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Poroelastodynamic Boundary Element Method in Time Domain: Numerical Aspects

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Based on Biot's theory the governing equations for a poroelastic continuum are given as a coupled set of partial differential equations (PDEs) for the unknowns solid displacements and pore pressure. Using the Convolution Quadrature Method (CQM) proposed by Lubich a boundary time stepping procedure is established based only on the fundamental solutions in Laplace domain.

To improve the numerical behavior of the CQM-based Boundary Element Method (BEM) dimensionless variables are introduced and different choices studied. This will be performed as a numerical study at the example of a poroelastic column.

Summarizing the results, the normalization to time and spatial variable as well as on Young's modulus yields the best numerical behavior.

1 Biot's theory

Following Biot's approach [1] to model the behavior of porous media, an elastic skeleton with a statistical distribution of interconnected pores saturated by a fluid is considered. The solid displacements u_i and the pore pressure p are chosen as a sufficient set of independent variables. For this choice, the dynamic governing differential equations can only be formulated in Laplace or Fourier domain. Here, a representation in Laplace domain (denoted by $\hat{f}(s) = \mathscr{L}\{f(t)\}(s)$) will be used. The governing PDE-System can be written in matrix form as

$$\mathbf{B}^* \begin{bmatrix} \hat{u}_j \\ \hat{p} \end{bmatrix} + \begin{bmatrix} \hat{F}_i \\ \hat{a} \end{bmatrix} = \mathbf{0}, \quad \mathbf{B}^* = \begin{bmatrix} \left(G\nabla^2 - s^2(\rho - \beta\rho_f)\right)\delta_{ij} + \left(K + \frac{1}{3}G\right)\partial_i\partial_j & -(\alpha-\beta)\,\partial_i \\ -s(\alpha-\beta)\,\partial_j & \frac{\beta}{s\rho_f}\nabla^2 - \frac{\phi^2 s}{R} \end{bmatrix} \tag{1}$$

with the non-self adjoint differential operator $\mathbf{B}^*$. In Eq. (1), Biot's effective stress coefficient $\alpha = \phi(1 + Q/R)$ and the abbreviation $\beta = \kappa\rho_f\phi^2 s^2 / \left(\phi^2 s + s^2\kappa(\rho_a + \phi\rho_f)\right)$ are introduced. The parameters Q and R represent the coupling between the solid and the fluid. The bulk material is defined by the compression modulus K, the shear modulus G, the porosity ϕ, and the permeability κ. Further, the bulk density $\rho = (1-\phi)\rho_s + \phi\rho_f$ with the solid's density ρ_s and the fluid's density ρ_f is used. The apparent mass density introduced by Biot is assumed to be $\rho_a = 0.66\ \phi\ \rho_f$. The load vector consists of bulk body forces $\hat{F}_i$ in the solid and sources $\hat{a}$ in the fluid (for details see [1]).

2 BEM Formulation

Under the assumption of vanishing body forces F_i and sources a the time dependent boundary integral equation corresponding to 1 can be written in the following form

$$\int_\Gamma \begin{bmatrix} U_{ij}^s & -P_j^s \\ U_i^f & -P^f \end{bmatrix} * \begin{bmatrix} t_i \\ q \end{bmatrix} \mathrm{d}\Gamma = \oint_\Gamma \begin{bmatrix} T_{ij}^s & Q_j^s \\ T_i^f & Q^f \end{bmatrix} * \begin{bmatrix} u_i \\ p \end{bmatrix} \mathrm{d}\Gamma + \begin{bmatrix} c_{ij} & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} u_i \\ p \end{bmatrix} \tag{2}$$

with the convolution $f * g = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$. This formulation is based on the unknown displacements u_i and the pore pressure p. The total stress vector is denoted by t_i and the normal flux by q. Capital letters represent the respective fundamental solutions. Unfortunately time-dependent fundamental solutions are not known in closed form for the given PDE-System. However, the CQM needs only the fundamental solutions in Laplace domain to evaluate the convolution integral, which are available [3]. According to the Boundary Element Method the boundary surface Γ is discretized by linear isoparametric elements. Further, the convolution integrals are approximated by the CQM, which then results in a boundary element time stepping formulation. To obtain a system of algebraic equations point collocation is used at every node and, finally, a direct equation solver is applied.

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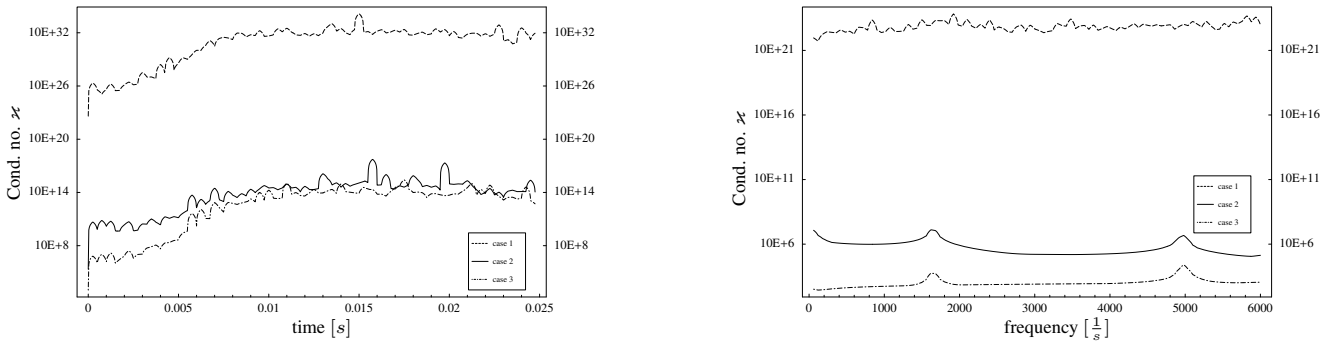

Fig. 1 Condition number $\varkappa$ for time domain and frequency domain

3 Dimensionless Variables

To improve the numerical behavior of the CQM-based poroelastodynamic BEM the following dimensionless variables are introduced

$$\tilde{x}_i = \frac{x_i}{A} \qquad \tilde{t} = \frac{t}{B} \qquad \tilde{K} = \frac{K}{C} \qquad \tilde{G} = \frac{G}{C} \qquad \tilde{\rho} = \frac{A^2}{B^2 C}\rho \qquad \tilde{\kappa} = \frac{BC}{A^2}\kappa \tag{3}$$

with real valued scalar parameters A, B, C. Here, three cases will be studied. First, without any transformation denoted by case 1: $A = 1, B = 1, C = 1$. The second transformation was suggested in [2] and is denoted by case 2: $A = \rho\kappa V, B = \rho\kappa, C = \rho V^2$, with $V = \sqrt{(K + 4G/3 + \alpha^2 R/\phi^2)/\rho}$. Finally, the last transformation is a quite simple one, which represents a normalization of the system by maximum values, i.e. the longest mesh distance $r_{\max}$, the total time $t_{\max}$ as well as Young's modulus E and is labeled as case 3: $A = r_{\max}, B = t_{\max}, C = E$.

4 Numerical Example

In order to study the influence of dimensionless variables a poroelastic column is considered as shown in Fig. 2. The material data corresponds to a water saturated rock ($E = 1.44\ 10^{10}\frac{\mathrm{N}}{\mathrm{m}^2}, \nu = 0, \rho = 2458\frac{\mathrm{kg}}{\mathrm{m}^3}, \rho_f = 1000\frac{\mathrm{kg}}{\mathrm{m}^3}, \phi = 0.19, R = 4.7\ 10^8\frac{\mathrm{N}}{\mathrm{m}^2}, \alpha = 0.86, \kappa = 1.9\ 10^{-10}\frac{\mathrm{m}^4}{\mathrm{Ns}}$).

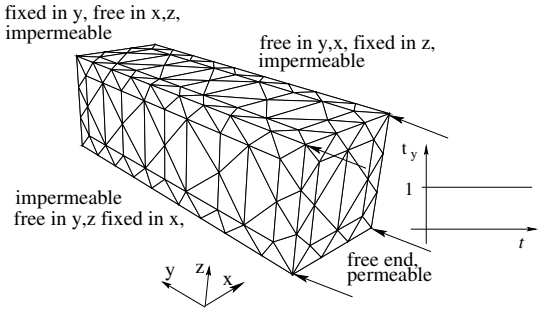

Fig. 2 Geometry, boundary conditions and discretization of a 3-d column

In Fig. 1 the condition number $\varkappa$ of the system matrices is depicted versus time and frequency. The condition number serves as a measurement of the solvability of the resulting linear system. It is obvious that a calculation with non-transformed variables (case 1) yields the worst condition numbers. It must be mentioned, that in many cases a calculation with non-transformed variables is not even possible. Further, it can be seen that the transformation suggested in [2] yields system matrices with improved condition numbers (case 2). In opposite to the non-transformed system an improvement of factor $10^{16} - 10^{18}$ can be observed in time domain and about 10^{15} in frequency domain. A further reduction of the condition number can be achieved if the system is normalized by maximum values (case 3). Although the improvement in this final step is not as large as from case 1 to case 2, it is sufficient to ensure the solvability of systems, which even can't be calculated with the transformation suggested in [2]. Therefore, it can be concluded that the case 3 yields the best numerical behavior. This has been confirmed in several other tests with other geometries and material data. Nevertheless, it should be noted that there exist few constellations where case 2 is superior.

References

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