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Closed-Form Stability Conditions and Differentiation Error Bounds for Levant's Arbitrary Order Robust Exact Differentiator

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Abstract: This paper proposes, for the first time, closed-form stability conditions and differentiation error upper bounds for arbitrary orders of Levant's robust exact differentiator. Based on these conditions, a tuning rule is provided to select the differentiator parameters. Numerical examples demonstrate the application of the proposed tuning rule and compare the conservativeness of the obtained conditions to existing results.

Keywords: Levant's differentiator; stability condition; measurement noise; Lyapunov function

1. INTRODUCTION

Differentiation in presence of measurement noise is an important task in many control applications, such as velocity estimation or fault detection, for example. An important class of differentiators are the robust exact differentiators first introduced by Levant (1998) and generalized to arbitrary differentiation order in (Levant, 2003). In the absence of noise, these differentiators yield the exact value of the input signal's derivatives. At the same time, they also exhibit robustness with respect to measurement noise in the form of an optimal asymptotic differentiation error bound as a function of the noise amplitude, cf. e.g. Levant et al. (2017).

When it comes to rigorous stability conditions for the robust exact differentiator, most existing works focus on the first order differentiator, also known as the super-twisting algorithm. Even for this algorithm, bounds on the differentiation error in presence of measurement noise are only scarcely studied, e.g., by Angulo et al. (2012). In contrast, Polyakov and Poznyak (2009); Moreno and Osorio (2012); Utkin (2013); Seeber and Horn (2017); Brogliato et al. (2020), among others, propose different Lyapunov functions and stability conditions for the super-twisting algorithm, and a necessary and sufficient stability condition is obtained in (Seeber and Horn, 2018). A notable breakthrough for higher differentiation orders has been achieved by Cruz-Zavala and Moreno (2019), who propose a family of Lyapunov function for all differentiation orders. Deriving stability conditions from these Lyapunov functions requires the (numerical) computation of maxima of homogeneous functions on the unit sphere, however. For faster and easier tuning, closed-form stability conditions and differentiation error bounds are clearly desirable. However, to the best of the author's knowledge, no such conditions or error bounds exist for the arbitrary order robust exact differentiator.

The present paper proposes, for the first time, *closed-form stability conditions* for arbitrary differentiation orders along with differentiation error upper bounds in presence of *measurement noise*. The results are based on a new Lyapunov function, which is constructed in the spirit of the one by Cruz-Zavala and Moreno (2019) but yields stability conditions and error bounds in closed form.

The paper is structured as follows: Section 2 introduces the considered differentiation problem and recapitulates the arbitrary order robust exact differentiator. The main result—stability conditions and differentiation error bounds for arbitrary differentiation orders—are stated in Section 3, along with a parameter tuning rule. Section 4 proposes a new Lyapunov function that is then used to formally prove the result. Finally, Section 5 applies the conditions and tuning rule in numerical examples, comparing the proposed result with existing conditions, and Section 6 draws conclusions and provides a brief outlook. Appendix A contains the proofs of all lemmata.

Notation: Boldface lowercase letters denote vectors, and $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ denote the reals, nonnegative reals, and positive reals. The abbreviation $\lfloor y \rceil^p = |y|^p \operatorname{sign}(y)$ is used for $y, p \in \mathbb{R}$, and $\lfloor y \rceil^0 = \operatorname{sign}(y)$. The jth time derivative of a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is written as $f^{(j)}$, and $f^{(0)} = f$.

2. PRELIMINARIES

In the following, the considered arbitrary order robust exact differentiator is introduced along with the corresponding differentiation problem.

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2.1 Arbitrary Order Differentiation

Consider a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$. For an integer $n > 1$, the problem of obtaining derivatives of f up to order $n-1$ is considered. For this purpose, the class of signals to be differentiated is restricted to functions f, for which these derivatives exist and the $(n-1)$th derivative is Lipschitz continuous, i.e., for which

$$|f^{(n)}(t)| \leq L \tag{1}$$

holds almost everywhere on $\mathbb{R}_{\geq 0}$ with some Lipschitz constant $L \in \mathbb{R}_{>0}$. The signal is furthermore assumed to be corrupted by a measurement noise η, such that the input u to the differentiator is given by $u = f + \eta$, with $|\eta(t)| \leq N$ for some noise amplitude $N \in \mathbb{R}_{\geq 0}$.

2.2 Robust Exact Differentiator

The $(n-1)$th order robust exact differentiator is given by the nth order system

$$\dot{z}_1 = \lambda_1 L^{\frac{1}{n}} \lfloor u - z_1 \rceil^{\frac{n-1}{n}} + z_2 \tag{2a}$$

$$\dot{z}_2 = \lambda_2 L^{\frac{2}{n}} \lfloor u - z_1 \rceil^{\frac{n-2}{n}} + z_3 \tag{2b}$$

$$\vdots$$

$$\dot{z}_{n-1} = \lambda_{n-1} L^{\frac{n-1}{n}} \lfloor u - z_1 \rceil^{\frac{1}{n}} + z_n \tag{2c}$$

$$\dot{z}_n = \lambda_n L \lfloor u - z_1 \rceil^{0} \tag{2d}$$

with input u, positive parameters $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$, and outputs $y_i = z_{i+1}$ $(i = 1, \ldots, n-1)$ corresponding to estimates for the respective derivatives $f^{(1)}, \ldots, f^{(n-1)}$ of the function f. Without any additional knowledge on the derivatives of f, it is usually prudent to choose the initial values as $z_1(0) = u(0)$ and $z_2(0) = \ldots = z_n(0) = 0$. Solutions of this system are understood in the sense of Filippov (1988).

3. MAIN RESULT

In order to state the main result, an additional restriction is imposed on the signal f and the measurement noise η to guarantee a certain well-posedness of the differentiation error trajectories. The following assumption states this restriction, along with the previously stated requirements.

Assumption 1. Let $L \in \mathbb{R}_{>0}$ and $N \in \mathbb{R}_{\geq 0}$. The signal f and the noise η satisfy the following three conditions:

(a) f is $n-1$ times differentiable, and $f^{(n-1)}$ is Lipschitz continuous, i.e., $|f^{(n)}(t)| \leq L$ almost everyhwere;
(b) η is Lebesgue measureable and uniformly bounded according to $|\eta(t)| \leq N$;
(c) f and η are such that, for all $i = 1, \ldots, n$, the signal $z_i - f^{(i-1)}$ is equal to zero only on a set comprised of *countably* many time intervals or time instants.

According to the item (c), signals such as the tracking error $z_1 - f$ or the differentiation error $z_2 - \dot{f}$ can exhibit an infinite sequence of isolated zeros due to oscillations followed by being zero on the remaining time interval, but may not have zeros that coincide with some fractal set such as the Cantor set, for example. This restriction is a technical requirement for dealing with the fact that the Lyapunov function proposed in this paper, unlike the one by Cruz-Zavala and Moreno (2019), is not smooth. While this makes the formal results presented here somewhat less general, this assumption is not an actual restriction for practical applications.

The reason for this is that, in a practical implementation, any continuous-time solution of the differentiator has to be approximated by a discrete-time system using sampled measurements of u, cf. e.g. Livne and Levant (2014). By a slight modification of f and η in between the sampling time instants, every continous-time solution with uncountably many zero crossings can conceivably be approximated by a modified solution with countably many zeros without changing the sampled measurements. Hence, although the discrete-time case is not formally considered here, it is reasonable to conjecture that the presented analysis is valid (in an asymptotic sense) also for a discrete-time differentiator implementation without the restriction in Assumption 1, item (c).

The following theorem states the main result: closed-form stability conditions for the arbitrary order robust exact differentiator and upper bounds for its differentiation error in the presence of measurement noise. It is proven in Section 4.3.

Theorem 2. Let $a_1, \ldots, a_{n-1} \in (1, 2)$ and recursively define $\beta_1, \ldots, \beta_n$ and $\gamma_0, \gamma_1, \ldots, \gamma_n$ via

$$\beta_{j+1} = \left(\beta_j^j + \frac{a_j}{\gamma_j^j} \right)^{\frac{1}{j}}, \qquad \gamma_{j+1} = \left(\frac{2}{2 - a_j} \right)^{\frac{1}{j}} \gamma_j \tag{3}$$

with $\beta_1 = 1$ and $\gamma_0 = \gamma_1 = 2$. Suppose that the parameters $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ satisfy $\lambda_n > 1$ and

$$\frac{\lambda_i}{\lambda_{i-1}} > \frac{\lambda_{i+1}}{\lambda_i} \mu_{n-i+1} \tag{4a}$$

for $i = 1, \ldots, n-1$ with the abbreviation $\lambda_0 = 1$ and

$$\mu_j = \frac{j}{j-1} \cdot \frac{\gamma_{j-1}^{j-1}}{\gamma_{j-2}^{j-2}} \cdot \frac{\beta_j}{a_{j-1} - 1} \qquad (j = 2, \ldots, n). \tag{4b}$$

Consider the differentiator (2) with input $u = f + \eta$ satisfying Assumption 1. Then, for every given initial condition $z_1(0), \ldots, z_n(0)$, there exists a finite convergence time $\tau \in \mathbb{R}_{\geq 0}$ such that the differentiation errors are bounded from above according to

$$|y_i(t) - f^{(i)}(t)| \leq \left(\frac{\beta_{n-i} \gamma_n}{\sqrt[n]{2}} \right)^{n-i} \lambda_i N^{\frac{n-i}{n}} L^{\frac{i}{n}} \tag{5}$$

for all $t \geq \tau$ and $i = 1, \ldots, n-1$.

Remark 3. (Tuning Rule). To obtain a set of parameters $\lambda_1, \ldots, \lambda_n$ satisfying the stability condition (4), first choose positive constants $a_1, \ldots, a_{n-1} \in (1, 2)$ and compute $\mu_2, \ldots, \mu_n$ by using (4b) and the recursions (3). Then, select $\bar{\mu}_1 = 1$ and $\bar{\mu}_2, \ldots, \bar{\mu}_n$ such that $\bar{\mu}_i > \mu_i$ for $i = 2, \ldots, n$. For any given $\lambda_n > 1$, computing the remaining differentiator parameters $\lambda_1, \ldots, \lambda_{n-1}$ according to

$$\lambda_j = \lambda_n^{\frac{j}{n}} \frac{\prod_{k=n-j+1}^{n} \prod_{i=1}^{k} \bar{\mu}_i}{\left(\prod_{k=1}^{n} \prod_{i=1}^{k} \bar{\mu}_i \right)^{\frac{j}{n}}} \qquad (j = 1, \ldots, n-1) \tag{6}$$

then guarantees condition (4) to be satisfied.

4. LYAPUNOV FUNCTION

Consider the error variables $x_i = z_i - f^{(i-1)}$, $i = 1, \dots, n$. These are governed by

$$\dot{x}_1 = -\lambda_1 L^{\frac{1}{n}} \lfloor x_1 - \eta \rceil^{\frac{n-1}{n}} + x_2 \tag{7a}$$

$$\dot{x}_2 = -\lambda_2 L^{\frac{2}{n}} \lfloor x_1 - \eta \rceil^{\frac{n-2}{n}} + x_3 \tag{7b}$$

$$\vdots$$

$$\dot{x}_{n-1} = \lambda_{n-1} L^{\frac{n-1}{n}} \lfloor x_1 - \eta \rceil^{\frac{1}{n}} + x_n \tag{7c}$$

$$\dot{x}_n = -\lambda_n L \lfloor x_1 - \eta \rceil^{0} - f^{(n)}. \tag{7d}$$

In the following, this error system is first transformed to a recursive form which is also used to construct the Lyapunov function in (Cruz-Zavala and Moreno, 2019), and which is similar to the original recursive representation of the differentiator in (Levant, 2003). Then, a new Lyapunov function is proposed which is used to formally prove Theorem 2.

4.1 Transformed Error System

Using the abbreviations $\lambda_0 = \lambda_{n+1} = 1$, define transformed parameters

$$\kappa_j = \frac{\lambda_{n-j+1}}{\lambda_{n-j}} \qquad (j = 0, \dots, n) \tag{8}$$

and consider the state transform

$$\xi_j = \frac{x_{n-j+1}}{\lambda_{n-j} L} \qquad (j = 1, \dots n) \tag{9}$$

as in (Cruz-Zavala and Moreno, 2019), with an additional renumbering of the states for greater convenience.

With the abbreviations $\eta_{n+1} = -\frac{\eta}{L}$ and $\xi_0 = -\kappa_0 \frac{f^{(n)}}{L}$, the scaled error variables ξ_i are then governed by

$$\dot{\xi}_1 = -\kappa_1 \eta_1, \qquad \eta_1 = \lfloor \xi_1 + \eta_2 \rceil^0 - \xi_0 \tag{10a}$$

$$\dot{\xi}_2 = -\kappa_2 \eta_2, \qquad \eta_2 = \lfloor \xi_2 + \eta_3 \rceil^{\frac{1}{2}} - \xi_1 \tag{10b}$$

$$\dot{\xi}_3 = -\kappa_3 \eta_3, \qquad \eta_3 = \lfloor \xi_3 + \eta_4 \rceil^{\frac{2}{3}} - \xi_2 \tag{10c}$$

$$\vdots \qquad\qquad \vdots$$

$$\dot{\xi}_n = -\kappa_n \eta_n, \qquad \eta_n = \lfloor \xi_n + \eta_{n+1} \rceil^{\frac{n-1}{n}} - \xi_{n-1}. \tag{10d}$$

Note that, due to Assumption 1, $|\eta_{n+1}| \leq \frac{N}{L}$ and $|\xi_0| \leq \kappa_0$ holds almost everywhere.

4.2 Lyapunov Function

Introduce the state vector $\boldsymbol{\xi} = [\xi_1 \quad \dots \quad \xi_n]^{\mathrm{T}}$ and consider functions $V_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ recursively defined as

$$V_1(\boldsymbol{\xi}) = |\xi_1|, \tag{11a}$$

$$V_j(\boldsymbol{\xi}) = \max\left\{ V_{j-1}(\boldsymbol{\xi}), \alpha_j^{-\frac{1}{j-1}} W_j(\boldsymbol{\xi}) \right\}. \tag{11b}$$

for $j = 2, \dots, n$ with

$$W_j(\boldsymbol{\xi}) = \left| \lfloor \xi_j \rceil^{\frac{j-1}{j}} - \xi_{j-1} \right|^{\frac{1}{j-1}} \tag{11c}$$

and positive constants $\alpha_2, \dots, \alpha_n$.

The following lemma provides a guideline for the choice of the parameters α_j and states an upper bound on the states ξ_j using the functions V_j thus defined. Its proof is performed by induction over j and is given in the appendix.

Lemma 4. Suppose that $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}_{>0}$ $(j = 1, \dots, n)$ satisfy $\alpha_1 = 0$, $\beta_1 = 1$, $\gamma_1 = 2$, and $\alpha_{j+1}\gamma_j^j \in (1, 2)$ as well as

$$\beta_{j+1} = (\beta_j^j + \alpha_{j+1})^{\frac{1}{j}}, \quad \gamma_{j+1} = \left(\frac{2\gamma_j^j}{2 - \alpha_{j+1}\gamma_j^j} \right)^{\frac{1}{j}} \tag{12}$$

for $j = 1, \dots, n-1$. Then, the functions $V_j : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ $(j = 1, \dots, n)$ defined in (11) satisfy

$$|\xi_j| \leq \beta_j^j V_j(\boldsymbol{\xi})^j \tag{13}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^n$.

As a Lyapunov function candidate, the function V_n is now considered, which is positive definite and radially unbounded. It will be shown that, for each j, the larger one of the two functions in (11b) is always strictly decreasing provided that V_j is larger than some function of a bound on η_{j+1}. By induction, this ensures ultimate boundedness of V_n in terms of the bound N on $\eta = -L\eta_{n+1}$. Since $V_1, \dots, V_n$ are not everywhere differentiable, some further technical arguments are required to complete the formal proof later on. For the cases (i.e., time instants) when they *are* differentiable, the following lemma allows to bound their time derivative by a negative constant. The proof is performed by induction over j and is given in the appendix.

Lemma 5. Let $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}_{>0}$ $(j = 1, \dots, n)$ satisfy the conditions of Lemma 4. Suppose that $\kappa_1, \dots, \kappa_n \in \mathbb{R}_{>0}$ satisfy $\kappa_0 \in (0, 1)$, $\kappa_1 > 0$, and

$$\frac{\kappa_m}{m} \frac{\alpha_m \gamma_m^{m-1} - 2}{\beta_m} > \frac{\kappa_{m-1}}{m-1} \Big(\alpha_{m-1}\gamma_m^{m-1} + 2^{\frac{1}{m-1}} \gamma_m \left(\alpha_m \gamma_m^{m-1} + 2 \right)^{\frac{m-2}{m-1}} \Big) \tag{14}$$

for $m = 2, \dots, n$. Consider system (10) and the functions $V_j : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined in (11). Then, there exist $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}_{>0}$ such that for all $N \in \mathbb{R}_{>0}$ and all integers $j = 1, \dots, n$, the three inequalities $V_j(\boldsymbol{\xi}(t)) > \gamma_j \left(\frac{N}{2L}\right)^{\frac{1}{j}}$, $|\eta_{j+1}(t)| \leq \frac{N}{L}$, and $|\xi_0(t)| \leq \kappa_0$ imply $\frac{\mathrm{d}}{\mathrm{d}t} V_j(\boldsymbol{\xi}(t)) \leq -\varepsilon_j$ for all trajectories $\boldsymbol{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (10) and all $t \in \mathbb{R}_{\geq 0}$ where the derivative exists.

Since V_n is not everywhere differentiable, it is not yet possible to conclude from the previous lemma that V_n is non-increasing with respect to time. To deal with this issue, the following technical lemma is used. It relies on the technical assumption introduced in Assumption 1, item (c) to show, essentially, that $V_n(\boldsymbol{\xi}(t))$ is generalized absolutely continuous in the restricted sense (ACG_*), as defined in (Gordon, 1994, Definition 6.1), to conclude differentiability almost everywhere and, when V_n is furthermore non-increasing with respect to time, to conclude also absolute continuity. The formal proof is provided in the appendix.

Lemma 6. Suppose that the function $\boldsymbol{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is absolutely continuous and that the set

$$Z_j = \{t \in \mathbb{R}_{\geq 0} : \xi_j(t) = 0\} \qquad (j = 1, \dots, n) \tag{15}$$

is comprised of countably many time intervals or time instants. Consider the function $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as $V(t) = V_n(\boldsymbol{\xi}(t))$ with $V_n : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as defined in (11). Let $I \subseteq \mathbb{R}_{\geq 0}$ be an interval, and suppose that $V(t) > 0$ for all $t \in \bar{I}$. Then, V is differentiable almost everywhere on I. Moreover, if $\dot{V}(t) \leq 0$ holds almost everyhwere on I, then V is absolutely continuous on I.

Using the introduced lemmata, the following proposition will now be proven. It shows that all trajectories of the transformed error system (10) enter a sublevel set of V_n in finite time, whose size depends on the noise amplitude N.

Proposition 7. Let Assumption 1 hold with $L \in \mathbb{R}_{>0}$, $N \in \mathbb{R}_{\geq 0}$. Suppose that the conditions of Lemma 5 are fulfilled, and let $\varepsilon_n \in \mathbb{R}_{>0}$ and $V_n : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be as in that lemma. Then, every trajectory $\boldsymbol{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of system (10) satisfies

$$V_n(\boldsymbol{\xi}(t)) \leq \gamma_n \left(\frac{N}{2L}\right)^{\frac{1}{n}} \tag{16}$$

for all $t \geq \tau = V_n(\boldsymbol{\xi}(0))/\varepsilon_n$.

Proof. Define $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as $V(t) = V_n(\boldsymbol{\xi}(t))$. Denote by $Z_j = \{t \in \mathbb{R}_{\geq 0} : \xi_j(t) = 0\}$ the set of zeros of ξ_j. Due to Assumption 1, item (c), each Z_j is the union of countably many time intervals or time instants. Hence, $\dot{V}(t)$ exists almost everyhwere on time intervals where V is positive due to Lemma 6.

To prove the claim, assume that $V(t) > \gamma_n \left(\frac{N}{2L}\right)^{\frac{1}{n}}$ on some interval $I = (\tau_1, \tau_2)$ with either $\tau_1 = 0$, $\tau_2 > V(0)/\varepsilon$, or $\tau_2 > \tau_1 \geq V(0)/\varepsilon$ and $V(\tau_1) = \gamma_n \left(\frac{N}{2L}\right)^{\frac{1}{n}}$. Since $\dot{V}(t)$ exists almost everyhwere on I due to Lemma 6, $\dot{V}(t) \leq -\varepsilon_n < 0$ holds almost everywhere on I due to Lemma 5, and hence V is absolutely continuous on I again due to Lemma 6. Hence, the comparison lemma, cf. (Khalil, 2002), may be applied, yielding the contradiction $V(\tau_2) \leq V(0) - \varepsilon_n \tau_2 < 0$ in the case $\tau_1 = 0$ or the contradiction $V(\tau) < V(\tau_1) = \gamma_n \left(\frac{N}{2L}\right)^{\frac{1}{n}}$ for all $\tau \in (\tau_1, \tau_2)$ in the case $\tau_1 > 0$. This concludes the proof. □

With this proposition, the main theorem is now proven.

4.3 Proof of Theorem 2

In order to prove Theorem 2, first note that with

$$\alpha_j = \frac{a_{j-1}}{\gamma_{j-1}^{j-1}} \qquad (j = 2, \ldots, n) \tag{17}$$

the constants $\alpha_j, \beta_j, \gamma_j$ satisfy the conditions of Lemmata 4 and 5. Moreover, (4) is equivalent to condition (14) of Lemma 5; to see this, note that (14) may be written as $\kappa_m > \frac{m\beta_m}{m-1}\tilde{\mu}_m \kappa_{m-1}$ with

$$\tilde{\mu}_m = \frac{\alpha_{m-1}\gamma_m^{m-1} + 2^{\frac{1}{m-1}}\gamma_m(\alpha_m\gamma_m^{m-1} + 2)^{\frac{m-2}{m-1}}}{\alpha_m\gamma_m^{m-1} - 2} \tag{18}$$

Using the recursion (12), the numerator may be rewritten as

$$\begin{aligned}\tilde{\mu}_m &= \frac{\gamma_m^{m-1}(\alpha_{m-1} + 2^{\frac{1}{m-1}}(\alpha_m + 2\gamma_m^{-(m-1)})^{\frac{m-2}{m}})}{\alpha_m\gamma_m^{m-1} - 2} \\ &= \frac{\gamma_m^{m-1}(\alpha_{m-1} + 2\gamma_{m-1}^{-(m-2)})}{\alpha_m\gamma_m^{m-1} - 2} = \frac{\alpha_{m-1} + 2\gamma_{m-1}^{-(m-2)}}{\alpha_m - 2\gamma_m^{-(m-1)}}.\end{aligned} \tag{19}$$

Substituting the recursion again (verifying separately that equality holds also for $m = 2$), and using (17) yields

$$\tilde{\mu}_m = \frac{\alpha_{m-1} + \frac{2-\alpha_{m-1}\gamma_{m-2}^{m-2}}{\gamma_{m-2}^{m-2}}}{\alpha_m - \frac{2-\alpha_m\gamma_{m-1}^{m-1}}{\gamma_{m-1}^{m-1}}} = \frac{\gamma_{m-1}^{m-1}}{\gamma_{m-2}^{m-2}} \frac{1}{\alpha_m\gamma_{m-1}^{m-1} - 1} \tag{20}$$

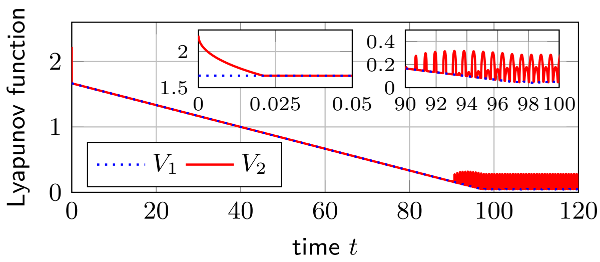

Fig. 1. Numerically obtained time evolution of $V_1(\mathbf{x}(t))$ and Lyapunov function $V_2(\mathbf{x}(t))$ in (25) with parameters $L = 1$, $N = \frac{1}{32}$, $\lambda_1 = 6$, $\lambda_2 = 1.1$, $a_1 = \frac{3}{2}$; the corresponding ultimate bound for V_2 from Lemma 5 is $\gamma_2\sqrt{N/(2L)} = 1$.

from which equivalence to (4) is obtained using (17), noting that $\mu_m = \frac{m\beta_m}{m-1}\tilde{\mu}_m$, and substituting $m = n-i+1$. For $N > 0$, Proposition 7 and Lemma 4 may thus be used to conclude that

$$|\xi_j(t)| \leq \beta_j^j V_j(\boldsymbol{\xi}(t))^j \leq \beta_j^j V_n(\boldsymbol{\xi}(t))^j \leq \beta_j^j \gamma_n^j \left(\frac{N}{2L}\right)^{\frac{j}{n}} \tag{21}$$

and hence

$$\begin{aligned}|y_i(t) - f^{(i)}(t)| &= |x_{i+1}(t)| = \lambda_i L |\xi_{n-i}(t)| \\ &\leq \left(\frac{\beta_{n-i}\gamma_n}{\sqrt[n]{2}}\right)^{n-i} \lambda_i N^{\frac{n-i}{n}} L^{\frac{i}{n}}\end{aligned} \tag{22}$$

holds after a finite time depending on the initial condition. For $N = 0$, finally, validity of the claim follows from the case $N > 0$ and the homogeneity of system (7). □

5. NUMERICAL EXAMPLES

To assess the conservativeness of the proposed stability condition, it is first evaluated for the super-twisting algorithm, i.e., for the first-order robust exact differentiator with $n = 2$. In this case, the only free parameter in Theorem 2 is $a_1 \in (1, 2)$, from which

$$\beta_1 = 1, \quad \beta_2 = 1 + \frac{a_1}{2}, \quad \gamma_1 = 2, \quad \gamma_2 = \frac{4}{2 - a_1} \tag{23}$$

are obtained. The stability condition (4) is then given by

$$\lambda_1^2 > \mu_2\lambda_2, \qquad \lambda_2 > 1 \tag{24a}$$

with

$$\mu_2 = 2\gamma_1\frac{\beta_2}{a_1 - 1} = 4\frac{1 + \frac{a_1}{2}}{a_1 - 1} > 8. \tag{24b}$$

The least conservative condition is obtained in the limit $a_1 \to 2$, yielding $\mu_2 \to 8$.

With $\alpha_2 = \frac{a_1}{\gamma_1} = \frac{a_1}{2}$, $\xi_1 = \frac{x_2}{\lambda_1 L}$, $\xi_2 = \frac{x_1}{L}$, relation (11) yields functions V_1, V_2 in original coordinates, i.e., in $\mathbf{x}$, as

$$V_1(\mathbf{x}) = \frac{|x_2|}{\lambda_1 L}, \quad V_2(\mathbf{x}) = \frac{\max\{|x_2|, \frac{2}{a_1}|\lambda_1 \lfloor x_1 \rceil^{\frac{1}{2}} - x_2|\}}{\lambda_1 L}. \tag{25}$$

Consider differentiation of the signal $f(t) = Lt^2/2 + 10t$ with noise $\eta(t) = N \sin 10t$ and parameters chosen as $N = \frac{1}{32}$, $L = 1$, $a_1 = \frac{3}{2}$, $\lambda_1 = 6$, $\lambda_2 = 1.1$ to satisfy (24). Fig. 1 depicts the time evolution of the Lyapunov function V_2 as well as V_1 obtained using a numerical forward Euler simulation with step size 10^{-5} and initial values $z_1(0) = z_2(0) = 0$. One can see that V_2 complies with the ultimate bound $\gamma_2\sqrt{N/(2L)} = 1$ from Lemma 5. Note that obtaining a bound for V_1 from the same lemma requires computing a bound on $\eta_2 = \lfloor \frac{x_1-\eta}{L} \rceil^{\frac{1}{2}} - \frac{x_2}{\lambda_1 L}$ first.

In contrast to (24), the Lyapunov functions proposed by Cruz-Zavala and Moreno (2019) allow to conclude stability already for $\lambda_1^2 > 4\lambda_2$ with $\lambda_2 > 1$. Thus, the Lyapunov function presented in this paper yields somewhat more conservative conditions than the one proposed by Cruz-Zavala and Moreno (2019). This increased conservativeness is balanced by the simpler construction of the Lyapunov function, which here allows to obtain stability conditions and differentiation error bounds in closed form.

To demonstrate the tuning rule from Remark 3, consider now the case $n = 3$ and select $a_1 = a_2 = \frac{3}{2}$. This yields

$$\beta_1 = 1, \qquad \beta_2 = \frac{7}{4}, \qquad \beta_3 = \frac{1}{8}\sqrt{\frac{395}{2}}, \tag{26a}$$

$$\gamma_1 = 2, \qquad \gamma_2 = 8, \qquad \gamma_3 = 16 \tag{26b}$$

and

$$\mu_2 = 14, \qquad \mu_3 = 12\sqrt{395/2}. \tag{27}$$

Selecting $\bar{\mu}_2 = 22 > \mu_2$ and $\bar{\mu}_3 = 176 > \mu_3$ the parameters

$$\lambda_1 = \lambda_3^{\frac{1}{3}} \frac{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}{(\bar{\mu}_1^3\bar{\mu}_2^2\bar{\mu}_3)^{\frac{1}{3}}} = 88\lambda_3^{\frac{1}{3}} \tag{28a}$$

$$\lambda_2 = \lambda_3^{\frac{2}{3}} \frac{\bar{\mu}_1^2\bar{\mu}_2^2\bar{\mu}_3}{(\bar{\mu}_1^3\bar{\mu}_2^2\bar{\mu}_3)^{\frac{2}{3}}} = 44\lambda_3^{\frac{2}{3}} \tag{28b}$$

are then obtained for any $\lambda_3 > 1$.

6. CONCLUSIONS AND OUTLOOK

Closed-form stability conditions and differentiation error bounds for the arbitrary order robust exact differentiator were presented for the first time. The conditions and bounds are obtained from a novel Lyapunov function that is constructed in a recursive fashion for arbitrary differentiation orders. A disadvantage of the proposed approach is its increased conservativeness compared to an already existing family of Lyapunov functions. The latter yield stability conditions only numerically, however, by solving optimization problems. Hence, future work may focus on reducing the conservativeness of the proposed approach while still maintaining sufficient simplicity in the construction of the Lyapunov function, in order to possibly obtain less conservative closed-form stability conditions.

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Appendix A. PROOFS

In order to prove the lemmata in this paper, the following auxiliary lemma from (Cruz-Zavala et al., 2018) is used.

Lemma 8. The inequality

$$\left| \lfloor x-\eta \rceil^{\frac{m-1}{m}} - \lfloor x \rceil^{\frac{m-1}{m}} \right| \le 2^{\frac{1}{m}} |\eta|^{\frac{m-1}{m}}. \tag{A.1}$$

holds for all $x, \eta \in \mathbb{R}$ and all positive integers m.

Proof. According to (Cruz-Zavala et al., 2018, Lemma 7), $|\lfloor x_1 \rceil^p + \lfloor x_2 \rceil^p|^{1/p} \le 2^{\frac{1}{p}-\frac{1}{q}} |\lfloor x_1 \rceil^q + \lfloor x_2 \rceil^q|^{1/q}$ holds for all $x_1, x_2 \in \mathbb{R}$ and all $q \ge p > 0$. Set $p = \frac{m-1}{m}$, $q = 1$, $x_1 = x - \eta$, $x_2 = -x$ to obtain the claimed inequality. □

A.1 Proof of Lemma 4

For $j = 1$, the statement is obvious from $V_1(\boldsymbol{\xi}) = |\xi_1|$ and $\beta_1 = 1$. Suppose that the statement is true for $j = m-1$ for some $2 \le m \le n$. Then,

$$\begin{aligned} |\xi_m| &\le \left(|\xi_{m-1}| + \left| \lfloor \xi_m \rceil^{\frac{m-1}{m}} - \xi_{m-1} \right| \right)^{\frac{m}{m-1}} \\ &\le \left(\beta_{m-1}^{m-1} V_{m-1}(\boldsymbol{\xi})^{m-1} + \alpha_m V_m(\boldsymbol{\xi})^{m-1} \right)^{\frac{m}{m-1}} \\ &\le (\beta_{m-1}^{m-1} + \alpha_m)^{\frac{m}{m-1}} V_m(\boldsymbol{\xi})^m = \beta_m^m V_m(\boldsymbol{\xi})^m, \end{aligned} \tag{A.2}$$

showing that the statement holds also for $j = m$. □

A.2 Proof of Lemma 5

Consider a time instant $t \in \mathbb{R}_{\geq 0}$. For simplicity, the time argument is suppressed in the following, writing ξ_j, η_j, V_j, and $\dot{V}_j$ instead of $\xi_j(t), \eta_j(t), V_j(\boldsymbol{\xi}(t)$, and $\frac{\mathrm{d}}{\mathrm{d}t} V_j(\boldsymbol{\xi}(t))$, respectively. The proof is performed by induction over j. For $j = 1$, $V_1 = |\xi_1| > N/L$ implies $\lfloor \xi_1 + \eta_2 \rceil^0 = \lfloor \xi_1 \rceil^0$; hence, $\dot{V}_1 \leq -\kappa_1(1-\kappa_0) =: -\varepsilon_1$. Suppose now that the statement is true for $j = m-1$ for some $2 \leq m \leq n$, define $\tilde{N}_{m+1} = N/L$, and note that $|\eta_{m+1}| \leq \tilde{N}_{m+1}$ by assumption. Then, according to Lemma 8,

$$|\eta_m| \leq \left| \lfloor \xi_m \rceil^{\frac{m-1}{m}} - \xi_{m-1} \right| + 2^{\frac{1}{m}} |\eta_{m+1}|^{\frac{m-1}{m}} \leq \alpha_m V_m^{m-1} + 2(\tilde{N}_{m+1}/2)^{\frac{m-1}{m}} =: \tilde{N}_m, \tag{A.3}$$

wherein $\tilde{N}_m$ is introduced as an abbreviation, and

$$\begin{aligned} |\eta_{m-1}| &\leq \alpha_{m-1} V_{m-1}^{m-2} + 2^{\frac{1}{m-1}} |\eta_m|^{\frac{m-2}{m-1}} \\ &\leq \alpha_{m-1} V_m^{m-2} + 2\left(\frac{\alpha_m}{2} V_m^{m-1} + (\tilde{N}_{m+1}/2)^{\frac{m-1}{m}}\right)^{\frac{m-2}{m-1}} \\ &= \alpha_{m-1} V_m^{m-2} + 2\left(\tilde{N}_m/2\right)^{\frac{m-2}{m-1}} = \tilde{N}_{m-1}. \end{aligned} \tag{A.4}$$

Note that this is true even for $m = 2$, because then

$$|\eta_1| = \left| \lfloor \xi_1 - \eta_2 \rceil^0 - \delta_k \right| \leq 1 + \kappa_0 \leq 2. \tag{A.5}$$

It will be shown that there exists a constant $\varepsilon_m > 0$ independent of t such that $V_m = V_{m-1}$ (at some time instant) implies $\dot{V}_{m-1} \leq -\varepsilon_m$ and $V_m = \alpha_m^{-\frac{1}{m-1}} W_m$ implies $\dot{W}_m \leq -\alpha_m^{\frac{1}{m-1}} \varepsilon_m$. Distinguishing these two cases, consider first the case $V_m = V_{m-1}$. Then, $V_m > \gamma_m (\tilde{N}/2)^{\frac{1}{m}}$ implies

$$(2 - \gamma_{m-1}^{m-1} \alpha_m) V_m^{m-1} > 2\gamma_{m-1}^{m-1} (\tilde{N}/2)^{\frac{m-1}{m}} \tag{A.6}$$

which is equivalent to

$$V_m^{m-1} > \frac{\gamma_{m-1}^{m-1}}{2} (\alpha_m V_m^{m-1} + 2(\tilde{N}/2)^{\frac{m-1}{m}}) = \gamma_{m-1}^{m-1} \frac{\tilde{N}_m}{2} \tag{A.7}$$

and yields $V_{m-1} = V_m > \gamma_{m-1} (\tilde{N}_m/2)^{\frac{1}{m-1}}$. Hence, using the induction assumption for $N = L\tilde{N}_m$, obtain the inequality $\dot{V}_{m-1} \leq -\varepsilon_{m-1} \leq -\varepsilon_m$ for $\varepsilon_m \in (0, \varepsilon_{m-1}]$. In the second case, define $w_m := \lfloor \xi_m \rceil^{\frac{m-1}{m}} - \xi_{m-1}$ with $W_m = |w_m|^{\frac{1}{m-1}}$ to obtain

$$\dot{W}_m = \frac{W_m^{2-m}}{m-1} \left(-\frac{m-1}{m |\xi_m|^{\frac{1}{m}}} \kappa_m \eta_m + \kappa_{m-1} \eta_{m-1} \right) \lfloor w_m \rceil^0. \tag{A.8}$$

Since $|\eta_m - w_m| \leq 2(\tilde{N}_{m+1}/2)^{\frac{m-1}{m}}$ according to Lemma 8 and $|\eta_{m-1}| \leq \tilde{N}_{m-1}$,

$$\dot{W}_m \leq \frac{W_m^{2-m}}{m-1} \Bigg(-\frac{m-1}{m |\xi_m|^{\frac{1}{m}}} \kappa_m (W_m^{m-1} - 2(\tilde{N}_{m+1}/2)^{\frac{m-1}{m}}) + \kappa_{m-1} \tilde{N}_{m-1} \Bigg). \tag{A.9}$$

Further utilizing $|\xi_m|^{\frac{1}{m}} \leq \beta_m V_m = \beta_m \alpha_m^{-\frac{1}{m-1}} W_m$ yields

$$\dot{W}_m \leq -\kappa_m \frac{1 - 2W_m^{1-m} (\tilde{N}_{m+1}/2)^{\frac{m-1}{m}}}{m \beta_m \alpha_m^{-\frac{1}{m-1}}} + \kappa_{m-1} \frac{W_m^{2-m} \tilde{N}_{m-1}}{m-1}, \tag{A.10}$$

provided that the right-hand side of (A.10) is negative, which will be verified in the end. In (A.10), substitute

$$\tilde{N}_{m-1} = \alpha_{m-1} \alpha_m^{-\frac{m-2}{m-1}} W_m^{m-2} + 2\left(\frac{\alpha_m}{2} \alpha_m^{-1} W_m^{m-1} + (\tilde{N}_{m+1}/2)^{\frac{m-1}{m}}\right)^{\frac{m-2}{m-1}} \tag{A.11}$$

and introduce the abbreviation $R_m = W_m^{1-m} (\tilde{N}_{m+1}/2)^{\frac{m-1}{m}}$ to obtain

$$\dot{W}_m \leq -\kappa_m \frac{1 - 2R_m}{m \beta_m \alpha_m^{-\frac{1}{m-1}}} + \kappa_{m-1} \frac{\alpha_{m-1} \alpha_m^{-\frac{m-2}{m-1}} + 2^{\frac{1}{m-1}} (1 + 2R_m)^{\frac{m-2}{m-1}}}{m-1}. \tag{A.12}$$

Since $(\tilde{N}_{m+1}/2)^{\frac{m-1}{m}} < \gamma_m^{1-m} V_m^{m-1} = \gamma_m^{1-m} \alpha_m^{-1} W_m^{m-1}$, the abbreviation R_m satisfies $R_m < \alpha_m^{-1} \gamma_m^{1-m}$, which finally yields

$$\alpha_m^{\frac{m-2}{m-1}} \gamma_m^{m-1} \dot{W}_m \leq -\kappa_m \frac{\alpha_m \gamma_m^{m-1} - 2}{m \beta_m} + \kappa_{m-1} \frac{\alpha_{m-1} \gamma_m^{m-1} + 2^{\frac{1}{m-1}} \gamma_m (\alpha_m \gamma_m^{m-1} + 2)^{\frac{m-2}{m-1}}}{m-1} \tag{A.13}$$

whose right-hand side can be bounded by some negative constant $-\varepsilon_m$ due to condition (14). □

A.3 Proof of Lemma 6

By assumption, the boundary ∂Z_j of Z_j consists of countably many points. Hence, $\lfloor \xi_j(t) \rceil^{\frac{j-1}{j}}$ is differentiable with respect to t on the set $J = \mathbb{R}_{\geq 0} \backslash \cup_{j=1}^n \partial Z_j$, because for each $t \in J$ there exists a compact interval containing t on which $\xi_j(t)$ is either constant or strictly positive. For each compact interval $\bar{I} \subseteq I$, define $\mu = \min_{t \in \bar{I}} V(t) > 0$, where the minimum exists due to continuity of V and compactness of $\bar{I}$, and choose for each j an absolutely continuous function $g_j : \bar{I} \to [\mu/2, \mu]$ such that the functions $h_j : \bar{I} \to \mathbb{R}_{\geq 0}$ defined as $h_1(t) = |\xi_1(t)|$ and

$$h_j(t) = \max\{g_j(t), \alpha_j^{-\frac{1}{j-1}} \left| \lfloor \xi_j(t) \rceil^{\frac{j-1}{j}} - \xi_{j-1}(t) \right|^{\frac{1}{j-1}}\} \tag{A.14}$$

for $j = 2, \ldots, n$ are differentiable on $J \cap \bar{I}$. Then, since $V(t) \geq \mu$ and $g_j(t) \leq \mu$, the function V may equivalently be written as

$$V(t) = \max_{1 \leq j \leq n} h_j(t) \tag{A.15}$$

on $\bar{I}$. Since each h_j is differentiable nearly everywhere on $\bar{I}$ (i.e., everwhere except at countably many time instants), it is generalized absolutely continuous in the restricted sense (ACG$_*$) on $\bar{I}$ as defined in (Gordon, 1994, Definition 6.1) due to (Gordon, 1994, Corollary 6.23). Their pointwise maximum, i.e., V is then also ACG$_*$ on $\bar{I}$. Finally, since I may be written as a countable union of compact intervals $\bar{I}$, the function V is ACG$_*$ also on I, and differentiability almost everywhere on I follows from (Gordon, 1994, Corollary 6.19). Finally, if $\dot{V}(t) \leq 0$, then it is nonincreasing due to (Gordon, 1994, Theorem 6.25), and hence has bounded variation on I, implying absolute continuity, cf. (Gordon, 1994, Exercise 6.8). □