_n)$$

*as well.*

*(ii) Assume otherwise that there are universal constants $c_{\pi,\sigma}$ such that the expansion*

$$\varphi^{\mathcal{S}}_\pi(X_1, X_2, \dots, X_n) = \sum_{\sigma<\pi} c_{\pi,\sigma}\, \varphi_{(\sigma)}(X_1, X_2, \dots, X_n) \tag{4.19}$$

*holds for any tuple of random variables $X_i \in \mathcal{A}$. Then $K^{\mathcal{S}}_\pi(X_1, X_2, \dots, X_n) = 0$ identically.*

*Proof.* Since the number of blocks of any partition $\sigma < \pi$ is strictly larger than the number of blocks of $\pi$, this is an immediate consequence of Theorem 4.5 and Definition 4.6. □

**Proposition 4.19.** *(i) Let $K^{\mathrm{T}}_\pi$ be the cumulants associated to the tensor spreadability system $\mathcal{S}_{\mathrm{T}}$. Then*

$$K^{\mathrm{T}}_\pi = K^{\mathrm{T}}_{(\pi)}, \qquad \pi \in \mathcal{OP}_n. \tag{4.20}$$

*(ii) Let $K_\pi^{\mathrm{F}}$ be the cumulants associated to the free spreadability system $\mathcal{S}_{\mathrm{F}}$. Then*

$$K_\pi^{\mathrm{F}} = \begin{cases} 0, & \pi \notin \mathcal{ONC}_n, \\ K_{(\pi)}^{\mathrm{F}}, & \pi \in \mathcal{ONC}_n. \end{cases} \tag{4.21}$$

*(iii) Let $K_\pi^{\mathrm{T}}$ be the cumulants associated to the Boolean spreadability system $\mathcal{S}_{\mathrm{B}}$. Then*

$$K_\pi^{\mathrm{B}} = \begin{cases} 0, & \pi \notin \mathcal{OI}_n, \\ K_{(\pi)}^{\mathrm{B}}, & \pi \in \mathcal{OI}_n. \end{cases} \tag{4.22}$$

*(iv) Let $K_\pi^{\mathrm{M}}$ be the cumulants associated to the monotone spreadability system $\mathcal{S}_{\mathrm{M}}$. Then*

$$K_\pi^{\mathrm{M}} = \begin{cases} 0, & \pi \notin \mathcal{M}_n, \\ K_{(\pi)}^{\mathrm{M}}, & \pi \in \mathcal{M}_n. \end{cases} \tag{4.23}$$

*Proof.* Items (i), (ii) and (iii) satisfy exchangeability and are covered in [Leh04] (note that our cumulants are invariant under the permutation of the blocks of $\pi$, see Proposition 4.9); alternatively these cases can be shown using Lemma 4.18.

We are left with case (iv). Assume first that $\pi \in \mathcal{M}_n$, then it follows from the very definition of monotone independence in section 3.3.4 that (4.18) holds and thus $K_\pi$ vanishes identically.

Assume now to the contrary that $\pi \in \mathcal{OP}_n \setminus \mathcal{M}_n$, then it follows from [Mur02, Prop. 3.2] that (4.19) holds and thus $K_\pi$ vanishes identically. □

**Remark 4.20.** Other examples of noncrossing cumulants come from the c-free exchangeability systems of [BLS96], see [Leh04, Section 4.7]. Again Lemma 4.18 provides a new proof.

We conclude this section with a generalized variant of monotone cumulants coming from c-monotone spreadability system $\mathcal{S}_{\mathrm{CM}}$ as we shall see next. Recall that we consider two states $\varphi, \psi$ on an algebra $\mathcal{A}$ to define $\mathcal{S}_{\mathrm{CM}}$.

**Proposition 4.21.** *The cumulant functional $K_\pi^{\mathrm{CM}}$ associated to the c-monotone spreadability system $\mathcal{S}_{\mathrm{CM}}$ is given by*

$$K_\pi^{\mathrm{CM}}(X_1, X_2, \dots, X_n) = \begin{cases} \prod_{P \in \mathrm{Outer}(\pi)} K_{|P|}^{\mathrm{CM}}(X_P) \prod_{P \in \mathrm{Inner}(\pi)} K_{|P|}^{\mathrm{M},\psi}(X_P), & \pi \in \mathcal{M}_n, \\ 0, & \pi \notin \mathcal{M}_n, \end{cases}$$

*where $K_n^{\mathrm{M},\psi}$ is the $n^{\mathrm{th}}$ monotone cumulant with respect to the linear map $\psi : \mathcal{A} \to \mathbb{C}$.*

*Proof.* We extend the linear map $\psi$ on $\mathcal{A}$ to $\mathcal{U}$ by taking the monotone product of $\psi$'s, and so we have two spreadability systems: the c-monotone spreadability system $\mathcal{S}_{\mathrm{CM}} = (\mathcal{U}, (\iota^{(i)})_{i=1}^\infty, \widetilde{\varphi})$ for $(\mathcal{A}, \varphi)$ and the monotone spreadability system $\mathcal{S}_{\mathrm{M}} = (\mathcal{U}, (\iota^{(i)})_{i=1}^\infty, \widetilde{\psi})$ for $(\mathcal{A}, \psi)$. The multiplicative extension $\varphi_{(\pi)}$ of $\varphi$ is now replaced by

$$\varphi_{(\pi),\psi}(X_1, X_2, \dots, X_n) := \prod_{P \in \mathrm{Outer}(\pi)} \varphi_{|P|}(X_P) \prod_{P \in \mathrm{Inner}(\pi)} \psi_{|P|}(X_P), \quad \pi \in \mathcal{ONC}_n, \tag{4.24}$$

which coincides with $\varphi_{(\pi)}$ if $\varphi = \psi$. Similar to the monotone case, for every ordered pair $\rho, \tau \in \mathcal{P}_n$ with $\rho < \tau$ there exist a universal constant $c_{\mathrm{CM}}(\rho, \tau) \in \mathbb{C}$ and a subset $\mathcal{V}(\rho, \tau) \subseteq \rho$, such that

$$\varphi_\pi(Y_1, Y_2, \dots, Y_n) = \begin{cases} \varphi_{(\pi),\psi}(Y_1, Y_2, \dots, Y_n) + \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma < \bar{\pi}}} c_{\mathrm{CM}}(\sigma, \bar{\pi}) \left( \prod_{S \in \mathcal{V}(\sigma,\pi)} \varphi_{|S|}(Y_S) \right) \left( \prod_{S \in \mathcal{V}(\sigma,\pi)^c} \psi_{|S|}(Y_S) \right), & \pi \in \mathcal{M}_n, \\ \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma < \bar{\pi}}} c_{\mathrm{CM}}(\sigma, \bar{\pi}) \left( \prod_{S \in \mathcal{V}(\sigma,\pi)} \varphi_{|S|}(Y_S) \right) \left( \prod_{S \in \mathcal{V}(\sigma,\pi)^c} \psi_{|S|}(Y_S) \right), & \pi \notin \mathcal{M}_n \end{cases}$$

for $Y_j \in \bigcup_{i=1}^\infty \iota^{(i)}(\mathcal{A}), j \in [n]$ and the same idea of Lemma 4.18 and Proposition 4.19 applies. □

**Remark 4.22.** Similar arguments apply for the cumulants introduced in [Has10].

## 5. Central limit theorem

Cumulants provide a natural framework to understand central limit theorems. Speicher and Waldenfels studied central limit theorems in a general setting of noncommutative probability assuming a certain *singleton condition* [SW94] (see also [AHO98]). In this section we will see that a similar approach also applies in the setting of a spreadability system, provided that an appropriate singleton condition holds.

**Definition 5.1.** (i) An element $k \in [n]$ is called a singleton of $\pi \in \mathcal{OP}_n$ if $\{k\} \in \pi$.

(ii) Let $\mathcal{S} = (\mathcal{U}, (\iota^{(j)})_{j\geq 1}, \widetilde{\varphi})$ be a spreadability system for a noncommutative probability space $(\mathcal{A}, \varphi)$. We will say that the *singleton condition holds for* $\mathcal{S}$ if $\varphi_\pi(X_1, X_2, \dots, X_n) = 0$ for every tuple $(X_1, X_2, \dots, X_n)$ and every partition $\pi$ containing a singleton $\{k\}$ such that $\varphi(X_k) = 0$.

Under this assumption we can show the following type of central limit theorem.

**Theorem 5.2.** *Assume that a spreadability system $(\mathcal{U}, (\iota^{(j)})_{j\geq 1}, \widetilde{\varphi})$ for $(\mathcal{A}, \varphi)$ satisfies the singleton condition. Assume $\varphi(X) = 0$ and let $Y_N := \frac{N.X}{\sqrt{N}}$. Then, for each $n \in \mathbb{N}$ and $\rho \in \mathcal{P}_n$,*

$$\lim_{N\to\infty} \varphi_\rho(Y_N, Y_N, \dots, Y_N) = \begin{cases} \sum_{\substack{\pi\in\mathcal{OP}_n^{(2)}\\ \pi\leq\rho}} \frac{1}{|\pi|!}\varphi_\pi(X, X, \dots, X), & n \text{ is even},\\ 0, & n \text{ is odd}.\end{cases}$$

*where $\mathcal{OP}_n^{(2)}$ is the set of pair ordered set partitions, i.e. every block of $\pi \in \mathcal{OP}_n^{(2)}$ contains exactly 2 entries.*

*Proof.* Note that the following holds: if $\pi$ has a singleton at $k$ and $\varphi(X_k) = 0$, then

$$K_\pi(X_1, X_2, \dots, X_n) = 0. \tag{5.1}$$

This holds because $K_\pi(X_1, X_2, \dots, X_n)$ is the coefficient of $N$ in $\varphi_\pi(N.X_1, N.X_2, \dots, N.X_n)$, and

$$\varphi_\pi(N.X_1, N.X_2, \dots, N.X_n) = \sum_{i_1,i_2,\dots,i_n\in[n]} \varphi_{\pi\wedge\kappa(i_1,i_2,\dots,i_n)}(X_1, X_2, \dots, X_n) = 0 \tag{5.2}$$

because each $\pi \wedge \kappa(i_1, i_2, \dots, i_n)$ has a singleton at $k$.

Now multilinearity and extensivity of cumulants imply

$$K_\pi(Y_N, Y_N, \dots, Y_N) = N^{-\frac{n}{2}+|\pi|} K_\pi(X, X, \dots, X). \tag{5.3}$$

If $\pi$ has a singleton, this is zero. If $\pi$ does not have a singleton nor $\pi$ is not a pair ordered set partition, then $|\pi| < \frac{n}{2}$. Therefore,

$$\lim_{N\to\infty} K_\pi(Y_N, Y_N, \dots, Y_N) = \begin{cases} K_\pi(X, X, \dots, X), & \text{if } n \text{ is even and } \pi \in \mathcal{OP}_n^{(2)},\\ 0, & \text{otherwise}.\end{cases} \tag{5.4}$$

If $\pi \in \mathcal{OP}_n^{(2)}$, then $K_\pi(X, X, \dots, X) = \varphi_\pi(X, X, \dots, X)$ from Theorem 4.7 because the expectations $\varphi_\sigma(X, X, \dots, X)$ all vanish for $\sigma < \pi$ from the singleton condition. Finally, from Theorem 4.8, we obtain the conclusion. ∎

Thus the use of cumulants simplifies the proof of the central limit theorem. It may happen that the moments of the limit distribution are not uniquely determined only by the variance of $X$ alone, because in general $\varphi_\pi(X, X, \dots, X)$ cannot be written in terms of $\varphi(X^2)$. For example, the limit distribution for the c-monotone spreadability system is characterized by the moments

$$\lim_{N\to\infty} \varphi(Y_N^n) = \begin{cases} \sum_{\pi\in\mathcal{M}_n^{(2)}} \frac{1}{|\pi|!}\alpha^{2|\mathrm{Outer}(\pi)|}\beta^{2|\mathrm{Inner}(\pi)|}, & n \text{ is even},\\ 0, & n \text{ is odd},\end{cases} \tag{5.5}$$

where $\alpha^2 = \varphi(X^2)$, $\beta^2 = \psi(X^2)$ and $\mathcal{M}_n^{(2)}$ is the set of monotone pair partitions; the reader is referred to Theorems 4.7, 5.1 of [Has11]. The limit moments also depend on the second linear map $\psi$ and they are not determined by $\alpha^2$. To recover uniqueness, we have to introduce an additional assumption. One possibility is a universal calculation rule [Mur02, Spe97] which also works for spreadability systems:

**Definition 5.3.** Let $(\mathcal{U}, (\iota^{(j)})_{j\ge 1}, \widetilde{\varphi})$ be a spreadability system for a ncps $(\mathcal{A}, \varphi)$. It is said to satisfy a universal calculation rule if there exist constants $s(\tau,\pi) \in \mathbb{R}$ for each $\pi \in \mathcal{OP}_n, \tau \in \mathcal{P}_n, \tau \le \bar{\pi}$, such that the equality

$$\varphi_\pi = \sum_{\substack{\tau\in\mathcal{P}_n,\\ \tau\le\bar{\pi}}} s(\tau,\pi)\,\varphi_{(\tau)}$$

as functionals on $\mathcal{A}^n$.

For our purpose, it suffices to consider a universal calculation rule for $\pi \in \mathcal{OP}_n^{(2)}$ and $\tau \le \bar{\pi}$ (the case $\tau < \bar{\pi}$ in fact guarantees the singleton condition). Then the limit distribution of the central limit theorem is determined only by the variance and by the universal constants $s(\pi,\bar{\pi})$. More precisely, $\varphi_\pi(X,X,\dots,X) = s(\bar{\pi},\pi)\alpha^{2|\pi|}$ if $\pi \in \mathcal{OP}_n^{(2)}$, $\alpha^2 = \varphi(X^2)$ and $\varphi(X) = 0$. Hence the limit moments can be written as

$$\lim_{N\to\infty} \varphi_\rho(Y_N, Y_N, \dots, Y_N) = \begin{cases} \sum_{\substack{\pi\in\mathcal{OP}_n^{(2)}\\ \pi\le\rho}} \frac{s(\bar{\pi},\pi)}{|\pi|!}\alpha^{2|\pi|}, & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases} \tag{5.6}$$

Tensor, free, monotone and Boolean spreadability/exchangeability systems satisfy universal calculation rules. On the other hand, the result of Maassen and Leeuwen showed that the exchangeability system for the $q$-Fock space does not have a universal calculation rule [vLM96]. A natural question is if there exist more spreadability systems which have universal calculation rules. The answer to this question seems to be positive; Muraki recently constructed some examples [Mur13].

## 6. Partial cumulants and differential equations

Neither the defining formula (Definition 4.6) nor the Möbius formula (Theorem 4.7) are suitable for the efficient calculation of cumulants of higher orders. In the case of exchangeability systems recursive formulas are available which are more adequate for this purpose; see [Leh04, Proposition 3.9]. In the classical case, the recursion reads as follows:

$$K^{\mathrm{T}}(X_1, X_2, \dots, X_n) = \mathbf{E}\, X_1 X_2 \cdots X_n - \sum_{\substack{A\subsetneq[n]\\ 1\in A}} K^{\mathrm{T}}_{|A|}(X_i : i\in A)\,\mathbf{E} \prod_{j\in A^c} X_j.$$

In the univariate case this is the familiar formula

$$\kappa_n = m_n - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \kappa_k m_{n-k}$$

which for normal random variables specifies to *Stein's method.*

In the free case the recursive formula reads

$$K^{\mathrm{F}}_n(X_1, X_2, \dots, X_n) = \varphi(X_1 X_2 \cdots X_n) - \sum_{\substack{A\subsetneq[n]\\ 1\in A}} K^{\mathrm{F}}_{|A|}(X_i : i\in A)\,\varphi_{\tilde{\iota}_{\max(A^c)}}(X_j \mid j\in A^c) \tag{6.1}$$

(see Definition 2.3) and specifies to the free Schwinger-Dyson equation [MS13]. From the point of view of combinatorial Hopf algebras this has been recently considered under the name of "splitting process" [EFP16].

Turning to our general setting we note that already in the case of monotone probability we lack a simple recursive formula; however the first author and Saigo [HS11a, HS11b] found a good replacement in terms of differential equations. Differential equations also play a major role in free probability, for example the complex Burgers' equation and its generalizations appear in the context of free Lévy processes [Voi86]. In this section we will unify these differential equations from the viewpoint of spreadability systems.

For $\pi = (P_1, \dots, P_p) \in \mathcal{OP}_n$ we have observed in Theorem 4.5 that $\varphi_\pi(N_{\pi(1)}.X_1, \dots, N_{\pi(n)}.X_n)$ is a polynomial in $N_1, \dots, N_p$ and we may formally replace $N_1, \cdots, N_p$ with real numbers $t_1, \dots, t_p$. Thus we obtain formal multivariate moment polynomials

$$\varphi_\pi^{\underline{t}}(X_1, \dots, X_n) := \varphi_\pi(t_{\pi(1)}.X_1, \dots, t_{\pi(n)}.X_n), \qquad \underline{t} \in \mathbb{R}^p. \tag{6.2}$$

In this section we derive recursive differential equations for these moment polynomials. By Definition 4.6, our cumulants are given by

$$K_\pi(X_1,\dots,X_n) = \frac{\partial^p}{\partial t_1\partial t_2\cdots\partial t_p}\bigg|_{\underline{t}=(0,\dots,0)} \varphi_\pi^{\underline{t}}(X_1,X_2,\dots,X_n). \tag{6.3}$$

In order to get recursive differential equations we need a refinement of cumulants which we call *partial cumulants*. They are obtained by taking the derivatives in (6.3) one at a time.

**Definition 6.1.** Let $\pi=(P_1,P_2,\dots,P_p)\in\mathcal{OP}_n, \underline{t}=(t_1,\dots,t_p)\in\mathbb{R}^p, j\in[p]$. We define the *partial cumulant* to be the polynomial

$$K_{\pi,P_j}^{(t_1,\dots,t_{j-1},1,t_{j+1},\dots,t_p)}(X_1,\dots,X_n) = \frac{\partial}{\partial t_j}\bigg|_{t_j=0} \varphi_\pi^{\underline{t}}(X_1,\dots,X_n).$$

Applying the binomial formula to specific blocks similar to the proofs of Theorems 4.5 and 4.7 it is easy to derive the following explicit expression for the partial cumulants.

**Proposition 6.2.** *Let* $\pi=(P_1,P_2,\dots,P_p)\in\mathcal{OP}_n$, $\underline{t}=(t_1,\dots,t_p)\in\mathbb{R}^p$, $j\in[p]$. *Then*

$$K_{\pi,P_j}^{(t_1,\dots,t_{j-1},1,t_{j+1},\dots,t_p)}(X_1,\dots,X_n) = \sum_{\sigma\in\mathcal{OP}_{P_j}} \varphi_{(P_1,P_2,\dots,P_{j-1},\sigma,P_{j+1},\dots,P_p)}^{(t_1,\dots,t_{j-1},1,t_{j+1},\dots,t_p)}(X_1,\dots,X_n)\,\widetilde{\mu}(\sigma,\hat{1}_{P_j}).$$

We are now ready to establish partial differential equations for the evolution of moments.

**Theorem 6.3.** *For* $\pi=(P_1,P_2,\dots,P_p)\in\mathcal{OP}_n$, $\underline{t}=(t_1,t_2,\dots,t_p)$ *and* $j\in[p]$ *we have*

$$\frac{\partial}{\partial t_j}\varphi_\pi^{\underline{t}}(X_1,\dots,X_n) = \sum_{\emptyset\neq A\subseteq P_j} K_{(P_1,\dots,P_{j-1},A,P_j\setminus A,P_{j+1},\dots,P_p),A}^{(t_1,\dots,t_{j-1},1,t_j,t_{j+1},\dots,t_p)}(X_1,\dots,X_n) \tag{6.4}$$

$$= \sum_{\emptyset\neq A\subseteq P_j} K_{(P_1,\dots,P_{j-1},P_j\setminus A,A,P_{j+1},\dots,P_p),A}^{(t_1,\dots,t_{j-1},t_j,1,t_{j+1},\dots,t_p)}(X_1,\dots,X_n). \tag{6.5}$$

*Proof.* The main ingredient here is the invariance principle of Lemma 4.3. Recall the delta operation from Definition 4.1(1), tensor notations from Section 3.1 and recall that for any partition $\pi=(P_1,P_2,\dots,P_p)\in\mathcal{OP}_n$ the multilinear map $\varphi_\pi^{\underline{t}}\colon\mathcal{A}^n\to\mathbb{C}$ is identified with the linear lifting $\tilde{\varphi}_\pi^{\underline{t}}\colon\mathcal{A}^{\otimes n}\to\mathbb{C}$. Let $\mathbf{X}=X_1\otimes X_2\otimes\cdots\otimes X_n$, let $k+[m]:=\{k+1,k+2,\dots,k+m\}$ and let $e_j\in\mathbb{R}^p$ be the $j$th unit vector. Then, for each $j\in[p]$,

$$\varphi_\pi^{\underline{N}+me_j}(X_1,\dots,X_n) = \varphi_\pi^{\underline{N}+me_j}(\mathbf{X}) = \varphi_\pi\left(\left(\delta_{[N_j+m]}^{\otimes P_j}\prod_{i\neq j}\delta_{[N_i]}^{\otimes P_i}\right)(\mathbf{X})\right). \tag{6.6}$$

Now by (3.3) we have

$$\delta_{[N_j+m]}^{\otimes P_j} = (\delta_{[N_j]}+\delta_{N_j+[m]})^{\otimes P_j} = \sum_{A\subseteq P_j}\delta_{[N_j]}^{\otimes P_j\setminus A}\delta_{N_j+[m]}^{\otimes A} \tag{6.7}$$

and thus (6.6) is equal to

$$= \sum_{A\subseteq P_j}\varphi_\pi\left(\left(\delta_{[N_j]}^{\otimes P_j\setminus A}\delta_{N_j+[m]}^{\otimes A}\prod_{i\neq j}\delta_{[N_i]}^{\otimes P_i}\right)(\mathbf{X})\right). \tag{6.8}$$

This is a sum of $\varphi_\pi$'s with entries $X_k^{(i_k)}, k\in[n]$, where the indices $i_k$ with $k\in A$ are strictly larger than the indices $i_k$ with $k\in P_j\setminus A$, and therefore we may split the block $P_j\in\pi$ into two parts:

$$= \sum_{A\subseteq P_j}\varphi_{(P_1,P_2,\dots,P_{j-1},P_j\setminus A,A,P_{j+1},\dots,P_p)}\left(\left(\delta_{[N_j]}^{\otimes P_j\setminus A}\delta_{N_j+[m]}^{\otimes A}\prod_{i\neq j}\delta_{[N_i]}^{\otimes P_i}\right)(\mathbf{X})\right). \tag{6.9}$$

Now by Lemma 4.3 the local shift in $A$ can be omitted without changing the value and we obtain

$$= \sum_{A\subseteq P_j}\varphi_{(P_1,P_2,\dots,P_{j-1},P_j\setminus A,A,P_{j+1},\dots,P_p)}\left(\left(\delta_{[N_j]}^{\otimes P_j\setminus A}\delta_{[m]}^{\otimes A}\prod_{i\neq j}\delta_{[N_i]}^{\otimes P_i}\right)(\mathbf{X})\right) \tag{6.10}$$

$$= \sum_{A\subseteq P_j}\varphi_{(P_1,P_2,\dots,P_{j-1},P_j\setminus A,A,P_{j+1},\dots,P_p)}^{(N_1,N_2,\dots,N_{j-1},N_j,m,N_{j+1},\dots,N_p)}(\mathbf{X}). \tag{6.11}$$

The analytic extension of this identity is

$$\begin{aligned}\varphi_\pi^{t+se_j}(\mathbf{X}) &= \sum_{A\subseteq P_j} \varphi_{(P_1,P_2,\dots,P_{j-1},P_j\smallsetminus A,A,P_{j+1},\dots,P_p)}^{(t_1,t_2,\dots,t_{j-1},t_j,s,t_{j+1},\dots,t_p)}(\mathbf{X}) \\ &= \varphi_\pi^{t}(\mathbf{X}) + \sum_{\varnothing\neq A\subseteq P_j} \varphi_{(P_1,P_2,\dots,P_{j-1},P_j\smallsetminus A,A,P_{j+1},\dots,P_p)}^{(t_1,t_2,\dots,t_{j-1},t_j,s,t_{j+1},\dots,t_p)}(\mathbf{X}),\end{aligned}$$

and the derivative satisfies the derived identity (6.5):

$$\frac{\partial}{\partial t_j}\varphi_\pi^{t}(\mathbf{X}) = \frac{\partial}{\partial s}\Big|_{s=0} \varphi_\pi^{t+se_j}(\mathbf{X}) = \sum_{\varnothing\neq A\subseteq P_j} K_{(P_1,P_2,\dots,P_{j-1},P_j\smallsetminus A,A,P_{j+1},\dots,P_p)}^{(t_1,t_2,\dots,t_{j-1},t_j,1,t_{j+1},\dots,t_p)}(\mathbf{X}).$$

In order to prove the first differential equation (6.4) we replace (6.7) by the complementary expansion

$$\delta_{[N_j+m]}^{\otimes P_j} = (\delta_{[m]} + \delta_{m+[N_j]})^{\otimes P_j} = \sum_{A\subseteq P_j} \delta_{[m]}^{\otimes A}\delta_{m+[N_j]}^{\otimes P_j\smallsetminus A}, \tag{6.12}$$

and following the lines of (6.8)–(6.10) we obtain

$$\begin{aligned}\varphi_\pi^{\underline{N}+me_j}(X_1,\dots,X_n) &= \sum_{A\subseteq P_j} \varphi_\pi\Bigg(\Bigg(\delta_{[m]}^{\otimes A}\delta_{m+[N_j]}^{\otimes P_j\smallsetminus A}\prod_{i\neq j}\delta_{[N_i]}^{\otimes P_i}\Bigg)(\mathbf{X})\Bigg) \\ &= \sum_{A\subseteq P_j} \varphi_{(P_1,P_2,\dots,P_{j-1},A,P_j\smallsetminus A,P_{j+1},\dots,P_p)}\Bigg(\Bigg(\delta_{[m]}^{\otimes A}\delta_{[N_j]}^{\otimes P_j\smallsetminus A}\prod_{i\neq j}\delta_{[N_i]}^{\otimes P_i}\Bigg)(\mathbf{X})\Bigg) \\ &= \sum_{A\subseteq P_j} \varphi_{(P_1,P_2,\dots,P_{j-1},A,P_j\smallsetminus A,P_{j+1},\dots,P_p)}^{(N_1,N_2,\dots,N_{j-1},m,N_j,N_{j+1},\dots,N_p)}(\mathbf{X}).\end{aligned} \tag{6.13}$$

By analytic continuation, we may replace $m$ and $N_i$ with $s$ and $t_i$ respectively. Taking the derivative with respect to $s$ at 0 we obtain (6.4). □

**Remark 6.4.** In the case of exchangeability both differential equations coincide.

In the remainder of this section we consider specializations to the differential equations of Theorem 6.3 to various spreadability systems. In all these examples $\varphi_\pi^{t}$ factorizes into products along to the blocks of $\pi$ and therefore it suffices to consider $\pi = \hat{1}_n$, i.e., the expectation $\varphi^t(X_1,\dots,X_n) := \varphi((t.X_1)(t.X_2)\cdots(t.X_n))$.

**Example 6.5** (Tensor independence)**.** Consider the tensor spreadability system $\mathcal{S}_{\mathrm{T}}$. Then

$$\varphi_{(A,A^c)}^{(t_1,t_2)}(X_1,\dots,X_n) = \varphi^{t_1}(X_A)\,\varphi^{t_2}(X_{A^c}), \tag{6.14}$$

so we get

$$K_{(A,A^c),A}^{(1,t_2)}(X_1,X_2,\dots,X_n) = \frac{\partial}{\partial t_1}\Big|_{t_1=0} \varphi^{t_1}(X_A)\,\varphi^{t_2}(X_{A^c}) = K_{|A|}^{\mathrm{T}}(X_A)\,\varphi^{t_2}(X_{A^c}). \tag{6.15}$$

The identities in Theorem 6.3 (for $\pi = \hat{1}_n$) read

$$\frac{d}{dt}\varphi^t(X_1,\dots,X_n) = \sum_{\varnothing\neq A\subseteq[n]} K_{|A|}^{\mathrm{T}}(X_A)\,\varphi^t(X_{A^c}). \tag{6.16}$$

This differential equation can be translated to (exponential) generating functions as follows. Let $\underline{u} = (u_1,u_2,\dots,u_n)$ be a vector of *commuting* indeterminates and $\underline{X} = (X_1,X_2,\dots,X_n)$ a vector of random variables. We define the exponential moment generating function

$$\begin{aligned}\mathcal{F}_{\underline{X}}^t(\underline{u}) &:= 1 + \sum_{\substack{(p_1,\dots,p_n)\in(\mathbb{N}\cup\{0\})^n\\(p_1,\dots,p_n)\neq 0}} \frac{u_1^{p_1}\cdots u_n^{p_n}}{p_1!\cdots p_n!}\,\varphi^t(\underbrace{X_1,\dots,X_1}_{p_1\text{ times}},\dots,\underbrace{X_n,\dots X_n}_{p_n\text{ times}}) \\ &= [\varphi(e^{u_1X_1}\cdots e^{u_nX_n})]^t\end{aligned} \tag{6.17}$$

and the exponential cumulant generating function as the logarithm of the previous

$$\begin{aligned}\mathcal{L}_{\underline{X}}(\underline{u}) &:= \sum_{\substack{(p_1,\dots,p_n)\in(\mathbb{N}\cup\{0\})^n\\(p_1,\dots,p_n)\neq 0}} \frac{u_1^{p_1}\cdots u_n^{p_n}}{p_1!\cdots p_n!}\,K_{p_1+\dots+p_n}^{\mathrm{T}}(\underbrace{X_1,\dots,X_1}_{p_1\text{ times}},\dots,\underbrace{X_n,\dots X_n}_{p_n\text{ times}}) \\ &= \log[\mathcal{F}_{\underline{X}}^1(\underline{u})].\end{aligned} \tag{6.18}$$

Then one can prove that

$$\frac{d}{dt}\mathcal{F}^t_{\underline{X}}(\underline{u}) = \mathcal{L}_{\underline{X}}(\underline{u})\,\mathcal{F}^t_{\underline{X}}(\underline{u}), \tag{6.19}$$

which is equivalent to (6.16). Note that the functions $\mathcal{L}_{\underline{X}}(\underline{u})$ and $\mathcal{F}^t_{\underline{X}}(\underline{u})$ commute.

**Example 6.6** (Boolean independence)**.** In the Boolean spreadability system $\mathcal{S}_{\mathrm{B}}$ we have

$$\varphi^{(t_1,t_2)}_{(A,A^c)}(X_1,X_2,\dots,X_n) = \prod_{P\in\iota_{\max}(A)} \varphi^{t_1}(X_P) \prod_{Q\in\iota_{\max}(A^c)} \varphi^{t_2}(X_Q). \tag{6.20}$$

where $\iota_{\max}(A)$ is the interval partition constructed in Definition 2.3 and consists of the contiguous subintervals of $A$. It follows that

$$K^{(1,t_2)}_{(A,A^c),A}(X_1,X_2\dots,X_n) = \begin{cases} K^{\mathrm{B}}_{|A|}(X_A)\prod_{Q\in\iota_{\max}(A^c)}\varphi^{t_2}(X_Q), & \text{if } |\iota_{\max}(A)| = 1,\\ 0, & \text{if } |\iota_{\max}(A)| > 1.\end{cases} \tag{6.21}$$

The identities in Theorem 6.3 (for $\pi = \hat{1}_n$) coincide and read

$$\frac{d}{dt}\varphi^t(X_1,X_2,\dots,X_n) = \sum_{A\text{: interval of }[n]} K^{\mathrm{B}}_{|A|}(X_A) \prod_{Q\in\iota_{\max}(A^c)} \varphi^t(X_Q). \tag{6.22}$$

This differential equation can be interpreted in terms of generating functions in *noncommuting* indeterminates $z_1,\dots,z_n$. To this end we define noncommutative formal power series

$$M^t_{\underline{X}}(\underline{z}) = 1 + \sum_{m=1}^{\infty}\sum_{i_1,\cdots,i_m=1}^{n} \varphi^t(X_{i_1},X_{i_2},\dots,X_{i_m})\, z_{i_1}\cdots z_{i_m} \tag{6.23}$$

and

$$\begin{aligned} K^{\mathrm{B}}_{\underline{X}}(\underline{z}) &:= \frac{\partial}{\partial t}\Big|_0 M^t_{(X_1,X_2,\dots,X_n)}(\underline{z}) \\ &= \sum_{m=1}^{\infty}\sum_{i_1,\cdots,i_m=1}^{n} K^{\mathrm{B}}_m(X_{i_1},\dots,X_{i_m})\, z_{i_1}\cdots z_{i_m}. \end{aligned} \tag{6.24}$$

Then the differential equation (6.22) is equivalent to the identity

$$\frac{d}{dt}M^t_{\underline{X}}(\underline{z}) = M^t_{\underline{X}}(\underline{z})\,K^{\mathrm{B}}_{\underline{X}}(\underline{z})\,M^t_{\underline{X}}(\underline{z}). \tag{6.25}$$

Note that $M^t_{\underline{X}}(\underline{z})$ and $K^{\mathrm{B}}_{\underline{X}}(\underline{z})$ do not commute.

**Example 6.7** (Monotone independence)**.** Consider the monotone spreadability system $\mathcal{S}_{\mathrm{M}}$. With the notation $\iota_{\max}(A)$ introduced in Definition 2.3 we have

$$\varphi^{(t_1,t_2)}_{(A,A^c)}(X_1,X_2,\dots,X_n) = \varphi^{t_1}(X_A) \prod_{B\in\iota_{\max}(A^c)} \varphi^{t_2}(X_B), \tag{6.26}$$

using monotone independence. Thus

$$K^{(1,t_2)}_{(A,A^c),A}(X_1,X_2,\dots,X_n) = \frac{\partial}{\partial t_1}\Big|_{t_1=0} \varphi^{t_1}(X_A) \prod_{B\in\iota_{\max}(A^c)} \varphi^{t_2}(X_B) = K^{\mathrm{M}}_{|A|}(X_A) \prod_{B\in\iota_{\max}(A^c)} \varphi^{t_2}(X_B). \tag{6.27}$$

The first identity in Theorem 6.3 (for $\pi = \hat{1}_n$) reads

$$\frac{d}{dt}\varphi^t(X_1,X_2,\dots,X_n) = \sum_{\emptyset\neq A\subseteq[n]} K^{\mathrm{M}}_{|A|}(X_A) \prod_{B\in\iota_{\max}(A^c)} \varphi^{t_2}(X_B), \tag{6.28}$$

which is exactly the first identity in [HS11a, Corollary 5.2].

On the other hand

$$\begin{aligned} K^{(t_1,1)}_{(A^c,A),A}(X_1,X_2,\dots,X_n) &= \frac{\partial}{\partial t_2}\Big|_{t_2=0} \varphi^{t_1}(X_A) \prod_{B\in\iota_{\max}(A^c)} \varphi^{t_2}(X_B) \\ &= \begin{cases} \varphi^{t_1}(X_A)\,K^{\mathrm{M}}_{|A^c|}(X_{A^c}), & \text{if } |\iota_{\max}(A^c)| = 1,\\ 0, & \text{if } |\iota_{\max}(A^c)| > 1.\end{cases} \end{aligned} \tag{6.29}$$

The condition $|\iota_{\max}(A^c)| = 1$ holds if and only if $A^c$ is an interval. Therefore, the second equality in Theorem 6.3 (for $\pi = \hat{1}_n$) reads

$$\frac{d}{dt}\varphi^t(X_1, X_2, \dots, X_n) = \sum_{B\colon \text{interval of } [n]} K^{\mathrm{M}}_{|B|}(X_B)\, \varphi^t(X_{B^c}), \tag{6.30}$$

which is exactly the second equality in [HS11a, Corollary 5.2].

Results on generating functions in [HS11a] correspond to these differential equations. For noncommutative indeterminates $z_1, \dots, z_n$, we define the cumulant generating function

$$\begin{aligned} K^{\mathrm{M}}_{\underline{X}}(\underline{z}) &:= \frac{\partial}{\partial t}\Big|_0 M^t_{\underline{X}}(\underline{z}) \\ &= \sum_{m=1}^{\infty} \sum_{i_1,i_2,\cdots,i_m=1}^{n} K^{\mathrm{M}}_m(X_{i_1}X_{i_2}\cdots X_{i_m})\, z_{i_1}z_{i_2}\cdots z_{i_m}. \end{aligned} \tag{6.31}$$

Then it is shown in [HS11a, Theorem 6.3] that

$$M^{s+t}_{\underline{X}}(\underline{z}) = M^t_{\underline{X}}(\underline{z})\, M^s_{\underline{X}}(z_1 M^t_{\underline{X}}(\underline{z}), \dots, z_n M^t_{\underline{X}}(\underline{z})). \tag{6.32}$$

The partial derivatives of (6.32) regarding $s$ at 0 and $t$ at 0 become (6.28) and (6.30), respectively.

**Example 6.8** (Free independence)**.** Consider the free spreadability system $\mathcal{S}_{\mathrm{F}}$. One can use the formula for products of free random variables [NS06, Theorem 14.4] to show that for a nonempty subset $A \subseteq [n]$ we can expand

$$\varphi_{(A,A^c)}(X_1, X_2, \dots, X_n) = \varphi(X_A) \prod_{P \in \tilde{\iota}_{\max}(A^c)} \varphi(X_P) + R, \tag{6.33}$$

where $\tilde{\iota}_{\max}(A^c)$ is the partition defined in Definition 2.3 (ii) and every term in $R$ has at least two factors from $A$, i.e., factors of the form $\varphi(X_{k_1}X_{k_2}\cdots X_{k_m})$, $k_1, k_2, \dots, k_m \in A$. For example $\varphi_{\{\{2,4\},\{1,3,5\}\}}(X_1, X_2, \dots, X_5) = \varphi(X_2X_4)\,\varphi(X_1X_5)\,\varphi(X_3) + R$, where every term in $R$ has the factor $\varphi(X_2)\,\varphi(X_4)$. This implies that

$$\varphi^{(t_1,t_2)}_{(A,A^c)}(X_1, X_2, \dots, X_n) = \varphi^{t_1}(X_A) \prod_{P \in \tilde{\iota}_{\max}(A^c)} \varphi^{t_2}(X_P) + O(t_1^2) \qquad (t_1 \to 0), \tag{6.34}$$

so by taking the partial derivative $\frac{\partial}{\partial t_1}\big|_{t_1=0}$ we get

$$K^{(1,t_2)}_{(A,A^c),A}(X_1, X_2, \dots, X_n) = K^{\mathrm{F}}_{|A|}(X_A) \prod_{P \in \tilde{\iota}_{\max}(A^c)} \varphi^{t_2}(X_P). \tag{6.35}$$

This yields the differential equation

$$\frac{d}{dt}\varphi^t(X_1, X_2, \dots, X_n) = \sum_{\emptyset \neq A \subseteq [n]} K^{\mathrm{F}}_{|A|}(X_A) \prod_{P \in \tilde{\iota}_{\max}(A^c)} \varphi^t(X_P), \tag{6.36}$$

which is similar to the monotone case (6.28).

Let $\underline{z} = (z_1, z_2, \dots, z_n)$ be a vector of noncommuting indeterminates and let $\underline{X} = (X_1, X_2, \dots, X_n)$ be a random vector. In order to obtain a differential equation we need in addition the two-sided generating function

$$\tilde{M}^t_{\underline{X}}(\underline{z}, w) := \sum_{p,q=0}^{\infty} \sum_{\substack{i_1,i_2,\dots,i_p \in [n] \\ j_1,j_2,\dots,j_q \in [n]}} \varphi^t(X_{i_1}, X_{i_2}, \dots, X_{i_p}, X_{j_1}, X_{j_2}, \dots, X_{j_q})\, z_{i_1}z_{i_2}\cdots z_{i_p} w z_{j_1}z_{j_2}\cdots z_{j_q} \tag{6.37}$$

as well as the $R$-transform

$$R_{\underline{X}}(\underline{z}) = \sum_{j=1}^{\infty} \sum_{i_1,\cdots,i_j=1}^{n} K^{\mathrm{F}}_j(X_{i_1}, X_{i_2}, \cdots, X_{i_j})\, z_{i_1}z_{i_2}\cdots z_{i_j}. \tag{6.38}$$

After some computations we infer from (6.36) that

$$\frac{\partial}{\partial t} M^t_{\underline{X}}(\underline{z}) = \tilde{M}^t_{\underline{X}}(\underline{z}, R(z_1 M^t_{\underline{X}}(\underline{z}), z_2 M^t_{\underline{X}}(\underline{z}), \dots, z_n M^t_{\underline{X}}(\underline{z}))(M^t_{\underline{X}}(\underline{z}))^{-1}). \tag{6.39}$$

Note that by [NS06, Corollary 16.16] the second argument can be written as

$$R(z_1 M^t_{\underline{X}}(\underline{z}), z_2 M^t_{\underline{X}}(\underline{z}), \dots, z_n M^t_{\underline{X}}(\underline{z}))\,(M^t_{\underline{X}}(\underline{z}))^{-1} = \frac{1 - M^t_{\underline{X}}(\underline{z})^{-1}}{t}. \tag{6.40}$$

When $n = 1$ the differential equation (6.39) is equivalent to the generalized complex Burgers equation (see [Voi86, p. 343])

$$\frac{\partial}{\partial t} G^t_X(z) + \frac{R_X(G^t_X(z))}{G^t_X(z)} \frac{\partial}{\partial z} G^t_X(z) = 0, \tag{6.41}$$

where $G^t_X$ is the Cauchy transform

$$G^t_X(z) = \sum_{n=0}^{\infty} \varphi^t(\underbrace{X, \dots, X}_{n \text{ fold}})\, z^{-n-1}. \tag{6.42}$$

## 7. Mixed cumulants and sums of independent random variables

7.1. **Vanishing of mixed cumulants in exchangeability systems.** In the case of exchangeability systems independence is characterized by the vanishing of mixed cumulants [Leh04, Prop. 2.10]. That is, if the arguments of $K_n(X_1, X_2, \dots, X_n)$ can be split into two mutually independent families then the cumulant vanishes; more generally, $K_\pi(X_1, X_2, \dots, X_n) = 0$ whenever the entries of one of the blocks of $\pi$ splits into two mutually independent subsets. This is the content of the following proposition.

**Proposition 7.1.** *Let $(\mathcal{A}, \varphi)$ be a ncps and $\mathcal{E} = (\mathcal{U}, \tilde{\varphi})$ an exchangeability system for $\mathcal{A}$. Given a partition $\pi \in \mathcal{P}_n$ and a family $X_1, X_2, \dots, X_n \in \mathcal{A}$ such that there is a block $P \in \pi$ which can be partitioned into $P = P_1 \dot\cup P_2$ such that $\{X_i : i \in P_1\}$ and $\{X_i : i \in P_2\}$ are independent in the sense of Definition 3.6, we have $K_\pi(X_1, X_2 \dots, X_n) = 0$.*

For further reference let us reproduce here a short proof, due to P. Zwiernik [Zwi12], which is based on *Weisner's Lemma* (see [Sta97, Cor. 3.9.3] for a simple version and [BBR86] for the full version). Its generalization will be essential for the understanding of mixed cumulants in the spreadable setting.

**Lemma 7.2** (Weisner's Lemma). *In any lattice $(P, \leq)$ the Möbius function satisfies the identity*

$$\sum_{\substack{x \\ x \wedge a = c}} \mu(x, b) = \begin{cases} \mu(c, b) & \text{if } a \geq b \\ 0 & \text{if } a \ngeq b. \end{cases}$$

*Proof of Proposition 7.1.* Let $\rho$ be the partition obtained from $\pi$ by splitting the block $P$ as indicated in the proposition, then $\rho < \pi$ and by assumption $\varphi_\sigma(X_1, X_2, \dots, X_n) = \varphi_{\sigma\wedge\rho}(X_1, X_2, \dots, X_n)$ for any $\sigma \in \mathcal{P}_n$; hence

$$\begin{aligned} K_\pi(X_1, X_2, \dots, X_n) &= \sum_{\substack{\sigma, \pi \in \mathcal{P}_n \\ \sigma \leq \pi}} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \mu_{\mathcal{P}}(\sigma, \pi) \\ &= \sum_{\substack{\sigma, \pi \in \mathcal{P}_n \\ \sigma \leq \pi}} \varphi_{\sigma\wedge\rho}(X_1, X_2, \dots, X_n)\, \mu_{\mathcal{P}}(\sigma, \pi) \\ &= \sum_{\substack{\tau, \pi \in \mathcal{P}_n \\ \tau \leq \pi}} \varphi_\tau(X_1, X_2, \dots, X_n) \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma\wedge\rho = \tau}} \mu_{\mathcal{P}}(\sigma, \pi). \end{aligned} \tag{7.1}$$

Now $\rho \ngeq \pi$ and the second case of Weisner's lemma applies. □

7.2. **Partial vanishing of mixed cumulants in spreadability systems.** Since vanishing of mixed cumulants implies additivity of cumulants (1.1) for sums of independent random variables, it cannot hold for general spreadability systems, e.g., monotone convolution is noncommutative and therefore monotone cumulants are not additive [HS11a].

In this section we investigate what remains true in the general setting and we provide a formula expressing mixed cumulants in terms of lower order cumulants. This question is intimately related to the question of convolution - determining the distribution of the sum of independent random variables, which is not commutative in general. As we shall see, the *Goldberg coefficients* will appear in our formulas.

To obtain this result in full generality, it turns out that certain statistics of multiset permutations play a central role. The first systematic study of permutation statistics is contained in the seminal work of MacMahon [Mac15], for a modern treatment see [Bón12]. These statistics play a major role in the theory of free Lie algebras [Reu93], see Section 8 below.

**Definition 7.3.** A *multiset* is a pair $(A, f)$ where $A$ is the *underlying set* and $f : A \to \mathbb{N}$ is a function. The value $f(a)$ is called the *multiplicity* of the element $a \in A$. Informally, a multiset is a set which contains multiple indistinguishable copies of each of its elements. In the present paper multisets will always be based on integer segments $A = [n]$ and in this case the tuple $(f(1), f(2), \dots, f(n))$ is called the *type* of the multiset. A *permutation of a multiset* is a rearrangement of all its elements where all multiplicities are preserved, i.e., a word with a prescribed total number of occurences of each letter. The number of distinct permutations of a multiset of type $(f_1, f_2, \dots, f_n)$ is given by the multinomial coefficient

$$\binom{f_1 + f_2 + \dots + f_n}{f_1, f_2, \dots, f_n}.$$

A *proper set* is a multiset of type $(1, 1, \dots, 1)$ and we recover the number of its permutations as $n!$.

We will be interested in the following statistics of multiset permutations.

**Definition 7.4.** Let $\sigma = w_1 w_2 \dots w_s$ be a permutation of a multiset of type $(f_1, f_2, \dots, f_n)$ where $s = f_1 + f_2 + \dots + f_n$. An index $1 \le i \le s-1$ is called a

(i) *descent* (or *drop* or *fall*) if $w_i > w_{i+1}$.
(ii) *plateau* (or *level*) if $w_i = w_{i+1}$.
(iii) *ascent* (or *rise*) if $w_i < w_{i+1}$.

We denote by

$$\begin{aligned}\mathrm{Des}(\sigma) &= \mathrm{Des}((w_i)_{i=1}^s) = \{j \in [s-1] \mid w_j > w_{j+1}\},\\ \mathrm{Pla}(\sigma) &= \mathrm{Pla}((w_i)_{i=1}^s) = \{j \in [s-1] \mid w_j = w_{j+1}\},\\ \mathrm{Asc}(\sigma) &= \mathrm{Asc}((w_i)_{i=1}^s) = \{j \in [s-1] \mid w_j < w_{j+1}\}\end{aligned}$$

these sets and by $\mathrm{des}(\sigma) = \mathrm{des}((w_i)_{i=1}^s) := |\mathrm{Des}((w_i)_{i=1}^s)|$, $\mathrm{pla}(\sigma) = \mathrm{pla}((w_i)_{i=1}^s) := |\mathrm{Pla}((w_i)_{i=1}^s)|$ and $\mathrm{asc}(\sigma) = \mathrm{asc}((w_i)_{i=1}^s) := |\mathrm{Asc}((w_i)_{i=1}^s)|$ the respective cardinalities, i.e., the number of descents (ascents, plateaux respectively) of the multiset permutation $\sigma$.

**Remark 7.5.** 1. Note that some authors also count $i = s$ as a descent and $i = 0$ as an ascent.
2. Counting permutations by descents and ascents is a classic subject in combinatorics. In the case of descents of multiset permutations this is also known as *Simon Newcomb's problem* [Rio58, DR69].

Clearly every element except the last is either a descent, an ascent, or a plateau, and therefore

$$|\sigma| = \mathrm{des}(\sigma) + \mathrm{pla}(\sigma) + \mathrm{asc}(\sigma) + 1. \tag{7.2}$$

**Definition 7.6.** To any pair of ordered set partitions $\tau, \eta \in \mathcal{OP}_n$ such that $\bar{\tau} \le \bar{\eta}$, we associate a multiset on $[|\eta|]$ by setting the multiplicity of $k \in \{1, 2, \dots, |\eta|\}$ to be equal to the number of blocks of $\tau$ contained in the $k$-th block of $\eta$. Replacing every block of $\tau$ by the label of the block of $\eta$ containing it we obtain a permutation of this multiset. We denote by $\mathrm{des}_\eta(\tau)$, $\mathrm{asc}_\eta(\tau)$ and $\mathrm{pla}_\eta(\tau)$ its statistics as defined in Definition 7.4. More precisely, let $\tau, \eta \in \mathcal{OP}_n$ such that $\bar{\tau} \le \bar{\eta}$. If $\eta = (E_1, E_2, \dots, E_e)$, then the blocks of $\bar{\tau}$ can be arranged

$$\bar{\tau} = \{E_{1,1}, E_{1,2}, \dots, E_{1,l_1}, \dots, E_{e,1}, \dots, E_{e,l_e}\},$$

where each $E_i = \bigcup_{j=1}^{l_i} E_{i,j}$ is a disjoint union taken in canonical order of the subsets (i.e., sorted according to their minimal elements). So $\tau$ can be written as

$$\tau = (E_{m_1,n_1}, E_{m_2,n_2}, \dots, E_{m_s,n_s}),$$

which is a permutation of the blocks of $\bar{\tau}$. Then we denote by $\mathrm{Des}_\eta(\tau), \mathrm{Pla}_\eta(\tau)$ and $\mathrm{Asc}_\eta(\tau)$ respectively, the sets $\mathrm{Des}((m_i)_{i=1}^s), \mathrm{Pla}((m_i)_{i=1}^s), \mathrm{Asc}((m_i)_{i=1}^s)$ and by $\mathrm{des}_\eta(\tau), \mathrm{pla}_\eta(\tau), \mathrm{asc}_\eta(\tau)$ the respective cardinalities of the latter.

**Example 7.7.** Consider the partitions $\eta = (E_1, E_2, E_3, E_4, E_5) \in \mathcal{OP}_{10}$ with blocks $E_1 = \{5,8\}$, $E_2 = \{9,10\}$, $E_3 = \{3,6\}$, $E_4 = \{1,2,4\}$, $E_5 = \{7\}$, and $\tau = (T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8) \in \mathcal{OP}_{10}$ with blocks $T_1 = \{3\}$, $T_2 = \{6\}$, $T_3 = \{1,4\}$, $T_4 = \{7\}$, $T_5 = \{10\}$, $T_6 = \{5,8\}$, $T_7 = \{9\}$, $T_8 = \{2\}$. Then $\bar{\tau} \le \bar{\eta}$. Now $T_1 \subseteq E_3$ and thus $w_1 = 3$, $T_2 \subseteq E_3$ and thus $w_2 = 3$, etc.; the multiset permutation thus induced on $\tau$ by $\eta$ is

$$(w_i)_{i=1}^8 = (3,3,4,5,2,1,2,4),$$

and the statistics are $\operatorname{des}((w_i)_{i=1}^8) = 2$, $\operatorname{pla}((w_i)_{i=1}^8) = 1$ and $\operatorname{asc}((w_i)_{i=1}^8) = 4$.

**Definition 7.8.** Let $\sigma = (w_i)_{i=1}^s$ be a multiset permutation. An *ascending (*resp. *descending) run* is a maximal contiguous subsequence $(w_i)_{i=l}^k$ which is strictly increasing (resp. decreasing). Similarly, *a level run* is a maximal subsequence $(w_i)_{i=j}^k$ such that $w_j = w_{j+1} = \cdots = w_k$.

**Proposition 7.9.** *Let $\sigma = (w_i)_{i=1}^s$ be a multiset permutation.*

*(i) $\sigma$ can be decomposed uniquely into ascending runs separated by descents and plateaux. The number of ascending runs is equal to $s - \operatorname{asc}(\sigma)$.*
*(ii) $\sigma$ can be decomposed uniquely into descending runs separated by ascents and plateaux. The number of descending runs is equal to $s - \operatorname{des}(\sigma)$.*
*(iii) $\sigma$ can be decomposed uniquely into level runs separated by ascents and descents. The number of level runs is equal to $s - \operatorname{pla}(\sigma)$.*

*Proof.* We only prove (i). The claim is clearly true when $\operatorname{asc}\sigma = s-1$, i.e., when the sequence is monotone increasing. Otherwise replacing any ascent by a descent or plateau splits an ascending run into two, i.e., increases the number of ascending runs by one. □

**Remark 7.10.** Assume $\bar{\tau} = \bar{\eta}$, i.e., if $\eta = (E_1, E_2, \dots, E_e)$ and $\tau$ is a permutation of the blocks of $\eta$ in the sense that there is a permutation $h \in \mathfrak{S}_e$ such that $\tau = h(\eta) = (E_{h(1)}, E_{h(2)}, \dots, E_{h(e)})$. Then $\operatorname{des}_\eta(\tau)$ and $\operatorname{asc}_\eta(\tau)$ coincide with $\operatorname{des}(h)$ and $\operatorname{asc}(h)$, where the latter quantities are the numbers of descents and ascents of the permutation $h$, respectively. See [Bón12, Bre93, Reu93] and [Sta97, p. 25] for the uses of $\operatorname{des}(h)$ and $\operatorname{asc}(h)$ in the context of symmetric groups.

**Example 7.11.** Consider $(m_i)_{i=1}^9 = (1,1,3,5,5,5,4,1,4)$. Then $\operatorname{des}((m_i)_{i=1}^9) = 2$, $\operatorname{pla}((m_i)_{i=1}^9) = 3$ and $\operatorname{asc}((m_i)_{i=1}^9) = 3$. The decomposition into ascending runs is given by

$$(1), (1,3,5), (5), (5), (4), (1,4),$$

the decomposition into level runs is

$$(1,1), (3), (5,5,5), (4), (1), (4)$$

and the decomposition into descending runs is

$$(1), (1), (3), (5), (5), (5,4,1), (4).$$

**Lemma 7.12.** *Let $\eta, \tau \in \mathcal{OP}_n$ such that $\bar{\tau} \le \bar{\eta}$, then there is an ordered set partition $\sigma_{\max}^{\text{asc}}(\tau, \eta)$ such that*

$$\{\sigma \in \mathcal{OP}_n \mid \sigma \wedge \eta = \tau\} = [\tau, \sigma_{\max}^{\text{asc}}(\tau, \eta)].$$

*and the restriction of the mapping $\Psi$ in Proposition 2.14 establishes a poset isomorphism*

$$\{\sigma \in \mathcal{OP}_n \mid \sigma \wedge \eta = \tau\} \to \mathcal{I}_{p_1} \times \mathcal{I}_{p_2} \times \cdots \times \mathcal{I}_{p_t},$$

*where $1 \le p_i \le n$ are the lengths of the ascending runs of the sequence $(m_i)_{i=1}^s$ from Definition 7.6 and $t = |\tau| - \operatorname{asc}_\eta(\tau)$.*

*Proof.* Let $\sigma \in \mathcal{OP}_n$ be a partition such that $\sigma \wedge \eta = \tau$ and write $\tau = (E_{m_1,n_1}, E_{m_2,n_2}, \dots, E_{m_s,n_s})$ and $\eta = (E_1, E_2, \dots, E_e)$ as in Definition 7.6. From Proposition 2.12 we infer that $\sigma \ge \tau$ and thus by Proposition 2.14 there is an interval partition $\lambda = (L_1, L_2, \dots, L_l) \in \mathcal{I}_s$ such that

$$\sigma = \Big(\bigcup_{i\in L_1} E_{m_i,n_i}, \bigcup_{i\in L_2} E_{m_i,n_i}, \dots, \bigcup_{i\in L_l} E_{m_i,n_i}\Big).$$

In order that $\sigma \wedge \eta = \tau$ it is necessary and sufficient that every block $L \in \lambda$, say $L = \{a+1, a+2, \dots, a+b\}$, induces a strictly increasing sequence $m_{a+1} < m_{a+2} < \cdots < m_{a+b}$.

Let $(m_i)_{i=1}^{i_1}, (m_i)_{i=i_1+1}^{i_2}, \dots, (m_i)_{i=i_{t-1}+1}^{s}$ be the decomposition of $(m_i)_{i=1}^{s}$ into ascending runs. This decomposition defines the interval blocks $A_j := \{i_{j-1}+1, i_{j-1}+2, \dots, i_j\}$ ($i_0 = 0, i_t = s$) and hence defines an interval partition $\alpha = (A_1, A_2, \dots, A_t)$. The interval partition $\lambda$ consists of increasing intervals and therefore is finer than $\alpha$. Thus we have an isomorphism

$$\{\sigma \in \mathcal{OP}_n \mid \sigma \wedge \eta = \tau\} \to \mathcal{I}_{A_1} \times \mathcal{I}_{A_2} \times \cdots \times \mathcal{I}_{A_t}$$

via the restriction of the map $\Psi\rfloor_{[\tau, \hat{1}_n]}$ from Proposition 2.14. The number $t = |\alpha|$ is equal to $|\tau| - \operatorname{asc}_\eta(\tau)$, the integers $p_j$ are the cardinalities of $A_j$ and $\sigma_{\max}^{\text{asc}}(\tau, \eta)$ is the ordered set partition corresponding to $\lambda = \alpha$. □

Before discussing the question of mixed cumulants we prove an analogue of Weisner's lemma for ordered set partitions.

**Proposition 7.13** (Weisner coefficients). *(i) For $\tau, \eta \in \mathcal{OP}_n$, we have*

$$w(\tau, \eta) := \sum_{\substack{\sigma \in \mathcal{OP}_n \\ \sigma \wedge \eta = \tau}} \widetilde{\mu}(\sigma, \hat{1}_n)$$

$$= \begin{cases} \displaystyle\int_{-1}^{0} x^{|\tau| - \operatorname{asc}_\eta(\tau) - 1} (1+x)^{\operatorname{asc}_\eta(\tau)}\, dx = \frac{(-1)^{|\tau| - \operatorname{asc}_\eta(\tau) - 1}}{|\tau| \binom{|\tau|-1}{\operatorname{asc}_\eta(\tau)}}, & \bar{\tau} \le \bar{\eta}, \\ 0, & \bar{\tau} \nleq \bar{\eta}. \end{cases} \tag{7.3}$$

*(ii) More generally, for $\tau, \eta, \pi \in \mathcal{OP}_n$, we have*

$$w(\tau, \eta, \pi) := \sum_{\substack{\sigma \in \mathcal{OP}_n \\ \sigma \wedge \eta = \tau \\ \sigma \le \pi}} \widetilde{\mu}(\sigma, \pi) = \begin{cases} \displaystyle\prod_{P \in \pi} w(\tau\rfloor_P, \eta\rfloor_P), & \bar{\tau} \le \bar{\eta},\ \tau \le \pi, \\ 0, & \textit{otherwise.} \end{cases}$$

*We call the numbers $w(\tau, \eta, \pi)$ and $w(\tau, \eta) = w(\tau, \eta, \hat{1}_n)$* Weisner coefficients.

*Proof.* (i) If $\bar{\tau} \nleq \bar{\eta}$, then there is no $\sigma$ such that $\sigma \wedge \eta = \tau$ and the sum is empty. Let us therefore assume henceforth that $\bar{\tau} \le \bar{\eta}$. We take up the end of the proof of Lemma 7.12 where we established the poset isomorphism

$$\begin{aligned} \{\sigma \in \mathcal{OP}_n \mid \sigma \wedge \eta = \tau\} &\cong \mathcal{I}_{p_1} \times \mathcal{I}_{p_2} \times \cdots \times \mathcal{I}_{p_t} \\ &\cong \mathcal{B}_{p_1 - 1} \times \mathcal{B}_{p_2 - 1} \times \cdots \times \mathcal{B}_{p_t - 1} \\ &\cong \mathcal{B}_{p_1 + \cdots + p_t - t}, \end{aligned} \tag{7.4}$$

where the second isomorphism follows from Proposition 2.2 and $p_i$ denotes the length of the $i$-th ascending run. The latter contains $p_i - 1$ rises and therefore the total number of ascents is $\operatorname{asc}_\eta(\tau) = p_1 + p_2 + \cdots + p_t - t$. In the identification above, a partition $\sigma$ is mapped to a subset $A \subseteq [p_1 + p_2 + \cdots + p_t - t]$ with $|A| = |\sigma| - t$ elements and we have

$$\widetilde{\mu}(\sigma, \hat{1}_n) = \frac{(-1)^{|\sigma|-1}}{|\sigma|} = \frac{(-1)^{|A|+t-1}}{|A|+t}. \tag{7.5}$$

Performing the sum we obtain

$$\begin{aligned} \sum_{\substack{\sigma \in \mathcal{OP}_n \\ \sigma \wedge \eta = \tau}} \widetilde{\mu}(\sigma, \hat{1}_n) &= \sum_{A \subseteq [\operatorname{asc}_\eta(\tau)]} \frac{(-1)^{|A|+t-1}}{|A|+t} \\ &= \sum_{k=0}^{\operatorname{asc}_\eta(\tau)} \binom{\operatorname{asc}_\eta(\tau)}{k} \frac{(-1)^{k+t-1}}{k+t} \\ &= \int_{-1}^{0} \sum_{k=0}^{\operatorname{asc}_\eta(\tau)} \binom{\operatorname{asc}_\eta(\tau)}{k} x^{k+t-1}\, dx \\ &= \int_{-1}^{0} x^{t-1} (1+x)^{\operatorname{asc}_\eta(\tau)}\, dx \\ &= (-1)^{t-1} B(t, \operatorname{asc}_\eta(\tau) + 1), \end{aligned} \tag{7.6}$$

where $B$ is the beta function $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ which can be written in terms of binomial coefficients as desired.

(ii) In order for the set $\{\sigma \in \mathcal{OP}_n \mid \sigma \le \pi,\ \sigma \wedge \eta = \tau\}$ to be nonempty, it is necessary that $\tau \le \pi$ and $\bar\tau \le \bar\eta$. We adapt the notations from Definition 7.6 and infer from Proposition 2.14 that there exists an interval partition $\rho = (R_1, R_2, \dots, R_p) \in \mathcal{I}_s$ such that

$$\pi = (P_1, P_2, \dots, P_p) = (\bigcup_{i\in R_1} E_{m_i,n_i}, \bigcup_{i\in R_2} E_{m_i,n_i}, \dots, \bigcup_{i\in R_p} E_{m_i,n_i}). \tag{7.7}$$

If follows from Lemma 7.12 that $\sigma$ belongs to $[\tau, \sigma^{\mathrm{asc}}_{\max}(\tau,\eta)]$. In addition, $\sigma$ must satisfy $\sigma \le \pi$. Hence, the ascending runs considered in Lemma 7.12 are split by the blocks of $\pi$. More precisely, for each $k \in [p]$ we decompose $(m_i)_{i\in R_k}$ into ascending runs, which give rise to the interval partition $\gamma_k = (G_{k,1}, \dots, G_{k,u_k}) \in \mathcal{I}_{R_k}$ where $G_{k,j}$ consists of the indices $i$ of the $j$-th ascending run of $(m_i)_{i\in R_k}$. Then

$$\sigma^{\mathrm{asc}}_{\max}(\tau\rfloor_{P_k}, \eta\rfloor_{P_k}) := (\bigcup_{i\in G_{k,1}} E_{m_i,n_i}, \dots, \bigcup_{i\in G_{k,u_k}} E_{m_i,n_i}). \tag{7.8}$$

and for each $P \in \pi$, we pick an arbitrary $\sigma_P \in \mathcal{OP}_P$ from the interval $[\tau\rfloor_P, \sigma^{\mathrm{asc}}_{\max}(\tau\rfloor_P, \eta\rfloor_P)]$, concatenate them and obtain $\sigma = \sigma_{P_1}\sigma_{P_2}\cdots\sigma_{P_p} \in \mathcal{OP}_n$. Since $\widetilde{\mu}(\sigma,\pi)$ is the product of $\frac{(-1)^{\#(\sigma\rfloor_P)-1}}{\#(\sigma\rfloor_P)}$ over $P \in \pi$, the conclusion follows. □

Examples of the Weisner coefficients will be given in Example 7.25. we are now ready to proceed with the investigation of cumulants with independent entries.

**Proposition 7.14.** *Let $\pi, \eta \in \mathcal{OP}_n$ and $(i_1, i_2, \dots, i_n) \in \mathbb{N}^n$ be a tuple with kernel $\kappa(i_1, i_2, \dots, i_n) = \eta$. Then*

$$K_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}) = \sum_{\tau\in\mathcal{OP}_n} \varphi_\tau(X_1, X_2, \dots, X_n)\, w(\tau,\eta,\pi).$$

*Proof.* We proceed as in the proof of Proposition 7.1. Expressing cumulants in terms of moments (see Theorem 4.7), we have

$$\begin{aligned} K_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}) &= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_\sigma(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)})\, \widetilde{\mu}(\sigma,\pi) \\ &= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_{\sigma\wedge\eta}(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma,\pi) \\ &= \sum_{\tau\in\mathcal{OP}_n} \varphi_\tau(X_1, X_2, \dots, X_n) \left( \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\wedge\eta=\tau, \sigma\le\pi}} \widetilde{\mu}(\sigma,\pi) \right) \\ &= \sum_{\tau\in\mathcal{OP}_n} \varphi_\tau(X_1, X_2, \dots, X_n)\, w(\tau,\eta,\pi). \end{aligned} \tag{7.9}$$

□

Since $w(\tau,\eta,\pi) \ne 0$ for $\bar\tau \le \bar\eta, \tau \le \pi$ and in general the expectation values $\varphi_\tau(X_1, X_2, \dots, X_n)$ among different $\tau$ do not cancel each other, we cannot expect the vanishing of cumulants without further assumptions. However we can express cumulants with independent entries also in terms of cumulants of lower orders. In this case it turns out that the coefficients are determined by the number of plateaux.

**Lemma 7.15.** *Let $\eta, \tau \in \mathcal{OP}_n$ such that $\bar\tau \le \bar\eta$, then the restriction of the map $\Psi\rfloor_{[\tau,\hat 1_n]}$ from Proposition 2.14 establishes a poset isomorphism*

$$\{\sigma \in \mathcal{OP}_n \mid \sigma \ge \tau, \bar\sigma \le \bar\eta\} \to \mathcal{I}_{q_1} \times \mathcal{I}_{q_2} \times \dots \times \mathcal{I}_{q_r}, \tag{7.10}$$

*where $1 \le q_i \le n$ are the lengths of the level runs and the number $r$ is equal to $|\tau| - \mathrm{pla}_\eta(\tau)$. In particular, there is an ordered set partition $\sigma^{\mathrm{pla}}_{\max}(\tau,\eta)$ such that*

$$\{\sigma \in \mathcal{OP}_n \mid \sigma \ge \tau, \bar\sigma \le \bar\eta\} = [\tau, \sigma^{\mathrm{pla}}_{\max}(\tau,\eta)].$$

*Proof.* The proof is similar to that of Lemma 7.12. Write $\tau = (E_{m_1,n_1}, E_{m_2,n_2}, \dots, E_{m_s,n_s})$ and $\eta = (E_1, E_2, \dots, E_e)$ as in Definition 7.6. Let $\sigma \geq \tau$, then from Proposition 2.14 we infer that there is an interval partition $\lambda = (L_1, L_2, \dots, L_l) \in \mathcal{I}_s$ such that

$$\sigma = \Big(\bigcup_{i\in L_1} E_{m_i,n_i}, \dots, \bigcup_{i\in L_l} E_{m_i,n_i}\Big). \tag{7.11}$$

Let $(m_i)_{i=1}^{i_1}, (m_i)_{i=i_1+1}^{i_2}, \dots, (m_i)_{i=i_{r-1}+1}^{s}$ be the decomposition of $(m_i)_{i=1}^{s}$ into level runs. This decomposition determines interval blocks $B_j := (i_{j-1}+1, i_{j-1}+2, \dots, i_j)$ ($i_0 = 0, i_r = s$) and hence gives rise to an interval partition $\beta = (B_1, B_2, \dots, B_r)$. In order that $\bar\sigma \leq \bar\eta$, each $L_i$ connects only plateaux, which is equivalent to the condition that $\lambda \leq \beta$. Denoting by $q_j = |B_j| = i_j - i_{j-1}$, we get the isomorphism (7.10). The ordered set partition $\sigma^{\mathrm{pla}}_{\max}(\tau,\eta)$ corresponds to the choice $\lambda = \beta$. □

**Proposition 7.16** (Goldberg coefficients). *(i) For $\tau, \eta \in \mathcal{OP}_n$, we have*

$$g(\tau,\eta) := \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\geq\tau}} \widetilde{\zeta}(\tau,\sigma)\, w(\sigma,\eta) = \begin{cases} \dfrac{1}{q_1!q_2!\cdots q_r!} \displaystyle\int_{-1}^{0} x^{\mathrm{des}_\eta(\tau)} (1+x)^{\mathrm{asc}_\eta(\tau)} \prod_{j=1}^{r} P_{q_j}(x)\, dx, & \bar\tau \leq \bar\eta,\\ 0, & \bar\tau \nleq \bar\eta, \end{cases}$$

*where $q_j$ are the integers in Lemma 7.15 and*

$$P_q(x) = \sum_{k=1}^{q} k!\, S(q,k)\, x^{k-1} = E_q(x, x+1)$$

*where $S(q,k)$ are the Stirling numbers of the second kind and $E_n(x,y) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}\,\sigma} y^{\mathrm{asc}\,\sigma}$ are the* homogeneous Eulerian polynomials [Reu93, p. 62]*. The first few polynomials are*

$$P_1(x) = 1, \quad P_2(x) = 2x+1, \quad P_3(x) = 6x^2+6x+1, \quad \dots$$

*(ii) More generally, for $\tau, \eta, \pi \in \mathcal{OP}_n$, we have*

$$g(\tau,\eta,\pi) := \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\in[\tau,\pi]}} \widetilde{\zeta}(\tau,\sigma)\, w(\sigma,\eta,\pi) = \begin{cases} \prod_{P\in\pi} g(\tau\rfloor_P, \eta\rfloor_P), & \bar\tau \leq \bar\eta,\ \tau \leq \pi,\\ 0, & \textit{otherwise.} \end{cases}$$

**Remark 7.17.** The name "Goldberg coefficients" originates from the Campbell-Baker-Hausdorff formula, see Section 8 below, in particular Theorem 8.17. Some examples of $g(\tau,\eta)$ will be computed in Example 7.24.

**Remark 7.18.** The expansion of $P_q(x)$ in terms of Stirling coefficients was first proved by Frobenius [Com74, Theorem E, p. 244].

*Proof.* (i) If $\bar\tau \nleq \bar\eta$, then $w(\sigma,\eta) = 0$ for all $\sigma \geq \tau$ and so $g(\tau,\eta) = 0$. Assume hereafter that $\bar\tau \leq \bar\eta$. From Lemma 7.15 we have the isomorphism

$$\begin{aligned} \{\sigma \in \mathcal{OP}_n \mid \sigma \geq \tau, \bar\sigma \leq \bar\eta\} &\cong \mathcal{I}_{q_1} \times \mathcal{I}_{q_2} \times \cdots \times \mathcal{I}_{q_r},\\ \sigma &\mapsto (\sigma_1, \sigma_2, \dots, \sigma_r). \end{aligned} \tag{7.12}$$

Since $r = |\tau| - \mathrm{pla}_\eta(\tau) = \mathrm{des}_\eta(\tau) + \mathrm{asc}_\eta(\tau) + 1$, we have $|\sigma| - \mathrm{asc}_\eta(\tau) - 1 = \sum_{i=1}^{r}(|\sigma_i| - 1) + \mathrm{des}_\eta(\tau)$. Note also that $[\tau:\sigma]! = \prod_{i=1}^{r}\prod_{S\in\sigma_i}|S|!$. Since $\sigma$ just connects the blocks of a level run of $(m_i)_{i=1}^{|\tau|}$, it does not change the number of ascents: $\operatorname{asc}_\eta(\sigma) = \operatorname{asc}_\eta(\tau)$ and we have

$$\begin{aligned}
&\sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\geq\tau}} \widetilde{\zeta}(\tau,\sigma)\, w(\sigma,\eta)\\
&\qquad = \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\geq\tau,\bar\sigma\leq\bar\eta}} \frac{1}{[\tau:\sigma]!}\int_{-1}^0 x^{|\sigma|-\operatorname{asc}_\eta(\tau)-1}(1+x)^{\operatorname{asc}_\eta(\tau)}\,dx\\
&\qquad = \sum_{(\sigma_1,\sigma_2,\dots,\sigma_r)\in\mathcal{I}_{q_1}\times\mathcal{I}_{q_2}\times\dots\times\mathcal{I}_{q_r}} \int_{-1}^0 x^{\operatorname{des}_\eta(\tau)}(1+x)^{\operatorname{asc}_\eta(\tau)}\prod_{i=1}^r \frac{x^{|\sigma_i|-1}}{\prod_{S\in\sigma_i}|S|!}dx\\
&\qquad = \int_{-1}^0 x^{\operatorname{des}_\eta(\tau)}(1+x)^{\operatorname{asc}_\eta(\tau)}\prod_{i=1}^r\left(\sum_{\rho\in\mathcal{I}_{q_i}}\frac{x^{|\rho|-1}}{\prod_{R\in\rho}|R|!}\right)dx.
\end{aligned} \tag{7.13}$$

For $q\in\mathbb{N}$ we have

$$\sum_{\rho\in\mathcal{I}_q}\frac{x^{|\rho|-1}}{\prod_{R\in\rho}|R|!} = \sum_{\substack{n_1+\dots+n_k=q\\ n_i\geq1,k\geq1}}\frac{x^{k-1}}{n_1!\cdots n_k!}. \tag{7.14}$$

For each fixed $k$, the sum $\sum_{\substack{n_1+\dots+n_k=q\\ n_i\geq1}}\frac{q!}{n_1!\cdots n_k!}$ is the number of ways of distributing $q$ distinct objects among $k$ nonempty urns, so it equals $k!S(q,k)$ and the proof is complete.

(ii) The idea of the proof is similar to Proposition 7.13(ii) and we omit the proof. ☐

**Proposition 7.19.** *Let $\pi,\eta\in\mathcal{OP}_n$. Suppose that a tuple $(i_1,i_2,\dots,i_n)$ has kernel $\kappa(i_1,i_2,\dots,i_n)=\eta$. Then*

$$K_\pi(X_1^{(i_1)},X_2^{(i_2)},\dots,X_n^{(i_n)}) = \sum_{\tau\in\mathcal{OP}_n} K_\tau(X_1,X_2,\dots,X_n)\, g(\tau,\eta,\pi).$$

*Proof.* Using Proposition 7.14, we have

$$\begin{aligned}
K_\pi(X_1^{(i_1)},X_2^{(i_2)},\dots,X_n^{(i_n)}) &= \sum_{\sigma\in\mathcal{OP}_n}\varphi_\sigma(X_1,X_2,\dots,X_n)\, w(\sigma,\eta,\pi)\\
&= \sum_{\sigma\in\mathcal{OP}_n}\sum_{\substack{\tau\in\mathcal{OP}_n\\ \tau\leq\sigma}} K_\tau(X_1,X_2,\dots,X_n)\,\widetilde{\zeta}(\tau,\sigma)\, w(\sigma,\eta,\pi)\\
&= \sum_{\tau\in\mathcal{OP}_n} K_\tau(X_1,X_2,\dots,X_n)\left(\sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\in[\tau,\pi]}}\widetilde{\zeta}(\tau,\sigma)\, w(\sigma,\eta,\pi)\right)\\
&= \sum_{\tau\in\mathcal{OP}_n} K_\tau(X_1,X_2,\dots,X_n)\, g(\tau,\eta,\pi).
\end{aligned} \tag{7.15}$$

For the third equality, we use Proposition 7.13(ii) which asserts that $w(\sigma,\eta,\pi)=0$ if $\sigma\nleq\pi$. ☐

We can characterize $\mathcal{S}$-independence in terms of the above proposition, that is "semi-vanishing" of mixed cumulants.

**Theorem 7.20.** *Let $\mathcal{S}=(\mathcal{U},\tilde\varphi)$ be a spreadability system for a given ncs $(\mathcal{A},\varphi)$. A sequence of subalgebras $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{A}$, where $I\subseteq\mathbb{N}$, is $\mathcal{S}$-*independent *if and only if for any tuple $(i_1,i_2,\dots,i_n)\in I^n$, any random variables $(X_1,X_2,\dots,X_n)\in\mathcal{A}_{i_1}\times\mathcal{A}_{i_2}\times\dots\times\mathcal{A}_{i_n}$ and any ordered set partition $\pi\in\mathcal{OP}_n$, we have*

$$K_\pi(X_1,X_2,\dots,X_n) - \sum_{\tau\in\mathcal{OP}_n} K_\tau(X_1,X_2,\dots,X_n)\, g(\tau,\kappa(i_1,\dots,i_n),\pi) = 0. \tag{7.16}$$

**Remark 7.21.** We can also formulate the theorem in terms of moments; we only need to replace (7.16) by the equation

$$K_\pi(X_1,X_2,\dots,X_n) - \sum_{\tau\in\mathcal{OP}_n} \varphi_\tau(X_1,X_2,\dots,X_n)\, w(\tau,\kappa(i_1,\dots,i_n),\pi) = 0. \tag{7.17}$$

*Proof.* We fix a tuple $(i_1,\dots,i_n)$ with kernel $\eta:=\kappa(i_1,\dots,i_n)$. It suffices to show that for any tuple of random variables $(X_1,X_2,\dots,X_n)\in\mathcal{A}_{i_1}\times\mathcal{A}_{i_2}\times\dots\times\mathcal{A}_{i_n}$ the following are equivalent:

(i) $\varphi_\pi(X_1, X_2, \dots, X_n) = \varphi_{\pi\wedge\eta}(X_1, X_2, \dots, X_n)$ for all $\pi \in \mathcal{OP}_n$,
(ii) $K_\pi(X_1, X_2, \dots, X_n) = \sum_{\tau\in\mathcal{OP}_n} K_\tau(X_1, X_2, \dots, X_n)\, g(\tau, \eta, \pi)$ for all $\pi \in \mathcal{OP}_n$.

Assume (i), then

$$
\begin{aligned}
K_\pi(X_1, X_2, \dots, X_n) &= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi)\\
&= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_{\sigma\wedge\eta}(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi)\\
&= K_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}),
\end{aligned}
\tag{7.18}
$$

where (7.9) was used in the last step. We conclude (i) from Proposition 7.19.

Conversely, assume (ii), then Proposition 7.19 implies that

$$K_\pi(X_1, X_2, \dots, X_n) = K_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}). \tag{7.19}$$

On one hand Theorem 4.7 implies

$$K_\pi(X_1, X_2, \dots, X_n) = \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi), \tag{7.20}$$

and on the other hand (7.9) implies

$$K_\pi(X_1^{(i_1)}, X_2^{(i_2)}, \dots, X_n^{(i_n)}) = \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_{\sigma\wedge\eta}(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi). \tag{7.21}$$

Thus we get

$$\sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_\sigma(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi) = \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\le\pi}} \varphi_{\sigma\wedge\eta}(X_1, X_2, \dots, X_n)\, \widetilde{\mu}(\sigma, \pi) \tag{7.22}$$

for all $\pi \in \mathcal{OP}_n$. Finally Möbius inversion yields (i). ☐

It is not obvious that Theorem 7.20 generalizes Proposition 7.1, the vanishing of mixed cumulants for $\mathcal{E}$-independent subalgebras. In fact it implies a nontrivial identity: for any $\tau, \eta, \pi \in \mathcal{OP}_n$ such that $\eta\rfloor_P \ne \hat{1}_P$ for some $P \in \pi$,

$$\sum_{h\in\mathfrak{S}_{|\tau|}} g(h(\tau), \eta, \pi) = 0, \tag{7.23}$$

where $h(\tau)$ is the action of the permutation $h$ on the blocks of $\tau$. Similarly, Remark 7.21 (or Proposition 7.14) generalizes the vanishing of mixed cumulants for $\mathcal{E}$-independent subalgebras. Consequently we must have

$$\sum_{h\in\mathfrak{S}_{|\tau|}} w(h(\tau), \eta, \pi) = 0 \tag{7.24}$$

under the same assumptions on $\tau, \eta, \pi$.

Some Goldberg coefficients $g(\tau, \eta)$ are known to vanish even when $\bar{\tau} \le \bar{\eta}$ (see [VBV16]), while the Weisner coefficients $w(\tau, \eta)$ do not vanish whenever $\bar{\tau} \le \bar{\eta}$. The following Proposition describes two sufficient criteria for vanishing Goldberg coefficients .

**Proposition 7.22.** *Suppose $\tau, \eta \in \mathcal{OP}_n$ are such that $\bar{\tau} \le \bar{\eta}$.*

*(i) If $\mathrm{des}_\eta(\tau) = \mathrm{asc}_\eta(\tau)$ and $|\tau|$ is even then $g(\tau, \eta) = 0$.*
*(ii) If $|\tau|$ is prime then $g(\tau, \eta) \ne 0$.*

*Proof.* (i) By [Reu93, Corollary 3.15], the coefficient $g(\tau, \eta)$ of $K_\tau(X_1, \dots, X_n)$ vanishes if $q_1 + \dots + q_r + \mathrm{des}_\eta(\tau) + \mathrm{asc}_\eta(\tau) - r$ is odd and $\mathrm{des}_\eta(\tau) = \mathrm{asc}_\eta(\tau)$. Since $\sum_{i=1}^r q_i = r + \mathrm{pla}_\eta(\tau)$ and $|\tau| = \mathrm{des}_\eta(\tau) + \mathrm{asc}_\eta(\tau) + \mathrm{pla}_\eta(\tau) + 1$, the conclusion follows.

(ii) The idea of the proof is taken from [VBV16]. Let $p := |\tau|$. By definition and Proposition 7.13 we have

$$(7.25)\quad \begin{aligned} g(\tau,\eta) &= \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\geq\tau}} \widetilde{\zeta}(\tau,\sigma)\, w(\sigma,\eta) = \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma\geq\tau}} \frac{1}{[\tau:\sigma]!}\frac{(-1)^{|\sigma|-\mathrm{asc}_\eta(\sigma)-1}}{|\sigma|\binom{|\sigma|-1}{\mathrm{asc}_\eta(\sigma)}} \\ &= \frac{(-1)^{p-\mathrm{asc}_\eta(\tau)-1}}{p\binom{p-1}{\mathrm{asc}_\eta(\tau)}} + \sum_{\substack{\sigma\in\mathcal{OP}_n\\ \sigma>\tau}} \frac{1}{[\tau:\sigma]!}\frac{(-1)^{|\sigma|-\mathrm{asc}_\eta(\sigma)-1}}{|\sigma|\binom{|\sigma|-1}{\mathrm{asc}_\eta(\sigma)}}. \end{aligned}$$

Since the number $[\tau:\sigma]!\,|\sigma|\binom{|\sigma|-1}{\mathrm{asc}_\eta(\sigma)}$ never contains $p$ as a factor for any $\sigma>\tau$, $g(\tau,\eta)$ is nonzero. ☐

**Remark 7.23.** It is a difficult problem to characterize vanishing Goldberg coefficients. The criterion (i) from Proposition 7.22 above does not cover all cases, for example one can show that $g(\tau,\eta)=0$ for $\tau=(\{3\},\{4\},\{2\},\{1\}), \eta=(\{1,2,3\},\{4\})$, although the pair $(\tau,\eta)$ does not satisfy the assumption of the criterion. More information about Goldberg coefficients can be found in [Reu93, Tho82] and in particular [VBV16, Section IV] concerning the question of vanishing coefficients.

**Example 7.24** (Goldberg coefficients and partial vanishing of cumulants)**.** We will write the ordered kernel set partition $\kappa(i_1,\dots,i_n)$ simply as $i_1 i_2 \dots i_n$.

1. Take $\eta=12$. We compute $g(12,12)$. Now $\tau=\eta$ so $m_1=1, m_2=2, r=2, q_1=q_2=1$. Hence $\mathrm{des}_\eta(\tau)=0, \mathrm{asc}_\eta(\tau)=1$ and so

$$(7.26)\quad g(12,12) = \frac{1}{1!1!}\int_{-1}^{0}(1+x)\,dx = \frac{1}{2}.$$

Similarly we get $g(21,12)=-\frac{1}{2}, g(12,21)=-\frac{1}{2}, g(21,21)=\frac{1}{2}$ and hence

$$\begin{aligned} K_{11}(X^{(1)},Y^{(2)}) &= \frac{1}{2}K_{12}(X,Y) - \frac{1}{2}K_{21}(X,Y), \\ K_{11}(X^{(2)},Y^{(1)}) &= -\frac{1}{2}K_{12}(X,Y) + \frac{1}{2}K_{21}(X,Y). \end{aligned}$$

In the most important spreadability systems, like the tensor, free, Boolean or monotone spreadability systems, partitioned cumulants factorize, e.g., $K_{12}(X,Y)=K_1(X)K_1(Y)$. Hence we get $K_{11}(X^{(1)},Y^{(2)}) = K_{11}(X^{(2)},Y^{(1)}) = 0$.

2. We then consider the case $\eta=112=(\{1,2\},\{3\})$. The Goldberg coefficients can be nonzero only when $\bar{\tau}\leq\bar{\eta}$, so $\tau$ is one of

$$112, 221, 123, 132, 213, 231, 312, 321.$$

If $\tau=112$ then $g(112,112)=\frac{1}{2}$ by the same calculation as $g(12,12)$. If $\tau=221$ then again $g(221,112)=-\frac{1}{2}$. If $\tau=123$ then $m_1=1, m_2=1, m_3=2, r=2, q_1=2, q_2=1$. So

$$(7.27)\quad g(123,112) = \frac{1}{1!2!}\int_{-1}^{0}(1+x)P_2(x)\,dx = \frac{1}{12}.$$

If we take $\tau=132$ then $m_1=1, m_2=2, m_3=1$, so $r=3, q_1=q_2=q_3=1$, $\mathrm{asc}=1, \mathrm{des}=1$. Thus we get

$$(7.28)\quad g(132,112) = \frac{1}{1!1!1!}\int_{-1}^{0}x(1+x)\,dx = -\frac{1}{6}.$$

If we take $\tau=231$ then $m_1=2, m_2=1, m_3=1$. So $r=2, q_1=1, q_2=2$, $\mathrm{asc}=0, \mathrm{des}=1$. Hence

$$(7.29)\quad g(231,112) = \frac{1}{1!2!}\int_{-1}^{0}xP_2(x)\,dx = \frac{1}{2}\int_{-1}^{0}x(2x+1)\,dx = \frac{1}{12}.$$

Similarly we can compute the remaining Goldberg coefficients and get

$$(7.30)\quad \begin{aligned} K_{111}(X^{(1)},Y^{(1)},Z^{(2)}) &= \frac{1}{2}K_{112} - \frac{1}{2}K_{221} + \frac{1}{12}K_{123} - \frac{1}{6}K_{132} \\ &\quad + \frac{1}{12}K_{213} + \frac{1}{12}K_{231} - \frac{1}{6}K_{312} + \frac{1}{12}K_{321}, \end{aligned}$$

where $X, Y, Z$ are omitted for simplicity. We can see that (7.23) holds (now $\pi = 111$):

$$\frac{1}{2} - \frac{1}{2} = 0, \qquad \frac{1}{12} - \frac{1}{6} + \frac{1}{12} + \frac{1}{12} - \frac{1}{6} + \frac{1}{12} = 0. \tag{7.31}$$

Again if the cumulants factorize then $K_{112} = K_{221} = K_{11}(X,Y)K_1(Z)$ and $K_{123} = \cdots = K_{321} = K_1(X)K_1(Y)K_1(Z)$, so the mixed cumulant $K_{111}(X^{(1)}, Y^{(1)}, Z^{(2)})$ vanishes.

Similarly one can compute $g(\tau, 121)$ for all $\tau$ such that $\bar{\tau} \leq \overline{121}$ and get

$$\begin{aligned} K_{111}(X^{(1)}, Y^{(2)}, Z^{(1)}) = \frac{1}{2}K_{121} - \frac{1}{2}K_{212} - \frac{1}{6}K_{123} + \frac{1}{12}K_{132} \\ + \frac{1}{12}K_{213} + \frac{1}{12}K_{231} + \frac{1}{12}K_{312} - \frac{1}{6}K_{321}. \end{aligned} \tag{7.32}$$

Now in the tensor, free or Boolean spreadability system the mixed cumulant vanishes. However in the monotone spreadability system it does not: $K_{212}$ vanishes identically since 212 is not a monotone partition (see Proposition 4.19) and therefore, in the monotone case we have

$$K_{111}(X^{(1)}, Y^{(2)}, Z^{(1)}) = \frac{1}{2}K_{121}(X,Y,Z) = \frac{1}{2}K_{11}(X,Z)K_1(Y)$$

which does not vanish in general.

The calculation of these cumulants in terms of moments is easier.

**Example 7.25** (Weisner coefficients and partial vanishing of cumulants)**.** We can reuse some results from Example 7.24.

1. If we take $\tau = \eta = 12$ then $\mathrm{asc}_\eta(\tau) = 1$ and so

$$w(12,12) = \frac{(-1)^{2-1-1}}{2\binom{1}{1}} = \frac{1}{2}. \tag{7.33}$$

2. Similarly if $\tau = 21$ and $\eta = 12$ then $m_1 = 2, m_1 = 1$ so $\mathrm{asc}_\eta(\tau) = 0$. Therefore we get $w(21,12) = -\frac{1}{2}$. Similarly, $w(12,21) = -\frac{1}{2}, w(21,21) = \frac{1}{2}$. So

$$\begin{aligned} K_{11}(X^{(1)}, Y^{(2)}) &= \frac{1}{2}\varphi_{12}(X,Y) - \frac{1}{2}\varphi_{21}(X,Y), \\ K_{11}(X^{(2)}, Y^{(1)}) &= -\frac{1}{2}\varphi_{12}(X,Y) + \frac{1}{2}\varphi_{21}(X,Y). \end{aligned}$$

In factorizing spreadability systems we have $\varphi_{12}(X,Y) = \varphi_{21}(X,Y) = \varphi_1(X)\varphi_1(Y)$ and hence $K_{11}(X^{(1)}, Y^{(2)}) = K_{11}(X^{(2)}, Y^{(1)}) = 0$.

3. One can show that $w(112,112) = \frac{1}{2} = -w(221,112)$ by the same calculation as $w(12,12)$ and $w(21,12)$. If $\tau = 123, \eta = 112$ then $m_1 = 1, m_2 = 1, m_3 = 2$, so asc = 1 and

$$w(123,112) = \frac{(-1)^{3-1-1}}{3\binom{2}{1}} = -\frac{1}{6}. \tag{7.34}$$

If we take $\tau = 231$ then $m_1 = 2, m_2 = 1, m_3 = 1$ and so asc = 0. Hence

$$w(231,112) = \frac{(-1)^{3-0-1}}{3\binom{2}{0}} = \frac{1}{3}. \tag{7.35}$$

Similarly we can compute the remaining Weisner coefficients and get

$$\begin{aligned} K_{111}(X^{(1)}, Y^{(1)}, Z^{(2)}) = \frac{1}{2}\varphi_{112} - \frac{1}{2}\varphi_{221} - \frac{1}{6}\varphi_{123} - \frac{1}{6}\varphi_{132} \\ - \frac{1}{6}\varphi_{213} + \frac{1}{3}\varphi_{231} - \frac{1}{6}\varphi_{312} + \frac{1}{3}\varphi_{321}, \end{aligned} \tag{7.36}$$

where $X, Y, Z$ are omitted for simplicity. We can see that (7.24) holds (now $\pi = 111$):

$$\frac{1}{2} - \frac{1}{2} = 0, \qquad -\frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{3} - \frac{1}{6} + \frac{1}{3} = 0. \tag{7.37}$$

In factorizing spreadability systems the mixed cumulant $K_{111}(X^{(1)}, Y^{(1)}, Z^{(2)})$ vanishes by using the factorization of partitioned moments.

4. Similarly one can compute $w(\tau,121)$ for all $\tau$ such that $\bar{\tau}\le\overline{121}$ and get

$$\begin{aligned} K_{111}(X^{(1)},Y^{(2)},Z^{(1)}) &= \frac{1}{2}\varphi_{121}-\frac{1}{2}\varphi_{212}-\frac{1}{6}\varphi_{123}-\frac{1}{6}\varphi_{132} \\ &\quad +\frac{1}{3}\varphi_{213}-\frac{1}{6}\varphi_{231}+\frac{1}{3}\varphi_{312}-\frac{1}{6}\varphi_{321}. \end{aligned} \tag{7.38}$$

In the tensor, free or Boolean spreadability system the mixed cumulant vanishes, but if it is monotone, then $\varphi_{212}=\varphi(X)\varphi(Y)\varphi(Z)$ while $\varphi_{121}=\varphi(XZ)\varphi(Y)$. Therefore, in the monotone case we have

$$K_{111}(X^{(1)},Y^{(2)},Z^{(1)})=\frac{1}{2}(\varphi_{121}-\varphi_{212})=\frac{1}{2}(\varphi(XZ)-\varphi(X)\varphi(Z))\varphi(Y)$$

which does not vanish in general.

## 8. Campbell-Baker-Hausdorff formula and Lie polynomials

The material of the preceding section resembles some results from the theory of free Lie algebras, cf. the book by C. Reutenauer [Reu93] already cited above. In particular, the *Goldberg coefficients* $g(\tau,\eta)$ from Proposition 7.16 coincide with the coefficients of the Campbell-Baker-Hausdorff series, i.e.,

$$\log(e^{a_1}e^{a_2}\cdots e^{a_n})=\sum_{w:\text{word}} g_w w$$

when it is expanded in the ring of noncommutative formal power series, see [Gol56]. Indeed the following paragraphs will provide a new interpretation of the coefficients of the CBH formula in terms of a certain spreadability system.

8.1. **The noncommutative tensor spreadability system $\mathcal{S}_{\mathrm{NCT}}$.** Given an algebra $\mathcal{A}$ we introduce an operator-valued spreadability system with the following ingredients.

- Put $\varphi=\mathrm{Id}:\mathcal{A}\to\mathcal{A}$.
- $\mathcal{U}:=\otimes_{i=1}^{\infty}\mathcal{A}$ is the algebraic tensor product, cf. Section 3.3.1.
- $\iota^{(j)}:\mathcal{A}\to\mathcal{U}$ is the natural embedding of $\mathcal{A}$ into the $j^{\text{th}}$ component of $\mathcal{U}$:

$$\iota^{(j)}(X):=1^{\otimes(j-1)}\otimes X\otimes 1^{\otimes\infty}.$$

- $\widetilde{\varphi}:=\mathrm{conc}_{\infty}:\mathcal{U}\to\mathcal{A}$ is the concatenation product. This can be alternatively defined by

$$\widetilde{\varphi}(X_1^{(i_1)}X_2^{(i_2)}\cdots X_n^{(i_n)}):=\varphi_\pi(X_1,X_2,\dots,X_n)=X_{P_1}X_{P_2}\cdots X_{P_k},$$

  where $\pi=\kappa(i_1,i_2,\dots,i_n)=(P_1,P_2,\dots,P_k)$ and $X_P$ is defined in (3.1).

Thus $X_1,X_2,\dots,X_n$ are "shuffled" by $\widetilde{\varphi}$ accordingly to the upper indices $(i_1,i_2,\dots,i_n)$. For example

$$\widetilde{\varphi}(X_1^{(5)}X_2^{(2)}X_3^{(3)}X_4^{(2)}X_5^{(3)}):=X_2X_4X_3X_5X_1.$$

We call the triple $\mathcal{S}_{\mathrm{NCT}}=(\mathcal{U},(\iota^{(j)})_{j=1}^{\infty},\mathrm{conc}_{\infty})$ the *noncommutative tensor spreadability system.*

**Proposition 8.1.** *The above defined triple $\mathcal{S}_{\mathrm{NCT}}$ is a spreadability system for $(\mathcal{A},\mathrm{Id})$.*

*Proof.* Clearly $\widetilde{\varphi}\circ\iota^{(i)}=\mathrm{Id}$ on $\mathcal{A}$. The symmetry condition (3.5) holds too since the value of $\widetilde{\varphi}$ only depends on the ordered kernel set partition of the upper indices. □

The corresponding NCT cumulants $K_\pi^{\mathrm{NCT}}$ satisfy (ordered) multiplicativity like the tensor, free and Boolean cumulants.

**Proposition 8.2** (Multiplicativity of partitioned cumulants). *For $\pi=(P_1,P_2,\dots,P_p)\in\mathcal{OP}_n$, we have*

$$K_\pi^{\mathrm{NCT}}(X_1,\dots,X_n)=K_{|P_1|}^{\mathrm{NCT}}(X_{P_1})\,K_{|P_2|}^{\mathrm{NCT}}(X_{P_2})\cdots K_{|P_p|}^{\mathrm{NCT}}(X_{P_p}).$$

*Proof.* For $\pi = (P_1, \dots, P_p) \in \mathcal{OP}_n$,

$$\begin{aligned}
\varphi_\pi(N.X_1, \dots, N.X_n) &= \sum_{i_1,\dots,i_n \in [N]} \varphi_{\pi \wedge \kappa(i_1,\dots,i_n)}(X_1, \dots, X_n) \\
&= \sum_{i_1,\dots,i_n \in [N]} X_{\kappa(i_1,\dots,i_n)\rfloor_{P_1}} X_{\kappa(i_1,\dots,i_n)\rfloor_{P_2}} \cdots X_{\kappa(i_1,\dots,i_n)\rfloor_{P_p}} \\
&= \sum_{\substack{i_k \in [N] \\ k \in P_1}} X_{\kappa(i_1,\dots,i_n)\rfloor_{P_1}} \sum_{\substack{i_k \in [N] \\ k \in P_2}} X_{\kappa(i_1,\dots,i_n)\rfloor_{P_2}} \cdots \sum_{\substack{i_k \in [N] \\ k \in P_p}} X_{\kappa(i_1,\dots,i_n)\rfloor_{P_p}} \qquad (8.1)\\
&= \widetilde{\varphi}\left(\prod_{i \in P_1} N.X_i\right) \cdots \widetilde{\varphi}\left(\prod_{i \in P_p} N.X_i\right).
\end{aligned}$$

The conclusion follows by comparing the coefficients of $N^p$. □

After having established multiplicativity it suffices to compute $K_n^{\mathrm{NCT}}$. Theorem 4.7 reads as follows.

**Proposition 8.3.** *For $n \in \mathbb{N}$ we have*

$$K_n^{\mathrm{NCT}}(X_1, \dots, X_n) = \sum_{\pi = (P_1, P_2, \dots) \in \mathcal{OP}_n} \frac{(-1)^{|\pi|-1}}{|\pi|} X_{P_1} X_{P_2} \cdots X_{P_{|\pi|}}.$$

**Example 8.4.** The reader can easily verify that

$$K_1^{\mathrm{NCT}}(X) = X, \qquad (8.2)$$

$$K_2^{\mathrm{NCT}}(X_1, X_2) = \frac{1}{2}[X_1, X_2], \qquad (8.3)$$

$$\begin{aligned}
K_3^{\mathrm{NCT}}(X_1, X_2, X_3) &= \frac{1}{3}(X_1X_2X_3 + X_3X_2X_1) \qquad (8.4)\\
&\quad - \frac{1}{6}(X_1X_3X_2 + X_2X_1X_3 + X_2X_3X_1 + X_3X_1X_2).
\end{aligned}$$

**Remark 8.5.** It can be shown that $K_n^{\mathrm{NCT}}$ $(n \geq 2)$ can be expressed as a sum of commutators. For the proof it suffices to show that $K_n^{\mathrm{FL}}$ defined later can be written as a sum of commutators, which follows from the fact that $K_n^{\mathrm{FL}}(a_1, \dots, a_n)$ is a Lie polynomial (see Remark 8.14) and every Lie polynomial is a fixed point of the (linear extension of) map $a_1a_2\cdots a_n \mapsto n^{-1}[a_1, [a_2, [\cdots, [a_{n-1}, a_n]]\cdots]$ (see [Reu93, Theorem 1.4]).

The CBH formula on $\mathcal{A}$ can be expressed in terms of NCT cumulants.

**Theorem 8.6** (CBH formula)**.** *As formal power series on $\mathcal{A}$ we have the identity*

$$\begin{aligned}
&\log(e^{a_1}e^{a_2}\cdots e^{a_n}) \\
&= \sum_{\substack{(p_1,p_2,\dots,p_n) \in (\mathbb{N} \cup \{0\})^n, \\ (p_1,p_2,\dots,p_n) \neq (0,0,\dots,0)}} \frac{1}{p_1!p_2!\cdots p_n!} K^{\mathrm{NCT}}_{p_1+p_2+\cdots+p_n}(\underbrace{a_1, a_1, \dots, a_1}_{p_1 \text{ times}}, \dots, \underbrace{a_n, a_n, \dots, a_n}_{p_n \text{ times}}).
\end{aligned}$$

*Proof.* First observe that

$$\begin{aligned}
&1 + \sum_{\substack{(p_1,p_2,\dots,p_n) \in (\mathbb{N} \cup \{0\})^n, \\ (p_1,p_2,\dots,p_n) \neq (0,0,\dots,0)}} \frac{1}{p_1!p_2!\cdots p_n!} \widetilde{\varphi}((N.a_1)^{p_1}(N.a_2)^{p_2}\cdots(N.a_n)^{p_n}) \\
&= \widetilde{\varphi}(e^{N.a_1}\cdots e^{N.a_n}) \qquad (8.5)\\
&= \widetilde{\varphi}(e^{a_1^{(1)}+\cdots+a_1^{(N)}}\cdots e^{a_n^{(1)}+\cdots+a_n^{(N)}}).
\end{aligned}$$

Our construction implies that $a_i^{(j)}, j = 1, 2, 3, \dots$ commute for each fixed $i$, so

$$\begin{aligned}
&\widetilde{\varphi}(e^{a_1^{(1)}+\cdots+a_1^{(N)}}\cdots e^{a_n^{(1)}+\cdots+a_n^{(N)}}) \qquad (8.6)\\
&= \widetilde{\varphi}((e^{a_1^{(1)}}e^{a_1^{(2)}}\cdots e^{a_1^{(N)}})(e^{a_2^{(1)}}e^{a_2^{(2)}}\cdots e^{a_2^{(N)}})\cdots(e^{a_n^{(1)}}e^{a_n^{(2)}}\cdots e^{a_n^{(N)}})).
\end{aligned}$$

By the definition of $\widetilde{\varphi}$ the last expression equals

$$(e^{a_1}e^{a_2}\cdots e^{a_n})^N. \qquad (8.7)$$

We conclude by comparing the coefficient of $N$ in the identity

$$\begin{aligned} e^{N\log(e^{a_1}e^{a_2}\cdots e^{a_n})} &= (e^{a_1}e^{a_2}\cdots e^{a_n})^N \\ &= 1 + \sum_{\substack{(p_1,p_2,\dots,p_n)\in(\mathbb{N}\cup\{0\})^n,\\ (p_1,p_2,\dots,p_n)\neq(0,0,\dots,0)}} \frac{1}{p_1!p_2!\cdots p_n!}\widetilde{\varphi}((N.a_1)^{p_1}(N.a_2)^{p_2}\cdots(N.a_n)^{p_n}). \end{aligned} \tag{8.8}$$

□

**Remark 8.7.** Theorem 8.6 is similar to the well-known formula in probability theory,

$$\begin{aligned} &\log\mathbb{E}[e^{z_1X_1+\cdots+z_nX_n}] \\ &= \sum_{\substack{(p_1,p_2,\dots,p_n)\in(\mathbb{N}\cup\{0\})^n,\\ (p_1,p_2,\dots,p_n)\neq(0,0,\dots,0)}} \frac{z_1^{p_1}z_2^{p_2}\cdots z_n^{p_n}}{p_1!p_2!\cdots p_n!}K^{\mathrm{T}}_{p_1+p_2+\cdots+p_n}(\underbrace{X_1,X_1,\dots,X_1}_{p_1\text{ times}},\dots,\underbrace{X_n,X_n,\dots,X_n}_{p_n\text{ times}}), \end{aligned} \tag{8.9}$$

where $X_1,\dots,X_n$ are $\mathbb{C}$-valued classical random variables and $z_1,\dots,z_n$ are commutative indeterminates.

**Proposition 8.8.** *(i) A sequence $(\mathcal{A}_i)_{i=1}^\infty$ of subalgebras of $\mathcal{A}$ is $\mathcal{S}_{\mathrm{NCT}}$-independent if and only if the subalgebras $\mathcal{A}_1,\mathcal{A}_2,\dots$ commute mutually. This is obviously equivalent to $K_2^{\mathrm{NCT}}(X,Y)=0$ whenever $X\in\mathcal{A}_i, Y\in\mathcal{A}_j$ with $i\neq j$.*
*(ii) For fixed $n\geq 2$, if $\{X_1,\dots,X_n\}\subseteq\mathcal{A}$ splits into two mutually commutative families then $K_n^{\mathrm{NCT}}(X_1,\dots,X_n)=0$.*

**Remark 8.9.** 1. Additivity of Lie polynomials in commuting variables is well known, see [Reu93, p. 20]).
2. We have thus shown that

$\mathcal{S}_{\mathrm{NCT}}$-independence = commutativity = vanishing of mixed NCT cumulants.

This example illustrates that in general vanishing of mixed cumulants does not imply exchangeability.

*Proof.* (i) Suppose that $(\mathcal{A}_i)_{i=1}^\infty$ is $\mathcal{S}_{\mathrm{NCT}}$-independent. Let $\pi=\hat{1}_2, \rho=(\{2\},\{1\})\in\mathcal{OP}_2$ and let $X\in\mathcal{A}_i, Y\in\mathcal{A}_j$ for fixed $i>j$. Then $\kappa(i,j)=\rho$ and

$$\varphi_\pi(X,Y)=XY,\qquad \varphi_{\pi\wedge\rho}(X,Y)=YX, \tag{8.10}$$

and by $\mathcal{S}_{\mathrm{NCT}}$-independence these two must coincide, so $XY=YX$. This shows that $[\mathcal{A}_i,\mathcal{A}_j]=0$. Conversely, if the subalgebras $\mathcal{A}_1,\mathcal{A}_2,\dots$ are mutually commutative then for any $i_1,\dots,i_n\in\mathbb{N}$, any $X_k\in\mathcal{A}_{i_k}, k=1,\dots,n$ and any $\pi=(P_1,\dots,P_p)\in\mathcal{OP}_n$, let $\rho:=\kappa(i_1,\dots,i_n)=(R_1,\dots,R_r)$. Then

$$\varphi_{\pi\wedge\rho}(X_1,\dots,X_n)=(X_{P_1\cap R_1}X_{P_1\cap R_2}\cdots X_{P_1\cap R_r})\cdots(X_{P_p\cap R_1}X_{P_p\cap R_2}\cdots X_{P_p\cap R_r}) \tag{8.11}$$

where $X_\varnothing$ is understood as the unit $\hat{1}_{\mathcal{A}}$. By commutativity, $\{X_k\mid k\in R_i\}$ and $\{X_k\mid k\in R_j\}$ commute for distinct $i,j$ and consequently for each $i=1,\dots,p$ we have

$$X_{P_i\cap R_1}X_{P_i\cap R_2}\cdots X_{P_i\cap R_r}=X_{P_i}, \tag{8.12}$$

which shows that $\varphi_{\pi\wedge\rho}(X_1,\dots,X_n)=\varphi_\pi(X_1,\dots,X_n)$.

(ii) Suppose that $\{X_i\mid i\in I\}$ and $\{X_i\mid i\in I^c\}$ commute with each other and $\varnothing\subsetneq I\subsetneq\{1,\dots,n\}$. Then, for commutative indeterminates $z_1,\dots,z_n$, we have

$$\log(e^{z_1X_1}\cdots e^{z_nX_n})=\log\left(\prod_{i\in I}e^{z_iX_i}\right)+\log\left(\prod_{i\in I^c}e^{z_iX_i}\right), \tag{8.13}$$

where the products $\prod_{i\in I},\prod_{i\in I^c}$ preserve the natural orders on $I, I^c$. On the other hand, by Theorem 8.6, $\log(e^{z_1X_1}\cdots e^{z_nX_n})$ equals

$$\sum_{\substack{(p_1,p_2,\dots,p_n)\in(\mathbb{N}\cup\{0\})^n,\\ (p_1,p_2,\dots,p_n)\neq(0,0,\dots,0)}} \frac{z_1^{p_1}z_2^{p_2}\cdots z_n^{p_n}}{p_1!p_2!\cdots p_n!}K^{\mathrm{NCT}}_{p_1+p_2+\cdots+p_n}(\underbrace{X_1,X_1,\dots,X_1}_{p_1\text{ times}},\dots,\underbrace{X_n,X_n,\dots,X_n}_{p_n\text{ times}}). \tag{8.14}$$

If we compare the coefficients of $z_1z_2\cdots z_n$ in (8.13) and (8.14) then we get $K_n(X_1,\dots,X_n)=0$. □

8.2. **Specialization to free algebras.** We restrict the noncommutative tensor spreadability system $\mathcal{S}_{\mathrm{NCT}}$ to the case when the underlying algebra $\mathcal{A}$ is a free algebra. The aim of this section is to show that the NCT cumulants are Lie polynomials in the following sense (cf. [Reu93]). Let $\mathbf{A}$ be an alphabet, whose elements are typically denoted by $a_1, a_2, \ldots$, and let $\mathbf{A}^*$ be the free monoid generated by $\mathbf{A}$ endowed with the concatenation product. An element of $\mathbf{A}^*$ is called a *word*. If $w = a_1 a_2 \cdots a_n$, $a_i \in \mathbf{A}$, then the word $w$ has *length* $n$. The length of the unit 1 is understood to be 0. Let $\mathbb{C}[\mathbf{A}]$ be the monoid algebra of the monoid $\mathbf{A}^*$ over $\mathbb{C}$.

We construct the noncommutative tensor spreadability system for $(\mathbb{C}[\mathbf{A}], \mathrm{Id})$ as in Section 8.1. In this case we call it *free Lie spreadability system* and denote it by $\mathcal{S}_{\mathrm{FL}}$ instead of $\mathcal{S}_{\mathrm{NCT}}$.

Moreover, on free algebras we can introduce a bialgebra structure. Let $\delta_k : \mathbb{C}[\mathbf{A}] \to \mathbb{C}[\mathbf{A}]^{\otimes k}$ be the coproduct uniquely determined by the values

$$\delta_k(a) = \sum_{n=0}^{k-1} 1^{\otimes n} \otimes a \otimes 1^{\otimes (k-n-1)}, \qquad a \in \mathbf{A}$$

on the generators (compare Definition 4.1). In the language of Hopf algebras, this is the $k$-fold copower of the unique coproduct $\delta$ for which every generator is a primitive element. Moreover, in this case the dot operation corresponds to the coproduct, that is,

$$N.a = \delta_N(a) \otimes 1^{\otimes \infty}, \qquad a \in \mathbf{A}.$$

The *counit* is the unique linear map $\epsilon : \mathbb{C}[\mathbf{A}] \to \mathbb{C}$ such that

$$\epsilon(1) = 1, \qquad \epsilon(w) = 0, \quad w \in \mathbf{A}^* \setminus \{1\}.$$

Let $\mathrm{conc}_k : \mathbb{C}[\mathbf{A}]^{\otimes k} \to \mathbb{C}[\mathbf{A}]$ be the linear map defined by

$$\mathrm{conc}_k(X_1 \otimes X_2 \otimes \cdots \otimes X_k) = X_1 X_2 \cdots X_k, \qquad X_i \in \mathbb{C}[\mathbf{A}].$$

For endomorphisms $f_1, f_2, \ldots, f_k$ of $\mathbb{C}[\mathbf{A}]$, we define the convolution

$$f_1 * f_2 * \cdots * f_k := \mathrm{conc}_k \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_k) \circ \delta_k.$$

**Definition 8.10.** Let $\Pi : \mathbb{C}[\mathbf{A}] \to \mathbb{C}[\mathbf{A}]$ be the map defined by

$$\Pi = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\mathrm{Id} - \epsilon)^{*k},$$

where $\epsilon$ is regarded as an endomorphism of $\mathbb{C}[\mathbf{A}]$. It follows from Proposition 8.12 below that for any $a_i \in \mathbf{A}$ the value $\Pi(a_1 a_2 \cdots a_n)$ is a finite sum of words and that indeed $\Pi(\mathbb{C}[\mathbf{A}]) \subseteq \mathbb{C}[\mathbf{A}]$.

**Proposition 8.11.** *If* $a_1, a_2, \ldots, a_n \in \mathbf{A}$ *and* $k \in \mathbb{N}$, *then*

$$\delta_k(a_1 a_2 \cdots a_n) = \sum_{\substack{\pi = (P_1, P_2, \ldots) \in \mathcal{OPP}_n \\ |\pi| = k}} a_{P_1} \otimes a_{P_2} \otimes \cdots \otimes a_{P_k},$$

*where the sum runs over pseudopartitions, see Definition 2.8.*

*Proof.* Since $\delta_k$ is a homomorphism, we have

$$\begin{aligned} \delta_k(a_1 a_2 \cdots a_n) &= \delta_k(a_1)\, \delta_k(a_2) \cdots \delta_k(a_n) \\ &= (a_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a_1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a_1) \\ &\quad (a_2 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a_2 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a_2) \\ &\quad \cdots (a_n \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a_n \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a_n), \end{aligned} \tag{8.15}$$

which is equal to the desired expression in terms of ordered pseudopartitions. □

Now we have the following expression for $\Pi$.

**Proposition 8.12.** *Let* $a_1, a_2, \ldots, a_n \in \mathbf{A}$. *Then*

$$\Pi(a_1 a_2 \cdots a_n) = \sum_{\pi = (P_1, P_2, \ldots) \in \mathcal{OP}_n} \frac{(-1)^{|\pi|-1}}{|\pi|} a_{P_1} a_{P_2} \cdots a_{P_{|\pi|}}.$$

*Proof.* From Proposition 8.11 we have

$$\begin{aligned}\Pi(a_1a_2\cdots a_n) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\mathrm{Id}-\epsilon)^{*k}(a_1a_2\cdots a_n)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\mathrm{conc}_k\circ(\mathrm{Id}-\epsilon)^{\otimes k}\circ\delta_k)(a_1a_2\cdots a_n)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{\substack{\pi=(P_1,P_2,\dots)\in\mathcal{OPP}_n\\|\pi|=k}}(\mathrm{Id}-\epsilon)(a_{P_1})(\mathrm{Id}-\epsilon)(a_{P_2})\cdots(\mathrm{Id}-\epsilon)(a_{P_k}).\end{aligned}$$

Note that $(\mathrm{Id}-\epsilon)(1)=0$ and $(\mathrm{Id}-\epsilon)(w)=w$ for $w\in\mathbf{A}^*\smallsetminus\{1\}$. If $P_i=\varnothing$ for some $i$, then the product $(\mathrm{Id}-\epsilon)(a_{P_1})(\mathrm{Id}-\epsilon)(a_{P_2})\cdots(\mathrm{Id}-\epsilon)(a_{P_k})$ vanishes. Therefore only proper ordered set partitions contribute to the sum, and in particular $k\le n$. This proves the claim. ☐

Combining Proposition 8.12 with Proposition 8.3 we conclude the following identity.

**Theorem 8.13.** *Let $a_1,a_2,\dots,a_n\in\mathbf{A}$ and let $K_\pi^{\mathrm{FL}}$ be the cumulants associated to the spreadability system $\mathcal{S}_{\mathrm{FL}}$. Then*

$$\Pi(a_1a_2\cdots a_n)=K_n^{\mathrm{FL}}(a_1,a_2,\dots,a_n).$$

**Remark 8.14.** 1. The set of *Lie polynomials* $\mathcal{L}(\mathbf{A})$ is the smallest subspace of $\mathbb{C}[\mathbf{A}]$ that contains $\mathbf{A}$ and is closed with respect to the Lie bracket $[X,Y]=XY-YX$. It is well known that $\Pi$ is a Lie projector, i.e., $\mathcal{L}(\mathbf{A})=\Pi(\mathbb{C}[\mathbf{A}])$ [Reu93, Theorem 3.7]. So in the above setting cumulants with entries from $\mathbf{A}$ are exactly Lie polynomials.

2. The values at words of higher order are given by convolution powers of $\Pi$, i.e.,

$$\Pi_k=\frac{1}{k!}\Pi^{*k},\qquad k\in\mathbb{N}, \tag{8.16}$$

(see [Reu93]), and we can show that

$$\Pi_k(a_1a_2\cdots a_n)=\frac{1}{k!}\sum_{\substack{\pi\in\mathcal{OP}_n\\|\pi|=k}}K_\pi^{\mathrm{FL}}(a_1,a_2,\dots,a_n),\qquad a_i\in\mathbf{A},\ i\in[n]. \tag{8.17}$$

is the coefficient of $N^k$ appearing in $\widetilde{\varphi}((N.a_1)(N.a_2)\cdots(N.a_n))$ by Theorem 4.8 and Theorem 4.15. In other words,

$$\widetilde{\varphi}((N.a_1)(N.a_2)\cdots(N.a_n))=\sum_{k=1}^{n}N^k\Pi_k(a_1a_2\cdots a_n),\qquad a_i\in\mathbf{A},\ i\in[n]. \tag{8.18}$$

Now the combination of Theorem 8.6 and Theorem 8.13 provides yet another proof of the CBH formula

$$\log(e^{a_1}e^{a_2}\cdots e^{a_n})=\sum_{\substack{(p_1,p_2,\dots,p_n)\in(\mathbb{N}\cup\{0\})^n,\\(p_1,p_2,\dots,p_n)\ne(0,0,\dots,0)}}\Pi\left(\frac{a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n}}{p_1!p_2!\cdots p_n!}\right); \tag{8.19}$$

see [Reu93, Lemma 3.10].

8.3. **Coefficients of the Campbell-Baker-Hausdorff formula.** We adopt the notations and definitions in the previous subsection. The coefficients of the Campbell-Baker-Hausdorff formula, when written out in the monomial basis, were first computed using generating functions in [Gol56] and are called *Goldberg coefficients*; a combinatorial proof can be found in [Reu93, Theorem 3.11]. In the following we give another derivation of the Goldberg coefficients; see also a recent proof using the theory of noncommutative symmetric functions [FPT16].

**Lemma 8.15.** *Let $\eta=(E_1,E_2,\dots,E_e)$ be an interval partition of $[n]$ where the blocks are in canonical order, i.e. if $s<t$ then $i<j$ for all $i\in E_s, j\in E_t$. Then for any $X_i\in\mathbb{C}[\mathbf{A}]$ and any $\pi\in\mathcal{OP}_n$, we have*

$$\varphi_{\pi\wedge\eta}(X_1,X_2,\dots,X_n)=\varphi_\pi(X_1,X_2,\dots,X_n).$$

*Proof.* The statement holds by definition because every block $P\in\pi$ is the concatenation of $P\cap E_1,P\cap E_2,\dots,P\cap E_e$. ☐

**Remark 8.16.** Thus a sequence of distinct letters satisfies some partial independence, but it is not $\mathcal{S}_{\mathrm{FL}}$-independent. Indeed, $(a_1, a_2) \in \mathbf{A} \times \mathbf{A}$ is $\mathcal{S}_{\mathrm{FL}}$-independent if and only if $a_1 = a_2$ by Proposition 8.8.

**Theorem 8.17** ([Gol56, Reu93]). *Let $a_1, a_2, \dots, a_n$ be distinct letters from the alphabet $\mathbf{A}$, let $r \in \mathbb{N}$ and $q_j, i_j \in \mathbb{N}$ for $j \in [m]$ such that $i_j \neq i_{j+1}$ for $j \in [m-1]$. Then the coefficient of the monomial $a_{i_1}^{q_1} a_{i_2}^{q_2} \cdots a_{i_m}^{q_m}$ appearing in $\log(e^{a_1} \cdots e^{a_n})$ is given by*

$$\frac{1}{q_1! q_2! \cdots q_m!} \int_{-1}^{0} x^{\operatorname{des}(\underline{i})} (1+x)^{\operatorname{asc}(\underline{i})} \prod_{j=1}^{m} P_{q_j}(x)\, dx, \tag{8.20}$$

*where $P_q(x)$ are the homogeneous Euler polynomials already encountered in Proposition 7.16.*

*Proof.* In order for a monomial $v = a_{i_1}^{q_1} a_{i_2}^{q_2} \cdots a_{i_m}^{q_r}$ to occur as a term in $\Pi(a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n})$ it necessarily has to be a rearrangement of the word (= permutation of the multiset) $w = a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}$, since the projector in the CBH formula (8.19) does not change multiplicities and therefore every letter must occur the same number of times in $v$ and $w$.

Thus $p_k = \sum_{j \in B_k} q_j$ where $B_k = \{j : i_j = k\}$. Let $p = p_1 + p_2 + \dots + p_n$ be the total length of $w$ and $\eta = (A_1, A_2, \dots, A_n) \in \mathcal{OI}_p$ be the ordered interval partition corresponding to the composition $(p_1, p_2, \dots, p_n)$, i.e., $A_j = \{p_1 + p_2 + \dots + p_{j-1} + 1, p_1 + p_2 + \dots + p_{j-1} + 2, \dots, p_1 + p_2 + \dots + p_j\}$ and $|A_j| = p_j$.
From Theorem 8.13 we infer

$$\Pi(a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n}) = K_p^{\mathrm{FL}}(\underbrace{a_1, a_1, \dots, a_1}_{p_1 \text{ times}}, \underbrace{a_2, a_2, \dots, a_2}_{p_2 \text{ times}}, \dots, \underbrace{a_n, a_n, \dots, a_n}_{p_n \text{ times}}) =: K_p^{\mathrm{FL}}(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n}).$$

Note that by Lemma 8.15 we have $\varphi_\pi(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n}) = \varphi_{\pi \wedge \eta}(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n})$ for any $\pi \in \mathcal{OP}_p$. Thus

$$\begin{aligned} K_p^{\mathrm{FL}}(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n}) &= \sum_{\sigma \in \mathcal{OP}_p} \varphi_\sigma(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n})\, \widetilde{\mu}(\sigma, \hat{1}_p) \\ &= \sum_{\sigma \in \mathcal{OP}_p} \varphi_{\sigma \wedge \eta}(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n})\, \widetilde{\mu}(\sigma, \hat{1}_p) \\ &= \sum_{\substack{\tau \in \mathcal{OP}_p \\ \bar{\tau} \leq \bar{\eta}}} \varphi_\tau(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n})\, w(\tau, \eta) \end{aligned} \tag{8.21}$$

by Proposition 7.13. Thus in order to determine the coefficient of $v$ we must collect all ordered set partitions $\tau$ such that

$$T_v = \{\tau \in \mathcal{OP}_p : \bar{\tau} \leq \bar{\eta}, \varphi_\tau(a_1^{\cdots p_1}, a_2^{\cdots p_2}, \dots, a_n^{\cdots p_n}) = v\} \tag{8.22}$$

and then sum up the corresponding values of $w(\tau, \eta)$. Let us now investigate the structure of this set. First note that $T_v$ is an order ideal: if $\tau \in T_v$ and $\tau' \leq \tau$, then $\tau' \in T_v$, because every block of $\tau$ contains only repetitions of one letter and further refinement of $\tau$ leaves the end result $v$ invariant. Moreover $T_v$ is the disjoint union of the principal ideals $\downarrow\tau_0 = \{\tau : \tau \leq \tau_0\}$ where $\tau_0 \in T_v$ is maximal.

The maximal partition $\tau_0$ arises as follows: After application of $\varphi_\tau$ for $\tau \in T_v$ each factor $a_j^{p_j}$ is divided into pieces $a_j^{q_l}$, $l \in B_j$ and the number of such subdivisions is the multinomial coefficient

$$\binom{p_j}{q_l : l \in B_j}.$$

These subdivisions are in one-to-one correspondence with ordered set partitions of $A_j$ which, pieced together in the order of $j$, give rise to a maximal ordered set partition $\tau_0 \in T_v$. Thus in total there are as many maximal ordered set partitions as there are subdivisions, namely

$$\binom{p_1}{q_l : l \in B_1} \binom{p_2}{q_l : l \in B_2} \cdots \binom{p_n}{q_l : l \in B_n} = \frac{p_1! p_2! \cdots p_n!}{q_1! q_2! \cdots q_m!}. \tag{8.23}$$

By Proposition 2.13 the principal ideal $\downarrow\tau_0$ generated by a maximal ordered set partition $\tau_0$ is isomorphic to

$$\mathcal{OP}_{q_1} \times \mathcal{OP}_{q_2} \times \cdots \mathcal{OP}_{q_m}$$

and in particular, all principal ideals are isomorphic. Moreover the number of ascents does not depend on the choice of $\tau \in T_v$, indeed it is equal to $\operatorname{asc}_\eta(\tau) = \operatorname{asc}(\underline{i})$. Thus all ideals $\downarrow\tau_0$ deliver the same contribution and as a consequence of the discussion above we are left with

$$\sum_{\tau\in T_v} w(\tau,\eta) = \frac{p_1!p_2!\cdots p_n!}{q_1!q_2!\cdots q_m!} \sum_{\tau\le\tau_0} w(\tau,\eta)$$

for one fixed maximal element $\tau_0 \in T_v$.

A canonical representative $\tau_0$ is obtained by concatenating consecutive subintervals of length $q_j$ from $A_{i_j}$, $j = 1,2,\dots,m$. Let us now turn to the value of $w(\tau,\eta)$. As seen above, the numbers of both ascents and descents of $\tau$ only depend on those of the sequence $\underline{i}$ and since $\underline{i}$ has no plateaux we infer from (7.2) that $m = \operatorname{des}(\underline{i}) + \operatorname{asc}(\underline{i}) + 1$. On the other hand, if we denote by $(\tau_1,\tau_2,\dots,\tau_m)$ the image of $\tau \in \downarrow\tau_0$ under the isomorphism of Proposition 2.13 then the first exponent in formula (7.3) of Proposition 7.13 becomes

$$|\tau| - \operatorname{asc}_\eta(\tau) - 1 = \sum_{j=1}^m |\tau_j| - \operatorname{asc}(\underline{i}) - 1 = \sum_{j=1}^m (|\tau_j| - 1) + \operatorname{des}(\underline{i})$$

and thus

$$w(\tau,\eta) = \int_{-1}^0 x^{\operatorname{des}(\underline{i})}(1+x)^{\operatorname{asc}(\underline{i})} \prod_{j=1}^m x^{|\tau_j|-1}\,dx$$

and summing over the cartesian product yields the total value

$$\sum_{\tau\in T_v} w(\tau,\eta) = \frac{p_1!p_2!\cdots p_n!}{q_1!q_2!\cdots q_m!} \int_{-1}^0 x^{\operatorname{des}(\underline{i})}(1+x)^{\operatorname{asc}(\underline{i})} \prod_{j=1}^m \left( \sum_{\rho\in\mathcal{OP}_{q_j}} x^{|\rho|-1} \right) dx.$$

Finally note that

$$\sum_{\rho\in\mathcal{OP}_q} x^{|\rho|-1} = \sum_{\sigma\in\mathcal{P}_q} |\sigma|!\, x^{|\sigma|-1} = \sum_{k=1}^q k!\, S(q,k) x^{k-1}. \tag{8.24}$$

is indeed the homogeneous Euler polynomial as claimed, see Remark 7.18. □

**Remark 8.18.** The authors were not able to prove Theorem 8.17 as a corollary of Proposition 7.19, although Goldberg coefficients appear in both formulas.

## 9. Open Problems

(a) Find a proof of Theorem 8.17 based on Proposition 7.19, or vice versa.
(b) Is there an application of spreadability systems to noncommutative quasi-symmetric functions (see [BZ09]) or vice versa?

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