Department of Mathematical Structure Theory

Free multiplicative central limit theorems and the free multiplicative normal distribution on the unit circle

Christian Sattlecker

TU Graz
Graz University of Technology

Introduction and notations

Let $\mathcal{A}$ be a finite von Neumann algebra with a normal, faithful trace φ. Let $u \in \mathcal{A}$ be a unitary and x a self–adjoint operator affiliated with $\mathcal{A}$. Then we can assign a probability measure supported on the unit circle $\mu \in \mathcal{M}(\mathbb{T})$ to u and one supported on the real line $\nu \in \mathcal{M}(\mathbb{R})$ to x. The measures are called spectral distributions of u or x and determined by

$$\varphi(f(u)) = \int_{\mathbb{T}} f(z)\, d\mu(z), \qquad \varphi(g(x)) = \int_{\mathbb{R}} g(z)\, d\nu(z)$$

for all bounded Borel functions f, g on the spectrum of u or x, respectively.

We study free multiplicative convolution of unitary operators. Let $u, v \in \mathcal{A}$ be unitaries with spectral distributions $\mu, \nu \in \mathcal{M}(\mathbb{T})$. If u and v are free in the sense of D. Voiculescu [9], then the spectral distribution of the product uv is determined by μ and ν. Hence, we write $\mu \boxtimes \nu$ for this measure.

For measures $\mu \in \mathcal{M}(\mathbb{T})$ with non–vanishing expectation value $m_1(\mu) \neq 0$ we define the Ψ- and S-transform

$$\Psi_\mu(z) := \sum_{n=1}^{\infty} \mathrm{m}_n(\mu) z^n, \qquad S_\mu(z) := \frac{z+1}{z} \Psi_\mu^{(-1)}(z),$$

where $\mathrm{m}_n(\mu) = \int z^n \, d\mu(z)$ and $\Psi_\mu^{(-1)}$ denotes the composition inverse of Ψ_μ. The spectral distribution of a product of free unitaries can be computed with

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) \cdot S_\nu(z).$$

Let $u \in \mathcal{U}(H)$ be a unitary on a Hilbert space and $(v_t)_{t \in \mathbb{R}} \subseteq \mathcal{U}(H)$ a one–parameter group of unitaries, i.e. $v_s v_t = v_{s+t}$. If $v_1 = u$, we can define a n-th root of u by $u^{1/n} := v_{1/n}$. A theorem of M. H. Stone describes a one–to–one correspondence

$$x \leftrightarrow (v_t = \exp(itx))$$

between self–adjoint operators x and strongly continuous, one–parameter groups of unitaries $(v_t)_{t \in \mathbb{R}}$. Hence, we also consider self–adjoint operators.

If the measure $\mu \in \mathcal{M}(\mathbb{R})$ is the spectral distribution of a self–adjoint x, then $R(\mu) \in \mathcal{M}(\mathbb{T})$ denotes the spectral distribution of $\exp(ix)$, i.e.

$$\int_{\mathbb{T}} f(z)\, dR(\mu)(z) := \int_{\mathbb{R}} f(e^{it})\, d\mu(t).$$

We define a right inverse $R^{-1} : \mathcal{M}(\mathbb{T}) \to \mathcal{M}([-\pi, \pi])$ by

$$\int_{-\pi}^{\pi} f(e^{iz})\, d\, R^{-1}(\mu)(z) := \int_{\mathbb{T}} f(z)\, d\mu(z)$$

Furthermore, we use the dilatation operator $D_c : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ for $c > 0$. If μ is the distribution of X, then $D_c(\mu)$ denotes the distribution of cX.

Central Limit Theorem

Let $(x_k)_{k \in \mathbb{N}}$ be free self–adjoint operators with the same spectral distribution μ. The unitaries $\exp(itx_k)$ are also free. We want to study the large n limit of products

$$\exp\left(\frac{i}{\sqrt{n}} x_1\right) \exp\left(\frac{i}{\sqrt{n}} x_2\right) \ldots \exp\left(\frac{i}{\sqrt{n}} x_n\right).$$

The limit distributions of the following CLT were defined by H. Bercovici and D. V. Voiculescu [2].

Free multiplicative normal distribution

The free multiplicative normal distributions are the unique measures $\sigma_t \in \mathcal{M}(\mathbb{T})$, $t \geq 0$, with S-transform

$$S_{\sigma_t}(z) := \exp\left(t\left(z + \frac{1}{2}\right)\right).$$

Central Limit Theorem

Let $\mu \in \mathcal{M}(\mathbb{R})$ with vanishing first moment $m_1(\mu) = 0$ and variance $t := m_2(\mu) < \infty$. Then we have

$$\left(R\left(D_{\frac{1}{\sqrt{n}}}(\mu)\right)\right)^{\boxtimes n} \xrightarrow{w} \sigma_t.$$

For the proof we use a limit theorem of G. P. Chistyakov and F. Götze [5].

Free multiplicative normal distribution σ_t

The measures σ_t of the last section appear also in other applications. The key point is, that σ_t is the large N limit of the heat kernel measures on matrix unitary groups $U(N)$ [3, 10]. This leads to many applications like a free multiplicative Brownian motion [3], and the large N limit of $U(N)$ Yang–Mills theories [1, 10]. The measures were also used for representation theory of symmetric groups [7].

Properties of σ_t are hard to handle. P. Biane [3] computed the moments of σ_t. A new formula using the confluent hypergeometric function

$${}_1\mathrm{F}_1(a, b, z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n$$

seems to be advantageous sometimes.

Theorem (moments and cumulants, [3])

The moments and free cumulants are given by

$$m_n(\sigma_t) = \exp\left(-\frac{nt}{2}\right) \sum_{k=0}^{n-1} \frac{(-1)^k n^{k-1}}{k!} \binom{n}{k+1} t^k$$

$$= \exp\left(-\frac{nt}{2}\right) {}_1\mathrm{F}_1(1-n, 2, nt) \quad \textit{and}$$

$$\kappa_n(\sigma_t) = \exp\left(-\frac{nt}{2}\right) \frac{(-nt)^{n-1}}{n!}.$$

Remark: The latter formulae are also valid for the free multiplicative normal distribution on the *positive real line*, i.e. $t < 0$. We can deduce from the cumulants, that σ_t, $t < 0$, are $\boxplus$–infinite divisible. See also [8].

Conjecture 1

The characteristic function of the uncoiled measure $R^{-1}(\sigma_t)$ for $t \leq 4$ is given by

$$\varphi_{R^{-1}(\sigma_t)}(z) \stackrel{?}{=} \exp\left(-\frac{zt}{2}\right) {}_1\mathrm{F}_1(1 - z, 2, zt).$$

The following theorem describes the density function of σ_t. Some aspects have already been discovered by P. Biane [4].

Theorem (density function, [4])

The uncoiled measure $R^{-1}(\sigma_t)$ is absolutely continuous and the density function is Hölder–continuous, even, and monotone raising on $[-\pi, 0]$.

$$\operatorname{supp}\left(R^{-1}(\sigma_t)\right) = \begin{cases} [-c, c] & t < 4 \\ [-\pi, \pi] & t \geq 4 \end{cases}$$

$$\textit{with} \quad c = 2 \arctan\left(\sqrt{\frac{t}{4-t}}\right) + \sqrt{t\left(1 - \frac{t}{4}\right)}.$$

The density function can be written as a uniform converging Fourier series

$$\frac{dR^{-1}(\sigma_t)}{dx}(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-\frac{nt}{2}} {}_1\mathrm{F}_1(1-n, 2, nt) \cos(nx) \qquad -\pi < x < \pi.$$

Numerical approximation of the moments of $R^{-1}(\sigma_t)$ using Fourier polynomials led us to the following conjecture.

Conjecture 2

The free cumulants of the uncoiled measure $R^{-1}(\sigma_t)$ for $t \leq 4$ are

$$\kappa_n\left(R^{-1}(\sigma_t)\right) \stackrel{?}{=} \begin{cases} 0 & n \text{ odd} \\ t - \frac{t^2}{12} & n = 2 \\ \frac{(it)^n}{n!} B_n & n \geq 4 \end{cases}$$

where B_n denotes the n-th Bernoulli number. In terms of the $\mathcal{R}$-transform this means

$$\mathcal{R}(z) \stackrel{?}{=} \frac{it}{1 - e^{-itz}} - \frac{1}{z} - \frac{it}{2} + tz.$$

We have hoped, that the two conjectures would shed some light on the question, wether there is a nice way to uncoil the measure σ_t, $t > 4$, on a larger interval than $[-\pi, \pi]$ or on the whole real line. Maybe this would help to understand the "shock" of the free unitary Brownian motion, when the spectrum collide at $-1 = e^{\pm i\pi}$, i.e. when the time parameter reaches $t = 4$. Unfortunately, the conjectures fail for $t > 4$. The functions are not a characteristic function or a $\mathcal{R}$–transform, respectively.

The Cauchy distribution and the real Poisson kernel

The Cauchy distribution plays an important role for additive convolution in non–commutative probability theory [6]. It has also nice properties for multiplicative convolution on the unit circle.

Let $\nu_t \in \mathcal{M}(\mathbb{R})$, $t > 0$, be the Cauchy distribution with density function

$$d\nu_t(x) = \frac{t}{\pi(t^2 + x^2)} dx \qquad -\infty < x < \infty.$$

The rolled–up measure $\rho_t := R(\nu_t) \in \mathcal{M}(\mathbb{T})$ is the real Poisson kernel with density function

$$dR^{-1}(\rho_t)(\theta) = \frac{\sinh(t)}{2\pi(\cosh(t) - \cos(\theta))} d\theta \qquad -\pi < \theta < \pi.$$

and S-transform $S_{\rho_t}(z) = e^t$.

The free and classical convolution of an arbitrary $\mu \in \mathcal{M}(\mathbb{R})$ and a Cauchy distribution coincide

$$R(\mu) \boxtimes \rho_t = R(\mu) \circledast \rho_t = R(\mu * \nu_t) = R(\mu \boxplus \nu_t).$$

In especially, we have

$$R(D_s(\nu_1)) \boxtimes R(D_t(\nu_1)) = R(D_{s+t}(\nu_1)).$$

So we can call the Cauchy distribution 1–stable with respect to the free multiplicative convolution on the unit circle. We do not know more stable distribution in this sense, than $\delta_c \boxplus \nu_t$ with $c \in \mathbb{R}$ and $t \geq 0$.

We guess, that there exists also a central limit theorem for measures with heavy tails similar to [6]. The set $\mathcal{M}^* \subseteq \mathcal{M}(\mathbb{R})$ of applicable probability measures may be characterised as follows. Every $\mu \in \mathcal{M}^*$ can be written as a sum of two positive measures $\mu = \tau + \lambda$.

- τ has finite second moment $m_2(\tau) < \infty$
- There exists a function $f(z) = \sum_{n=2}^{\infty} \frac{a_n}{z^n}$ and constants $0 < r < R$, such that $f(z)$ is analytic for $|z| > r$ and non–negative for all real $|z| > R$ and

$$d\lambda(x) = f(x) \mathbb{1}_{|x|>R}\, dx.$$

Conjecture

Let $\mu \in \mathcal{M}^*$ and

$$t \stackrel{?}{=} \Im \int_{\Gamma} z f(z)\, dz$$

where f is the function from the definition of $\mathcal{M}^*$ and $\Gamma = \{Re^{i\theta} \,|\, \theta \in [0, \pi]\}$ equipped with counter–clockwise direction. We conjecture, that

$$\left(R\left(D_{\frac{1}{n}}(\mu)\right)\right)^{\boxtimes n} \xrightarrow{w} \rho_t$$

If $t = 0$, then we treat ρ_0 as the Dirac measure δ_1.

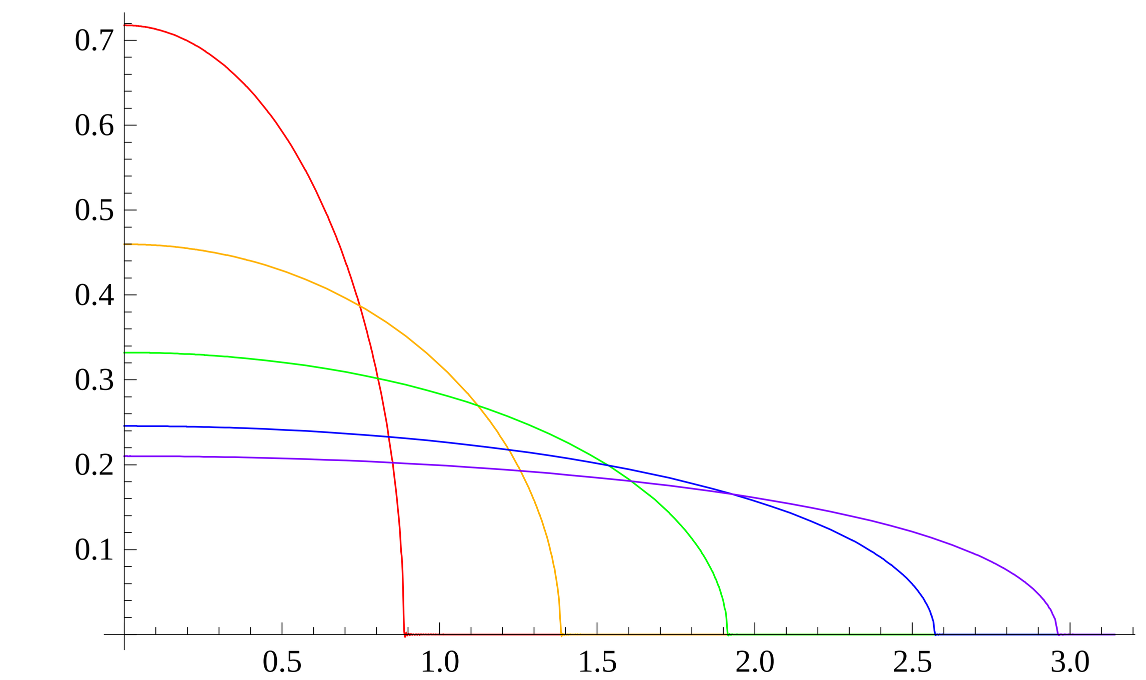

Figure: Approximated density functions of the free multiplicative normal distribution σ_t for $t = 0.1$, $t = 0.5$, $t = 1$, $t = 2$, and $t = 3$.

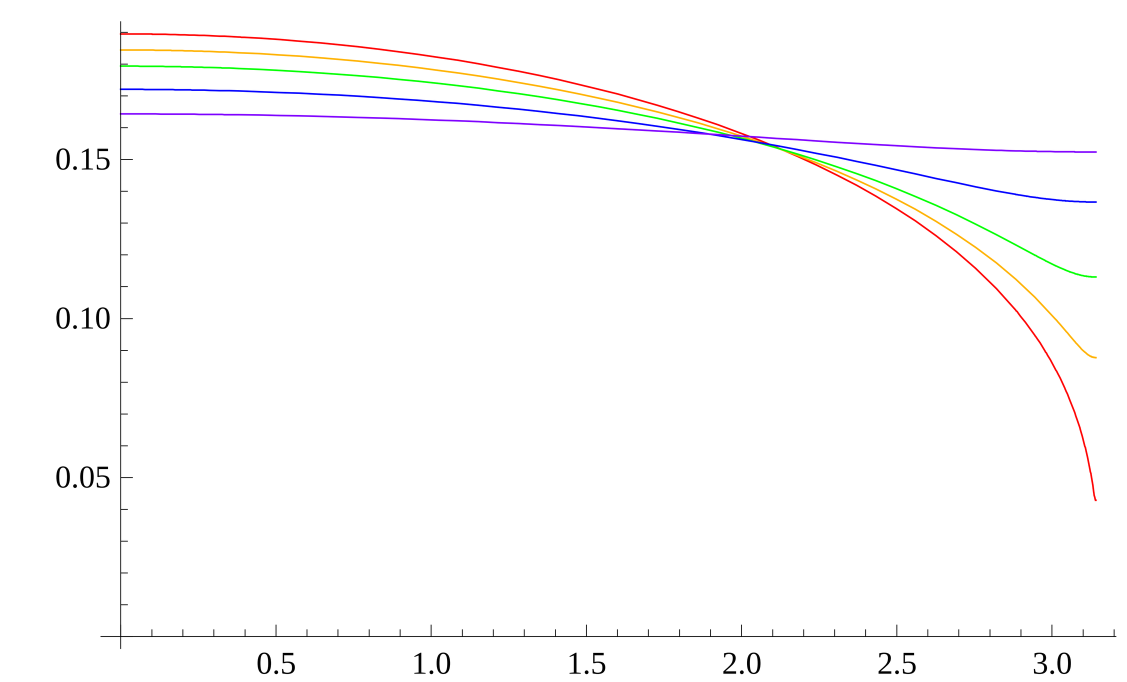

Figure: Approximated density functions of the free multiplicative normal distribution σ_t for $t = 4.1$, $t = 4.5$, $t = 5$, $t = 6$, and $t = 8$.

Bibliography

[1] Michael Anshelevich and Ambar N. Sengupta.
Quantum Free Yang–Mills on the Plane.
arXiv:1106.2107, 2011.

[2] Hari Bercovici and Dan Voiculescu.
Lévy-Hinčin type theorems for multiplicative and additive free convolution.
Pacific J. Math., 153(2):217–248, 1992.

[3] Philippe Biane.
Free Brownian motion, free stochastic calculus and random matrices.
In *Free probability theory (Waterloo, ON, 1995)*, volume 12 of *Fields Inst. Commun.*, pages 1–19. Amer. Math. Soc., Providence, RI, 1997.

[4] Philippe Biane.
Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems.
J. Funct. Anal., 144(1):232–286, 1997.

[5] Gennadii P. Chistyakov and Friedrich Götze.
Limit theorems in free probability theory. II.
Cent. Eur. J. Math., 6(1):87–117, 2008.

[6] Takahiro Hasebe.
Analytic continuations of Fourier and Stieltjes transforms and generalized moments of probability measures.
J. Theor. Probability, 2011.
in Vorbereitung, arXiv:1009.1510v2.

[7] Thierry Lévy.
Schur-Weyl duality and the heat kernel measure on the unitary group.
Adv. Math., 218(2):537–575, 2008.

[8] Noriyoshi Sakuma and Hiroaki Yoshida.
New limit theorems related to free multiplicative convolution.
2011.
arXiv:1103.6156v1.

[9] D. V. Voiculescu, K. J. Dykema, and A. Nica.
Free random variables, volume 1 of *CRM Monograph Series*.
American Mathematical Society, Providence, RI, 1992.
A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[10] Feng Xu.
A random matrix model from two-dimensional Yang-Mills theory.
Comm. Math. Phys., 190(2):287–307, 1997.

sattlecker@math.tugraz.at