Discrete Applied Mathematics 331 (2023) 88–103

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On k-bend and monotonic ℓ-bend edge intersection graphs of paths on a grid

Eranda Çela [a,*], Elisabeth Gaar [b,1]

[a] TU Graz, Steyrergasse 30, Graz A-8010, Austria
[b] Johannes Kepler University Linz, Altenberger Strasse 69, Linz A-4040, Austria

ARTICLE INFO

Article history:
Received 21 September 2021
Received in revised form 30 December 2022
Accepted 12 January 2023
Available online 1 February 2023

Keywords:
Paths on a grid
EPG graph
(Monotonic) bend number
Complete bipartite graph

ABSTRACT

If a graph G can be represented by means of paths on a grid, such that each vertex of G corresponds to one path on the grid and two vertices of G are adjacent if and only if the corresponding paths share a grid edge, then this graph is called EPG and the representation is called EPG representation. A k-bend EPG representation is an EPG representation in which each path has at most k bends. The class of all graphs that have a k-bend EPG representation is denoted by B_k. B_ℓ^m is the class of all graphs that have a monotonic ℓ-bend EPG representation, i.e. an ℓ-bend EPG representation, where each path is ascending in both columns and rows.

It is trivial that $B_k^m \subseteq B_k$ for all k. Moreover, it is known that $B_k^m \subsetneq B_k$, for $k = 1$. By investigating the B_k-membership and the B_k^m-membership of complete bipartite graphs we prove that the inclusion is also proper for $k \in \{2, 3, 5\}$ and for $k \geqslant 7$. In particular, we derive necessary conditions for this membership that have to be fulfilled by m, n and k, where m and n are the number of vertices on the two partition classes of the bipartite graph. We conjecture that $B_k^m \subsetneq B_k$ holds also for $k \in \{4, 6\}$.

Furthermore, we show that $B_k \not\subseteq B_{2k-9}^m$ holds for all $k \geqslant 5$. This implies that restricting the shape of the paths can lead to a significant increase of the number of bends needed in an EPG representation. So far no bounds on the amount of that increase were known. We prove that $B_1 \subseteq B_3^m$ holds, providing the first result of this kind.

1. Introduction and definitions

In 2009 Golumbic, Lipshteyn and Stern [16] introduced edge intersection graphs of paths on a grid. If a graph G can be represented by means of paths on a grid, such that each vertex of G corresponds to one path on the grid and two vertices of G are adjacent if and only if the corresponding paths share a grid edge, then this graph is called *edge intersection graph of paths on a grid (EPG)* and the representation is called *EPG representation*. Here the term *edge intersection of paths* refers to the fact that the paths share a grid edge.

A *k-bend EPG representation* or *B_k-EPG representation* is an EPG representation in which each path has at most k bends. A graph that has a B_k-EPG representation is called *B_k-EPG* and the class of all B_k-EPG graphs is denoted by B_k. We consider the following natural ordering of grid lines: the columns increase from the left to the right and the rows increase from

* Corresponding author.
E-mail addresses: cela@math.tugraz.at (E. Çela), elisabeth.gaar@jku.at (E. Gaar).

[1] The second author acknowledges support by the Austrian Science Fund (FWF): I 3199-N31. We are very thankful to the anonymous reviewers for their careful reading and their valuable suggestions.

https://doi.org/10.1016/j.dam.2023.01.010

the bottom to the top. A path on a grid is called *monotonic*, if it is ascending in both columns and rows, i.e. it has the shape of a staircase that is going upwards from the left to the right. The graphs that have a B_ℓ-EPG representation in which each path is monotonic are called B_ℓ^m-*EPG* and the class of all these graphs is denoted by B_ℓ^m. The bend number $b(G)$ of a graph G is the minimum k such that G is B_k-EPG. The monotonic bend number $b^m(G)$ of graph G is defined as the minimum ℓ such that G is B_ℓ^m-EPG. Note that already Golumbic, Lipshteyn and Stern [16] showed that each graph is B_k-EPG and B_ℓ^m-EPG for some k and ℓ.

As described in [16] the motivation for investigating EPG graphs was initially related to applications from circuit layout setting and chip manufacturing. In the knock-knee circuit layout model the wires can be seen as paths on a grid which can cross and bend at a grid point but are not allowed to share a grid edge, see [8,19]. The wires can be put in multiple layers each of them being a grid and such that the wires of each layer do not share a grid edge. In this setting the minimum number of layers needed to accommodate all wires would be equal to the chromatic number of the corresponding graph. Consider now that a so-called transition hole is needed, whenever a wire bends. If a large number of transition holes is included, the layout area and consequently, the cost of the chip, may increase. Therefore, it might be desirable to find a circuit layout setting which minimizes the largest number of bends used in each wire. In our notation this corresponds to finding the minimum k such that the corresponding graph is in B_k.

Similar graph classes known in the literature include *edge intersection graphs of paths on a tree* (EPT) (see [15]), *vertex intersection graph of paths on a tree (VPT)* (see [14]) and *vertex intersection graphs of paths on a grid* (*VPG*) (see [1]). In this paper we will only deal with EPG graphs.

There has been a lot of research on EPG graphs since their introduction. One of the topics of interest is the recognition problem of B_k-EPG graphs, i.e. to determine for a given k and a given graph whether this graph is in B_k (B_k^m). Currently it is known that the recognition problem is NP-hard for B_1 (Heldt, Knauer and Ueckerdt [18]), B_1^m (Cameron, Chaplick and Hoàng [10]), B_2 and B_2^m (Pergel and Rzążewski [20]).

Recently a number of results on combinatorial optimization problems on specific B_k-EPG graphs have been published. Subject of investigation are certain NP-hard combinatorial optimization problems which turn out to be tractable, i.e. polynomially solvable or approximable within a guaranteed approximation ratio, for B_k-EPG graphs, see [5–7,12]. Thus, the computation of the bend number and the monotonic bend number of graphs or related upper bounds is a relevant research question in this context. However, this appears to be a challenging task, considering that even the recognition of B_k (B_k^m) graphs is NP-hard for $k = 1$ and $k = 2$, as mentioned above.

A related and more viable line of research is the determination of (upper bounds on) the (monotonic) bend number of special graph classes. Among the first graph class for which an upper bound on the bend number was given were planar graphs. The first upper bound was 5 and it was obtained in 2009 by Biedl and Stern [4]. In 2012 Heldt, Knauer and Ueckerdt [17] improved the bound to 4 and also showed that 2 is an upper bound on the bend number of outerplanar graphs. Çela and Gaar [9] showed recently that 2 is also an upper bound on the monotonic bend number of outerplanar graphs. Moreover, they give a full characterization of any maximal outerplanar graph and any cactus[2] with (monotonic) bend number equal to 0, 1 and 2 in terms of forbidden induced subgraphs.

Also other graph classes were considered. Recently Francis and Lahiri [13] proved that Halin graphs are in B_2^m and Deniz, Nivelle, Ries and Schindl [11] provided a characterization of split graphs for which there exists a B_1-EPG representation which uses only L-shaped paths on the grid, i.e. paths consisting of a vertical top-bottom segment followed by a horizontal left–right segment.

Another line of research on EPG graphs concerns the mutual relationship between the classes B_k and the classes B_ℓ^m. Our paper is a contribution in this direction. The chains of inclusions $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ and $B_0^m \subseteq B_1^m \subseteq B_2^m \subseteq \dots$ trivially hold. Furthermore, $B_0 = B_0^m \subseteq B_1^m$ and $B_k^m \subseteq B_k$, for every k, are obvious. In [18] Heldt, Knauer and Ueckerdt dealt with the question whether the complete bipartite graph $K_{m,n}$ on m and n vertices in the two partition classes is in B_k. They identified several sufficient conditions which have to be fulfilled by m, n and k to guarantee that $K_{m,n}$ is in B_k or $K_{m,n}$ is not in B_k. They used this kind of results to prove that $B_k \subsetneqq B_{k+1}$ holds for every $k \geqslant 0$. In this paper, we will derive new results of this type, especially for the monotonic case. It is still not known whether $B_k^m \subsetneqq B_{k+1}^m$ also holds.

The relationship between B_k and B_k^m has already been considered in the literature. Golumbic, Lipshteyn and Stern [16] conjectured that $B_1^m \subsetneqq B_1$, which was confirmed in [10]. In this paper, we show that $B_k^m \subsetneqq B_k$ also holds for $k \in \{2, 3, 5\}$ and $k \geqslant 7$, while the cases $k = 4$ and $k = 6$ remain open.

Furthermore, we are interested in the gap between the bend number $b(G)$ and the monotonic bend number $b^m(G)$ of a graph. More precisely we pose the question whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $b^m(G) \leqslant f(b(G))$ holds for every graph G. As a first step towards answering this question we show that $B_k \not\subseteq B_{2k-9}^m$ holds for any $k \in \mathbb{N}$, $k \geqslant 5$, which implies the existence of graphs for which $b^m(G) \geqslant 2k - 8$ and $b(G) \leqslant k$, for any $k \in \mathbb{N}$, $k \geqslant 5$. Moreover, we show that $b(G) \leqslant 1$ implies $b^m(G) \leqslant 3$.

The rest of the paper is organized as follows. Section 2 deals with the (monotonic) bend number of $K_{m,n}$. First we review some results from the literature on the bend number of $K_{m,n}$, where $m \leqslant n$. In particular, we discuss a theorem from [18] and point out that the proof of the theorem does not work out for $m = 4$ and $m = 5$. Further, we show that the statement of the theorem holds for $m = 4$, while we do not know whether it holds for $m = 5$. However, we only exploit the statement of the theorem for $m \geqslant 7$ in our later work. In Section 2.2, we derive two inequalities on m, n and k which

2 A connected graph is called a cactus iff any two simple cycles in it share at most one vertex.

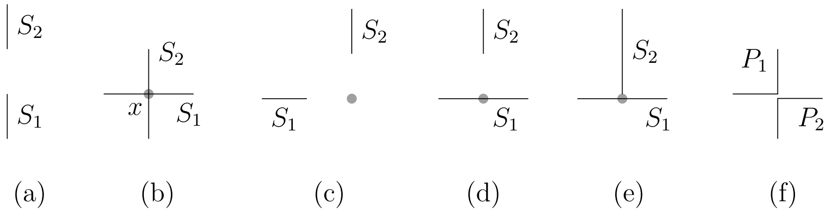


Fig. 1. (a) An alignment (S_1, S_2). (b) A crossing (S_1, S_2). (c)–(e) Different pseudocrossings (S_1, S_2). (f) Two paths P_1 and P_2 containing two alignments and two pseudocrossings.

have to be fulfilled if $K_{m,n}$ is in B_k^m. In Section 2.3 we show that $b^m(K_{m,n}) \leqslant 2m-2$ for every $m, n \in \mathbb{N}$, $m \leqslant n$, which implies that $b^m(G) \leqslant 2m-2$ holds for every graph G that is an induced subgraph of $K_{m,n}$. Moreover, we show that this upper bound on $b^m(K_{m,n})$ is best possible, i.e. for each $m \in \mathbb{N}$ there exists an $n_m \in \mathbb{N}$, $n_m \geqslant m$, such that $b^m(K_{m,n_m}) = 2m-2$. An analogous behavior of $b(K_{m,n})$ has been already shown in literature (see [18]). However, we will see that this maximum bend number is attained already for smaller values of n_m in the monotonic case.

In Section 3.1, we present a graph which is in B_2 and not in B_2^m in order to prove $B_k^m \subsetneq B_k$ for $k = 2$. In Section 3.2, we use the results of Section 2.2 to prove that $B_k^m \subsetneq B_k$ also for $k \in \{3, 5\}$ and $k \geqslant 7$, thus answering an open question posed in [16] for almost all values of k.

Finally, in Section 4 we investigate the relationship between B_k and B_ℓ^m for $\ell > k$. In Section 4.1 we show that for odd $k \geqslant 5$ there is a graph in B_k which is not in B_{2k-8}^m and for even $k \geqslant 5$ there is a graph in B_k which is not in B_{2k-9}^m. Then in Section 4.2 we prove that $B_1 \subseteq B_3^m$, giving the first result of this kind. We summarize our results and discuss some open questions in Section 5.

Terminology and notation. Finally, we settle the terminology and the notations used throughout the paper. The crossings of two *grid lines* are called *grid points*. The part of a grid line between two consecutive grid points is called a *grid edge*. A grid edge can be *horizontal* or *vertical*.

A *path* on a grid consists of two grid points, called the *end points* of the path, and a number of consecutive grid edges connecting the end points. If the two end-points lay on different vertical grid lines, we call the left-most point the *start point* and the other one the *terminal point*. Otherwise, we call the lower point the *start point* and the other one the *terminal point*. A turn of a path on the grid is called *bend* and a grid point, at which the path turns, is called a *bend point*.

The part of a path between two consecutive bend points is called a *segment*. Also the part of the path from the start point to the first bend point is called a *segment*. This is called the *first segment* of the path. Analogously, the part of the path from the last bend point to the terminal point is also called a *segment*. This is the *last segment* of the path. We consider the intermediate segments in their natural order: the segment of the path following the first one is the *second segment*, and so on.

The grid points contained in a segment of a path which are neither bend points nor end points of that path build the *interior* of that segment. Clearly any segment consists either entirely of horizontal grid edges or entirely of vertical grid edges. We call such segments *horizontal* and *vertical segments*, respectively. Paths without bends correspond to (horizontal or vertical) segments.

We say that two paths on a grid *intersect*, if they have at least one common grid edge. If two segments S_1, S_2 lie on the same grid line but do not intersect (if considered as paths), then we call them *aligned*; such a pair (S_1, S_2) is called an *alignment*. Fig. 1(a) depicts two aligned segments S_1 and S_2.

A pair (S_1, S_2) of segments is called a *crossing* if one of the two segments lies on a horizontal grid line, the other segment lies on a vertical grid line, and there is a grid point which belongs to the interior of both segments. Fig. 1(b) depicts a crossing (S_1, S_2) with grid point x belonging to the interior of both segments.

A pair (S_1, S_2) of segments is called a *pseudocrossing* if one of the two segments lies on a horizontal grid line, the other segment lies on a vertical grid line, and there is no grid point which belongs to the interior of each of the segments. Fig. 1(c)–(e) depict different pseudocrossings.

Given a set $\mathcal{P}$ of pairwise non-intersecting paths on a grid we define the alignments (crossings, pseudocrossings) of $\mathcal{P}$ as the set of all alignments (crossings, pseudocrossings) (S_1, S_2) for which there exist two distinct paths $P_1, P_2 \in \mathcal{P}$ such that S_i is a segment of P_i, for $i \in \{1, 2\}$. Fig. 1(f) depicts two paths P_1 and P_2 containing two alignments (a horizontal one and a vertical one) and two pseudocrossings.

Finally, notice that in an EPG representation of a graph G with vertex set V we will denote by P_v the path on the grid corresponding to the vertex $v \in V$.

2. Complete bipartite graphs

The aim of this section is to summarize existing results on the B_k-EPG representation of complete bipartite graphs and derive new upper and lower bounds on their (monotonic) bend number. We start by investigating some results from the literature in Section 2.1. Then we derive two Lower-Bound-Lemmas in Section 2.2. Eventually, in Section 2.3, we give an upper bound on the monotonic bend number of $K_{m,n}$ for every $m, n \in \mathbb{N}$, $m \leqslant n$. The results obtained in this section will be used in Section 3.2, where the relationship between B_k^m and B_k for $k \geqslant 3$ is investigated.

Throughout this section we consider the complete bipartite graph $K_{m,n}$ with $m \leqslant n$. We denote the two partition classes of $K_{m,n}$ by A and B, where $|A| = m$ and $|B| = n$. In an EPG representation we denote the set of all paths that correspond to vertices of A and B by $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively; so $\mathcal{P}_A = \{P_v : v \in A\}$ and $\mathcal{P}_B = \{P_w : w \in B\}$.

2.1. Upper bounds on the bend number

First of all notice that the bend number of $K_{m,n}$ for $m \in \{0, 1, 2\}$ is known. The trivial case $m = 0$ corresponds to a graph without any edges and hence $b(K_{0,n}) = b^m(K_{0,n}) = 0$, for all $n \in \mathbb{N}$.

The other trivial case $m = 1$ corresponds to a star graph with $n + 1$ vertices. A B_0-EPG representation of this graph consists of a horizontal path P with n grid edges to represent the central vertex, and the pairwise different grid edges of P represent the other vertices. Thus $b(K_{1,n}) = b^m(K_{1,n}) = 0$, for all $n \in \mathbb{N}$.

The bend number of $K_{2,n}$ has been determined by Asinowski and Suk [2] for all $n \in \mathbb{N}$: $b(K_{2,n}) = 2$ if and only if $n \geqslant 5$, $b(K_{2,n}) = 1$ if and only if $2 \leqslant n \leqslant 4$, and $b(K_{2,n}) = 0$ if and only if $n \leqslant 1$. The EPG representations for $K_{2,n}$ given in [2] are monotonic, therefore $b^m(K_{2,n}) = b(K_{2,n})$ holds for all $n \in \mathbb{N}$.

The more general case $m \geqslant 3$ has been considered by Heldt, Knauer and Ueckerdt in [18]. We first discuss the following result of these authors.

Theorem 2.1 (*Heldt, Knauer, Ueckerdt [18]*). *If $m \geqslant 4$ is even and $n = \frac{1}{4}m^3 - \frac{1}{2}m^2 - m + 4$, then $K_{m,n}$ is in B_{m-1} but not in B_{m-2}. If $m \geqslant 7$ is odd and $n = \frac{1}{4}m^3 - m^2 + \frac{3}{4}m$, then $K_{m,n}$ is in B_{m-1} but not in B_{m-2}.*

The above theorem makes no statement for the cases $m = 3$ and $m = 5$. However, in [18] the authors claim that the statement for odd m holds also for $m = 5$ (see [18, Theorem 4.4.]). But the proof provided in [18] is not correct for $m = 5$ and we do not know whether the statement is true in this case. Also for the case $m = 4$ the proof provided in [18] is not correct, however in this case the statement is true as argued below.

To be more precise, in [18] on the one hand the authors provide a B_{m-1}-EPG representation for $K_{m,n}$ for $m \geqslant 3$ and n defined as in Theorem 2.1, i.e. a constructive proof for one part of [18, Theorem 4.4.]. On the other hand the Lower-Bound-Lemma I [18, Lemma 4.1] is used in order to show that $K_{m,n}$ is not in B_{m-2} for n defined as in Theorem 2.1. This Lower-Bound-Lemma I states that

$$(k + 1)(m + n) \geqslant mn + \sqrt{2k(m + n)}$$

holds for every B_k-EPG representation of $K_{m,n}$ with $n \geqslant m \geqslant 3$. Further they observe that for n defined as in Theorem 2.1 the inequality $n \geqslant (m - 1)^2$ holds, while the inequality of the Lower-Bound-Lemma I is not fulfilled for $n \geqslant (m - 1)^2$ and $k = m - 2$, thus negating the membership of the corresponding graphs in B_{m-2}. However, for n defined as in Theorem 2.1, the inequality $n \geqslant (m - 1)^2$ holds only if $m \geqslant 6$. Thus, the proof provided for [18, Theorem 4.4] only works for $m \geqslant 6$.

For $m = 4$ we have $n = 8$, and the construction in [18] proves that $K_{4,8}$ is in B_3. Furthermore, by applying the Lower-Bound-Lemma I for $m = 4$, $n = 6$ and $k = 2$ we get that $K_{4,6}$ is not in B_2. This implies that also $K_{4,8}$ is not in B_2. Therefore, the statement of Theorem 2.1 is also true for $m = 4$.

If $m = 5$ the construction in [18] yields that $K_{5,10}$ is in B_4. If we use the Lower-Bound-Lemma I, then we get that $K_{5,11}$ is not in B_3 and that the bend number of $K_{5,10}$ is at least 3. Therefore, the bend number of $K_{5,10}$ could be either 3 or 4.

2.2. Lower-Bound-Lemmas

In order to investigate the relationship between B_k^m and B_k for large values of k, we first derive a Lower-Bound-Lemma for B_k^m-EPG representations similarly to the Lower-Bound-Lemma I for B_k-EPG representations from [18]. To this end, we use an auxiliary result from [18, Lemma 4.6].

Lemma 2.2 (*Heldt, Knauer, Ueckerdt [18]*). *Let $3 \leqslant m \leqslant n$. Consider $K_{m,n}$ and denote by A the subset of vertices of cardinality m in the partition of the vertex set of $K_{m,n}$. Consider further a B_k-EPG representation of $K_{m,n}$ and denote by $\mathcal{P}_A$ be the set of the paths on the grid corresponding to the vertices of A in this representation. Let c be the total number of crossings of $\mathcal{P}_A$. Then, the following inequality holds:*

$$n(2m - k - 2) \leqslant 2c + 2(k + 1)m.$$

In the following we derive inequalities on m, n and k which hold whenever a $K_{m,n}$ is in B_k^m. The following lemma is a first step towards such a result. Note that $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $\lceil x \rceil$ is the least integer greater than or equal to x for any real number x.

Lemma 2.3. *Let $3 \leqslant m \leqslant n$. Consider $K_{m,n}$ and denote by A the subset of vertices of cardinality m in the partition of the vertex set of $K_{m,n}$. Consider further a B_k-EPG representation of $K_{m,n}$ and denote by $\mathcal{P}_A$ be the set of the paths on the grid corresponding*

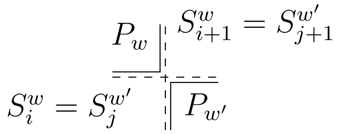


Fig. 2. The crossings (S_i^w, S_{i+1}^w) and $(S_j^{w'}, S_{j+1}^{w'})$ coincide.

to the vertices of A in this representation. Let a, c and p be the total number of alignments, crossing and pseudocrossings of $\mathcal{P}_A$, respectively. Then, the following inequality holds:

$$n\left(m - \left\lceil \frac{k+1}{2} \right\rceil\right) \leqslant a + 2c + p.$$

Proof. Let w be a vertex of B. For each vertex $v \in A$ we denote by e_v^w a fixed but arbitrarily chosen common grid edge of P_v and P_w. Such an edge exists, because P_w intersects P_v since w is adjacent to all vertices of A. The grid edges e_v^w for all $v \in A$ are pairwise disjoint, because the vertices of A are not adjacent to each other.

We order the vertices $A = \{v_1, \dots, v_m\}$ in such a way that $e_{v_i}^w$ precedes $e_{v_{i+1}}^w$ in the path P_w, for all $i \in \{1, 2, \dots, m-1\}$. Let x_w, y_w and z_w be the number of indices $i \in \{1, \dots, m-1\}$ such that $e_{v_i}^w$ and $e_{v_{i+1}}^w$ lie on the same segment of P_w, on consecutive segments of P_w, and neither on the same nor on consecutive segments of P_w, respectively. Then, clearly $x_w + y_w + z_w = m - 1$ holds.

If $e_{v_i}^w$ and $e_{v_{i+1}}^w$ lie neither on the same nor on consecutive segments of P_w, then the subpath of P_w between (and not including) the two segments of P_w containing $e_{v_i}^w$ and $e_{v_{i+1}}^w$ contains at least one segment and does not contain any $e_{v_{i'}}^w$ for $i' \in \{1, \dots, m\}$. Let us call such a subpath a *free subpath* of P_w. Since P_w has at most $k+1$ segments and each free subpath is preceded and also succeeded by a segment containing $e_{v_{i'}}^w$ for some $i' \in \{1, \dots, m\}$, the number of free subpaths is at most $\left\lfloor \frac{k}{2} \right\rfloor$ and hence $z_w \leqslant \left\lfloor \frac{k}{2} \right\rfloor$ holds. To summarize up to now we have shown that

$$m - \left\lceil \frac{k+1}{2} \right\rceil = m - 1 - \left\lfloor \frac{k}{2} \right\rfloor \leqslant m - 1 - z_w = x_w + y_w \tag{1}$$

holds.

It remains to determine an upper bound on $x_w + y_w$. Towards this end, let S_i^w be the segment of P_{v_i} that contains $e_{v_i}^w$ for $i \in \{1, 2 \dots, m\}$. Now we consider the pairs (S_i^w, S_{i+1}^w), $i \in \{1, 2, \dots, m-1\}$.

We denote by a_w the number of indices $i \in \{1, \dots, m-1\}$ such that $e_{v_i}^w$ and $e_{v_{i+1}}^w$ lie on the same segment of P_w and the pair (S_i^w, S_{i+1}^w) is an alignment. It is easy to see that if $e_{v_i}^w$ and $e_{v_{i+1}}^w$ lie on the same segment of P_w, then the corresponding segments S_i^w and S_{i+1}^w of P_{v_i} and $P_{v_{i+1}}$ lie on the same grid line and therefore (S_i^w, S_{i+1}^w) is an alignment. Thus, $a_w = x_w$ holds.

Furthermore, let c_w (p_w) denote the number of $i \in \{1, \dots, m-1\}$ such that $e_{v_i}^w$ and $e_{v_{i+1}}^w$ lie on consecutive segments of P_w and the pair (S_i^w, S_{i+1}^w) is a crossing (pseudocrossing). If $e_{v_i}^w$ and $e_{v_{i+1}}^w$ lie on consecutive segments of P_w, then one of the corresponding segments S_i^w and S_{i+1}^w is horizontal and the other one is vertical. Hence (S_i^w, S_{i+1}^w) is either a crossing or a pseudocrossing. Therefore, $c_w + p_w = y_w$ holds.

As a result, we can use (1) to deduce that

$$m - \left\lceil \frac{k+1}{2} \right\rceil \leqslant x_w + y_w = a_w + c_w + p_w$$

holds. Summing this up over all vertices $w \in B$ yields

$$n\left(m - \left\lceil \frac{k+1}{2} \right\rceil\right) \leqslant \sum_{w \in B} (a_w + c_w + p_w).$$

It remains to determine an upper bound on $\sum_{w \in B}(a_w + c_w + p_w)$. Towards this end, let $a_B = \sum_{w \in B} a_w$, $c_B = \sum_{w \in B} c_w$ and $p_B = \sum_{w \in B} p_w$. Clearly, an alignment (crossing, pseudocrossing) (S_i^w, S_{i+1}^w), for $w \in B$ and for $i \in \{1, 2, \dots, m-1\}$, is an alignment (crossing, pseudocrossing) of $\mathcal{P}_A$, since S_i^w is a segment of P_{v_i}, for $i \in \{1, 2, \dots, m\}$. This implies that $a_B \leqslant a$ and $p_B \leqslant p$ because the alignments and pseudocrossings counted in a_B and p_B are pairwise distinct due to the fact that the paths in $\mathcal{P}_A$ are pairwise non-intersecting and also the paths in $\mathcal{P}_B$ are pairwise non-intersecting.

The crossings counted in c_B are not necessarily pairwise distinct because a crossing (S_i^w, S_{i+1}^w) can also appear as a crossing $(S_j^{w'}, S_{j+1}^{w'})$, for some $w, w' \in B$, $w \neq w'$ and some $i, j \in \{1, 2, \dots, m-1\}$, see Fig. 2. (Notice that in this case the vertices v_i and v_j coincide.) However, the same crossing cannot be counted more than twice in c_B because the paths in $\mathcal{P}_B$ are pairwise non-intersecting, so $c_B \leqslant 2c$ holds.

Finally, we can deduce

$$n\left(m-\left\lceil\frac{k+1}{2}\right\rceil\right) \leqslant \sum_{w\in B}(a_w+c_w+p_w) = a_B+c_B+p_B \leqslant a+2c+p. \quad \square$$

The next lemma gives bounds on the number of alignments, crossings and pseudocrossings.

Lemma 2.4. *Consider two paths P_1, P_2 in a B_k-EPG representation that do not intersect (i.e. have no grid edge in common). Let a, c and p be the number of alignments, crossings and pseudocrossings of $\{P_1, P_2\}$, respectively. If one path starts horizontally and the other one starts vertically, then*

(a) $c+p \leqslant 2\left\lfloor\frac{k+1}{2}\right\rfloor\left\lceil\frac{k+1}{2}\right\rceil+\left\lceil\frac{k+1}{2}\right\rceil-\left\lfloor\frac{k+1}{2}\right\rfloor$ *and*
(b) if the paths are monotonic $a+c\leqslant k+1$ *holds.*

If both paths start horizontally or both paths start vertically, then

(c) $c+p \leqslant 2\left\lfloor\frac{k+1}{2}\right\rfloor\left\lceil\frac{k+1}{2}\right\rceil$ *and*
(d) if the paths are monotonic $a+c\leqslant k$ *holds.*

Proof. First we consider (a) and (c). In a crossing or a pseudocrossing (S_1, S_2) of $\{P_1, P_2\}$ one of the segments is horizontal and the other one is vertical. Notice that a path that starts with a horizontal segment has at most $\left\lceil\frac{k+1}{2}\right\rceil$ horizontal and at most $\left\lfloor\frac{k+1}{2}\right\rfloor$ vertical segments, whereas a path that starts with a vertical segment has at most $\left\lfloor\frac{k+1}{2}\right\rfloor$ horizontal and at most $\left\lceil\frac{k+1}{2}\right\rceil$ vertical segments. If one of the paths starts horizontally and the other path starts vertically this implies that

$$\begin{aligned}
c+p &\leqslant \left\lfloor\frac{k+1}{2}\right\rfloor^2+\left\lceil\frac{k+1}{2}\right\rceil^2 \\
&= 2\left\lfloor\frac{k+1}{2}\right\rfloor\left\lceil\frac{k+1}{2}\right\rceil+\left(\left\lceil\frac{k+1}{2}\right\rceil-\left\lfloor\frac{k+1}{2}\right\rfloor\right)^2 \\
&= 2\left\lfloor\frac{k+1}{2}\right\rfloor\left\lceil\frac{k+1}{2}\right\rceil+\left\lceil\frac{k+1}{2}\right\rceil-\left\lfloor\frac{k+1}{2}\right\rfloor,
\end{aligned}$$

where we can omit the square because the squared value is either 0 or 1, and hence (a) holds. With the same arguments we obtain

$$c+p \leqslant 2\left\lfloor\frac{k+1}{2}\right\rfloor\left\lceil\frac{k+1}{2}\right\rceil$$

for paths that start in the same direction. Thus (c) is satisfied.

Next we consider (b), so assume the paths are monotonic. It is easy to see that each segment of P_1 cannot cross two or more segments of P_2 and cannot be aligned with two or more segments of P_2. Furthermore, whenever a segment of P_1 crosses a segment of P_2, it cannot be aligned with another segment of P_2. Moreover, whenever a segment of P_1 is aligned with a segment of P_2, it cannot cross another segment of P_2. Hence each segment of P_1 can be part of at most one crossing or alignment. This implies (b) as P_1 has at most $k+1$ segments.

In order to prove (d) assume without loss of generality that both paths start horizontally. The arguments of (b) imply that each segment of each of the paths can appear in at most one crossing or one alignment. We distinguish two cases. If one of the paths starts in a lower grid line than the other, then the first segment of this path can neither be aligned to nor cross the other path. Therefore, alignments and crossings can only occur on the remaining k segments of the path and hence $a+c\leqslant k$ holds. If both paths start on the same grid line, then let without loss of generality the first segment of P_1 lie to the left of the first segment of P_2. It is easy to see that the second segment of P_1 can neither be aligned to nor cross any segment of P_2. Therefore, also in this case we have $a+c\leqslant k$. This proves (d). □

Next we combine the bounds on the number of crossings derived in Lemma 2.4 with Lemma 2.2 in the following result.

Lemma 2.5. *Let $3\leqslant m\leqslant n$. In every B_k^m-EPG representation of $K_{m,n}$*

$$n(2m-k-2)\leqslant k(m-1)m+\frac{1}{2}m^2+2(k+1)m$$

holds.

Proof. Let c denote the number of crossings of the paths in $\mathcal{P}_A$. Every B_k^m-EPG representation is also a B_k-EPG representation. Therefore, it follows from Lemma 2.2 that

$$n(2m-k-2)\leqslant 2c+2(k+1)m \tag{2}$$

holds for every B_k^m-EPG representation of $K_{m,n}$. Now we give an upper bound on c. Let ℓ be the number of paths in $\mathcal{P}_A$ which start with a horizontal segment. Then, $m-\ell$ paths of $\mathcal{P}_A$ start with a vertical segment. Since the paths in $\mathcal{P}_A$ are pairwise non-intersecting, the number c of crossings of $\mathcal{P}_A$ can be calculated as $c = \sum_{\{v,v'\}\subseteq A} c_{v,v'}$, where $c_{v,v'}$ is the number of crossings of $\{P_v, P_{v'}\}$.

If both P_v and $P_{v'}$ start with a horizontal (vertical) segment, then $c_{v,v'} \leqslant k$ by Lemma 2.4(d). If one of the paths P_v and $P_{v'}$ starts with a horizontal segment and the other one starts with a vertical segment, then $c_{v,v'} \leqslant k+1$ by Lemma 2.4(b). Notice that there are exactly $\ell(m-\ell)$ pairs of paths P_v and $P_{v'}$ with the latter property and $\binom{m}{2} - \ell(m-\ell)$ pairs of paths P_v and $P_{v'}$ both starting with a horizontal (vertical) segment. In total we get

$$c = \sum_{\{v,v'\}\subseteq A} c_{v,v'} \leqslant k\binom{m}{2} + \ell(m-\ell).$$

Since $\ell(m-\ell) \leqslant \left(\frac{m}{2}\right)^2$ for all $0 \leqslant \ell \leqslant m$ we get

$$c \leqslant k\binom{m}{2} + \frac{m^2}{4} = \frac{1}{2}\left(k(m-1)m + \frac{1}{2}m^2\right),$$

which in combination with (2) completes the proof. □

Next we combine the bounds on the number of crossings derived in Lemmas 2.4 and 2.3 as follows.

Lemma 2.6. *Let $3 \leqslant m \leqslant n$. In every B_k^m-EPG representation of $K_{m,n}$*

$$n\left(m - \left\lceil\frac{k+1}{2}\right\rceil\right) \leqslant \binom{m}{2}\left(2\left\lfloor\frac{k+1}{2}\right\rfloor\left\lceil\frac{k+1}{2}\right\rceil + k\right) + \frac{1}{4}m^2\left(1 + \left\lceil\frac{k+1}{2}\right\rceil - \left\lfloor\frac{k+1}{2}\right\rfloor\right)$$

holds.

Proof. We combine Lemmas 2.3 and 2.4 by proceeding analogously as in the proof of Lemma 2.5.

In particular, let a, c and p be the number of alignments, crossings and pseudocrossings of $\mathcal{P}_A$, respectively. As done in the proof of Lemma 2.5 we can compute c as the sum of the number of crossings $c_{v,v'}$ of $\{P_v, P_{v'}\}$ over all pairs $\{v, v'\} \subseteq A$. Similarly we write p and a as the sum of the number of pseudocrossings $p_{v,v'}$ (alignments $a_{v,v'}$) of $\{P_v, P_{v'}\}$ over all pairs $\{v, v'\} \subseteq A$. Thus, we obtain $a + 2c + p = \sum_{\{v,v'\}\subseteq A}(a_{v,v'} + c_{v,v'}) + \sum_{\{v,v'\}\subseteq A}(c_{v,v'} + p_{v,v'})$.

Then, we use Lemma 2.4 (b) and (d) to bound each summand of the first sum from above and Lemma 2.4 (a) and (c) to bound each summand of the second sum from above. Then we transform the sum of these upper bounds analogously as in the proof of Lemma 2.5 and finally use Lemma 2.3 to bound $a + 2c + p$ from below. This completes the proof. □

To summarize Lemmas 2.5 and 2.6 provide inequalities on m, n and k which hold whenever a $K_{m,n}$ with $3 \leqslant m \leqslant n$ is in B_k^m. These inequalities are used in Sections 2.3 and 3.2.

2.3. Upper bounds on the monotonic bend number

In [18] a lot of work has been done to determine the bend number of $K_{m,n}$ in dependence of m and n. In particular, it was proven that $b(K_{m,n}) = 2m-2$ for $m \geqslant 3$ and $n \geqslant m^4 - 2m^3 + 5m^2 - 4\,m + 1$. We deduce a similar result for the monotonic case.

We first generalize a result of [4]. There it was shown by slightly modifying a construction of [16] that $K_{m,n} \in B_{2m-2}$ for all n. We modify the construction of [4] and give an analogous result for the monotonic case.

Theorem 2.7. *It holds that $K_{m,n} \in B_{2m-2}^m$.*

Proof. In order to prove this, it is enough to give a B_{2m-2}^m-EPG representation of $K_{m,n}$, which can be found in Fig. 3. Each vertex of $K_{m,n}$ belonging to the partition class A of size m is represented in the grid by a path consisting of just one horizontal segment. Each of the n vertices of the other partition class B is represented in the grid by a staircase with $2m-2$ bends. The staircases have pairwise empty intersections. □

Note that Theorem 2.7 implies that $b^m(G) \leqslant 2m-2$ holds for every graph G that is an induced subgraph of $K_{m,n}$. Furthermore, Theorem 2.7 shows that for fixed m and varying n, $b(K_{m,n}) \leqslant b^m(K_{m,n}) \leqslant 2m-2$ holds. Hence, the upper bound on the number of bends needed for an EPG representation of $K_{m,n}$ with $3 \leqslant m \leqslant n$ is the same, namely $2m-2$, no matter whether all kind of bends or only monotonic bends are allowed. This fact is even more surprising if we take into account Theorem 4.1, which states the existence of graphs for which the gap between the bend number and the monotonic bend number can be arbitrarily large.

However, it turns out that the upper bound on $b^m(K_{m,n})$ is already reached for a smaller n than the upper bound on $b(K_{m,n})$. In particular, the above stated result from [18] implies that $b(K_{m,n}) = 2m-2$ for $n \geqslant N_1$ for some $N_1 \in \Theta(m^4)$. As a consequence of the next result it follows that $b^m(K_{m,n}) = 2m-2$ for $n \geqslant N_2$ already for some $N_2 \in \Theta(m^3)$.

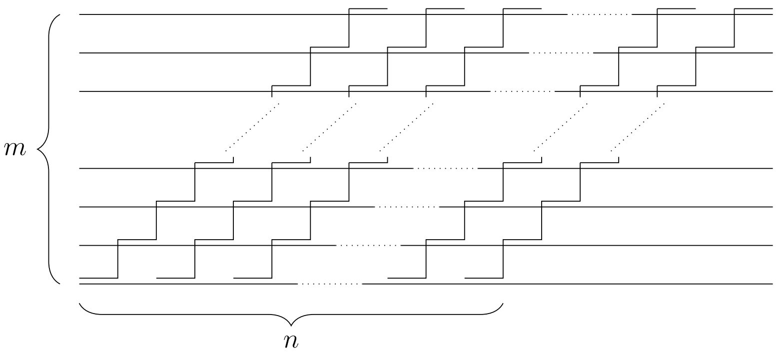


Fig. 3. A B^m_{2m-2}-EPG representation of $K_{m,n}$.

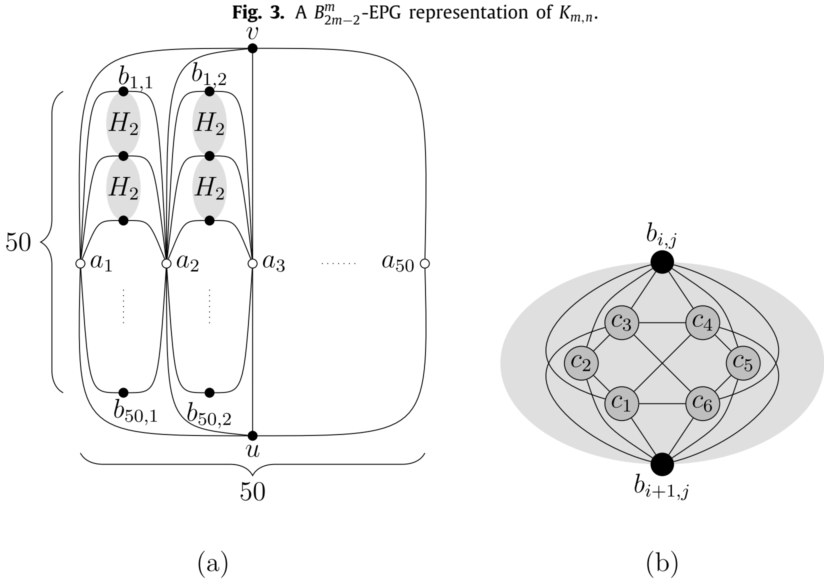


Fig. 4. (a) The graph H_1. (b) The graph H_2 contained in every gray area of H_1.

Theorem 2.8. *Let* $3 \leqslant m$. *If* $n \geqslant 2m^3 - \frac{1}{2}m^2 - m + 1$ *then* $K_{m,n} \notin B^m_{2m-3}$.

Proof. Suppose, in order to derive a contradiction, that $K_{m,n} \in B^m_{2m-3}$. By applying Lemma 2.5 for $k = 2m - 3$ we get that

$$n(2\ m - (2m - 3) - 2) \leqslant (2m - 3)(m - 1)m + \frac{1}{2}m^2 + 2(2m - 2)m$$

Then, doing the maths operations we have that $n \leqslant 2m^3 - \frac{1}{2}m^2 - m$ has to hold. This contradicts $n \geqslant 2m^3 - \frac{1}{2}m^2 - m + 1$. □

3. Relationship between B^m_k and B_k

It is an open question of [16] to determine the relationship between B^m_k and B_k for $k \geqslant 1$. Obviously $B^m_k \subseteq B_k$ holds for every k. In [16] Golumbic, Lipshteyn and Stern conjectured that $B^m_1 \subsetneqq B_1$. This conjecture was confirmed by Cameron, Chaplick and Hoàng in [10] by showing that the graph S_3, which was known to be in B_1 from [16], is not in B^m_1. The graph S_3 is isomorphic to the subgraph induced by the vertices $\{a, b, c, d, e, f\}$ in the graph represented in Fig. 8(a).

In this section we consider the question whether $B^m_k \subsetneqq B_k$ holds also for $k \geqslant 2$. We first consider the case $k = 2$ in Section 3.1 and then the remaining cases $k \geqslant 3$ in Section 3.2. The case distinction is due to the different methods used in the investigations.

3.1. Relationship between B^m_2 and B_2

The aim of this section is to prove that $B^m_2 \subsetneqq B_2$ holds. For this purpose we show that the graph H_1 represented in Fig. 4 is in B_2 but not in B^m_2. H_1 is defined as follows.

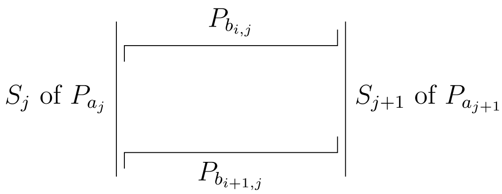


Fig. 5. A part of the hypothetical B_2^m-EPG representation of H_1.

Definition 3.1. The graph H_1 depicted in Fig. 4 is constructed in the following way. The vertices $\{u, v\}$ and $\{a_1, \ldots, a_{50}\}$ form a $K_{2,50}$. Furthermore, for every $1 \leqslant j < 50$ the vertices $\{a_j, a_{j+1}\}$ and $\{b_{1,j}, \ldots, b_{50,j}\}$ form a $K_{2,50}$. Additional to that for every $1 \leqslant j < 50$ and for every $1 \leqslant i < 50$ there is the graph H_2 of Fig. 4(b) placed between the vertices $b_{i,j}$ and $b_{i+1,j}$.

The next result follows from a proof of Heldt, Knauer and Ueckerdt given in [17]. In Proposition 1 of [17] they use a similar construction in order to prove that there is a planar graph with treewidth at most 3 which is not in B_2. Their construction builds also on the graph H_1 (called G in their paper) but the graph suspended between any two vertices $b_{i,j}$, $b_{i+1,j}$, for $1 \leqslant i, j < 50$, (called H in their paper) is a 29-vertex graph different from H_2. In the first part of the proof of Proposition 1 Heldt, Knauer and Ueckerdt prove some properties of B_2-EPG representations of the subgraph of H_1 as summarized in the following lemma.

Lemma 3.2 (*Heldt, Knauer, Ueckerdt [17]*)**.** *In any B_2-EPG representation of the graph H_1 depicted in Fig. 4 there exist two indices i and j, $1 \leqslant i, j \leqslant 49$, with the following properties:*

(a) the paths $P_{b_{i,j}}$ and $P_{b_{i+1,j}}$ consist of three segments each,
(b) there is a segment S_j of the path P_{a_j} which completely contains one end segment of $P_{b_{i,j}}$ and one end segment of $P_{b_{i+1,j}}$,
(c) there is a segment S_{j+1} of the path $P_{a_{j+1}}$ which completely contains the other end segments of $P_{b_{i,j}}$ and $P_{b_{i+1,j}}$,
(d) S_j and S_{j+1} are either both vertical segments or both horizontal segments.

With this auxiliary result we are able to prove the following lemma.

Lemma 3.3. *The graph H_1 is not in B_2^m.*

Proof. Suppose, in order to derive a contradiction, that H_1 is in B_2^m. Every B_2^m-EPG representation is a B_2-EPG representation as well, therefore Lemma 3.2 holds also for any B_2^m-EPG representation of H_1. Assume without loss of generality that the center segment (i.e. the second segment) of $P_{b_{i,j}}$ is a horizontal segment, that it is above the center segment of $P_{b_{i+1,j}}$ and that the segment S_j of P_{a_j} is on the left side of the segment S_{j+1} of $P_{a_{j+1}}$. Then the positioning of the segments of the paths has to look like in Fig. 5.

Each vertex c_ℓ, $1 \leqslant \ell \leqslant 6$, of the copy of H_2 between $b_{i,j}$ and $b_{i+1,j}$ is adjacent to both $b_{i,j}$ and $b_{i+1,j}$, but neither to a_j nor to a_{j+1}. Therefore, each of the six paths $P_{c_1}, \ldots, P_{c_6}$ has to share a grid edge with the center segments of both $P_{b_{i,j}}$ and $P_{b_{i+1,j}}$. As a result, P_{c_i} starts with a first horizontal segment intersecting the center segment of $P_{b_{i+1,j}}$, continues with a second vertical segment and ends with a third horizontal segment intersecting the center segment of $P_{b_{i,j}}$, for every for $1 \leqslant i \leqslant 6$.

Now consider the vertices c_1, c_3 and c_5. They are pairwise nonadjacent, so P_{c_1}, P_{c_3}, P_{c_5} are non-intersecting. Therefore, the three vertical segments of these paths are disjoint and can be ordered from the left to the right. Let P_L, P_M and P_R be the path in $\{P_{c_1}, P_{c_3}, P_{c_5}\}$ with the left-most, the middle and the right-most center segment, respectively. In the following we say that a path P_{c_i} lies to the left of, to the right of and on another path P_{c_j} if the center segment of P_{c_i} lies to the left of, to the right of and on the center segment of P_{c_j} for some $1 \leqslant i \neq j \leqslant 6$, respectively.

Next take a closer look at the paths P_{c_4} and P_{c_6}. Each of them intersects each of the three paths P_L, P_M and P_R, since both vertices c_4, c_6 are adjacent to each of c_1, c_3 and c_5. Since c_4 and c_6 are not adjacent to each other, P_{c_4} and P_{c_6} do not intersect and hence the vertical segments of P_{c_4} and P_{c_6} are disjoint. Assume without loss of generality that P_{c_4} is to the left of P_{c_6}.

If P_{c_4} lies to the right of or on P_L, then P_{c_6} cannot intersect P_L on the first or second segment of P_L, because P_{c_6} is to the right of P_{c_4} and does not intersect P_{c_4}. Therefore, P_{c_6} intersects P_L on its third segment. This implies that P_{c_6} lies to the left of or on P_M. But P_{c_4} is to the left of P_{c_6}, which is to the left of or on P_M. Thus, no point of P_{c_4} can lie to the right of the center segment of P_M, which implies that P_{c_4} does not intersect P_R, a contradiction. Analogously, it follows that P_{c_6} cannot lie to the left of or on P_R.

As a result P_{c_4} lies to the left of P_L and P_{c_6} lies to the right of P_R. P_{c_4} has to intersect P_R, so the third segment of P_L and P_M are completely contained in the third segment of P_{c_4}. Similarly P_{c_6} has to intersect P_L, so the first segment of P_M and P_R are completely contained in the first segment of P_{c_6}. For an illustration of this configuration see Fig. 6.

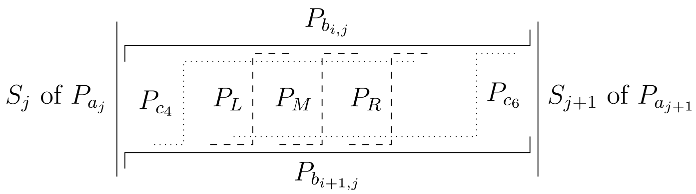
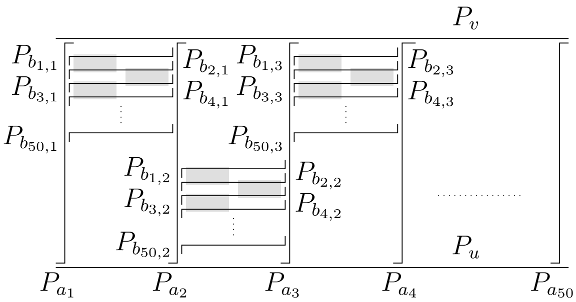
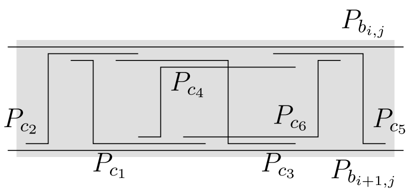


Fig. 6. The only possible placement of paths P_{c_4}, P_{c_6} and $\{P_L, P_M, P_R\} = \{P_{c_1}, P_{c_3}, P_{c_5}\}$ in the hypothetical B_2^m-EPG representation of H_1.



(a)



(b)

Fig. 7. (a) A B_2-EPG representation of the graph H_1 of Fig. 4. Every gray area represents the B_2-EPG representation of H_2 depicted in (b).

Now consider the path P_{c_2}. Note that if P_{c_2} intersects the middle segment of P_M, then P_{c_2} is on P_M, in which case P_{c_2} also intersects the first and third segments of P_M. So we can conclude that if P_{c_2} intersects P_M, then P_{c_2} intersects either the first segment of P_M or the third segment of P_M. But the former is completely contained in P_{c_6} and the latter is completely contained in P_{c_4}, which means that P_{c_2} intersects either P_{c_4} or P_{c_6}, which is a contradiction to the fact that c_2 is nonadjacent to c_4 and c_6. We can therefore conclude that P_{c_2} does not intersect P_M. Since c_2 is adjacent to both c_1 and c_3, this means that $P_M = P_{c_5}$ and $\{P_L, P_R\} = \{P_{c_1}, P_{c_3}\}$. But now it is not possible for P_{c_2} to intersect both P_L and P_R without intersecting P_M. Hence, H_1 cannot have a B_2^m-EPG representation. □

After proving that H_1 is not in B_2^m, we observe that H_1 is in B_2 and obtain the following theorem.

Theorem 3.4. *It holds that $B_2^m \subsetneq B_2$.*

Proof. The fact that $B_2^m \subseteq B_2$ follows by definition. In order to see that strict inclusion holds, we consider the graph H_1 depicted in Fig. 4.

We have already seen in Lemma 3.3 that the graph H_1 is not in B_2^m. So it is enough to show that H_1 is in B_2. To this end, consider a B_2-EPG representation of H_1 given in Fig. 7. □

Summarizing $B_k^m \subsetneq B_k$ holds for $k = 1$ as shown in [10] and also for $k = 2$ as shown in this paper.

3.2. Relationship between B_k^m and B_k for $k \in \{3, 5\}$ and $k \geqslant 7$

In this section we use the results from Section 2 in order to investigate the relationship between B_k^m and B_k for $k \in \{3, 5\}$ and $k \geqslant 7$.

We start with $k = 3$ and prove that $B_3^m \subsetneq B_3$ holds. To this end we use a result of [18] to show that a particular graph is in B_3, and then use results of Section 2 to prove that this graph is not in B_3^m.

Lemma 3.5. *It holds that* $B_3^m \subsetneqq B_3$.

Proof. Since $B_3^m \subseteq B_3$ obviously holds, it is enough to show that $B_3^m \subsetneqq B_3$. Heldt, Knauer, Ueckerdt [18] showed that $b(K_{3,36}) = 3$, hence $K_{3,36}$ belongs to B_3. Now assume that $K_{3,36}$ is in B_3^m. Then by Lemma 2.6 we have

$$36\left(3 - \left\lceil \frac{4}{2} \right\rceil\right) \leqslant 3\left(2\left\lfloor \frac{4}{2} \right\rfloor \left\lceil \frac{4}{2} \right\rceil + 3\right) + \frac{1}{4}3^2,$$

That is, $36 \leq 35.25$, a contradiction. Hence, $K_{3,36}$ is not in B_3^m. □

Now we know that $B_k^m \subsetneqq B_k$ holds for $k \leqslant 3$. Next we show $B_5^m \subsetneqq B_5$. Similarly as in the case of $k = 3$ we use a result of [18] to show that a particular graph is in B_5 and then use results of Section 2 to prove that this graph is not in B_5^m.

Lemma 3.6. *It holds that* $B_5^m \subsetneqq B_5$.

Proof. Since $B_5^m \subseteq B_5$ obviously holds, it is enough to show $B_5^m \neq B_5$.

Heldt, Knauer, Ueckerdt [18] showed that $K_{m,n} \in B_{2m-3}$ if $n \leqslant m^4 - 2m^3 + \frac{5}{2}m^2 - 2\,m - 4$ (see Theorem 4.5 in [18]). For $m = 4$ this implies that $K_{4,156} \in B_5$. Assume that $K_{4,156} \in B_5^m$. Then, by Lemma 2.5 we get

$$156(2 \cdot 4 - 5 - 2) \leqslant 5 \cdot 3 \cdot 4 + \frac{1}{2}4^2 + 2 \cdot 6 \cdot 4$$

That is, $156 \leq 116$, a contradiction. So $K_{4,156} \notin B_5^m$ but $K_{4,156} \in B_5$, hence $B_5^m \neq B_5$. □

Finally we show $B_k^m \subsetneqq B_k$ for $k \geqslant 7$. To this end, we use Lemma 2.5 and Theorem 2.1.

Lemma 3.7. *It holds that* $B_k^m \subsetneqq B_k$ *for* $k \geqslant 7$.

Proof. We first prove the statement for odd k. Theorem 2.1 implies that $K_{k+1,\frac{1}{4}(k+1)^3-\frac{1}{2}(k+1)^2-(k+1)+4} = K_{k+1,\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}} \in B_k$ for $k \geqslant 3$. Suppose, in order to derive a contradiction, that this graph is in B_k^m. Then, by Lemma 2.5 with $m = k+1$ and $n = \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}$ it follows that

$$\begin{aligned} & \left(\frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}\right)(2(k+1) - k - 2) \leqslant k^2(k+1) + \frac{1}{2}(k+1)^2 + 2(k+1)^2 \\ \Leftrightarrow \quad & k\left(\frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}\right) \leqslant k^3 + \frac{7}{2}k^2 + 5k + \frac{5}{2} \\ \Leftrightarrow \quad & k^4 - 3k^3 - 19k^2 - 9k - 10 \leqslant 0, \end{aligned}$$

which is a contradiction for $k \geqslant 7$. Hence, for odd $k \geqslant 7$ there is a graph in B_k which is not in B_k^m and therefore $B_k^m \subsetneqq B_k$ holds for odd $k \geqslant 7$.

Now consider the complementary case of even k. Theorem 2.1 implies that the graph $K_{k+1,\frac{1}{4}(k+1)^3-(k+1)^2+\frac{3}{4}(k+1)} = K_{k+1,\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k} \in B_k$ for $k \geqslant 6$. Suppose, in order to derive a contradiction, that this graph is in B_k^m. Then, by Lemma 2.5 with $m = k+1$ and $n = \frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k$ we get

$$\begin{aligned} & \left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right)(2(k+1) - k - 2) \leqslant k^2(k+1) + \frac{1}{2}(k+1)^2 + 2(k+1)^2 \\ \Leftrightarrow \quad & k\left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right) \leqslant k^3 + \frac{7}{2}k^2 + 5k + \frac{5}{2} \\ \Leftrightarrow \quad & k^4 - 5k^3 - 16k^2 - 20k - 10 \leqslant 0, \end{aligned}$$

which is a contradiction for $k \geqslant 8$. Hence, for even $k \geqslant 8$ there is a graph in B_k which is not in B_k^m. Therefore $B_k^m \subsetneqq B_k$ for even $k \geqslant 8$ and this completes the proof. □

Lemmas 3.5–3.7 imply the following theorem.

Theorem 3.8. *It holds that* $B_k^m \subsetneqq B_k$ *for* $k = 3$, $k = 5$ *and* $k \geqslant 7$.

Summarizing, in Theorems 3.4 and 3.8 we have shown that $B_k^m \subsetneqq B_k$, for $k \in \{2, 3, 5\}$ and for $k \geqslant 7$, addressing herewith a question raised in [16]. Recall that $B_1^m \subsetneqq B_1$ was already shown in [10]. Thus, the only open cases are $k = 4$ and $k = 6$. We conjecture that $B_k^m \subsetneqq B_k$ holds also for these two remaining cases.

Conjecture 3.9. $B_k^m \subsetneqq B_k$ *holds also for* $k = 4$ *and* $k = 6$.

However, this remains an open question.

4. Relationship between B_k and B_ℓ^m for $\ell > k$

Recall that the inclusions chains $B_i \subseteq B_{i+1}$ and $B_i^m \subseteq B_{i+1}^m$ trivially hold for all $i \in \mathbb{N}$. In other words the size of the classes of graphs that have a (monotonic) k-bend EPG representation increases with increasing k. Also the relationships $B_0 = B_0^m$ and $B_0 \subseteq B_1^m$ are trivial. Moreover, $B_k^m \subsetneq B_k$ holds for almost all $k \in \mathbb{N}$, as shown in Section 3.

This means that in general the minimum number of bends needed for an EPG representation of a graph increases when the representing paths on the grid are required to be monotonic. Analogously, in general the minimum number of bends needed for an EPG representation of a graph decreases as compared to the minimum number of bends needed in a monotonic EPG representation. Quantifying the magnitude of such an increase (decrease) arises as a natural question in this context. More generally, it would be interesting to investigate the existence of non-trivial functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $b^m(G) \leqslant f(b(G))$ and $b(G) \leqslant g(b^m(G))$ hold for all graphs G, or only for all G belonging to some particular class of graphs.

To the best of our knowledge questions of this kind have not been addressed in the literature so far. In this section we present some related results. In particular, in Section 4.1 we show that the increase (decrease) of the number of bends as mentioned above cannot be bounded by one, in general. More precisely, by combining the results of Theorems 4.1 and 2.7 with some result known in the literature we show that none of the inclusions $B_k \subseteq B_{k+1}^m$, $B_{k+1}^m \subseteq B_k$ holds, a result not known so far in the literature. Then in Section 4.2 we show that $B_1 \subseteq B_3^m$ holds.

4.1. Relationship between B_k and B_{2k-9}^m

Theorem 4.1. *Let $k \geqslant 5$. If k is odd, then there is a graph which is in B_k but not in B_{2k-8}^m. If k is even, there is a graph which is in B_k but not in B_{2k-9}^m.*

Proof. Consider first the case where k is odd. In this case Theorem 2.1 implies that $G_k := K_{k+1,\frac{1}{4}(k+1)^3-\frac{1}{2}(k+1)^2-(k+1)+4} = K_{k+1,\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}}$ is in B_k for $k \geqslant 3$.

Assume that G_k belongs to B_{2k-8}^m for $k \geqslant 5$. Then, Lemma 2.5 implies

$$\begin{aligned}
&\left(\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}\right)8 \leqslant (2k-8)(k+1)k+\frac{1}{2}(k+1)^2+2(2k-7)(k+1)\\
\Leftrightarrow\quad & 8\left(\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}\right) \leqslant 2k^3-\frac{3}{2}k^2-17k-\frac{27}{2}\\
\Leftrightarrow\quad & 7k^2+14k+71 \leqslant 0\,,
\end{aligned}$$

which is a contradiction for $k \geqslant 0$. So G_k is not in B_{2k-8}^m. Hence, for odd $k \geqslant 5$, there is a graph in B_k which is not in B_{2k-8}^m.

Consider now the case where k is even. Theorem 2.1 implies $G_k' := K_{k+1,\frac{1}{4}(k+1)^3-(k+1)^2+\frac{3}{4}(k+1)} = K_{k+1,\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k} \in B_k$ for $k \geqslant 6$. If we assume that G_k' is in B_{2k-9}^m for $k \geqslant 6$, we obtain the following inequality by applying Lemma 2.5

$$\begin{aligned}
&\left(\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k\right)9 \leqslant (2k-9)(k+1)k+\frac{1}{2}(k+1)^2+2(2k-8)(k+1)\\
\Leftrightarrow\quad & 9\left(\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k\right) \leqslant 2k^3-\frac{5}{2}k^2-20k-\frac{31}{2}\\
\Leftrightarrow\quad & k^3+k^2+62k+62 \leqslant 0\,,
\end{aligned}$$

which is a contradiction for $k \geqslant 0$. Hence, G_k' is in B_k but not in B_{2k-9}^m for even $k \geqslant 6$. □

Theorem 4.1 reveals that $B_k \not\subseteq B_{2k-8}^m$ for odd $k \geqslant 5$ and that $B_k \not\subseteq B_{2k-9}^m$ for even $k \geqslant 5$. Thus, restricting the paths of the EPG representation to be monotonic is a significant limitation. Theorem 4.1 clearly implies that $B_k \subseteq B_{k+1}^m$ does not hold in general.

We can also settle the question whether $B_{k+1}^m \subseteq B_k$ holds in general. Indeed, in [18] it was proven that $b(K_{m,n}) = 2m-2$ for $m \geqslant 3$ and $n \geqslant m^4 - 2m^3 + 5m^2 - 4\,m + 1$. Hence, in particular, $K_{m,m^4-2m^3+5m^2-4\,m+1}$ is in B_{2m-2}, but it is not in B_{2m-3}. On the other hand, Theorem 2.7 implies that $K_{m,m^4-2m^3+5m^2-4\,m+1}$ is in B_{2m-2}^m, so $B_{2m-2}^m \not\subseteq B_{2m-3}$ for all $m \geqslant 3$. Thus, $B_{k+1}^m \subseteq B_k$ does not hold in general.

4.2. Relationship between B_1 and B_3^m

As mentioned at the beginning of Section 4, in general the minimum number of bends needed for an EPG representation of a graph increases when the paths on the grid are required to be monotonic. In order to quantify the amount of this increase we would like to find the minimum ℓ such that $B_k \subseteq B_\ell^m$. Theorem 4.1 shows that $2k-9$ is a lower bound for ℓ, i.e. $\ell \geqslant 2k-9$ for $k \geqslant 5$.

In the following we focus on small values of k. Since $B_0 = B_0^m$ holds, 1 is the smallest value of k for which ℓ and/or bounds on it are not known. In the following we show that $B_1 \subseteq B_3^m$, i.e. 3 is an upper bound on the minimum value of ℓ for which $B_1 \subseteq B_\ell^m$.

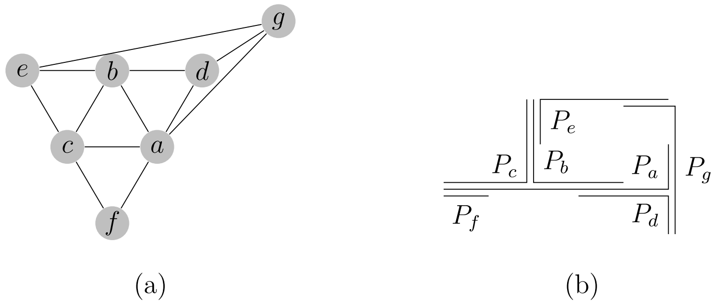


Fig. 8. (a) A graph G. (b) The B_1-EPG representation R of G.

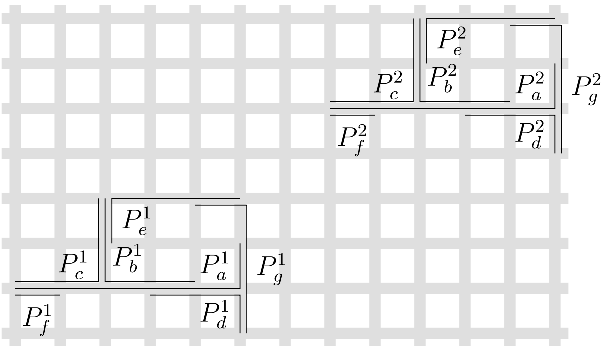


Fig. 9. The grid with the two copies R_1 and R_2 of R shown in Fig. 8(b).

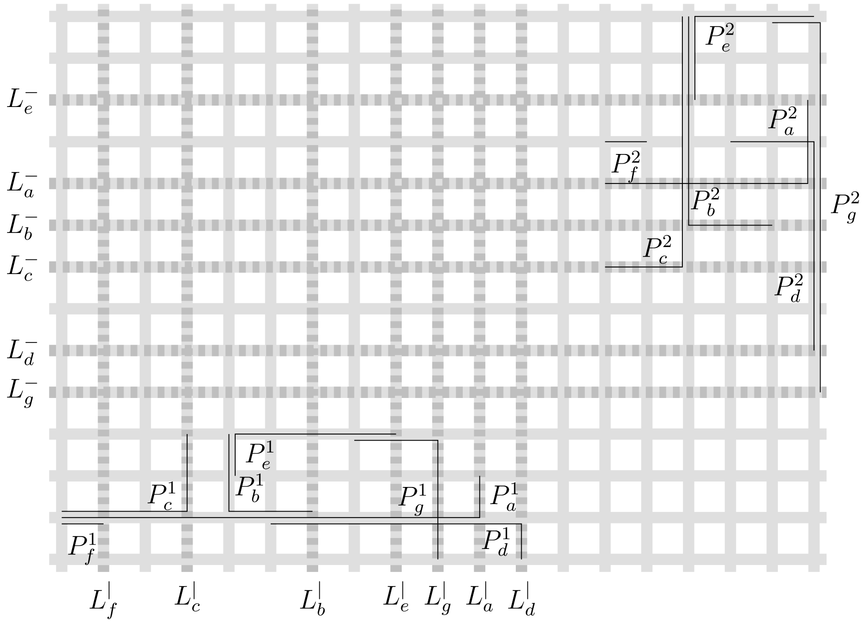


Fig. 10. The final status of the modifications R_1 and R_2 of the B_1-EPG representation R shown in Fig. 8(b) and its copy. This final status is obtained for any order $\prec$ of vertices in which $c \prec b$ and $e \prec g \prec a \prec d$.

Theorem 4.2. *The inclusion $B_1 \subseteq B_3^m$ holds.*

Proof. Let G be a graph in B_1. We show that G is in B_3^m by presenting a monotonic B_3-EPG representation of G. The latter is constructed by transforming a B_1-EPG representation of G into a B_3^m-EPG representation of G as described below. The transformation is illustrated by means of an example; Figs. 8(a) and 8(b) show a graph G and a B_1-EPG representation of it, respectively, whereas Fig. 11 shows the corresponding B_3^m-EPG representation obtained as a result of the transformation mentioned above.

Let R be an arbitrary B_1-EPG representation of G. We place another copy of the same B_1-EPG representation to the top right of R, see Fig. 9, and then step by step modify both the original B_1-EPG representation and its copy as described below. At any point in time during this modification process we denote by R_1 and R_2 the current modified B_1-EPG representation and the current modified copy of the original B_1-EPG representation, respectively. For a vertex v of G we denote by P_v, P_v^1 and P_v^2 the path corresponding to v in R, R_1 and R_2, respectively. At the beginning of the modification process R_1 and R_2 coincide with the original B_1-EPG representation and its copy, respectively, as in Fig. 9.

Now consider the vertices of G one by one in an arbitrary order and for every vertex perform the modifications described below. Let v be the currently considered vertex. The modification of R_1 is driven by the horizontal segment of the path P_v, if any, whereas the modification of R_2 is driven by the vertical segment of P_v, if any. If P_v has a horizontal segment, we modify R_1 as follows. We introduce a new vertical grid line $L_v^|$ directly to the left of the vertical grid line containing the right end point of the horizontal segment of P_v^1 in R_1 and shorten the horizontal segment of P_v^1 to end in $L_v^|$ instead of ending at the original right end point. Then, if the path P_v contains a vertical segment which starts at the original right end point of the horizontal segment mentioned above, we modify P_v^1 in R_1 by shifting its vertical segment to lie on $L_v^|$.

If P_v has a vertical segment, we modify R_2 as follows. We introduce a new horizontal grid line L_v^- directly beneath the horizontal grid line containing the lower end point of the vertical segment of P_v^2 in R_2 and extend the vertical segment of P_v^2 until L_v^-. Then, if the path P_v contains a horizontal segment which starts at the original lower end point of the vertical segment mentioned above, we modify P_v^2 in R_2 by shifting its horizontal segment to lie on L_v^-. An example of the modified grid and paths and the final R_1, R_2 for the graph in Fig. 8(a) can be seen in Fig. 10.

Now we construct a B_3^m-EPG representation of G with a path Q_v for every vertex v in the following way. If the path P_v consists of a single horizontal segment, we define Q_v as the horizontal segment of P_v^1 in R_1 and call this segment the *lower* segment of Q_v. If the path P_v consists of a single vertical segment, we define Q_v as the vertical segment of P_v^2 in R_2 and call this segment the *upper* segment of Q_v. If the path P_v contains a horizontal and a vertical segment, then the path Q_v starts with the horizontal segment of P_v^1 in R_1; this segment is called the *lower* segment of Q_v. Further the path Q_v continues with a vertical segment lying on the vertical grid line $L_v^|$ and ending at the intersection of $L_v^|$ and L_v^-. This intersection is the upper end point of this segment. Starting at this grid point Q_v proceeds with a horizontal segment lying on L_v^- until it reaches the vertical grid line containing the vertical segment of P_v^2 in R_2. Finally Q_v ends with the vertical segment of P_v^2 in R_2; this segment is called the *upper* segment of Q_v. The result of this construction for the graph given in Fig. 8(a) and its B_1-EPG representation R is depicted in Fig. 11.

Observe that this construction has the following properties. If P_v contains two segments, then Q_v contains 4 segments, the lower one being the horizontal segment of P_v^1 in R_1 and the upper one being the vertical segment of P_v^2 in R_2. The two remaining segments, a vertical and a horizontal one, are contained in the two additionally introduced grid lines that are used by no other path, because every path Q_v uses only the additional grid lines $L_v^|$ and L_v^- introduced exclusively for the vertex v. If P_v consists of one horizontal (vertical) segment, then Q_v consists also of one horizontal (vertical) segment which coincides with the corresponding segment of P_v^1 (P_v^2) in R_1 (R_2) and is a lower (upper) segment. It is easy to see that every path Q_v in this construction is monotonic and bends at most 3 times.

What is left to show is that the above construction indeed leads to an EPG representation of G, i.e. that any two paths Q_v and $Q_{v'}$ intersect if and only if the vertices v and v' are adjacent in G. To this end, it is enough to show that two paths Q_v and $Q_{v'}$ intersect, if and only if the paths P_v and $P_{v'}$ intersect in the original B_1-EPG representation R.

Assume Q_v and $Q_{v'}$ intersect. First consider the case that at least one of Q_v and $Q_{v'}$ consists of only one segment. Assume without loss of generality that Q_v consists of one horizontal segment. Due to the properties of the construction this segment of Q_v is a lower segment and hence the unique segment of P_v^1 in R_1. Consequently, again due to the properties of the construction, the segment of $Q_{v'}$ intersecting Q_v is the horizontal segment of $P_{v'}^1$ in R_1. Hence P_v^1 and $P_{v'}^1$ intersect in the final R_1 on their horizontal segments. By construction this is only the case if P_v and $P_{v'}$ intersect on their horizontal segments in R, because during the update of R_1 only vertical segments of paths are moved into new grid lines in such a way that no new intersections are created.

Now assume that both paths Q_v and $Q_{v'}$ consist of more than one segment. There are no intersections of the paths in any additionally introduced grid lines because every additionally introduced grid line is related to one vertex and the additionally introduced grid line related to different vertices are different. Moreover, by construction every additionally introduced vertical grid line contains at most one segment of the path P_v^1 in R_1 representing the vertex v to which the line is related. Analogously every additionally introduced horizontal grid line contains at most one segment of the path P_v^2 in R_2 representing the vertex v to which the line is related. These considerations together with the fact that R_1 and R_2 do not share any grid lines imply that the intersection of Q_v and $Q_{v'}$ involves either the lower segments of each path, or it involves the upper segments of each path. Consequently, according to the properties of the construction, the paths Q_v and $Q_{v'}$ intersect in their lower segments (in R_1) or in their upper segments (in R_2). In both situations we can proceed as in the previous case.

Next we show the other direction of the equivalence, that is we assume that P_v and $P_{v'}$ intersect in the original B_1-EPG representation R of G and show that also Q_v and $Q_{v'}$ intersect. By construction, if P_v and $P_{v'}$ intersect in a horizontal grid

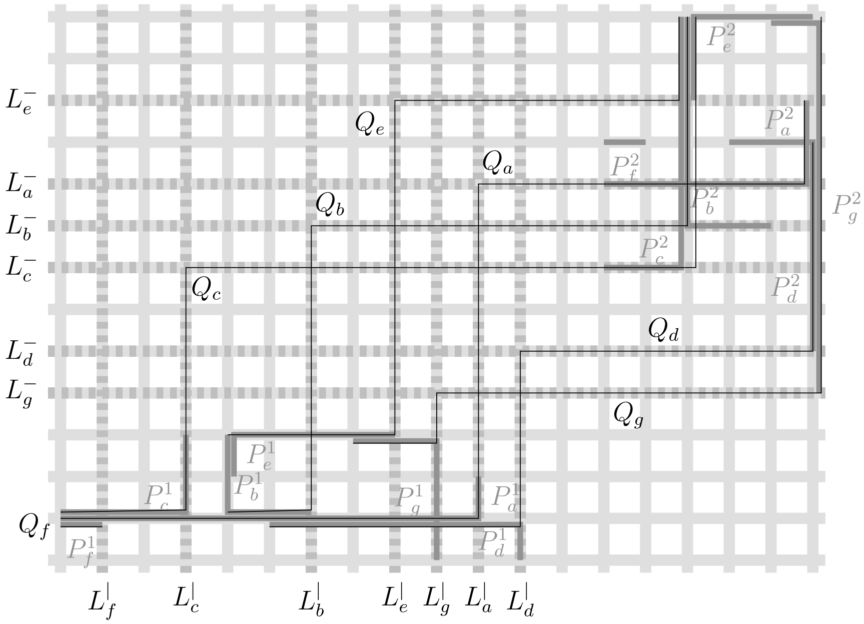


Fig. 11. The obtained B_3^m-EPG representation of the graph given in Fig. 8(a).

line, then the modified paths P_v^1 and $P_{v'}^1$ intersect in a horizontal grid line in R_1 at all times. Thus, the properties of the construction imply the intersection of the lower segments of Q_v and $Q_{v'}$. Analogously, if P_v and $P_{v'}$ intersect in a vertical grid line, then the modified paths P_v^2 and $P_{v'}^2$ intersect in a vertical grid line in R_2 at all times, and the properties of the construction imply the intersection of the upper segments of Q_v and $Q_{v'}$. □

Notice that it is an open question whether the result of Theorem 4.2 is the best possible, that is whether $\ell = 3$ is really the minimum ℓ such that $B_1 \subseteq B_\ell^m$ or whether even $B_1 \subseteq B_2^m$ holds.

We conclude this section with a few comments related to the size of the grid in EPG representations, that is the number of horizontal and vertical grid lines used by the paths in the EPG representation. Recently this question was investigated by Biedl, Derka, Dujmović and Morin [3]. The size of the B_3^m-EPG representation obtained by the construction in the proof of Theorem 4.2 depends on the size of the B_1-EPG representation of the graph; in the worst case the constructed B_3^m-EPG representation uses twice as many horizontal grid lines and twice as many vertical grid line as compared to the original B_1-EPG representation and an additional horizontal and vertical grid line for every vertex. This gives rise to the natural question whether the construction given in the proof of Theorem 4.2 is the best possible with respect to the size of the grid. Currently we cannot answer this question.

When considering the dependency of the grid size of the B_3^m-EPG representation on the grid size of the starting B_1-EPG representation in the construction given in [3], another natural question arises. What is the smallest possible size of the grid in a B_1-EPG representation of a B_1-EPG graph? In the small EPG representations dealt with in [3] no fixed number of bends is considered, so the question above is also open.

5. Conclusions and open problems

In this paper, we investigated the relationship between the classes B_k and B_ℓ^m for different values of $k, \ell \in \mathbb{N}$.

In particular, we considered the bend number and the monotonic bend number of complete bipartite graphs. We extended the already known result $b(K_{m,n}) \leqslant 2m - 2$ (see [18]) to the monotonic bend number, that is we proved $b^m(K_{m,n}) \leqslant 2m - 2$ for any $3 \leqslant m \leqslant n$, and showed that the upper bound $2m - 2$ is attained for smaller values of n in the monotonic case.

As auxiliary results we derived two different inequalities which hold whenever $K_{m,n}$ is in B_k^m. We used these inequalities to prove the strict inclusion $B_k^m \subsetneq B_k$ for $k \in \{3, 5\}$ and $k \geqslant 7$. Furthermore, we showed that $B_2^m \subsetneq B_2$ by specifying a particular graph which is in B_2 but not in B_2^m. Thus, we gave an almost complete answer to the open question on the correctness of $B_k^m \subsetneq B_k$ for $k > 1$, posed in [16]. We showed that $B_k^m \subsetneq B_k$ holds for all $k > 1$ except for $k \in \{4, 6\}$. Of course, it is a pressing question to prove $B_k^m \subsetneq B_k$ also for the remaining cases $k = 4$ and $k = 6$. In order to prove $B_4^m \subsetneq B_4$ by using Lemma 2.5 it would be enough to show that $K_{4,49} \in B_4$ or $K_{5,36} \in B_4$. In the case of $k = 6$ it would suffice to show that $K_{5,102}$, $K_{6,71}$ or $K_{7,63}$ is in B_6.

Additionally, we considered the relationship of B_k and B_ℓ^m for $\ell > k$. In this context the existence and the identification of non-trivial functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $b^m(G) \leqslant f(b(G))$ and $b(G) \leqslant g(b^m(G))$ hold for any graph G (or for any graph belonging to some particular class of graphs) is a general question the answer of which seems to be out of reach at the moment. However, we could deal with some specific problems related to that question.

In particular, we showed that for every $k \geqslant 5$ there is a graph in B_k which is not in B^m_{2k-9}, proving that $B_k \nsubseteq B^m_{2k-9}$ holds. In terms of the function f above this implies $f(x) \geqslant 2x - 8$ for all $x \geqslant 5$, $x \in \mathbb{N}$. Furthermore, we deduced that $B^m_{k+1} \subseteq B_k$ does not hold in general by providing a graph that is B^m_{2m-2} but not in B_{2m-3} for every $m \geqslant 3$. This implies that $g(2x) \geqslant 2x$ for all $x \geqslant 2$, $x \in \mathbb{N}$ for the above function g.

Further, we showed that $B_1 \subseteq B^m_3$, but we do not know whether this result is the best possible, i.e. whether there is a graph in B_1 which is not in B^m_2 or whether $B_1 \subseteq B^m_2$ holds.

Another natural question which seems to be simple but has not been answered yet concerns the inclusion $B^m_k \subseteq B^m_{k+1}$. We conjecture this inclusion to be strict, that is we conjecture that $B^m_k \subsetneq B^m_{k+1}$ holds. A possible approach to prove this conjecture for a given $k \in \mathbb{N}$ would be to specify a particular pair of natural numbers (m, n) with $3 \leqslant m \leqslant n$ for which (a) some Lower-Bound-Lemma implies $K_{m,n} \notin B^m_k$ and (b) a B^m_{k+1}-EPG representation can be constructed. The identification of such a pair (m, n), $3 \leqslant m \leqslant n$, would clearly prove the existence of a complete bipartite graph $K_{m,n}$ with monotonic bend number equal to k for any $k \geqslant 2$.

Finally, the size of (monotonic) EPG representations is another subject of interest. In particular, it would be interesting to determine the minimum number of grid lines needed for a B_k-EPG representation and B^m_ℓ-EPG representation of a graph G with $b(G) \leqslant k$ and $b^m(G) \leqslant \ell$, respectively.

Data availability

No data was used for the research described in the article.

References

[1] Andrei Asinowski, Elad Cohen, Martin Charles Golumbic, Vincent Limouzy, Marina Lipshteyn, Michal Stern, Vertex intersection graphs of paths on a grid, J. Graph Algorithms Appl. 16 (2) (2012) 129–150.
[2] Andrei Asinowski, Andrew Suk, Edge intersection graphs of systems of paths on a grid with a bounded number of bends, Discrete Appl. Math. 157 (14) (2009) 3174–3180.
[3] Therese Biedl, Martin Derka, Vida Dujmović, Pat Morin, EPG-representations with small grid-size, in: Graph Drawing and Network Visualization, in: Lecture Notes in Comput. Sci., vol. 10692, Springer, 2018, pp. 184–196.
[4] Therese Biedl, Michal Stern, Edge-intersection graphs of k-bend paths in grids, in: Computing and Combinatorics, Springer, 2009, pp. 86–95.
[5] Flavia Bonomo, María Pía Mazzoleni, Maya Stein, Clique coloring B_1-EPG graphs, Discrete Math. 340 (5) (2017) 1008–1011.
[6] M. Bougeret, S. Bessy, D. Gonçalves, C. Paul, On independent set on B1-EPG graphs, in: Approximation and Online Algorithms, in: Lecture Notes in Comput. Sci., vol. 9499, Springer, 2015, pp. 158–169.
[7] Nicolas Bousquet, Marc Heinrich, Computing maximum cliques in B_2-EPG graphs, in: Graph-Theoretic Concepts in Computer Science, in: Lecture Notes in Comput. Sci., vol. 10520, Springer, 2017, pp. 140–152.
[8] M.L. Brady, M. Sarrafzadeh, Stretching a knock-knee layout for multilayer wiring, IEEE Trans. Comput. 39 (1) (1990) 148–151.
[9] Eranda Çela, Elisabeth Gaar, Monotonic representations of outerplanar graphs as edge intersection graphs of paths on a grid, J. Graph Algorithms Appl. 26 (4) (2022) 519–552.
[10] Kathie Cameron, Steven Chaplick, Chính T. Hoàng, Edge intersection graphs of L-shaped paths in grids, Electron. Notes Discrete Math. 44 (2013) 363–369.
[11] Zakir Deniz, Simon Nivelle, Bernard Ries, David Schindl, On split B_1-EPG graphs, in: LATIN 2018: Theoretical Informatics, in: Lecture Notes in Comput. Sci., vol. 10807, Springer, 2018, pp. 361–375.
[12] Dror Epstein, Martin Charles Golumbic, Gila Morgenstern, Approximation algorithms for B_1-EPG graphs, in: Algorithms and Data Structures, Springer, 2013, pp. 328–340.
[13] Mathew C. Francis, Abhiruk Lahiri, VPG and EPG bend-numbers of Halin graphs, Discrete Appl. Math. 215 (2016) 95–105.
[14] Martin Charles Golumbic, Robert E. Jamison, Edge and vertex intersection of paths in a tree, Discrete Math. 55 (2) (1985a) 151–159.
[15] Martin Charles Golumbic, Robert E. Jamison, The edge intersection graphs of paths in a tree, J. Combin. Theory Ser. B 38 (1) (1985b) 8–22.
[16] Martin Charles Golumbic, Marina Lipshteyn, Michal Stern, Edge intersection graphs of single bend paths on a grid, Networks 54 (3) (2009) 130–138.
[17] Daniel Heldt, Kolja Knauer, Torsten Ueckerdt, On the bend-number of planar and outerplanar graphs, in: LATIN 2012: Theoretical Informatics, Springer, 2012, pp. 458–469.
[18] Daniel Heldt, Kolja Knauer, Torsten Ueckerdt, Edge-intersection graphs of grid paths: The bend-number, Discrete Appl. Math. 167 (2014) 144–162.
[19] Paul Molitor, A survey on wiring, J. Inf. Process. Cybern. 27 (1) (1991) 3–19.
[20] Martin Pergel, Paweł Rzążewski, On edge intersection graphs of paths with 2 bends, in: Graph-Theoretic Concepts in Computer Science, in: Lecture Notes in Comput. Sci., vol. 9941, Springer, Berlin, 2016, pp. 207–219.