108

FRACTAL BEHAVIOR OF TERRAIN TOPOGRAPHY

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EXTENDED ABSTRACT

The paper documents the concept of fractals to the study of terrain topography in Digital Elevation Models. The spatial distribution of the terrain elevation could be described using the difference variance of fractal theory, both on macroscopic and microscopic scale. The estimated fractal dimension is used to device proper sample spacing, to use proper interpolation methods and to study the accuracy of interpolation.

The terrain topography may exhibit a very complicated geometry which is too complicated to be described by deterministic methods. The variation of these parameters may seem unpredictable but when analyzing the data some systematic behavior is usually revealed. One way of characterizing these variables which are partly stochastic and partly deterministic in their behavior is by the concept of difference variance. It was introduced by de Wijs in 1954, see Mandelbrot (1982) and is now one of the basic concepts in studying stochastic variables. The difference variance is estimated as the mean square difference between two sample values a distance D apart:

$$V(D) = \frac{1}{N}\sum_{N} (Z(i) - Z(i + D))^2 . \tag{1}$$

The difference variance for many terrain types proves both from experiment and theory (de Wijs (1954)) to be of the form

$$V(D) = K.D^B \tag{2}$$

where K stands for the value V(D) of unit distance and B is a real number. On a log-log plot, the difference variance appears as a straight line with slope B. The straight line behavior of V(D) can be observed from the examples of actual terrain of figure 1. The slope B tells about the roughness of the curve. Slopes close to 0 indicate very irregular behavior, while very smooth surfaces have a large value B.

The modelling of elevation profiles by the difference variance (2) is equivalent to model their magnitude by the differential equation

$$Z^{(n)} = \varepsilon \qquad (3)$$

where (n) denotes the n-th derivative of elevation Z, and ε is an independent random variable (white noise). The coefficient n may also be non-integer. In that latter case this fractional n-th derivative is defined by continuous interpolation into the integer derivatives. The coefficient n can be related to the slope B of the log log difference variance plot, by

$$n = \tfrac{1}{2}(B + 1).$$

The differential equation (3) may now be used in finite element approximation of the topographic surface. The proper functional to be minimized in the finite element approach is

$$J = \int \left\{ \frac{\partial^{(n)} Z}{\partial^{(n)}} \right\}^2 dx \rightarrow \min$$

The result of the finite element method are identical to the result of the prediction method using the difference variance (2), as it was shown already by Kimeldorf and Wabha (1971). When interpolating with the prediction method (Wiener, 1949)

$$H(x) = V(x, x_i).V(x_i, x_j)^{-1}.Z(x_j) \ ,$$

with $Z(x_j)$ vector of of sample values, the numerical stability of the computation should be carefully controlled.

The error e of the interpolated value H(x) is:

$$e(x) = Z(x) - H(x)$$

and the variance of this error Var(e) can be evaluated using the difference variance $V(D) = K.D^B$. To study the accuracy of interpolation, the standard deviation $\sigma = \sqrt{Var(e)}$ is used with its maximum σ_{max} and mean σ_{mean} (mean square of all σ in the interpolated points). The standard deviation σ is expressed in units of the unit K. The maximum and mean standard deviations for prediction interpolation are obtained as

$$\frac{\sigma^2_{max}}{K} = D^B \left[\frac{2-2^{B-1}}{2^{B-1}}\right]$$

and

$$\frac{\sigma^2_{max}}{K} = \frac{D^B}{2}\left[-\frac{1}{3} + \frac{4}{(B+1)(B+2)}\right]$$

The formulae relate the sampling space D to the accuracy of interpolation, for different values B of the grade or width.

These studies of the fractal behavior of the terrain topography enable the planning of proper spacing of samples, such that pregiven accuracy criteria are met. The results of interpolation and accuracy prediction have been verified on a number of terrain, and the models proved to be valid and their deductions verifiable.

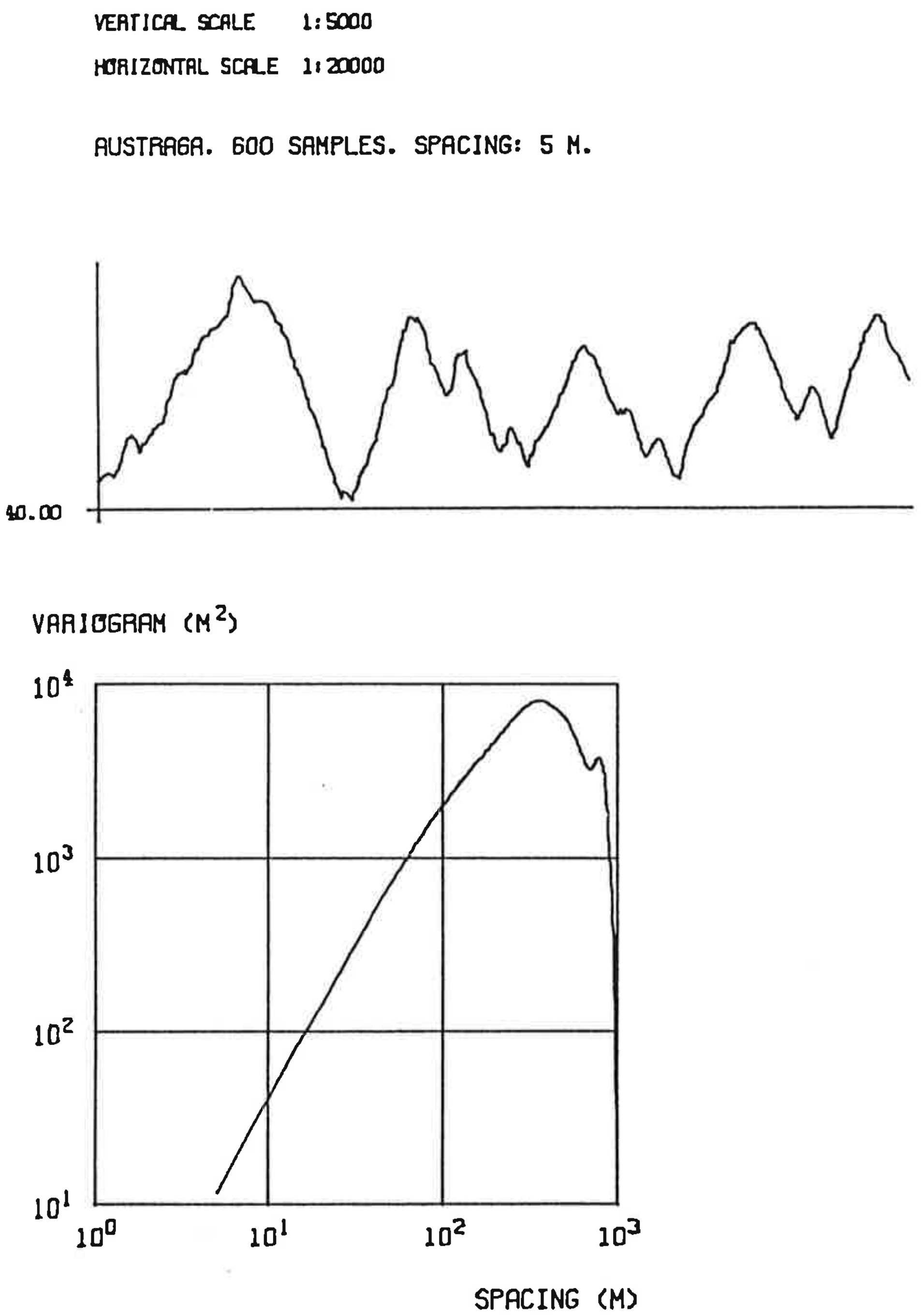

Figure 1. Terrain profile and its difference variance plot

REFERENCES

Kimeldorf, G.S. and G. Wabha, 1970. Spline Functions and Stochastic Processes. Sankhya, Series A. pp 173-180.

Mandelbrot, B., 1982. The Fractal Geometry of Nature. W.H. Freeman and Company, San Francisco.

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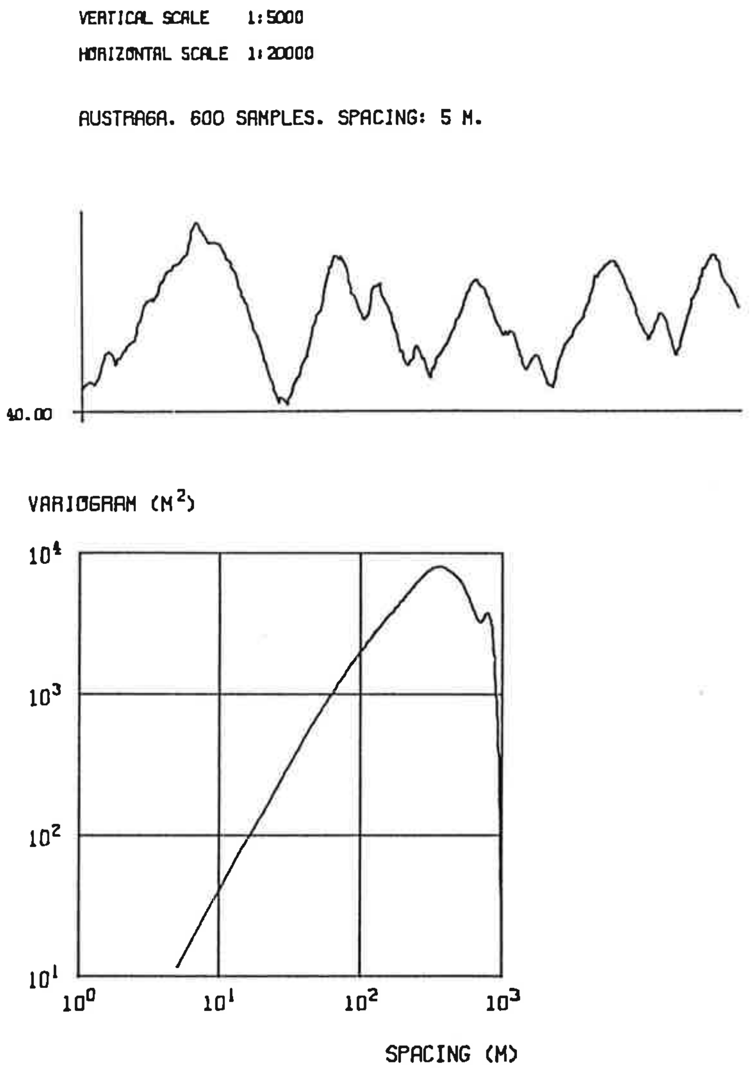

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