On lacunary series with random gaps

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Abstract

We prove Strassen's law of the iterated logarithm for sums $\sum_{k=1}^N f(n_k x)$, where f is a smooth periodic function on the real line and $(n_k)_{k\geq 1}$ is an increasing random sequence. Our results show that classical results of the theory of lacunary series remain valid for sequences with random gaps, even in the nonharmonic case and if the Hadamard gap condition fails.

1 Introduction

Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1. \tag{1.1}$$

Salem and Zygmund [14] proved that $(\sin 2\pi n_k x)_{k\geq 1}$ obeys the central limit theorem, i.e.

$$\frac{1}{\sqrt{N/2}} \sum_{k=1}^{N} \sin 2\pi n_k x \xrightarrow{d} N(0,1) \tag{1.2}$$

with respect to the probability space $(0,1)$ equipped with Borel sets and Lebesgue measure. The corresponding law of the iterated logarithm

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N \log\log N}} \sum_{k=1}^{N} \sin 2\pi n_k x = 1 \qquad \text{a.e.} \tag{1.3}$$

was proved by Erdős and Gál [7]. These results show that under the gap condition (1.1) the functions $\sin 2\pi n_k x$ behave like independent random variables. Erdős [6] showed that the CLT (1.2) remains valid if we weaken the Hadamard gap condition to

$$n_{k+1}/n_k \geq 1 + c_k k^{-1/2}, \qquad c_k \to \infty \tag{1.4}$$

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and this result is sharp, i.e. for any $c > 0$ there exists a sequence (n_k) satisfying

$$n_{k+1}/n_k \geq 1 + ck^{-1/2}, \qquad k = 1, 2, \dots$$

such that the CLT (1.2) is false. The corresponding LIL was proved by Takahashi [18]. For sequences $(n_k)_{k\geq 1}$ growing slower than the speed defined by (1.4), the asymptotic behavior of the partial sums of $\sin 2\pi n_k x$ depends sensitively on the number theoretic properties of n_k and deciding the validity of the CLT is requires giving precise asymptotic estimates for the number of solutions of Diophantine equations of the form

$$n_k \pm n_\ell = a, \qquad 1 \leq k, \ell \leq N, \tag{1.5}$$

which is a very difficult problem. (See Berkes [2] for a detailed discussion of this number-theoretic connection.) On the other hand, the equation becomes manageable for random n_k (see e.g. Halberstam and Roth [11], Chapter 3) and thus it is natural to investigate the asymptotic behavior of lacunary series with random gaps. Fukuyama [8], [9] used such series to show the existence of a sequence $(n_k)_{k\geq 1}$ with bounded gaps $n_{k+1} - n_k$, such that $(\sin n_k x)_{k\geq 1}$ satisfies the CLT (1.2) with a limiting variance less than 1. For a similar construction with slightly larger gaps, see Berkes [3]; for constructions for the law of the iterated logarithm and metric discrepancy results, see Aitleitner and Fukuyama [1], Fukuyama [8]. A simple model was investigated by Schatte [16] who assumed that $(n_k)_{k\geq 1}$ is an increasing random walk, i.e. $n_{k+1} - n_k$ are i.i.d. positive random variables. Schatte's main interest was the behavior of the discrepancy of $\{n_k x\}$; his results were extended by Weber [19] and Berkes and Weber [4]. The purpose of the present paper is to prove a functional law of the iterated logarithm for sums $\sum_{k=1}^N f(n_k x)$ for the model.

Let $X_1, X_2, \dots$ be i.i.d. positive random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and put $S_k = \sum_{j=1}^k X_j$. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)dx = 0, \qquad \int_0^1 f^2(x)dx = 1. \tag{1.6}$$

Put

$$A_x = 1 + 2\sum_{k=1}^{\infty} \mathbb{E} f(U) f(U + S_k x), \tag{1.7}$$

where U is a uniform $(0,1)$ random variable, independent of $(X_j)_{j\geq 1}$. Clearly, the existence of such an U can always be guaranteed by an enlargement of the probability space. The absolute convergence of the series in (1.7) will follow from the proof of our theorem below. We shall establish the following version of Strassen's law of the iterated logarithm.

Theorem. *Assume that X_1 is bounded with bounded density and assume that f is a Lipschitz function satisfying (1.6). For any $x \in \mathbb{R}$ define the sequence $(\Gamma_n^x)_{n\in\mathbb{N}}$ of* *functions on* $[0,1]$ *by*

$$\Gamma_n^x(0)=0, \quad \Gamma_n^x(k/n)=(2n\log\log n)^{-1/2}\sum_{j=1}^k f(S_jx) \quad (k=0,\dots,n)$$

and $\Gamma_n^x(t)$ *is linear on* $[k/n,(k+1)/n]$ $(k=0,\dots,n-1)$. *Then* $\mathbb{P}$*-almost surely* $(\Gamma_n^x)_{n\in\mathbb{N}}$ *is relatively compact in* $C[0,1]$ *for almost all all* x *and the set of its limit points coincides with the scaled Strassen set*

$$K_x=\{y(t): y \text{ is absolutely continuous in } [0,1],\ y(0)=0 \text{ and } \int_0^2 y'(t)^2dt\le A_x\}.$$

As an immediate consequence, we get $\mathbb{P}$-almost surely

$$\limsup_{n\to\infty}(2n\log\log n)^{-1/2}\sum_{k=1}^n f(S_kx)=A_x^{1/2} \qquad \text{for almost all } x. \tag{1.8}$$

Note that the limsup in (1.8) is not a constant as in the law of the iterated logarithm for i.i.d. random variables, but a function of x. In the nonrandom lacunary case, the existence of nonconstant limits in the LIL was discovered by Erdős and Fortet (1949), see [12], p. 655. On the other hand, Gaposhkin [10] proved that the LIL holds for $\sum_{k=1}^N f(n_kx)$ with a constant limsup provided $n_{k+1}/n_k\to\infty$ or if $\lim_{k\to\infty}n_{k+1}/n_k=\alpha>1$ where α^r is irrational for $r=1,2,\dots$. Very little is known in the nonrandom case if n_k grows slower than exponential. Nonconstant limits appear also in the LIL for lacunary orthogonal series, see e.g. Weiss [20]. Theorem 1 also leads, just as Strassen's LIL in the i.i.d. case, to a whole class of asymptotic results for $f(S_jx)$. For example we get, setting $T_k=\sum_{j=1}^k f(S_jx)$

$$\limsup_{n\to\infty} n^{-a/2}(2\log\log n)^{-1-a/2}\sum_{k=1}^n|T_k|^a=A_x^{a/2}$$

$\mathbb{P}$-almost surely for almost all $x\in\mathbb{R}$. Another consequence of Theorem 1 is $\mathbb{P}$-almost surely

$$\limsup_{n\to\infty}\frac{1}{n}\#\{k\le n:(2k\log\log k)^{-1/2}T_k\ge cA_x^{1/2}\}=1-\exp\left\{-4\left(c^{-2}-1\right)\right\} \qquad \text{a.e.}$$

describing how frequently the ratio $(2k\log\log k)^{-1/2}T_k$ gets close to its limsup $A_x^{1/2}$. For further consequences of Strassen type laws of the iterated logarithm see [17].

Note that we assumed the r.v.'s X_j to have a density, and thus the elements of the random sequence $S_k, k=1,2,\dots$ are not integers. Our method does not work for lattice valued X_j and the existing results in the field (see e.g. Schatte [16], Berkes and Weber [4]) are less precise.

2 Proof of the theorem

Lemma 1. *Fix $l \in \mathbb{N}$, $x \neq 0$ and define a sequence of sets by*

$$\begin{aligned}
I_1 &:= \{1, 2, \ldots, \beta\} \\
I_2 &:= \{p_1, p_1+1, \ldots, p_1+\beta_1\} \quad \text{where} \quad p_1 \geq \beta + l + 2 \\
&\vdots \\
I_n &:= \{p_{n-1}, p_{n-1}+1, \ldots, p_{n-1}+\beta_{n-1}\} \quad \text{where} \quad p_{n-1} \geq p_{n-2} + \beta_{n-2} + l + 2 \\
&\vdots
\end{aligned}$$

Then there exists a sequence $\delta_1^x, \delta_2^x, \ldots$ of random variables satisfying the following properties:

(i) $|\delta_n^x| \leq C_x e^{-\lambda_x l}$ *for all $n \in \mathbb{N}$, where λ_x and C_x are some positive quantities that depend on x only.*

(ii) *The random variables*

$$\sum_{i \in I_1} f(S_i x), \ \sum_{i \in I_2} f(S_i x - \delta_1^x), \ldots, \ \sum_{i \in I_n} f(S_i x - \delta_{n-1}^x), \ldots$$

are independent.

Proof. We will construct the sequence $(\delta_n^x)_{n \in \mathbb{N}}$ by induction. Define

$$\delta_1^x := \{(S_{\beta+l} - S_\beta)x\} - F_{\{(S_{\beta+l}-S_\beta)x\}}\left(\{(S_{\beta+l} - S_\beta)x\}\right).$$

By the assumptions of our theorem and Theorem 1 of Schatte [15] we have

$$\sup_t |F_{\{S_n x\}}(t) - t| \leq C_x e^{-\lambda_x n} \qquad n \in \mathbb{N}. \tag{2.9}$$

Since $S_{\beta+l} - S_\beta \stackrel{d}{=} S_l$ for all β and all l, it follows easily that $|\delta_1^x| \leq C_x e^{-\lambda_x l}$. Furthermore, letting $\{\cdot\}$ denote fractional part, we have

$$\begin{aligned}
\{S_{p_1} x - \delta_1^x\} &= \Big\{S_{p_1} x - \{(S_{\beta+l} - S_\beta)x\} + F_{\{(S_{\beta+l}-S_\beta)x\}}\left(\{(S_{\beta+l} - S_\beta)x\}\right)\Big\} \\
&= \big\{(X_1 + \cdots + X_\beta)x + (X_{\beta+l+1} + \cdots + X_p)\, x \\
&\quad + F_{\{(S_{\beta+l}-S_\beta)x\}}\left(\{(S_{\beta+l} - S_\beta)x\}\right)\big\},
\end{aligned}$$

since $\{x\}$ has period 1. Similarly,

$$\begin{aligned}
\{S_{p_1+1} x - \delta_1^x\} &= \big\{(X_1 + \cdots + X_\beta)x + (X_{\beta+l+1} + \cdots + X_{p_1+1})\, x \\
&\quad + F_{\{(S_{\beta+l}-S_\beta)x\}}\left(\{(S_{\beta+l} - S_\beta)x\}\right)\big\} \\
&\vdots
\end{aligned}$$

$$\{S_{p_1+\beta_1}x - \delta_1^x\} = \big\{(X_1 + \cdots + X_\beta)x + (X_{\beta+l+1} + \cdots + X_{p_1+1})\,x$$
$$+ F_{\{(S_{\beta+l}-S_\beta)x\}}\left(\{(S_{\beta+l} - S_\beta)x\}\right)\big\}.$$

Thus applying Lemma 1 of [16] with

$$\begin{aligned} X &= (X_1x, X_2x, \ldots, X_\beta x) \\ U &= F_{\{(S_{\beta+l}-S_\beta)x\}}\left(\{(S_{\beta+l} - S_\beta)x\}\right) \\ (W_1, \ldots, W_{p_1+\beta_1}) &= \left((X_{\beta+l+1} + \ldots X_{p_1})\,x, \ldots, (X_{\beta+l+1} + \cdots + X_{p_1+\beta_1})\,x\right) \\ W &= X_1x + \cdots + X_\beta x \end{aligned}$$

it follows that

$$\sum_{j\in I_1} f(S_jx) \text{ is independent of } \sum_{j\in I_2} f(S_jx - \delta_1^x).$$

Now suppose $\delta_1^x, \ldots, \delta_{n-1}^x$ have been constructed and define

$$Y_n^x = \left\{\left(S_{p_{n-1}+\beta_{n-1}+l} - S_{p_{n-1}+\beta_{n-1}}\right)x\right\}, \qquad \delta_n^x = Y_n^x - F_{Y_n^x}(Y_n^x).$$

As before, it follows easily that $|\delta_n^x| \le C_x e^{-\lambda_x l}$. We let

$$\begin{aligned} &X = \left(X_1x, \ldots, X_{p_{n-1}+\beta_{n-1}}, \delta_1^x, \ldots, \delta_{n-1}^x\right) \\ &U = F_{Y_n^x}(Y_n^x) \\ &W = X_1x + \cdots + X_{p_{n-1}+\beta_{n-1}}x \\ &(W_1, \ldots, W_{p_n+\beta_n}) \\ &= \left(X_{p_{n-1}+\beta_{n-1}+l+1}x + \cdots + X_{p_n}x, \ldots, X_{p_{n-1}+\beta_{n-1}+l+1} + \cdots + X_{p_n+\beta_n}x\right). \end{aligned}$$

Then, again by Lemma 1 of [16] it follows that

$$\sum_{i\in I_{n+1}} f(S_ix - \delta_n^x) \text{ is independent of } \left(\sum_{i\in I_1} f(S_ix), \ldots, \sum_{i\in I_n} f(S_ix - \delta_{n-1}^x)\right),$$

which completes the induction step and the proof of the lemma.

Put $\widetilde{m}_k = \sum_{j=1}^{k} \lfloor j^{1/2} \rfloor$, $\widehat{m}_k = \sum_{j=1}^{k} \lfloor j^{1/4} \rfloor$ and let $m_k = \widetilde{m}_k + \widehat{m}_k$. Using Lemma 1 we can construct sequences $(\Delta_k^x)_{k\in\mathbb{N}}$, $(\Pi_k^x)_{k\in\mathbb{N}}$ of random variables such that setting

$$\begin{aligned} T_k &:= \sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor} \left(f(S_jx - \Delta_{k-1}^x) - \mathbb{E}f(S_jx - \Delta_{k-1})\right) \\ T_k^* &:= \sum_{j=m_{k-1}+\lfloor\sqrt{k}\rfloor+1}^{m_k} \left(f(S_jx - \Pi_{k-1}^x) - \mathbb{E}f(S_jx - \Pi_{k-1}^x\right) \end{aligned}$$

we have

(i) $\Delta_0^x = 0$; $|\Delta_k^x| \le C_x e^{-\lambda_x \sqrt[4]{k}}$; $(T_k)_{k\in\mathbb{N}}$ is a sequence of independent random variables,

(ii) $\Pi_0^x = 0$; $|\Pi_k^x| \le C_x e^{-\lambda_x \sqrt{k}}$; $(T_k^*)_{k\in\mathbb{N}}$ is a sequence of independent random variables.

Lemma 2.
$$\sum_{k=1}^{n}\mathbb{V}\mathrm{ar}(T_k)\sim A_x\widetilde{m}_n,\qquad \sum_{k=1}^{n}\mathbb{V}\mathrm{ar}(T_k^*)\sim A_x\widehat{m}_n.$$
where A_x is defined by (1.7).

Proof. In what follows C_x and λ_x will denote positive numbers, possibly different at different places, depending (at most) on x and the Lipschitz constant α of the function f. Clearly

$$\begin{aligned}\mathbb{V}\mathrm{ar}(T_k)&=\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor}\mathbb{E}f^2(S_jx-\Delta_{k-1}^x)\\&+2\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1}\sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho}\mathbb{E}f(S_lx-\Delta_{k-1}^x)f(S_{l+\varrho}x-\Delta_{k-1}^x)-L_x^{(k)}\end{aligned}$$

where

$$L_x^{(k)}:=\left(\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor}\mathbb{E}f(S_jx-\Delta_{k-1}^x)\right)^2.$$

By relation (i), (2.9) and the Lipschitz property of f we have

$$\left|f(S_jx-\Delta_{k-1}^x)-f(S_jx)\right|\le C|\Delta_{k-1}^x|^\alpha\le C_xe^{-\lambda_x\sqrt[4]{k-1}}$$

and

$$\left|\mathbb{E}f(S_jx)\right|=\left|\mathbb{E}f\big(\{S_jx\}\big)-\mathbb{E}f\left(F_{\{S_jx\}}\big(\{S_jx\}\big)\right)\right|\le C_xe^{-\lambda_xj}.$$

Putting these together yields

$$L_x^{(k)}\le C_xke^{-\lambda_x\sqrt[4]{k-1}}.$$

Let now

$${\Lambda_x}^{(k)}:=\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor}\gamma_j^x,\qquad {O_x}^{(k)}:=\sum_{j=m_{k-1}+1}^{m_{k-1}+\sqrt{k}}\varepsilon_j^x,$$

where

$$\begin{aligned}\gamma_j^x&=\mathbb{E}f^2(S_jx-\Delta_{k-1}^x)-\mathbb{E}f^2(S_jx)\\\varepsilon_j^x&=\mathbb{E}f^2(S_jx)-\mathbb{E}f^2\left(F_{\{S_jx\}}(\{S_jx\})\right).\end{aligned}$$

Repeating the argument above for the function f^2-1, we get

$$\sum_{j=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor}\mathbb{E}f^2(S_jx-\Delta_{k-1}^x)=\lfloor\sqrt{k}\rfloor+{\Lambda_x}^{(k)}+{O_x}^{(k)}$$

and

$$|{\Lambda_x}^{(k)}|\le C_x\sqrt{k}e^{-\lambda_x(k-1)^{1/4}},\qquad |{O_x}^{k}|\le C_xe^{-\lambda_x(m_{k-1}+1)}.$$

We now turn to the cross terms. Define $T_\varrho^l = X_{l+1} + \cdots + X_{l+\varrho}$ and split the product expectation $\mathbb{E} f(S_l x - \Delta_{k-1}^x) f(S_{l+\varrho} x - \Delta_{k-1}^x)$ into the sum of terms

$$
\begin{aligned}
e_l^x &:= \mathbb{E} f(S_l x - \Delta_{k-1}^x) f(S_{l+\varrho} x - \Delta_{k-1}^x) - \mathbb{E} f(S_l x) f(S_{l+\varrho} x - \Delta_{k-1}^x) \\
g_l^x &:= \mathbb{E} f(S_l x) f(S_{l+\varrho} x - \Delta_{k-1} x) - \mathbb{E} f(S_l x) f(S_{l+\varrho} x) \\
h_l^x &:= \mathbb{E} f(S_l x) f(S_{l+\varrho} x) - \mathbb{E} f\big(F_{\{S_l x\}}(\{S_l x\})\big) f(S_{l+\varrho} x) \\
i_l^x &:= \mathbb{E} f\big(F_{\{S_l x\}}(\{S_l x\})\big) f(S_l x + {T_\varrho}^l x) - \mathbb{E} f\big(F_{\{S_l x\}}(\{S_l x\})\big) f\big(F_{\{S_l x\}}(\{S_l x\}) + {T_\varrho}^l x\big) \\
C_\varrho^{l,x} &:= \mathbb{E} f\big(F_{\{S_l x\}}(\{S_l x\})\big) f\big(F_{\{S_l x\}}(\{S_l x\}) + {T_\varrho}^l x\big).
\end{aligned}
$$

Here $F_{\{S_l x\}}(\{S_l x\})$ is a uniformly distributed variable independent of T_ϱ^l and thus letting U denote a uniform random variable independent of $(X_j)_{j\geq 1}$,

$$
C_\varrho^{l,x} = C_\varrho^x = \mathbb{E} f(U) f(U + S_\varrho)
$$

does not depend on l. Exactly as before,

$$
|e_l^x| \leq C_x e^{-\lambda_x (k-1)^{1/4}} \quad |g_l^x| \leq C_x e^{-\lambda_x (k-1)^{1/4}} \quad |h_l^x| \leq C_x e^{-\lambda_x l} \quad |i_l^x| \leq C_x e^{-\lambda_x l}.
$$

Thus letting

$$
{E_x}^{(k)} = 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} e_l^x, \qquad {G_x}^{(k)} = 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} g_l^x
$$

$$
{H_x}^{(k)} = 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} h_l^x, \qquad {I_x}^{(k)} = 2 \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} i_l^x
$$

we have

$$
\begin{aligned}
|{E_x}^{(k)}| &\leq C_x k e^{-\lambda_x (k-1)^{1/4}}, \qquad |{G_x}^{(k)}| \leq C_x k e^{-\lambda_x (k-1)^{1/4}} \\
|{H_x}^{(k)}| &\leq C_x \sqrt{k}\, e^{-\alpha\lambda_x (m_{k-1}+1)}, \qquad |{I_x}^{(k)}| \leq C_x \sqrt{k}\, e^{-\alpha\lambda_x (m_{k-1}+1)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} {C_\varrho}^{l,x} &= \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor - 1} \sum_{l=m_{k-1}+1}^{m_{k-1}+\lfloor\sqrt{k}\rfloor-\varrho} C_\varrho^x = \\
&= \lfloor\sqrt{k}\rfloor \sum_{\varrho=1}^{\infty} C_\varrho^x - \lfloor\sqrt{k}\rfloor \sum_{\varrho=\lfloor\sqrt{k}\rfloor}^{\infty} C_\varrho^x - \sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \varrho C_\varrho^x,
\end{aligned}
$$

Thus using the independence of the T_k we get

$$
\mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} T_k\right) = \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}(T_k) = O(1) + \sum_{k=4}^{n} \mathbb{V}\mathrm{ar}(T_k)
$$

$$= O(1) + \sum_{k=4}^{n}\Bigg(\lfloor\sqrt{k}\rfloor + \Lambda_x{}^{(k)} + O_x{}^{(k)} + 2E_x{}^{(k)} + 2G_x{}^{(k)} + 2H_x{}^{(k)} + 2I_x{}^{(k)}$$

$$+ \lfloor\sqrt{k}\rfloor \cdot 2\sum_{\varrho=1}^{\infty} C_\varrho^x - 2\lfloor\sqrt{k}\rfloor \sum_{\varrho=\lfloor\sqrt{k}\rfloor}^{\infty} C_\varrho^x - 2\sum_{\varrho=1}^{\lfloor\sqrt{k}\rfloor-1} \varrho C_\varrho^x - L_x{}^{(k)}\Bigg).$$

Using the same techniques as before, we get $|C_\varrho^x| \le C_x e^{-\lambda_x \varrho}$. Hence the previously established inequalities yield

$$\mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} T_k\right) \sim A_x \widetilde{m}_n \sim A_x m_n.$$

Similarly,

$$\mathbb{V}\mathrm{ar}\left(\sum_{k=1}^{n} T_k^*\right) \sim A_x \widetilde{m}_n,$$

completing the proof of Lemma 2.

We now turn to the proof of Theorem 1. Put $B_n^x = \sum_{k=1}^{n} \mathbb{V}\mathrm{ar}(T_k)$ and define sequences $(\Psi_n^x)_{n\in\mathbb{N}}$, $(\zeta_n^x)_{n\in\mathbb{N}}$, $(\Phi_n^x)_{n\in\mathbb{N}}$, $(\theta_n^x) - n \in \mathbb{N})$ of random functions on $[0,1]$ such that

$$\Psi_n^x(0) = 0, \quad \Psi_n^x\left(B_k^x/B_n^x\right) = (2B_n^x \log\log B_n^x)^{-1/2} \sum_{j=1}^{k} T_j,$$

$$\zeta_n^x(0) = 0, \quad \zeta_n^x\left(B_k^x/B_n^x\right) = (2B_n^x \log\log B_n^x)^{-1/2} \sum_{j=1}^{k} (T_j + T_j^*),$$

$$\Phi_n^x(0) = 0, \quad \Phi_n^x\left(B_k^x/B_n^x\right) = (2B_n^x \log\log B_n^x)^{-1/2} \sum_{j=1}^{m_k} f(S_j x)$$

$$\theta_n^x(0) = 0 \quad \theta_n^x(B_k^x/B_n^x) = (2A_x m_n \log\log m_n)^{-1/2} \sum_{j=1}^{m_k} f(S_j x)$$

for $k = 0, 1, \ldots, n$ and Ψ_n^x, ζ_n^x, Φ_n^x, θ_n^x are linear on $[B_k^x/B_n^x, B_{k+1}^x/B_n^x]$; $k = 0, \ldots, n-1$. Clearly $|T_j| \le C j^{1/2}$, $B_n^x \sim C n^{3/2}$ and thus Kolmogorov's condition on the LIL is satisfied for the sequence $(T_j)_{j\in\mathbb{N}}$. Thus by a result of Major [13] it follows that $(\Psi_n^x)_{n\in\mathbb{N}}$ is $\mathbb{P}$-almost surely relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set. Similarly, the LIL holds for $(T_j^*)_{j\in\mathbb{N}}$, implying that $\mathbb{P}$-almost surely

$$\sup_{1\le k\le n} \left| \sum_{j=1}^{k} T_j^* \right| = o(B_n^2 \log\log B_n)$$

and consequently $\sup_{0\le t\le 1} |\Psi_n^x(t) - \zeta_n^x(t)| \to 0$. On the other hand, the estimate for δ_n^x in Lemma 1 and the Lipschitz property of f imply that $\mathbb{P}$-almost surely

$$\left| \sum_{j=1}^{k} (T_j + T_j^*) - \sum_{j=1}^{m_k} f(S_j x) \right| = O(1)$$

and consequently $\sup_{0\le t\le 1}|\zeta_n^x(t)-\Phi_n^x(t)|\to 0$. Thus $(\Phi_n^x)_{n\in\mathbb{N}}$ is also $\mathbb{P}$-almost surely relatively compact in $C[0,1]$ and the set of its limit points agrees with the Strassen set. By the first relation of Lemma 2 this holds also for $(\theta_n^x)_{n\in\mathbb{N}}$.

Next we introduce the random function

$$\xi_n^x(0)=0, \qquad \xi_n^x(m_k/m_n)=(2A_x m_n \log\log m_n)^{-1/2}\sum_{j=1}^{m_k} f(S_j x)$$

for $k=0,1,\dots,n$ and ξ_n^x is linear on $[m_k/m_n, m_{k+1}/m_n]$, $k=0,\dots,n-1$. This is the analogue of θ_n^x when the breakpoints are the numbers m_k/m_n instead of B_k^x/B_n^x. Let $T_n:[0,1]\to[0,1]$ be the transformation that maps $[m_k/m_n, m_{k+1}/m_n]$ linearly to $[B_k^x/B_n^x, B_{k+1}^x/B_n^x]$ for $k=0,\dots,n$. Clearly $\xi_n^x(t)=\theta_n^x(T_n(t))$ and

$$\sup_{0\le t\le 1}|T_n(t)-t|=\max_{0\le k\le n}|B_k^x/B_n^x-m_k/m_n|.$$

Since $B_n^x\sim A_x m_n$ the right hand side of the last relation tends to 0 as $n\to\infty$. By the $\mathbb{P}$-a.s. equicontinuity of $(\theta_n^x)_{n\in\mathbb{N}}$ this implies that $\|\xi_n^x-\theta_n^x\|\to 0$ $\mathbb{P}$-almost surely, where $\|\cdot\|$ denotes the sup norm. Thus $(\xi_n^x)_{n\in\mathbb{N}}$ itself is $\mathbb{P}$-almost surely relatively compact in $C[0,1]$ and the set of its limit points coincides with the Strassen set. This proves the validity of the Theorem along the indices $n=m_k$.

To prove the theorem for all n, let us note that by the Arzela-Ascoli theorem the relative compactness of the sequence $(\Gamma_n^x)_{n\in\mathbb{N}}$ in $C[0,1]$ is equivalent to its equicontinuity, which, in turn, is equivalent to the statement that for any $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that for any $n\ge 1$ we have

$$\left|\sum_{k\le j\le \ell} f(S_j x)\right|\le \varepsilon(2n\log\log n)^{1/2}$$

provided $\ell-k\le\delta n$. By what we proved above, this statement is valid if n is of the form m_k for some k. But since

$$m_{k+1}/m_k\to 1 \quad\text{and}\quad (\log\log m_{k+1})/(\log\log m_k)\to 1, \tag{2.10}$$

this statement will be valid for all n, i.e. $(\Gamma_n^x)_{n\in\mathbb{N}}$ is $\mathbb{P}$-almost surely equicontinuous. Note also that for $3\le n\le n'$ we have, as observed in [5],

$$\Gamma_{n'}^x\left(\frac{n'}{n}t\right)=\left(\frac{n\log\log n}{n'\log\log n'}\right)^{1/2}\Gamma_n^x(t)$$

for points of the form $t=k/n$. Now if $m_k\le n<n'\le m_{k+1}$ then the last formula, (2.10) and the already proved equicontinuity of $(\Gamma_n^x)_{n\in\mathbb{N}}$ imply that for $k\ge k(\varepsilon)$ we have $|\Gamma_n^x(t)-\Gamma_{n'}^x(t)|\le\varepsilon$ for all $0\le t,t'\le 1$. Thus the class of limit points of $(\Gamma_n^x)_{n\in\mathbb{N}}$ in $C[0,1]$ along the whole sequence n is the same as along the subsequence $(m_k)_{k\in\mathbb{N}}$, i.e. K_x. This completes the proof of our theorem.

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