FREE INTEGRAL CALCULUS

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Abstract. We study the problem of conditional expectations in free random variables and provide closed formulas for the conditional expectation of a resolvent $\Psi(z) = (1 - zP(a_1, \ldots, a_n))^{-1}$ of an arbitrary non–commutative polynomial $P(a_1, \ldots, a_n)$ in free random variables $a_1, \ldots, a_n$ onto the subalgebra of an arbitray subset of the variables $a_1, \ldots, a_n$. More precisely, given a linearization of $\Psi(z)$, our methods allow to compute a linearization of its conditional expectation. The coefficients of the expressions obtained in this process involve certain Boolean cumulant functionals which can be computed by solving a system of equations. On the way towards the main result we introduce a non–commutative differential calculus which allows to evaluate conditional expectations and certain Boolean cumulant functionals β^b and β^δ. We conclude the paper with several examples which illustrate the working of the developed machinery. For completeness two appendices complement the paper. The first appendix contains a purely algebraic approach to Boolean cumulants and the second appendix provides a crash course on linearizations of rational series.

Contents

Date: Rev.342, November 8, 2023.
2020 *Mathematics Subject Classification.* Primary: 46L54. Secondary: 15B52.
Key words and phrases. Free Probability, subordination, Boolean cumulants, conditional expectation.
Supported by the Austrian Federal Ministry of Education, Science and Research and the Polish Ministry of Science and Higher Education, grants N^{os} PL 08/2016 and PL 06/2018.
FL: This research was funded in part by the Austrian Science Fund (FWF) grant I 6232-N BOOMER (WEAVE).
KSz: This research was funded in part by National Science Centre, Poland WEAVE-UNISONO grant BOOMER 2022/04/Y/ST1/00008.

1. Introduction

1.1. **Outline.** Free probability was introduced by Voiculescu 40 years ago [42] as a means to solve long-standing problems in von Neumann algebras. Over the years however deep connections to several branches of mathematics came to light, among others random matrix theory and representation theory of the symmetric group. Complementing Voiculescu's analytic approach, Speicher developed a systematic theory of free cumulants [35], already announced in [42]. In this approach, free independence is characterized by vanishing of mixed free cumulants, in analogy to classical independence, which can by characterized by vanishing of mixed classical cumulants. Indeed, most properties of free cumulants can be obtained from the corresponding properties of classical cumulants by replacing the lattice of all set partitions by the lattice of non–crossing partitions, following the general scheme of multiplicative functions on lattices [12], see the standard reference [31] for a detailed treatment of free cumulants. The discovery of free cumulants triggered a lot of progress in free probability, and it was the starting point of many deep combinatorial studies of various structures in free probability (see [30, 21, 22] and many other).

"Partial cumulants" were introduced by von Waldenfels in order to simplify certain calculations in mathematical physics [47]. Later they were called *Boolean cumulants* corresponding to the notion of *Boolean independence* which was introduced in [36]. They naturally appear in the guise of first return probabilities of random walks [48]. Combinatorially Boolean cumulants follow the pattern of classical and free cumulants by replacing the lattice of set partitions (resp. noncrossing partitions) with the lattice of interval partitions which is isomorphic to the Boolean lattice. From a combinatorial point of view Boolean cumulants are the simplest kind of cumulants.

Recently it was noticed that despite their simplicity Boolean cumulants are useful for non–commutative probability in general [18] and free probability in particular [5, 14, 23, 37]. Boolean cumulants were used for the first combinatorial solution to the problem of the free anti–commutator in [14] (an analytic solution was found earlier by Vasilchuk [41]; a solution in terms of free cumulants was presented recently in [32]), and for the identification of the coefficients of power series expansion of subordination functions [23] (implicitly also in [48]).

In the present paper we continue these investigations and show that Boolean cumulants may indeed be used for a systematic study of free random variables. The first step in this direction is a surprisingly simple characterization of freeness by the vanishing of *some* mixed Boolean cumulants.

Theorem 1.1 (Characterization of freeness in terms of Boolean cumulants)**.** *Let $(\mathcal{M}, \varphi)$ be a tracial non–commutative probability space. Subalgebras $\mathcal{A}$ and $\mathcal{B}$ are free if and only if $\beta_m(a_1, a_2, \ldots, a_n) = 0$ whenever $n > 1$ and $a_j \in \mathcal{A} \cup \mathcal{B}$ for $j = 1, 2, \ldots, n$ with a_1 and a_n coming from different subalgebras.*

This property turns out to be the key to an efficient calculation of conditional expectations in free random variables. In particular, for free random variables $a_1, a_2, \ldots, a_n$ we determine the conditional expectation of the resolvent $(1 - zP(a_1, a_2, \dots, a_n))^{-1}$ of an arbitrary non–commutative polynomial $P(a_1, a_2, \dots, a_n)$ onto a subalgebra generated by some subset of the variables $a_1, a_2, \dots, a_n$. To this end we employ the concept of linearizations, a method also used in [4, 16]. More precisely, given a linearization of the resolvent, we obtain a linearization of its conditional expectation. Our discussion is focused on non–commutative rational functions of the form $(1 - zP(a_1, a_2, \dots, a_n))^{-1}$, however an example at the end of this paper shows that the methods are applicable to a wider class of rational functions.

The first step of our approach is based on a recurrence which can be summarized as a free integration formula and which in the case of resolvents naturally leads to a system of linear equations for the conditional expectation which can be turned into a linearization. The coefficients appearing in said linearization are certain generating functions of mixed Boolean cumulants of free random variables.

For the sake of simplicity let us consider the case of non–commutative polynomials in two free random variables X and Y. It is known that for free random variables the conditional expectation of a polynomial is a polynomial again. We show how to obtain this result in a recursive way. The functional β^b_Y is defined on non–commutative polynomials and depends on the distributions of X and Y. The derivative $\overrightarrow{\delta}_X$ acts on non–commutative polynomials. For precise definitions we refer to Section 4.

Theorem 1.2 (Free integration formula)**.** *The conditional expectation of a non–commutative polynomial $P \in \mathbb{C}\langle X, Y\rangle$ satisfies the identity*

$$\mathbb{E}_X[P] = \beta^b_Y(P) + (\beta^b_Y \otimes \mathbb{E}_X)(\overrightarrow{\delta}_X(P)).$$

This formula has a certain resemblance to classical integration. $\mathbb{E}_X$ "integrates away" the variable Y and if we denote $\mathfrak{I}_Y = \beta^b_Y \otimes \mathbb{E}_X$, then the formula reads

$$\mathfrak{I}_Y(\overrightarrow{\delta}_X(P)) = \mathbb{E}_X[P] - \beta^b_Y(P) = \mathfrak{I}_Y(I \otimes P - P \otimes I).$$

The formula above allows to calculate conditional expectations in terms of Boolean cumulants. Thus we are faced with the problem to calculate Boolean cumulants of functions in free random variables. In order to do this we introduce an algebraic calculus of Boolean cumulants, based on a number of algebraic rules and devices which allow to establish equations for the generating functions arising from the previous calculations. More precisely we introduce two functionals β^b and β^δ on the free algebra which evaluate Boolean cumulants in two ways, *block-wise* and *fully factored*. We then establish mutual recursive equations between these with the help of the previously developed algebraic devices which lead to a closed system of algebraic equations. Although the functionals β^b and β^δ are defined in relation to conditional expectations, it appears that they will have much wider applications, and the calculus we introduce for them is of independent interest. The calculus for β^b and β^δ, based on generalizations of observations from [14, 23], subsumes the combinatorial case-by-case analysis of functions in free variables into a general algebraic machinery.

First steps towards a similar calculus in terms of free cumulants were done in [10] and we expect the unshuffle algebras of [13] to play a role here. Mixed free cumulants of free random variables vanish and they turn additive free convolution into a simple addition. However it turned out that when it comes to multiplicative free convolution, free cumulants offer no advantage as the formulas are actually identical [5]. The advantage of Boolean cumulants lies in their combinatorial simplicity. The main steps of our calculations are as follows:

(i) Turn the simple combinatorics of Boolean cumulants of products of free variables into an algebraic rule involving certain derivations which appeared earlier in free probability.
(ii) Apply these derivations to resolvents and use the fact that these play the role of "eigen-functions" analogous to the exponential function in classical calculus.
(iii) Use the formula for Boolean cumulants with free entries in order to separate the variables and obtain equations for the generating functions.

We do not prove any new combinatorial results, rather provide algebraic reformulations of known combinatorial identities and put them together into an effective machinery to make them available for free analysis. It allows to establish equations for generating functions of Boolean cumulants of functions in free variables which in some cases can be effectively solved and used to compute conditional expectations and distributions of arbitrary polynomials in free random variables.

In particular we present an algebraic interpretation of the formula for Boolean cumulants with products as entries and a characterization of freeness in terms of Boolean cumulants from [14] in terms of Voiculescu's free derivative. We refer to Theorem 5.5 for the precise statement and Definitions 4.10 and 5.2 for definitions of the functionals β^b and β^δ.

The paper is organized as follows.

The rest of the introduction is devoted to an exposition of the problem of conditional expectations.

Section 2 presents results about the basic ingredients: conditional expectations, Boolean cumulants and derivations.

In Section 3 we prove the characterization of freeness announced above in Theorem 1.1.

In Section 4 we provide a method to compute conditional expectations in terms of Boolean cumulants. In particular we prove Theorem 1.2.

In Section 5 we introduce a calculus for the Boolean functionals β^b and β^δ which is summarized in Theorem 5.5.

In Section 6 we show how linearizations allow to solve the problem for subordinations for general polynomials. In particular we prove Theorem 1.3 which we discuss below.

Section 7 contains examples.

Appendix A we give self-contained algebraic proofs of the basic results about Boolean cumulants as well as a reformulation of these in terms of tensor algebras.

Finally Appendix B contains the essential information required for the computation of linearizations.

1.2. **Subordination for general polynomials.** Let X and Y be classically independent random variables with distributions μ_X and μ_Y, then their joint distribution is $\mu_{X,Y} = \mu_X \otimes \mu_Y$, i.e., the expectation of any integrable function $f(X,Y)$ can in practice be computed as a double integral

$$\mathbb{E}\, f(X,Y) = \int \int f(x,y)\, d\mu_Y(y)\, d\mu_X(x).$$

In the non–commutative case there is no such integral representation, however the inner integral is in fact the conditional expectation

$$\int f(X,y)\, d\mu_Y(y) = \mathbb{E}[f(X,Y)|X] \tag{1.1}$$

and thus

$$\mathbb{E}\, f(X,Y) = \mathbb{E}[\mathbb{E}[f(X,Y)|X]]$$

which does have a non–commutative analogue. In the present paper we propose a method to compute this non–commutative conditional expectation

$$\mathbb{E}_X[P(X,Y)]$$

for arbitrary non–commutative polynomials P and more general rational functions in *free* random variables. It is the analogy with (1.1) which motivated us to call our endeavour *free integral calculus*.

This follows ideas of Voiculescu [43] and Biane [7] who showed that for the sum $a+b$ of two free random variables a, b there exists an analytic self map of $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that the conditional expectation of the resolvent onto the algebra generated by a is a resolvent again

$$\mathbb{E}_a[(z-a-b)^{-1}] = (\omega(z)-a)^{-1}, \tag{1.2}$$

which after application of φ yields results in the subordination relation $G_a(\omega(z)) = G_{a+b}(z)$. Moreover the function ω is known to satisfy a fixed point equation.

In the present paper we generalize this method to arbitrary polynomials using an algebraic-analytic method based on the observation from our previous work [14, 23] that Boolean cumulants turn out to be a convenient tool to "store" the results of the "partial integral" described above.

Fix a non–commutative probability space $(\mathcal{M}, \varphi)$. Given self-adjoint random variables $a_1, a_2, \ldots, a_n \in \mathcal{M}$ and a non-commutative polynomial $P \in \mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle$, our objective is an explicit formula for the conditional expectation of $\left(1 - zP(a_1, a_2, \ldots, a_n)\right)^{-1}$ onto the algebra generated by some subset of the variables $a_1, a_2, \ldots, a_n$. Without loss of generality we may assume that the subset consists of $a_1, a_2, \ldots, a_k$, where $k < n$.

In the first step we construct a *linearization* of the resolvent, i.e., matrices $C_1, C_2, \ldots, C_n \in M_N(\mathbb{C})$ such that for $L = C_1 \otimes a_1 + C_2 \otimes a_2 + \cdots + C_n \otimes a_n \in M_N(\mathbb{C}[z]) \otimes \mathcal{M}$ and z in some neighborhood of zero we have

$$(1 - z^m P(a_1, a_2, \ldots, a_n))^{-1} = u^t (I_N - zL)^{-1} v \tag{1.3}$$

for some vectors $u, v \in \mathbb{C}^N$, where m is the total degree of P. In general a polynomial may have many linearizations; in Appendix B we discuss in detail algorithms for finding linearizations which work for our purposes. It suffices to say for the moment that in contrast to [4] some technical issues force us to work with regular linearizations which are not self-adjoint and to restrict the calculations to the level of formal power series.

In the case when $\mathcal{A}$ is the von Neumann subalgebra freely generated by $a_1, a_2, \ldots, a_n$, we prove the following theorem. It follows by evaluating the formal expression from Theorem 6.12 below in the variables $a_1, a_2, \ldots, a_n$.

Theorem 1.3 (Subordination for general polynomials)**.** *Given a linearization* (1.3) *for a polynomial* $P \in \mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle$ *of degree* m *we have for* z *in some neighbourhood of* 0 *the following linearization for the conditional expectation of the resolvent*

$$\begin{aligned} &\mathbb{E}_{\mathcal{A}}\left[(1 - z^m P(a_1, a_2, \ldots, a_n))^{-1}\right] \\ &\quad = u^t \left(I_N - z\left(C_1 \otimes a_1 + C_2 \otimes a_2 + \cdots + C_k \otimes a_k + (C_{k+1}F_{k+1} + \cdots + C_n F_n) \otimes I\right)\right)^{-1} v \end{aligned} \tag{1.4}$$

where the matrices $F_1, F_2, \ldots, F_n$ *constitute the unique fixed point of the system of equations*

$$F_i = \widetilde{\eta}_{a_i}\left(z\Big(I_N - z\sum_{j\neq i} C_j F_j\Big)^{-1} C_i\right) \qquad \text{for } i = 1, 2, \ldots, n \tag{1.5}$$

with entries which are analytic at 0*. Here by* $\widetilde{\eta}_a(z) = \sum_{n=1}^{\infty} \beta_n(a) z^{n-1}$ *we denote the shifted Boolean cumulant generating function of a random variable* a*.*

Remark 1.4.

(i) From a practical point of view Theorem 1.3 asserts that the evaluation of the conditional expectation of the resolvent

$$\left(1 - z^m P(a_1, a_2, \ldots, a_n)\right)^{-1} = u^t \left(I_N - z\left(C_1 \otimes a_1 + C_2 \otimes a_2 + \cdots + C_n \otimes a_n\right)\right)^{-1} v$$

onto $\mathcal{A}$ (i.e., "integrating out" $a_{k+1}, \ldots, a_n$) amounts to replacing the corresponding summands $I_N \otimes a_i$ with the matrices $F_i \otimes I$.

(ii) Although only the matrices $F_{k+1}, F_{k+2}, \ldots, F_n$ explicitly appear in the final formula (1.4), the matrices $F_1, F_2, \ldots, F_k$ are required as well in order to extract the former from the solution of equation (1.5). Moreover observe that in general the equations cannot be effectively decoupled (unless $n = 2$, see example 1.6 below) and each matrix from $F_1, F_2, \ldots, F_n$ depends on the distributions of all variables $a_1, a_2, \ldots, a_n$.

Remark 1.5 (Subordination). The previous theorem generalizes the subordination phenomenon in the following sense. For simplicity consider the case of two free variables a, b and fix a polynomial $P \in \mathbb{C}\langle X, Y\rangle$ of degree m. We fix an $N \times N$ linearization

$$(1 - z^m P(X,Y))^{-1} = u^t(I_N - zC_1X - zC_2Y)^{-1}$$

then the above theorem says that the moment generating function $M_P(z) = \varphi\left((1 - zP(a,b))^{-1}\right)$ is obtained from the linearization via

$$M_{P(a,b)}(z^m) = u^t\varphi^{(N)}(I_N - zC_1a - zC_2b)^{-1}v$$

where $\varphi^{(N)}$ is the entry–wise application of φ. Let $H_1 = (I - zC_1F_1)^{-1}$, then the preceding identity can be written as

$$M_{P(a,b)}(z^m) = u^t\varphi^{(N)}(I_N - zH_1C_2b)^{-1}H_1v$$

Suppose that the matrix H_1C_2 is diagonalizable and write $zH_1C_2 = QDQ^{-1}$ with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then we obtain

$$M_{P(a,b)}(z^m) = (Pu)^t\varphi(I_N - Db)^{-1}P^{-1}H_1v = \widehat{u}^t \begin{bmatrix} M_b(\lambda_1) & 0 & \ldots & 0 \\ 0 & M_b(\lambda_2) & \ldots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_b(\lambda_n) \end{bmatrix} \widehat{v}$$

where $\widehat{u} = Pu$, $\widehat{v} = P^{-1}H_1v$ and $M_b(z) = \varphi((1 - zb)^{-1})$ is the moment generating function of b.

Therefore the eigenvalues $\lambda_i = \lambda_i(z)$ of zH_1C_2 can be understood as a generalization of the subordination functions from free additive convolution. At the time of this writing we do not know whether H_1C_2 is actually diagonalizable in general. If this turns out not to be the case, one can use the Jordan canonical form and the derivatives M_b', M_b'',… will populate the upper triangular parts fo the Jordan blocks.

Let us illustrate Theorem 1.3 with a quick derivation of the subordination function for additive free convolution and for anti–commutator. For further examples see Section 7.

Example 1.6. The polynomial $P(a, b) = a + b$ is already linear and we can apply Theorem 1.3 with the trivial linearization

$$\mathbb{E}_a\left[(1 - z(a + b))^{-1}\right] = (1 - za - zF_b(z))^{-1}$$

where

$$F_a(z) = \widetilde{\eta}_a\Big(\frac{z}{1 - zF_b(z)}\Big) \qquad F_b(z) = \widetilde{\eta}_b\Big(\frac{z}{1 - zF_a(z)}\Big).$$

Both formulas for the conditional expectation and the two equations can be easily checked to be equivalent to the well known subordination results for free additive convolution. In particular in this case it is straightforward to decouple the equations and obtain separate fixed point equations for F_1 and F_2 after a simple substitution of one equation into the other.

Example 1.7. Let $P(a, b) = ab + ba$ be the anti–commutator, then for z in some neighbourhood of zero the conditional expectations of the resolvent are

$$\mathbb{E}_a\left[(1 - z^2(ab + ba))^{-1}\right] = \left(1 - f_{2,43}z - z^2a\left(f_{2,33} + f_{2,44}\right) - f_{2,34}a^2z^3\right)^{-1},$$

$$\mathbb{E}_b\left[(1 - z^2(ab + ba))^{-1}\right] = \left(1 - f_{1,12}z - z^2b\left(f_{1,11} + fx_{22}\right) - f_{1,21}b^2z^3\right)^{-1}.$$

This result is obtained with the linearization involving the matrices

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Apriori the equations (1.5) involve a total of 32 variables $f_{k,ij}$, $k \in \{1,2\}, 1 \leq i,j \leq 4$. However it is easy to see with the help of the projection matrices onto $\ker C_1$ and $\ker C_2$ (cf. Remark 6.14) that many variables vanish and the solution matrices have the form

$$F_1 = \begin{bmatrix} f_{1,11} & f_{1,12} & 0 & 0 \\ f_{1,21} & f_{1,22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & f_{2,33} & f_{2,34} \\ 0 & 0 & f_{2,43} & f_{2,44} \end{bmatrix}$$

and satisfy the following system of equations

$$\begin{cases} F_1 &= \widetilde{\eta}_a \left(zQ_1(1 - zC_2F_2)^{-1}C_1 \right), \\ F_2 &= \widetilde{\eta}_b \left(zQ_2(1 - zC_1F_1)^{-1}C_2 \right), \end{cases}$$

where $I - Q_i$ is the projection onto the kernel of the matrix C_i, i.e., $C_iQ_i = C_i$. This system gives the analogue for the anti–commutator of the fixed point equation in the case of additive convolution stated in the previous example. Note that the explicit formulas for the matrices $H_1 = Q_1(1 - zC_2F_2)^{-1}C_1$ and $H_2 = Q_2(1 - zC_1F_1)^{-1}C_2$ are essentially the same as those for matrices H_a and H_b from Theorem 6.1 in [14], i.e., linearizations were already implicitly present in [14].

2. Preliminaries

In this section we introduce the main ingredients of the paper: conditional expectations, Boolean cumulants and derivations.

2.1. **Non–commutative probability spaces.** We assume that $\mathcal{M}$ is a unital $*$-algebra and $\varphi : \mathcal{M} \mapsto \mathbb{C}$ is a positive unital linear functional, commonly called a *state* and we usually assume it to be faithful. We will refer to the pair $(\mathcal{M}, \varphi)$ as a *non–commutative probability space.* For technical reasons we will mostly work with the free associative algebra.

Notation 2.1. Let $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$ be an alphabet. We denote by $\mathcal{X}^+ = \{X_{i_1}X_{i_2}\cdots X_{i_k} \mid k \in \mathbb{N}, i_j \in \{1, 2, \dots, n\}\}$ the free semigroup it generates and by $\mathcal{X}^* = \mathcal{X}^+ \cup \{1\}$ the free monoid.

We denote by $\mathbb{C}\langle\mathcal{X}\rangle = \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$ the free associative algebra generated by the variables $X_1, X_2, \dots, X_n$, i.e., the linear span of $\mathcal{X}^*$, also known as the algebra of non–commutative polynomials.

For elements $a_1, a_2, \dots, a_n \in \mathcal{M}$ and $P \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$ we denote by $P(a_1, a_2, \dots, a_n)$ the evaluation of a polynomial $P \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$, i.e., the element of $\mathcal{M}$ obtained after substituting every X_i with a_i for $i = 1, 2, \dots, n$.

The *joint distribution* of $a_1, a_2, \dots, a_n$ is the linear functional $\mu : \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle \to \mathbb{C}$ given by

$$\mu(P) = \varphi\left(P(a_1, a_2, \dots, a_n)\right).$$

Definition 2.2. A family of subalgebras $(\mathcal{A}_i)_{i\in I}$ of a ncps $(\mathcal{M}, \varphi)$ is called *free* or *free independent* if

$$\varphi(u_1u_2\cdots u_n) = 0$$

for any choice of $u_j \in \bigcup_i \mathcal{A}_i$ such that $\varphi(u_j) = 0$ and $u_j \in \mathcal{A}_{i_j}$ with $i_j \neq i_{j+1}$ for all $j \in \{1, 2, \dots, n-1\}$.

2.2. **Conditional expectations.** Fix a non–commutative probability space $(\mathcal{M}, \varphi)$ and let $\mathcal{A} \subseteq \mathcal{M}$ be a subalgebra. A *conditional expectation* is a state-preserving projection $\mathbb{E}_\mathcal{A} : \mathcal{M} \to \mathcal{A}$, i.e., such that $\varphi \circ \mathbb{E}_\mathcal{A} = \varphi$. In general such a map not necessarily needs to exists, unless $\mathcal{M}$ is a finite von Neumann algebra and φ is tracial [39, Proposition 5.2.36]. If it does exist and the state φ is faithful, then the conditional expectation $\mathbb{E}_\mathcal{A}[u]$ is the unique element $\tilde{u} \in \mathcal{A}$ such that for any $a \in \mathcal{A}$ one has $\varphi(ua) = \varphi(\tilde{u}a)$. $\mathbb{E}_\mathcal{A} : \mathcal{M} \to \mathcal{A}$ is a unital $\mathcal{A}$-bimodule map, i.e., $E_\mathcal{A}[a_1ua_2] = a_1E_\mathcal{A}[u]a_2$, for all $u \in \mathcal{M}$ and $a_1, a_2 \in \mathcal{A}$.

In the present paper we will always assume that the algebra $\mathcal{M}$ is freely generated by two of its subalgebras $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$, or more specifically, $\mathcal{M}$ is freely generated by some subalgebras $\mathcal{A}_i$, $i \in I$ and the subalgebra $\mathcal{A}$ is generated by some subset of these, i.e., $\mathcal{A} = \langle\mathcal{A}_j\rangle_{j\in J}$ where $J \subseteq I$, and $\mathcal{B} = \langle\mathcal{A}_i\rangle_{i\in I\setminus J}$. In this case the existence of the conditional expectation $\mathbb{E}_\mathcal{A}$ onto $\mathcal{A}$ is generally warranted by combinatorial arguments [28, §2.5]. Alternatively, in the C*-algebraic context, if the state is faithful, then $\mathcal{M}$ is isomorphic to the reduced free product $(\mathcal{A}, \varphi|_\mathcal{A}) * (\mathcal{B}, \varphi|_\mathcal{B})$ and the existence of the conditional expectation also follows from [1, Proposition 1.3].

More precisely, in order to find the conditional expectation of a non–commutative polynomial in variables $a_1, a_2, \dots, a_n$ onto the algebra $\mathcal{A}$ generated by $a_1, a_2, \dots, a_k$ one has to find suitable expressions for moments of the form

$$\varphi\left(a_{i_1}a_{i_2}\cdots a_{i_r}b\right),$$

where $i_1, i_2, \dots, i_r \in \{1, \dots, n\}$ and $b \in \mathcal{A}$. It is a fundamental property of freeness (as one of the universal notions of independence in the sense of [29]) that all joint moments of freely independent random variables are uniquely determined by the marginal moments of the variables in question. Thus for each moment of the form indicated above there is a universal formula (not depending on the particular choice of the distributions of $a_1, a_2, \dots, a_n$) which expresses any joint moment as a sum of products of marginal moments.

After fixing random variables $a_1, a_2, \ldots, a_n$ all relevant information we need for finding the conditional expectation is contained in the pair $(\mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle, \mu)$ defined in Notation 2.1 and therefore we will mostly work on a formal level and focus on this non–commutative probability space. Note that such μ need not to be faithful.

Working with non–commutative polynomials is very useful as it allows us to ignore algebraic relations satisfied by variables $a_1, a_2, \ldots, a_n$. Another advantage of $\mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle$ is that it is augmented (see below) and has a natural linear basis, hence we can easily define linear mappings and functional on this algebra by prescribing values on the basis elements. It is also straightforward then, again by linearity, to extend all those mappings to the algebra of formal power series in non–commuting variables $X_1, X_2, \ldots, X_n$, denoted by $\mathbb{C}\langle\langle X_1, X_2, \ldots, X_n\rangle\rangle$.

2.3. **Main idea – Boolean cumulants and conditional expectations.** Boolean cumulants appeared in various contexts and disguises in the literature [47, 40, 36]. In our previous work [23] we observed that Boolean cumulants appear naturally in connection with conditional expectations of functions in free variables. The present paper is an exploration of this connection based on a recursive reformulation which is suitable for explicit calculations in closed form. In addition we introduce a non–commutative differential calculus for Boolean cumulants of polynomials in free variables which reduces the combinatorial apparatus to a minimum.

Although Speicher's free cumulants are the tool of choice in free probability [35, 31], more recently it turned out that for certain questions Boolean cumulants are useful as well. Indeed some problems like the free anti–commutator [14] and subordination functions [23, 38] are easier to describe in terms of Boolean cumulants rather than free cumulants. The present paper extends and unifies these ideas. Before discussing this, let us start with a review of some basic facts about Boolean cumulants. An interval partition is a partition $\pi = \{B_1, B_2, \ldots, B_k\}$ of the set $\{1, 2, \ldots, n\}$ such that all blocks are intervals, i.e., for all $1 \leq i \leq k$ we have $B_i = \{k, k+1, \ldots, l\}$ for some $k \leq l$ in $\{1, 2, \ldots, n\}$. The set of all interval partitions of $\{1, 2, \ldots, n\}$ is denoted by $\mathrm{Int}(n)$.

For any tuple $a_1, a_2, \ldots, a_n \in \mathcal{M}$ we define the Boolean cumulant functional $\beta_n : \mathcal{M}^n \to \mathbb{C}$ implicitly via the formula

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{\pi \in \mathrm{Int}(n)} \beta_\pi(a_1, a_2, \ldots, a_n), \tag{2.1}$$

where $\beta_\pi(a_1, a_2, \ldots, a_n) = \prod_{B\in\pi} \beta_{|B|}\big((a_1, a_2, \ldots, a_n)|B\big)$, and by $\beta_{|B|}\big((a_1, a_2, \ldots, a_n)|B\big)$ we mean that we take the functional β_k such that $k = |B|$, and we evaluate it at those a_i for which $i \in B$, and the arguments a_i appear in the natural order (one should note that in general Boolean cumulants are not invariant under permutations of arguments). One way to inverting the formula (2.1) and to obtain an explicit formula for Boolean cumulants is Möbius inversion on the lattice of interval partitions. We refrain from doing so and rather base our calculations on a well known recurrence, namely

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{k=1}^{n} \beta_k(a_1, a_2, \ldots, a_k)\, \varphi(a_{k+1} a_{k+2} \cdots a_n) \tag{2.2}$$

or equivalently

$$\varphi(a_1 a_2 \cdots a_n) = \sum_{k=1}^{n} \varphi(a_1 a_2 \cdots a_{k-1})\, \beta_{n-k+1}(a_k, a_{k+1}, \ldots, a_n). \tag{2.3}$$

This elementary recurrence is the starting point for our investigation. It immediately implies a similar recurrence for conditional expectations of products of free random variables: Assume that $\{a_1, a_2, \ldots, a_n\} \in \mathcal{A}$ and $\{b_1, b_2, \ldots, b_{n-1}\} \in \mathcal{B}$ are two families of variables and assume that subalgebras $\mathcal{A}$ and $\mathcal{B}$ are free.

In order to calculate the conditional expectation of $a_1b_1a_2\cdots b_{n-1}a_n \in \mathcal{M}$ onto the subalgebra $\mathcal{B}$, we have to find an element $\mathbb{E}_\mathcal{B}[a_1b_1a_2\cdots b_{n-1}a_n] \in \mathcal{B}$ such that

$$\varphi(a_1b_1a_2\cdots b_{n-1}a_nb) = \varphi\left(\mathbb{E}_\mathcal{B}[a_1b_1a_2\cdots a_n]b\right)$$

for any $b \in \mathcal{B}$. If φ is faithful then this element is uniquely determined and can be found using the following recursive reformulation of [23, Proposition 1.1].

Proposition 2.3. *Assume that φ is faithful. Then for $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ the conditional expectation of the alternating product satisfies the recurrence*

$$\mathbb{E}_\mathcal{B}[a_1b_1a_2\cdots a_{n-1}b_{n-1}a_n] = \sum_{k=1}^{n} \beta_{2k-1}(a_1, b_1, a_2, \dots, a_k)\, b_k\, \mathbb{E}_\mathcal{B}\left[a_{k+1}b_{k+1}\cdots a_n\right]. \tag{2.4}$$

Proof. The recurrence (2.2) yields

$$\begin{aligned}\varphi&\left(\mathbb{E}_\mathcal{B}[a_1b_1a_1\cdots a_n]b\right) = \varphi(a_1b_1a_2\cdots a_nb)\\ &= \sum_{k=1}^{n} \beta_{2k-1}(a_1, b_1, a_2, \dots, a_k)\, \varphi(b_ka_{k+1}\cdots b) + \sum_{k=1}^{n} \beta_{2k}(a_1, b_1, a_2, \dots, b_k)\, \varphi(a_{k+1}b_{k+1}\cdots b)\end{aligned}$$

Now it follows from Theorem 1.1 (see also Definition 3.1 and Proposition 3.6 below) that for free random variables the mixed Boolean cumulant $\beta_{2k}(a_1, b_1, a_2, \dots, a_k, b_k)$ vanishes for $k = 1, 2, \dots, n$ and hence

$$\begin{aligned}\varphi\left(\mathbb{E}_\mathcal{B}[a_1b_1a_2\cdots a_n]b\right) &= \sum_{k=1}^{n} \beta_{2k-1}(a_1, b_1, a_2, \dots, a_k)\, \varphi(b_ka_{k+1}b_{k+1}\cdots a_nb)\\ &= \varphi\Big(\sum_{k=1}^{n} \beta_{2k-1}(a_1, b_1, a_2, \dots, a_k)\, b_ka_{k+1}b_{k+1}\cdots a_nb\Big)\\ &= \varphi\Big(\sum_{k=1}^{n} \beta_{2k-1}(a_1, b_1, a_2, \dots, a_k)\, b_k\, \mathbb{E}_\mathcal{B}\left[a_{k+1}b_{k+1}\cdots a_n\right] b\Big).\end{aligned}$$

□

Observe that vanishing of cumulants of the form $\beta_{2k}(a_1, b_1, a_2, \dots, a_k, b_k)$ is essential in this proof, because it eliminates $\beta_{2n}(a_1, b_1, a_2, \dots, a_{n-1}, b)$. Otherwise b would be trapped inside this term and the recurrence would fail.

We will also make use of the original non-recursive version of the formula for the conditional expectation found in [23].

Corollary 2.4. *Let $(\mathcal{A}, \varphi)$ be a ncps and $\mathcal{B} \subseteq \mathcal{A}$ a subalgebra such that the conditional expectation $\mathbb{E}_\mathcal{B} : \mathcal{A} \to \mathcal{B}$ exists. Let $\{b_1, b_2, \dots, b_{n-1}\} \subseteq \mathcal{B}$ and assume that the family $\{a_1, a_2, \dots, a_n\} \subseteq \mathcal{A}$ is free from $\mathcal{B}$.*

(i)

$$\begin{aligned}&\varphi\left(a_1b_1a_2\cdots b_{n-1}a_nb_n\right)\\ &\quad= \sum_{k=0}^{n-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} \varphi(b_{i_1}b_{i_2}\cdots b_{i_k}b_n \prod_{j=0}^{k} \beta_{2(i_{j+1}-i_j)-1}(a_{i_j+1}, b_{i_j+1}, a_{i_j+2}, \dots, a_{i_{j+1}}),\end{aligned} \tag{2.5}$$

(ii) Then the conditional expectation of alternating monomials can be evaluated as follows

$$\begin{aligned}&\mathbb{E}_\mathcal{B}\left[a_1b_1a_2\cdots b_{n-1}a_n\right]\\ &\quad= \sum_{k=0}^{n-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} b_{i_1}b_{i_2}\cdots b_{i_k} \prod_{j=0}^{k} \beta_{2(i_{j+1}-i_j)-1}(a_{i_j+1}, b_{i_j+1}, a_{i_j+2}, \dots, a_{i_{j+1}}),\end{aligned} \tag{2.6}$$

where in the above sum for each sequence $0 < i_1 < i_2 < \cdots < i_k < n$ *we set* $i_0 = 0$ *and* $i_{k+1} = n$.

2.4. **Boolean cumulants with products as entries.** In this subsection we present the basic tools preparing the non–commutative differential calculus for Boolean cumulants of polynomials in free variables.

The main tool is the formula for Boolean cumulants with products as entries in terms of cumulants of individual variables. Such formulas are known for classical cumulants [24, 34] and free cumulants [20] and follow a general pattern [22]. The analogous formula for Boolean cumulants is given below. The proof may go via a standard argument involving Möbius inversion (see Lecture 14 in [31]). In view of possible generalizations and for the sake of completeness we present alternative algebraic proofs based solely on the recurrence (2.2) in Appendix A.1.

Proposition 2.5. *Let* $a_1, a_2, \ldots, a_n \in \mathcal{A}$ *be random variables then*

$$\beta_{m+1}(a_1 a_2 \cdots a_{d_1}, a_{d_1+1} a_{d_1+2} \cdots a_{d_2}, \ldots, a_{d_m+1} a_{d_m+2} \cdots a_n) = \sum_{\substack{\pi \in \operatorname{Int}(n) \\ \pi \vee \rho = 1_n}} \beta_\pi(a_1, a_2, \ldots, a_n), \tag{2.7}$$

where $\rho = \{\{1, 2, \ldots, d_1\}, \{d_1 + 1, d_1 + 2, \ldots, d_2\}, \ldots, \{d_m + 1, \ldots, n\}\} \in \operatorname{Int}(n)$*, and* $\vee$ *is the join in the lattice of interval partitions. The condition* $\pi \vee \rho = 1_n$ *is equivalent to*

$$\pi \geq \{\{1\}, \{2\}, \ldots, \{d_1 - 1\}, \{d_1, d_1 + 1\}, \{d_1 + 2\}, \ldots, \{d_m - 1\}, \{d_m, d_m + 1\}, \ldots, \{d_n\}\}.$$

Proposition 2.5 can be proved by iteration of the following lemma and Corollary 2.7 below.

Lemma 2.6.

$$\begin{aligned} \beta_{n-1}(a_1, a_2, \ldots, a_p a_{p+1}, a_{p+2}, a_{p+3}, \ldots, a_n) \\ = \beta_p(a_1, a_2, \ldots, a_p)\, \beta_{n-p}(a_{p+1}, a_{p+2}, a_{p+3}, \ldots, a_n) + \beta_n(a_1, a_2, \ldots, a_n) \end{aligned} \tag{2.8}$$

We will actually mostly make use of the following recursive version of Proposition 2.5.

Corollary 2.7. *Let* $a_1, a_2, \ldots, a_n \in \mathcal{A}$ *be random variables consider the interval partition* $\rho = \{\{1, \ldots, d_1\}, \{d_1 + 1, \ldots, d_2\}, \ldots, \{d_m + 1, \ldots, n\}\} \in \operatorname{Int}(n)$*. We write* $\rho = \{B_1, B_2, \ldots, B_{m+1}\}$*, where blocks are ordered in natural order. For* $j \in \{1, \ldots, n\}$ *denote by* $\rho(j)$ *the number of block containing* j*, i.e. we have* $\rho(j) = k$ *if* $j \in B_k$*, then*

$$\begin{aligned} &\beta_{m+1}(a_1 a_2 \cdots a_{d_1}, a_{d_1+1} a_{d_1+2} \cdots a_{d_2}, \ldots, a_{d_m+1} a_{d_m+2} \cdots a_n) \\ &= \sum_{j \in \{1, \ldots, n\} \setminus \{d_1, d_2, \ldots, d_m\}} \beta_j(a_1, a_2, \ldots, a_j) \beta_{m-\rho(j)+1)}(a_{j+1} a_{j+2} \cdots a_{d_{\rho(j)}}, \ldots, a_{d_m+1} \cdots a_n). \end{aligned} \tag{2.9}$$

We record here one immediate consequence of Lemma 2.6 which allows to eliminate units. This also follows from Proposition 3.8 below.

Corollary 2.8 ([33, Proposition 3.3])**.** *For any* $n \geq 2$ *we have*

$$\beta_n(1, a_2, \ldots, a_n) = 0 \tag{2.10}$$

$$\beta_n(a_1, a_2, \ldots, a_{n-1}, 1) = 0 \tag{2.11}$$

$$\beta_n(a_1, a_2, \ldots, a_{k-1}, 1, a_{k+1}, \ldots, a_n) = \beta_{n-1}(a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \tag{2.12}$$

2.5. **Tensor products and tensor algebras.** The *tensor product* $U \otimes V$ of two vector spaces has the universal property that every bilinear map $B : U \times V \to W$ has a unique extension to a linear map $B : U \otimes V \to W$. Consequently, a family of multilinear maps $f_n : V^n \to W$, $n \geq 0$, corresponds uniquely to a linear map $f : T(V) \to W$ on the tensor algebra $T(V) = \oplus_{n=0}^\infty V^{\otimes n}$. Moreover, any linear map on V can be extended to a derivation of the tensor algebra (Lemma A.1).

Notation 2.9. It will be convenient to denote the product operation on the tensor algebra by $\odot$ and extend it to matrices as follows: Given a matrix $A = [a_{ij}] \in M_n(\mathcal{A})$ and $a \in \mathcal{A}$, let $a \odot A = [a \otimes a_{ij}] \in M_n(\mathcal{A} \otimes \mathcal{A})$. Similarly, for two matrices $A, B \in M_n(\mathcal{A})$ we denote by $A \odot B \in M_n(\mathcal{A} \otimes \mathcal{A})$ the matrix with entries

$$(A \odot B)_{ij} = \sum_k a_{ik} \otimes b_{kj}.$$

Note that associativity holds in connection with multiplication with scalar matrices $C \in M_n(\mathbb{C})$, in the sense that $A \odot CB = AC \odot B$.

2.6. **Free products of algebras.** A *coproduct* or (algebraic) *free product* of unital algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over a field $\mathbb{K}$ is a unital algebra $\mathcal{A}$ with embeddings $\iota_1 : \mathcal{A}_1 \to \mathcal{A}$ and $\iota_2 : \mathcal{A}_2 \to \mathcal{A}$ such that the images generate $\mathcal{A}$ and every pair of homomorphisms $h_1 : \mathcal{A}_1 \to \mathcal{B}$ and $h_2 : \mathcal{A}_2 \to \mathcal{B}$ has a unique extension to a homomorphism $\mathcal{A} \to \mathcal{B}$.

For details about free products and tensor algebras we refer to [2, §1.4] and [9].

Proposition 2.10.

(i) The free product is unique and is given by the quotient of the tensor algebra $T^+(A_1 \oplus A_2) = \sum_{n=1}^{\infty} (A_1 \oplus A_2)^{\otimes n}$ *with respect to the ideal generated by all elements of the form*

$$a_1 \otimes b_1 - a_1 b_1 \qquad a_2 \otimes b_2 - a_2 b_2 \qquad 1_{\mathcal{A}_1} - 1_{\mathcal{A}_2}, \qquad a_i, b_i \in \mathcal{A}_i.$$

The algebraic free product is denoted by $\mathcal{A}_1 \amalg \mathcal{A}_2$.

(ii) [2, Lemma 1.4.5] *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be algebras with 1 over a field* $\mathbb{K}$ *with respective* $\mathbb{K}$*-bases* $\{1\} \cup M_1$ *and* $\{1\} \cup M_2$*. Then* $\{1\} \cup M$ *is a* $\mathbb{K}$*-basis for* $\mathcal{A}_1 \amalg \mathcal{A}_2$ *where* M *is the set of alternating monomials in letters from* M_1 *and* M_2*.*

Definition 2.11. An algebra $\mathcal{A}$ is called *augmented* if it comes with an *augmentation map*, i.e., an algebra homomorphism $\epsilon : \mathcal{A} \to \mathbb{C}$. Its kernel $\bar{\mathcal{A}} = \ker \epsilon$ is called the *augmentation ideal.*

Example 2.12. Typical examples of augmented algebras are polynomial algebras (both commutative and non–commutative) where the augmentation map

$$\epsilon(P) = P(0) = \text{constant coefficient}$$

is clearly a homomorphism.

The free product of augmented algebras is clearly augmented. Moreover, it is isomorphic to a subalgebra of the tensor algebra and this fact will be helpful for the definition of certain functionals to be defined in Section 5 below.

Proposition 2.13 ([17]). *The free product of augmented algebras is isomorphic to*

$$\mathcal{A} \amalg \mathcal{B} \simeq \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} T_n(\bar{\mathcal{A}}, \bar{\mathcal{B}}) \oplus T_n(\bar{\mathcal{B}}, \bar{\mathcal{A}}) \tag{2.13}$$

where by

$$T_n(U, V) = U \otimes V \otimes U \otimes \cdots \quad (n \text{ factors})$$

we denote the alternating tensor product of two vector spaces. The multiplication is given by the tensor product modulo the identifications $a' \otimes a'' = a'a''$ *and* $b' \otimes b'' = b'b''$ *already present in Proposition 2.10.*

Example 2.14. The free associative algebra $\mathbb{C}\langle X, Y\rangle$ is the isomorphic to algebraic free product $\mathbb{C}[X] \amalg \mathbb{C}[X]$. The decomposition (2.13) corresponds to the decomposition into the constant term and alternating monomials $X^{k_1} Y^{l_1} X^{k_2} Y^{l_2} \cdots$ with exponents $k_j, l_j > 0$.

Notation 2.15. Extending the terminology of the preceding example to free products of general augmented algebras, alternating products of the form $w = a_1 b_1 a_2 b_2 \cdots$ (resp. $w = b_1 a_1 b_2 a_2 \cdots$) with $a_i \in \bar{\mathcal{A}}$ and $b_i \in \bar{\mathcal{B}}$, i.e., the images of elementary tensors from $T_n(\mathcal{A}, \mathcal{B})$ (resp. $T_n(\mathcal{B}, \mathcal{A})$), will be called *monomials*.

The reduced free product of ncps $(\mathcal{A},\varphi)$ and $(\mathcal{B},\psi)$ will be denoted by $(\mathcal{A},\varphi)*(\mathcal{B},\psi)$ or $\mathcal{A}*\mathcal{B}$ in short. It is realized as the image of the free product of their GNS representations, see [46, §1.5]. For our purpose the following subalgebra is sufficient.

Proposition 2.16 ([46, §1.5])**.** *Let $(\mathcal{A}_1,\varphi_1)$ and $(\mathcal{A}_2,\varphi_2)$ be ncps and denote by $\mathring{\mathcal{A}}_i = \ker\varphi_i$ their centered components. Then the orthogonal direct sum*

$$\mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \oplus \bigoplus_{i_1\neq i_2\neq\cdots\neq i_n} \mathring{\mathcal{A}}_{i_1}\mathring{\mathcal{A}}_{i_2}\cdots\mathring{\mathcal{A}}_{i_n}$$

is a dense subalgebra of the reduced free product.

Definition 2.17. Unital subalgebras $\mathcal{A}$, $\mathcal{B}$ of an algebra $\mathcal{M}$ are called *algebraically free* if they do not satisfy any mutual algebraic relation, i.e., if the free extension of the embeddings $\iota_A : \mathcal{A}\to\mathcal{M}$, $\iota_B : \mathcal{B}\to\mathcal{M}$ to a homomorphism $h : \mathcal{A}\amalg\mathcal{B}\to\mathcal{M}$ is injective.

Proposition 2.18. *Let $(\mathcal{M},\varphi)$ be a ncps with faitful state φ and let $\mathcal{A}$, $\mathcal{B}$ be freely independent subalgebras in the sense of Definition 2.2. Then $\mathcal{A}$ and $\mathcal{B}$ are algebraically free.*

Proof. The algebra generated by $\mathcal{A}$ and $\mathcal{B}$ is isomorphic to their reduced free product $\mathcal{A}*\mathcal{B}$ and thus by [1, Proposition 2.3] contains a faithful copy of the algebraic free product. □

2.7. **Derivations.** Coincidentally it turns out that similar to classical calculus, integration has strong ties to derivations. More precisely, we will be concerned with several derivations from an algebra $\mathcal{A}$ into the bimodule $\mathfrak{M}=\mathcal{A}\otimes\mathcal{A}$ with the natural action

$$a_1\cdot(x\otimes y)\cdot a_2 = (a_1\otimes 1)(x\otimes y)(1\otimes a_2) = a_1x\otimes ya_2 \tag{2.14}$$

Definition 2.19. Let $\mathcal{A}$ be an algebra and $\mathfrak{M}$ be an $\mathcal{A}$-bimodule. An $\mathfrak{M}$-*derivation* is a linear map $D:\mathcal{A}\to\mathfrak{M}$ satisfying the Leibniz rule

$$D(a_1a_2) = D(a_1)\cdot a_2 + a_1\cdot D(a_2) \tag{2.15}$$

If one modifies the left and right actions of $\mathcal{A}$ on $\mathcal{A}\otimes\mathcal{A}$ then homomorphisms become a rich source of derivations.

Proposition 2.20. *Let $\mathcal{A}$ be a unital algebra and $\Phi_1,\Phi_2:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ be two homomorphisms, then $D=\Phi_1-\Phi_2$ is a derivation in the sense that $D(ab)=D(a)\Phi_2(b)+\Phi_1(a)D(b)$*

Let us present some examples of derivations

Example 2.21. The "tensor commutators" $\overleftarrow{\nabla}:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ mapping $\overleftarrow{\nabla}a = a\otimes 1 - 1\otimes a$ is a derivation for any algebra $\mathcal{A}$. It appears in [45, Section 5.3] (see Section 3.1 below) and has the universal property that every derivation factors through it [9, p. III.132, Proposition 17]. It will be convenient to denote $\overrightarrow{\nabla} = -\overleftarrow{\nabla}$.

Example 2.22. The *free derivative* or *free difference quotient* on $\mathbb{C}[x]$ is the map $\partial:\mathbb{C}[x]\to\mathbb{C}[x]\otimes\mathbb{C}[x]$ given by

$$\partial x^n = \sum_{k=0}^{n-1} x^k\otimes x^{n-k-1}.$$

In the natural identification $\mathbb{C}[x]\otimes\mathbb{C}[x]\simeq\mathbb{C}[x,y]$ this coincides with the difference quotient

$$\partial p(x) = \frac{p(x)-p(y)}{x-y}$$

and for this reason is also known as the *Newtonian coproduct* [19, §XII]. It first appeared in free probability in connection with the non–commutative Hilbert transform approach to free entropy [44].

Example 2.23. A slight modification of the previous derivation comes from the *divided powers coproduct* [19, §VI]. Let again $\mathcal{A} = \mathbb{C}[X]$ be the polynomial algebra, then the deconcatenation coproduct is a homomorphism

$$\Delta_d(x^n) = \sum_{k=0}^{n} x^k \otimes x^{n-k}$$

and thus both

$$\overleftarrow{\delta} p(x) = \Delta_d p(x) - 1 \otimes p(x) = (x \otimes 1)\partial p(x)$$
$$\overrightarrow{\delta} p(x) = \Delta_d p(x) - p(x) \otimes 1 = (1 \otimes x)\partial p(x)$$

are derivations.

Example 2.24. For an augmented algebra, the map $\Delta(a) = a - \epsilon(a)$ is a derivation in the sense of Proposition 2.20 and so are the left and right "block derivatives"

$$\overrightarrow{\Delta} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \overleftarrow{\Delta} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$
$$a \mapsto 1 \otimes (a - \epsilon(a)) \qquad a \mapsto (a - \epsilon(a)) \otimes 1.$$

Next we discuss free products of derivations.

Proposition 2.25. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be algebras and $\mathcal{A} = \mathcal{A}_1 \amalg \mathcal{A}_2$ their (unital) free product. Let $\mathfrak{M}$ an $\mathcal{A}$-bimodule (equivalently, a bimodule for both $\mathcal{A}_1$ and $\mathcal{A}_2$) and $D_i : \mathcal{A}_i \to \mathfrak{M}$, $i = 1, 2$ be derivations. Then there exists a unique derivation $D_1 * D_2 : \mathcal{A}_1 \amalg \mathcal{A}_2 \to \mathfrak{M}$ extending D_1 and D_2. Moreover, we have the decomposition*

$$D_1 * D_2 = D_1 * 0 + 0 * D_2$$

where 0 *is the trivial derivation.*

Proof. We first extend $\tilde{D} = D_1 \oplus D_2 : \mathcal{A}_1 \oplus \mathcal{A}_2 \to \mathfrak{M}$ to the tensor algebra $T^+(\mathcal{A}_1 \oplus \mathcal{A}_2)$ by the Leibniz rule, see Lemma A.1:

$$\tilde{D}(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = \sum_{k=1}^{n} u_1 \otimes u_2 \otimes u_{k-1} \cdot \tilde{D}(u_k) \cdot u_{k+1} \otimes \cdots \otimes u_n$$

and then check that it passes to the quotient (see Proposition 2.10), i.e., the ideal generated by $1_\mathcal{A} - 1_\mathcal{B}$, $a' \otimes a'' - a'a''$ for $a', a'' \in \mathcal{A}_1 \cup \mathcal{A}_2$ is contained in $\ker \tilde{D}$. Clearly $\tilde{D}(1) = 0$ and on the other hand for $a', a'' \in \mathcal{A}_1$

$$\begin{aligned} \tilde{D}(u \otimes (a' \otimes a'' - a'a'') \otimes v) &= \tilde{D}(u) \cdot (a' \otimes a'' - a'a'') \otimes v) \\ &+ u \cdot (D_1(a') \cdot a'' + a' \cdot D_1(a'') - D_1(a'a'')) \cdot v \\ &\qquad + u \otimes (a' \otimes a'' - a'a'') \cdot \tilde{D}(v) \end{aligned}$$

which is mapped to 0 in the quotient space and the Leibniz rule is satisfied by definition. □

Example 2.26. If we identify the algebra of non–commutative polynomials with the unital free product $\mathbb{C}\langle x, y\rangle \simeq \mathbb{C}[x] * \mathbb{C}[x]$, then we can construct the partial derivatives $\overleftarrow{\delta}_x$, $\overrightarrow{\delta}_x$, ∂_x and $\overleftarrow{\nabla}_x$ as free products of the derivations $\overleftarrow{\nabla}$ (Example 2.21), $\overleftarrow{\delta}$, $\overrightarrow{\delta}$ (Example 2.23) and 0 on $\mathbb{C}[x]$:

$$\begin{aligned} \overleftarrow{\delta}_x &= \overleftarrow{\delta} * 0 & \overleftarrow{\delta}_y &= 0 * \overleftarrow{\delta} \\ \overrightarrow{\delta}_x &= \overrightarrow{\delta} * 0 & \overrightarrow{\delta}_y &= 0 * \overrightarrow{\delta} \\ \partial_x &= \partial * 0 & \partial_y &= 0 * \partial \\ \overleftarrow{\nabla}_x &= \overleftarrow{\nabla} * 0 & \overleftarrow{\nabla}_y &= 0 * \overleftarrow{\nabla} \\ \overrightarrow{\nabla}_x &= \overrightarrow{\nabla} * 0 & \overrightarrow{\nabla}_y &= 0 * \overrightarrow{\nabla} \end{aligned}$$

moreover, we obtain decompositions $\overleftarrow{\delta} = \overleftarrow{\delta}_x + \overleftarrow{\delta}_y$ and $\overleftarrow{\nabla} = \overleftarrow{\nabla}_x + \overleftarrow{\nabla}_y$.

Example 2.27. Let $\mathcal{A}$ and $\mathcal{B}$ be augmented algebras and set $\mathfrak{M} = (\mathcal{A} \amalg \mathcal{B}) \otimes (\mathcal{A} \amalg \mathcal{B})$. Denote by $\overleftarrow{\Delta}_{\mathcal{A}}$ and $\overleftarrow{\Delta}_{\mathcal{B}}$ (resp. $\overrightarrow{\Delta}_{\mathcal{A}}$ and $\overrightarrow{\Delta}_{\mathcal{B}}$) the left (resp. right) block derivatives from Example 2.24 into the corresponding submodules of $\mathfrak{M}$. The free products $\overrightarrow{\Delta}_{\mathcal{A}} * 0, 0 * \overrightarrow{\Delta}_{\mathcal{B}} : \mathcal{A} \amalg \mathcal{B} \to \mathfrak{M}$ are called *partial block derivatives*. By abuse of notation we will denote them by $\overrightarrow{\Delta}_{\mathcal{A}}$ and $\overrightarrow{\Delta}_{\mathcal{B}}$ as well. Their action on the augmentation ideal is deconcatenation on monomials

$$\overrightarrow{\Delta}_{\mathcal{A}}(a_1 b_1 a_1 a_2 b_2 \cdots a_n b_n) = \sum_{k=1}^{n} a_1 b_2 a_2 \cdots b_k \otimes a_k b_{k+1} a_{k+1} \cdots a_n b_n$$

$$\overleftarrow{\Delta}_{\mathcal{A}}(a_1 b_1 a_1 a_2 b_2 \cdots a_n b_n) = \sum_{k=1}^{n} a_1 b_2 a_2 \cdots a_k \otimes b_k a_{k+1} b_{k+1} \cdots a_n b_n$$

where $a_i \in \bar{\mathcal{A}}$ and $b_i \in \bar{\mathcal{B}}$. In contrast to the previous derivations, neither the partial block derivatives nor the deconcatenation operator $\overrightarrow{\Delta} = \overrightarrow{\Delta}_{\mathcal{A}} + \overrightarrow{\Delta}_{\mathcal{B}}$ satisfy the Leibniz rule with respect to the natural action (2.14), but the "compatibility relation" of unital infinitesimal bialgebras applies [25, Definition 2.1]

$$\overrightarrow{\Delta}(uv) = (\overrightarrow{\Delta}\, u)(1 \otimes v) + (u \otimes 1)\, \overrightarrow{\Delta}\, v - u \otimes v.$$

Note however that $\overrightarrow{\Delta}_{\mathcal{A}} - \overleftarrow{\Delta}_{\mathcal{A}} = \overrightarrow{\nabla}_{\mathcal{A}}$ is a derivation.

Let us observe that resolvents work nicely with any derivation

Remark 2.28. Let $\mathcal{A}$ be a unital algebra and $D : \mathcal{A} \to \mathfrak{M}$ be a derivation into an $\mathcal{A}$-bimodule $\mathfrak{M}$. Then for any invertible element $a \in \mathcal{A}$ we have as a consequence of the Leibniz rule

$$D(a^{-1}) = -a^{-1} D(a) a^{-1} \tag{2.16}$$

In particular, the derivation of a resolvent $R = (z - a)^{-1}$ satisfies

$$D(R) = R D(a) R$$

while for $\Psi = (1 - za)^{-1}$ we have

$$D(\Psi) = z \Psi D(a) \Psi. \tag{2.17}$$

3. Boolean cumulants of free random variables

3.1. **Characterization of freeness by Boolean cumulants.** In this section we elaborate on on the main property which allowed us to derive recurrence (2.4), i.e., the fact that mixed Boolean cumulants of free variables which start and end with mutually free variables vanishes. The main result is a surprisingly simple characterization of freeness which is inspired by an algebraic characterization of freeness found by Voiculescu in [45]. We are grateful to R. Speicher and J. Mingo for bringing this paper to our attention.

Definition 3.1.

(i) Subalgebras $\mathcal{A}, \mathcal{B}$ of a ncps $(\mathcal{M}, \varphi)$ have vanishing *cyclically alternating cumulants* if

$$\beta_n(u_1, u_2, \ldots, u_n) = 0$$

whenever $u_i \in \mathcal{A} \cup \mathcal{B}$ such that u_1 and u_n come from different algebras. We will call this *Property* (CAC).

(ii) Subalgebras $\mathcal{A}, \mathcal{B}$ of a ncps $(\mathcal{M}, \varphi)$ satisfy *Property* $(WCAC)$ (weak (CAC)), if they satisfy Property (CAC) for alternating words, i.e.

$$\beta_n(a_1, b_1, a_2, b_2, \ldots, a_n, b_n) = 0$$

and

$$\beta_n(b_1, a_1, b_2, a_2, \ldots, b_n, a_n) = 0$$

for any choice of $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$.

(iii) Subalgebras $\mathcal{A}, \mathcal{B}$ of a ncps $(\mathcal{M}, \varphi)$ satisfy property (∇) if

$$(\varphi \otimes \varphi) \circ \overleftarrow{\nabla}_{\mathcal{A}}(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0 \tag{3.1}$$

for any choice of $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$. Here the expression on the right hand side of (3.1) is meant to be evaluated as follows:

1. The formal derivative $\overleftarrow{\nabla}_x x_1 y_1 x_2 y_2 \cdots x_n y_n$ is computed in the free associative algebra $\mathbb{C}\langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \rangle$;
2. substitute $x_i = a_i$, $y_i = b_i$ to obtain an element of $\mathcal{M} \otimes \mathcal{M}$;
3. evaluate $\varphi \otimes \varphi$ on the latter.

Remark 3.2. Since $\overleftarrow{\nabla} = \overleftarrow{\nabla}_{\mathcal{A}} + \overleftarrow{\nabla}_{\mathcal{B}}$ and trivially $(\varphi \otimes \varphi) \circ \overleftarrow{\nabla} = 0$, identity (3.1) is equivalent to

$$(\varphi \otimes \varphi) \circ \overleftarrow{\nabla}_{\mathcal{B}}(a_1 b_1 a_2 b_2 \cdots a_n b_n) = 0$$

Lemma 3.3. *Property $(WCAC)$ is equivalent to Property (CAC).*

Proof. Clearly Property (CAC) implies Property $(WCAC)$. The converse can be seen as a special case of Lemma 3.10 below, but here is a proof by induction, assuming that Property $(WCAC)$ holds for all orders.

Suppose that Property (CAC) holds for all orders up to $n-1$ and pick a tuple $u_1, u_2, \ldots, u_n \subseteq \mathcal{A} \cup \mathcal{B}$ such that u_1 and u_n come from different algebras.

If the tuple is alternating then there is nothing to prove.

Therefore assume that it is not alternating. Without loss of generality we consider the following configuration (the proof of the other cases is analogous): $u_1 = a, u_k = a', u_{k+1} = a'' \in \mathcal{A}$ and $u_n = b \in \mathcal{B}$. Thus we have to show that the cumulant $\beta_n(a, u_2, u_2, \ldots, a', a'', u_{k+2}, \ldots, u_{n-1}, b)$ vanishes. We apply the product formula (2.8) in the reverse direction and obtain

$$\begin{aligned}\beta_n(a, u_2, \ldots, u_{k-1}, a', a'', u_{k+2}, \ldots, u_{n-1}, b)\\ = \beta_{n-1}(a, u_2, u_2, \ldots, u_{k-1}, a'a'', u_{k+2}, \ldots, u_{n-1}, b)\\ - \beta_k(a, u_2, u_2, \ldots, u_{k-1}, a')\, \beta_{n-k}(a'', u_{k+2}, \ldots, u_{n-1}, b)\end{aligned}$$

All cumulants on the right hand side are of lower order and all except the second one satisfy the assumptions of the induction hypothesis and therefore the right hand side vanishes. □

Proposition 3.4. *Property $(WCAC)$ is equivalent to Property (∇).*

Proof. We apply recurrence (2.3) to the first factor of the first sum and (2.2) to the second factor of the second sum in the following expression

$$
\begin{aligned}
&(\varphi\otimes\varphi)(\overleftarrow{\nabla}_{\mathcal{A}} a_1b_1\cdots a_nb_n)\\
&\quad= \sum_{k=1}^{n} \varphi(a_1b_1\cdots a_k)\,\varphi(b_ka_{k+1}\cdots a_nb_n) - \varphi(a_1b_1\cdots a_{k-1}b_{k-1})\,\varphi(a_kb_k\cdots a_nb_n)\\
&\quad= \sum_{k=1}^{n}\sum_{p=1}^{k} \varphi(a_1b_1\cdots b_{p-1})\,\beta_{2(k-p)+1}(a_p,b_p,\dots,a_k)\,\varphi(b_ka_{k+1}\cdots a_nb_n)\\
&\qquad\quad + \sum_{k=1}^{n}\sum_{p=1}^{k-1} \varphi(a_1b_1\cdots a_p)\,\beta_{2(k-p)}(b_p,\dots,a_k)\,\varphi(b_ka_{k+1}\cdots a_nb_n)\\
&\qquad - \sum_{k=1}^{n}\sum_{p=k}^{n} \varphi(a_1b_1\cdots b_{k-1})\,\beta_{2(p-k)+1}(a_k,b_k,\dots,a_p)\,\varphi(b_pa_{p+1}\cdots a_nb_n)\\
&\qquad - \sum_{k=1}^{n}\sum_{p=k}^{n} \varphi(a_1b_1\cdots b_{k-1})\,\beta_{2(p-k)+2}(a_k,b_k,\dots,b_p)\,\varphi(a_{p+1}\cdots a_nb_n)\\
&\quad= \sum_{1\le p\le k\le n} \varphi(a_1b_1\cdots b_{p-1})\,\beta_{2(k-p)+1}(a_p,b_p,\dots,a_k)\,\varphi(b_ka_{k+1}\cdots a_nb_n)\\
&\qquad\quad + \sum_{k=1}^{n}\sum_{p=1}^{k-1} \varphi(a_1b_1\cdots a_p)\,\beta_{2(k-p)}(b_p,\dots,a_k)\,\varphi(b_ka_{k+1}\cdots a_nb_n)\\
&\qquad - \sum_{1\le k\le p\le n} \varphi(a_1b_1\cdots b_{k-1})\,\beta_{2(p-k)+1}(a_k,b_k,\dots,a_p)\,\varphi(b_pa_{p+1}\cdots a_nb_n)\\
&\qquad - \sum_{k=1}^{n-1}\sum_{p=k}^{n} \varphi(a_1b_1\cdots b_{k-1})\,\beta_{2(p-k)+2}(a_k,b_k,\dots,b_p)\,\varphi(a_{p+1}\cdots a_nb_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \beta_{2n}(a_1,b_1,\dots,a_n,b_n)\\
&\quad= \sum_{k=1}^{n}\sum_{p=1}^{k-1} \varphi(a_1b_1\cdots a_p)\,\beta_{2(k-p)}(b_p,\dots,a_k)\,\varphi(b_ka_{k+1}\cdots a_nb_n)\\
&\qquad - \sum_{k=1}^{n-1}\sum_{p=k}^{n} \varphi(a_1b_1\cdots b_{k-1})\,\beta_{2(p-k)+2}(a_k,b_k,\dots,b_p)\,\varphi(a_{p+1}\cdots a_nb_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \beta_{2n}(a_1,b_1,\dots,a_n,b_n)
\end{aligned}
$$

Now all but the last term on the right hand side involve lower order cumulants which vanish by by induction hypothesis. Therefore the left hand side vanishes if and only if $\beta_{2n}(a_1,b_1,\dots,a_n,b_n) = 0$. □

Lemma 3.5. *Free subalgebras satisfy Property* (CAC).

Proof. This is an immediate consequence of the vanishing of mixed free cumulants and the formula expressing Boolean cumulants as a sum of free cumulants indexed by irreducible non–crossing partitions [21, 5], and is also a special case of Proposition 3.8 below.

Here is a direct self-contained proof using only the recurrence (2.2) and induction. By Lemma 3.3 it suffices to prove Property $(WCAC)$.

To begin with, in the case $n = 2$ the recurrence (2.2) immediately resolves into the covariance:

$$\beta_2(a,b) = \varphi(ab) - \varphi(a)\varphi(b) = 0.$$

For the induction step, we first verify that cumulants of alternating centered words vanish and then reduce the general case to this.

Assume that the assertion holds for any order up to $k \leq n-1$ and that the tuple $u_1, u_2, \dots, u_n$ is alternating, i.e., neigbouring elements come from different algebras, and that all elements are centered. Then it follows from recurrence (2.2) that

$$\beta_n(u_1, u_2, \dots, u_n) = \varphi(u_1 u_2 \cdots u_n) - \sum_{k=1}^{n} \beta_k(u_1, u_2, \dots, u_k)\, \varphi(u_{k+1} u_{k+2} \cdots u_n)$$

and all terms on the right hand said contain expectations of alternating words in centered elements. Freeness implies that they vanish and so does the cumulant on the left hand side.

It remains to show that we can replace the letters of an alternating word with centered elements. We will do this step by step using multilinearity and Corollary 2.8. The condition that the first and last arguments of the involved cumulants come from different algebras is not violated throughout the following manipulations.

$$\begin{aligned}
\beta_n(u_1, u_2, \dots, u_n) &= \beta_n(\mathring{u}_1, u_2, \dots, u_n) + \varphi(u_1)\, \beta_n(1, u_2, \dots, u_n) \\
&= \beta_n(\mathring{u}_1, u_2, \dots, u_n) && \text{by (2.10)} \\
&= \beta_n(\mathring{u}_1, \mathring{u}_2, u_3, \dots, u_n) + \varphi(u_2)\, \beta_n(\mathring{u}_1, 1, u_3, \dots, u_n) \\
&= \beta_n(\mathring{u}_1, \mathring{u}_2, u_3, \dots, u_n) + \varphi(u_2)\, \beta_{n-1}(\mathring{u}_1, u_3, \dots, u_n) && \text{by (2.12)} \\
&= \beta_n(\mathring{u}_1, \mathring{u}_2, u_3, \dots, u_n) && \text{by induction hypothesis} \\
&\ \ \vdots \\
&= \beta_n(\mathring{u}_1, \mathring{u}_2, \dots, \mathring{u}_n).
\end{aligned}$$

□

In conclusion, in the tracial case we have the following extension of Voiculescu's freeness criterion [45, §14.4].

Proposition 3.6 (Characterization of freeness in terms of Boolean cumulants)**.** *Let $(\mathcal{M}, \varphi)$ be a tracial non-commutative probability space and $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$ two subalgebras. Then the following are equivalent.*

(i) $\mathcal{A}$ and $\mathcal{B}$ are free.
(ii) $\mathcal{A}$ and $\mathcal{B}$ satisfy property (CAC).
(iii) $\mathcal{A}$ and $\mathcal{B}$ satisfy property (∇).

Proof. Items (ii) and (iii) are equivalent by Proposition 3.4 even without traciality.

Item (ii) implies (iii) by Lemma 3.5 and it remains to prove the converse in the tracial case, i.e., Property (CAC) implies freeness.

Recall that in [45] it is shown that in the tracial case for every n the original freeness condition $(F(n))$

$$\varphi(a_1 b_1 \cdots a_n b_n) = 0$$

whenever all elements a_i, b_i are centered can be strengthened to condition $(F(n)')$ allowing one of a_1 or b_n to have nonzero expectation.

We will use this equivalence and proceed by induction and show that for every n condition $(F(k)')$ for $k < n$ together with (CAC) implies condition $(F(n))$. To this end we pick centered elements $a_1, a_2, \dots, a_n \in \mathcal{A}$ and $b_1, b_2, \dots, b_n \in \mathcal{B}$ and and show that $\varphi(a_1 b_1 \cdots a_n b_n) = 0$.

The case $n = 1$ is obvious.

For the induction step, observe that we can rewrite recurrence (2.2) as follows

$$\varphi(a_1 b_1 \cdots a_n b_n) = \sum_{k=1}^{n} \beta_{2k-1}(a_1, b_1, \dots, a_k)\, \varphi(b_k a_{k+1} b_{k+1} \cdots b_n)$$

$$+\sum_{k=1}^{n} \beta_{2k}(a_1, b_1, \dots, b_k)\, \varphi(a_{k+1}b_{k+1}\cdots b_n).$$

All terms in the second sum vanish by property (CAC). In the first sum the term corresponding to $k = n$ vanishes because we assumed that $\varphi(b_n) = 0$. In order to see that the remaining terms corresponding to $k < n$ vanish as well, first note that by traciality $\varphi(b_k a_{k+1} \dots b_n) = \varphi(a_{k+1} \dots b_n b_k)$ which is an alterning product with at most $2(n-1)$ factors. Now the element $b_n b_k$ needs not to be centered but as discussed above, the freeness conditions ($F(n-1)'$) and ($F(n-1)$) are equivalent and thus the expectation vanishes. □

Remark 3.7. Note that traciality of the linear functional is essential for this characterization. Property (CAC) (and hence Property (∇)) also holds for Boolean independent and more generally conditionally free random variables.

3.2. **Mixed Boolean cumulants of free random variables.** The following characterization of freeness from [14] generalizes Property (CAC) and will provide an essential tool for later considerations.

Proposition 3.8 ([14, Theorem 1.2], [18])**.** *Subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_s \subseteq \mathcal{M}$ of a ncps $(\mathcal{M}, \varphi)$ are free if and only if for any colouring $c : \{1, \dots, n\} \to \{1, \dots, s\}$ we have*

$$\beta_n(a_1, a_2, \dots, a_n) = \sum_{\pi \in NC^{irr}(n) \text{ with VNRP}} \beta_\pi(a_1, a_2, \dots, a_n)$$

whenever $a_i \in \mathcal{A}_{c(i)}$. Here a partition $\pi \in NC^{irr}(n)$ is said to have $VNRP$ if $\pi \leq \ker c$ and every inner block covered nested by a block of different colour, i.e., c induces a proper coloring on the nesting tree of π.

We will only use the preceding theorem in the special case of alternating arguments, which was first explicitly stated in [38].

Proposition 3.9. *Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_{n-1}\}$ be free, $n \geq 1$. Then*

(3.2)
$$\begin{aligned}&\beta_{2n-1}(a_1, b_1, \dots, a_n, b_{n-1}, a_n)\\ &\quad= \sum_{k=2}^{n} \sum_{1=j_1<j_2<\dots<j_k=n} \beta_k(a_{j_1}, a_{j_2}, \dots, a_{j_k}) \prod_{\ell=1}^{k-1} \beta_{2(j_{\ell+1}-j_\ell)-1}(b_{j_l}, a_{j_\ell+1}, \dots, a_{j_{\ell+1}-1}, b_{j_{\ell+1}-1}).\end{aligned}$$

Next we record another property which will be important in the following, namely as we already know a Boolean cumulant which takes as arguments elements of two free algebras $\mathcal{A}$ and $\mathcal{B}$ vanishes unless it starts and ends with elements from the same algebra. Suppose this is the case, say the both the first and last argument come from $\mathcal{A}$, then the Boolean cumulant does not change under arbitrary splitting of thee arguments coming from the subalgebra $\mathcal{B}$ into factors. This will allow us to write any Boolean cumulant in free variables as an alternating cumulant.

Lemma 3.10. *Suppose $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ are free and let $a_1, a_2, \dots, a_n \in \mathcal{A}$ and $b_1, b_2, \dots, b_{n-1} \in \mathcal{B}$. Assume further that for each $j = 1, 2, \dots, n-1$ we have $b_i = c_1^{(i)} \cdots c_{j_i}^{(i)}$ with $c_1^{(i)}, \dots, c_{j_i}^{(i)} \in \mathcal{B}$, then we have*

$$\beta_{2n-1}(a_1, b_1, a_2, \dots, b_{n-1}, a_n) = \beta_{n+j_1+\dots+j_{n-1}}(a_1, c_1^{(1)}, \dots, c_{j_1}^{(1)}, a_2, \dots, c_1^{(n-1)}, \dots, c_{j_{n-1}}^{(n-1)}, a_n).$$

Proof. This follows from the formula for Boolean cumulants with products as entries (2.7) and property (CAC). More precisely, applying said formula to the Boolean cumulant

$$\beta_{n+j_1+\dots+j_{n-1}}(a_1, c_1^{(1)}c_2^{(1)}\cdots c_{j_1}^{(1)}, a_2, \dots, c_1^{(n-1)}c_2^{(n-1)}\cdots c_{j_{n-1}}^{(n-1)}, a_n)$$

recall that we have to sum over interval partitions π such that $\pi \vee \sigma = 1_{n+j_1+\dots+j_{n-1}}$, where σ is given by the product structure, hence we sum over π greater than $\{\{1, 2\}, \{3\} \dots, \{j_1\}, \{j_1 +$

$1, j_1+2\}, \ldots \{n-1, n\}\}$ or in other words are complementary to $\tilde{\rho} = (1, j_1, 1, j_2, \ldots, j_{n-1}, 1)$. Observe that the block of π containing a_1 cannot end in any c_i^l, as in that case the corresponding Boolean cumulant vanishes by property (CAC). On the other hand the block containing a_1 cannot finish in any of the variables $a_1, a_2, \ldots, a_{n-1}$ as this would violate the complementarity property. Thus the block which contains a_1 has to end in a_n and since we consider only interval partitions, there is only one such partition $\pi = 1_{n+j_1+\cdots+j_{n-1}}$. □

Remark 3.11. Lemma 3.10 allows to rewrite an arbitrary joint Boolean cumulant of free random variables as an alternating cumulant, by grouping together terms from the algebra $\mathcal{B}$ above. Indeed after regrouping the inner variables into free blocks, Corollary 2.8 allows us to fill the gaps between elements of $\mathcal{A}$ with units to obtain

$$\beta_k(a_1, a_2, \ldots, a_l, a_{l+1}, \ldots, a_n) = \beta_k(a_1, 1_{\mathcal{M}}, a_2, \ldots, a_l, 1_{\mathcal{M}}, a_{l+1}, \ldots, a_n).$$

4. Calculus for conditional expectations

In order to define several objects objects used in this and the next section we require additional structure on the ncps we work with.

Assumption 4.1. From this point we will assume that $(\mathcal{M}, \varphi)$ is a ncps, where $\mathcal{M}$ is an augmented algebra (see Definition 2.11). Furthermore we assume $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ are freely independent subalgebras which generate $\mathcal{M}$ as an algebra, that is, $\mathcal{M} = \mathcal{A} \amalg \mathcal{B}$. The latter is a moderate restriction since it is a dense subalgebra by Proposition 2.18.

Remark 4.2.

(i) This assumption is necessary to make sure that the functionals introduced below are well defined. Indeed in the augmented case the algebraic free product is canonically isomorphic to a subspace of the tensor algebra (see Proposition 2.13) and thus families of multilinear functionals can be identified with linear functionals on the latter.

(ii) The reader not familiar with these notions should think of the algebras as polynomial algebras from Example 2.12 in the following. More precisely,

$$\mathcal{M} = \mathbb{C}\langle X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n\rangle \qquad \mathcal{A} = \mathbb{C}\langle X_1, X_2, \dots, X_m\rangle \qquad \mathcal{B} = \mathbb{C}\langle Y_1, Y_2, \dots, Y_n\rangle$$

and $\varphi = \mu$ is some formal distribution, i.e., a unital linear functional $\mu : \mathcal{M} \to \mathbb{C}$. The augmentation is $\epsilon(P) = P(0)$.

In fact it is sufficient to understand the bivariate case

$$\mathcal{M} = \mathbb{C}\langle X, Y\rangle \qquad \mathcal{A} = \mathbb{C}\langle X\rangle \qquad \mathcal{B} = \mathbb{C}\langle Y\rangle$$

and the general case will be straightforward.

In the case of polynomial algebras $w \in \bar{\mathcal{M}}$ means that w has zero constant term and in fact it is sufficient to consider monomials $w = a_1 b_2 a_2 b_2 \cdots$ where a_i and b_i come from the monomial basis, i.e., in the case of $\mathbb{C}\langle X, Y\rangle$ this means that $a_i \in \{X^k | k \geq 1\}$ and $b_i \in \{Y^k | k \geq 1\}$.

Notation 4.3. It follows from Proposition 2.13 that in the free product of augmented algebras any monomial (i.e., simple tensor) $W \in \bar{\mathcal{M}}$ has a unique factorization into an alternating product of elements from $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ in one out of four types:

type $\mathcal{A}$–$\mathcal{A}$	if $w = a_1 b_1 a_2 \cdots b_{n-1} a_n$,
type $\mathcal{A}$–$\mathcal{B}$	if $w = a_1 b_1 a_2 \cdots a_n b_n$,
type $\mathcal{B}$–$\mathcal{A}$	if $w = b_1 a_1 b_2 \cdots b_n a_n$,
type $\mathcal{B}$–$\mathcal{B}$	if $w = b_1 a_1 b_2 \cdots a_{n-1} b_n$,

with $a_1, a_2, \dots, a_n \in \bar{\mathcal{A}}$ and $b_1, b_2, \dots, b_{n-1} \in \bar{\mathcal{B}}$. We will call such monomials $\mathcal{A}$–$\mathcal{A}$ monomials etc. and we will refer to the factorization as the $\mathcal{A}$–$\mathcal{B}$ *block factorization*.

Remark 4.4. At this point augmentation is essential. Otherwise allowing units in monomials leads to inconsistencies like $(1 + a)b = b + ab$ belonging to the image of $\mathcal{A} \otimes \mathcal{B}$ and $(\mathcal{B} \oplus \mathcal{A} \otimes \mathcal{B})$ simultaneously. However the direct sum decomposition (2.13) is essential to consistently extend the multilinear maps to be defined shortly (see Definition 4.10) to linear maps on the free product.

We now take formula (2.6) as a formal definition of a conditional expectation.

Definition 4.5. Define a linear mapping $\mathbb{E}_\mathcal{B} : \mathcal{M} \to \mathcal{B}$ via the following requirements:

(i) $\mathbb{E}_\mathcal{B}[1] = 1$,

(ii) $\mathbb{E}_\mathcal{B}[bwb'] = b\mathbb{E}_B[w]b'$ for any $w \in \mathcal{M}$ and any $b, b' \in \mathcal{B}$.

(iii) We define the conditional expectation of a monomial of type $\mathcal{A}$–$\mathcal{A}$ $w = a_1 b_1 a_2 \cdots b_{n-1} a_n$ with $a_i \in \bar{\mathcal{A}}$ and $b_i \in \bar{\mathcal{B}}$ as

$$\mathbb{E}_\mathcal{B}\left[a_1 b_1 a_2 \cdots b_{n-1} a_n\right] = \beta_{2n-1}\left(a_1, b_1, a_2, \dots, b_{n-1}, a_n\right) +$$

$$\sum_{k=1}^{n-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n-1} b_{i_1}b_{i_2}\cdots b_{i_k} \prod_{j=0}^{k} \beta_{2(i_{j+1}-i_j)-1}(a_{i_j+1}, b_{i_j+1}, a_{i_j+2}, \dots, a_{i_{j+1}}),$$

(iv) We define the *block cumulant* functional $\beta^b : \mathcal{M} \to \mathbb{C}$ by prescribing the values on monomials

$$\beta^b(1) = 1$$
$$\beta^b(u_1u_2\cdots u_n) = \beta_n(u_1, u_2, \dots, u_n)$$

whenever $u_1u_2\cdots u_n$ is an alternating word with $u_i \in \bar{\mathcal{A}} \cup \bar{\mathcal{B}}$.

Remark 4.6.
(i) Note that property (CAC) implies $\beta(u_1u_2\cdots u_n)$ vanishes unless n is odd.
(ii) Augmentation is essential here to make sure that the definition of β^b does not depend on the choice of basis. Indeed if instead we choose orthonormal bases $\{1\} \cup (a_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ and $\{1\} \cup (b_k)_{k\in\mathbb{N}} \subseteq \mathcal{B}$ then it follows from Proposition 2.16 that the alternating words in a_i and b_j together with 1 form an orthonormal basis of the reduced free product. However $\beta(u_1, u_2, \dots, u_n) = 0$ for any such word and thus the corresponding notion of β^b would coincide with φ in this case.

As an immediate consequence of Proposition 2.13 and (2.5) we have the following.

Proposition 4.7. *Consider the maps defined in Definition 4.5 with Assumption 4.1, φ not necessarily faithful.*
(i) The maps $\mathbb{E}_\mathcal{B}$ and β^b are well defined.
(ii) The map $\mathbb{E}_\mathcal{B}$ defined above is a conditional expectation.

Note that the above conditional expectation is universal and allows for calculations on the level of arbitrary algebras.

Proposition 4.8. *Let $(\mathcal{M}, \varphi)$ be an arbitrary ncps with faithful state φ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ be free subalgebras such that the conditional expectation $\mathbb{E}_\mathcal{B}$ exists (e.g., when $\mathcal{B}$ and $\mathcal{M}$ are von Neumann algebras or $\mathcal{M} = \mathcal{A}*\mathcal{B}$). Pick elements $a_1, a_2, \dots, a_m \in \mathcal{A}$ and elements $b_1, b_2, \dots, b_n \in \mathcal{B}$. Let*

$$\mu : \mathbb{C}\langle X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n\rangle \to \mathbb{C}$$

be the joint distribution of the tuple $(a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n)$, i.e.,

$$\mu(P(X_1, X_2, \dots, Y_n)) = \varphi(P(a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n))$$

and let $\mathbb{E}_Y^\mu : \mathbb{C}\langle X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_n\rangle \to \mathbb{C}\langle Y_1, Y_2, \dots, Y_n\rangle$ be the conditional expectation from Definition 4.5 onto the subalgebra $\mathbb{C}\langle Y_1, Y_2, \dots, Y_n\rangle$. Then

$$\mathbb{E}_\mathcal{B}[P(a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n)] = (\mathbb{E}_Y^\mu[P])(b_1, b_2, \dots, b_n),$$

where $(\mathbb{E}_Y^\mu[P])(b_1, b_2, \dots, b_n)$ is the polynomial $\mathbb{E}_Y^\mu[P] \in \mathbb{C}\langle Y_1, Y_2, \dots, Y_n\rangle$ evaluated in $b_1, b_2, \dots, b_n$.

Proof. This is an immediate consequence of the definition of μ and Corollary 2.4. □

It is no surprise that this formal conditional expectation satisfies the recurrence (2.4).

Proposition 4.9. *For any $\mathcal{A}$–$\mathcal{A}$ monomial w the mapping $\mathbb{E}_\mathcal{B}$ satisfies the following recurrence*

$$\mathbb{E}_\mathcal{B}[a_1b_1a_2\cdots b_{n-1}a_n] = \beta_{2n-1}(a_1, b_1, a_2, \dots, b_{n-1}, a_n) + \sum_{k=1}^{n-1} \beta_{2k-1}(a_1, b_1, a_2, \dots, a_k)\, b_k \mathbb{E}_\mathcal{B}[a_{k+1}b_{k+1}a_{k+2}\cdots a_n] \tag{4.1}$$

$$= \beta^b(a_1b_1a_2\cdots b_{n-1}a_n) + \sum_{k=1}^{n-1} \beta^b(a_1b_1a_2\cdots a_k)\, b_k \mathbb{E}_\mathcal{B}[a_{k+1}b_{k+1}a_{k+2}\cdots a_n]. \tag{4.2}$$

Proof. Observe that in Definition 4.5 (iii) on the right hand side of the equation we sum over all possible choices of subsets of $\{b_1, b_2, \dots, b_{n-1}\}$ and for variables between the chosen b_i's we apply Boolean cumulant, where we split b_i's from a_i's. In order to obtain the recurrence above we reorganize the sum appearing in the definition of $\mathbb{E}_{\mathcal{B}}$ according to the value of i_1, i.e., we have

$$\begin{aligned}\sum_{k=1}^{n-1} \sum_{1\leq i_1<\dots<i_k\leq n-1} & b_{i_1} b_{i_2}\cdots b_{i_k} \prod_{j=0}^{k} \beta_{2(i_{j+1}-i_j)-1}(a_{i_j+1}, b_{i_j+1}, a_{i_j+2}, \dots, a_{i_{j+1}}) \\ &= \sum_{i_1=1}^{n-1} \beta_{2i_k-1}(a_1, b_1, a_2, \dots, a_{i_1}) b_{i_1} \sum_{k=0}^{n-2} \sum_{i_1<i_2<\dots<i_k\leq n-1} b_{i_2} b_{i_3}\cdots b_{i_k} \\ &\qquad \times \prod_{j=1}^{k} \beta_{2(i_{j+1}-i_j)-1}(a_{i_j+1}, b_{i_j+1}, a_{i_j+2}, \dots, a_{i_{j+1}}).\end{aligned}$$

To conclude the proof observe that each fixed value of i_1 the inner summation over $i_2, \dots, i_{n-1}$ reproduces exactly the definition of $\mathbb{E}_{\mathcal{B}}[a_{i_1+1} b_{i_1+1} a_{i_1+2}\cdots a_n]$. ☐

Proposition 4.9 offers a nice recursive formula for calculations of conditional expectations in free variables, however a drawback is that it works only on monomials starting and ending elements of algebra $\mathcal{A}$. In order to make this formula useful for general calculations we need to have a recurrence which works on any given polynomial. To this end we split the functional β^b somewhat analogously to the partial block derivatives $\vec{\Delta} = \vec{\Delta}_{\mathcal{A}} + \vec{\Delta}_{\mathcal{B}}$. The latter turns out to be the right tool to capture the structure of the recurrence from Proposition 4.9.

Definition 4.10. Under Assumption 4.1 we define the *block cumulant functional* $\beta^b_{\mathcal{A}} : \mathcal{A} \amalg \mathcal{B} \to \mathbb{C}$ by prescribing its values on monomials as follows:

$$\begin{aligned}\beta^b_{\mathcal{A}}(1) &= 1 \\ \beta^b_{\mathcal{A}}(a_1 b_1 a_2 \cdots b_{n-1} a_n) &= \beta_{2n+1}(a_1, b_1, a_2, \dots, b_{n-1}, a_n)\end{aligned}$$

for any $\mathcal{A}$–$\mathcal{A}$ monomial w with $\mathcal{A}$–$\mathcal{B}$ factorization $w = a_1 b_1 a_2 \cdots b_{n-1} a_n$ and $a_i \in \bar{\mathcal{A}}$ and $b_i \in \bar{\mathcal{B}}$. For monomials w which are not of type $\mathcal{A}$–$\mathcal{A}$ we set $\beta^b_{\mathcal{A}}(w) = 0$.

We define the analogous functional $\beta^b_{\mathcal{B}}$, which produces the value of Boolean cumulants of "blocked" variables from $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$, which vanishes unless a words starts and ends with elements from $\bar{\mathcal{B}}$.

Remark 4.11. Note that because of Property (CAC) we have

$$\beta^b - \epsilon = \beta^b_{\mathcal{A}} - \epsilon + \beta^b_{\mathcal{B}} - \epsilon.$$

With the help of these functionals and the block derivatives from Example 2.27 we can now extend recurrence (4.2) for $\mathbb{E}_{\mathcal{B}}$ to arbitrary monomials and to compress it into an intuitive formula, which is the main result of this section.

Theorem 4.12. *Suppose $\mathcal{M} = \mathcal{A} \amalg \mathcal{B}$ and Assumption 4.1, then for any $w \in \mathcal{M}$*

$$\begin{aligned}\mathbb{E}_{\mathcal{B}}[w] &= \beta^b_{\mathcal{A}}(w) + (\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\vec{\Delta}_{\mathcal{B}}(w)] \\ &= \beta^b_{\mathcal{A}}(w) + (\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\vec{\nabla}_{\mathcal{B}}(w)] \qquad (4.3) \\ &= \beta^b_{\mathcal{A}}(w) + (\mathbb{E}_{\mathcal{B}} \otimes \beta^b_{\mathcal{A}})[\overleftarrow{\Delta}_{\mathcal{B}}(w)].\end{aligned}$$

Proof. Consider the first identity. By linearity it is enough to consider monomials. For a monomial of type $\mathcal{A}$–$\mathcal{A}$ the statement is equivalent to Proposition 4.9.

If w is a monomial of type $\mathcal{B}$–$\mathcal{B}$ or $\mathcal{B}$–$\mathcal{A}$ then by the definition of $\beta^b_{\mathcal{A}}$ only one term contributes on the RHS and we obtain the trivial identity

$$\mathbb{E}_{\mathcal{B}}[w] = (\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[1 \otimes w].$$

If w is a monomial of type $\mathcal{A}$–$\mathcal{B}$ then it can be written as $w = w'b$, where w' is of type $\mathcal{A}$–$\mathcal{A}$ and $b \in \bar{\mathcal{B}}$ and the right hand side is

$$\beta^b_{\mathcal{A}}(w'b) + (\beta^b_{\mathcal{A}} \otimes E_{\mathcal{B}})[\vec{\Delta}_{\mathcal{B}}(w')(1 \otimes b) + w' \otimes b] = 0 + \Big((\beta^b_{\mathcal{A}} \otimes E_{\mathcal{B}})[\vec{\Delta}_{\mathcal{B}}\, w'] + \beta^b_{\mathcal{A}}(w')\Big)b$$
$$= \mathbb{E}_{\mathcal{B}}[w']b,$$

where we observed that the term inside the parenthesis contains the recurrence (4.1) for the conditional expectation of the word w' which is of type $\mathcal{A}$–$\mathcal{A}$.

For the second identity recall that $\vec{\nabla}_{\mathcal{B}} = \vec{\Delta}_{\mathcal{B}} - \overleftarrow{\Delta}_{\mathcal{B}}$. and observe the additional terms produced by $\overleftarrow{\Delta}_{\mathcal{B}}$ are tensors whose left legs are monomials of type $*$–$\mathcal{B}$ which are annihilated by $\beta^b_{\mathcal{A}}$.

The last identity is proved using the mirrored version (2.3) of the Boolean recurrence. ☐

Remark 4.13.

(i) It is straightforward to extend the functional β^b and the recurrence (4.3) to formal power series (provided that the resulting series converge) and in particular compute conditional expectations of resolvents of the form $(1 - zw)^{-1}$ with $w \in \mathcal{M}$.

(ii) In the setting of Remark 4.2 (ii), i.e., $\mathcal{A} = \mathbb{C}\langle\mathcal{X}\rangle$ for some alphabet $\mathcal{X} = \{X_1, X_2, \dots, X_m\}$ and $\mathcal{B} = \mathbb{C}\langle\mathcal{Y}\rangle$ for some alphabet $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_n\}$ we can also use the partial divided power derivations from Example 2.23

$$\vec{\delta}_{\mathcal{Y}}(P) = \sum_{i=1}^{n} (1 \otimes Y_i)\partial_{Y_i}(P),$$
$$\overleftarrow{\delta}_{\mathcal{Y}}(P) = \sum_{i=1}^{n} (Y_i \otimes 1)\partial_{Y_i}(P),$$

where ∂_Y is the free different quotient from Example 2.22 and we can write

$$\mathbb{E}^{\mu}_{\mathcal{Y}}[P] = \beta^b_{\mathcal{X}}(P) + (\beta^b_{\mathcal{X}} \otimes \mathbb{E}^{\mu}_{\mathcal{Y}})[\vec{\delta}_{\mathcal{Y}}(P)]. \tag{4.4}$$

Again the additional terms are annihilated by $\beta^b_{\mathcal{X}}$.

(iii) The map $\vec{\Delta}_{\mathcal{B}}$ produces the least number of terms and therefore is convenient for explicit calculations. On the other hand, $\vec{\nabla}_{\mathcal{B}}$ (and $\vec{\delta}_{\mathcal{Y}}$ in the case of polynomial algebras) are derivations, which is a great advantage in connection with non–commutative formal power series, in particular resolvents and other rational functions. Indeed, for simplicity let us consider the resolvent $\Psi = (1 - zP(X,Y))^{-1}$ of a bivariate polynomial $P(X,Y) \in \mathbb{C}\langle X, Y\rangle$. From identity (2.16) we infer that

$$\vec{\nabla}_X \Psi = z(\Psi \otimes 1)\vec{\nabla}_X P(X,Y)(1 \otimes \Psi)$$

and a similar formula is true for $\vec{\delta}_X \Psi$. It does *not* hold for $\vec{\Delta}_X$, which does not obey the Leibniz rule, yet in the case of recurrence (4.3) we can pretend that it does and we have

$$\begin{aligned}\mathbb{E}_X[\Psi] &= \beta^b_Y(\Psi) + z(\beta^b_Y \otimes \mathbb{E}_X)[(\Psi \otimes 1)(\vec{\delta}_X w)(1 \otimes \Psi]\\ &= \beta^b_Y(\Psi) + z(\beta^b_Y \otimes \mathbb{E}_X)[(\Psi \otimes 1)(\vec{\nabla}_X w)(1 \otimes \Psi]\\ &= \beta^b_Y(\Psi) + z(\beta^b_Y \otimes \mathbb{E}_X)[(\Psi \otimes 1)(\vec{\Delta}_X\, w)(1 \otimes \Psi]\end{aligned}$$

because the extra terms arising in $\vec{\delta}_X w$ and $\vec{\nabla}_X w$ are annihilated by β^b_Y.

Corollary 4.14. *For any* $w \in \mathcal{M}$

$$(\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\vec{\Delta}_{\mathcal{A}}\, w] = (\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\overleftarrow{\Delta}_{\mathcal{A}}\, w].$$

Proof. Observe that from (4.3) we obtain the identity

$$\begin{aligned}(\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\vec{\nabla} w] &= \mathbb{E}_{\mathcal{B}}[w] - \beta^b_{\mathcal{A}}(w)\\ &= (\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\vec{\nabla}_{\mathcal{B}} w];\end{aligned}$$

on the other hand, $\vec{\nabla} = \vec{\nabla}_{\mathcal{A}} + \vec{\nabla}_{\mathcal{B}}$ and it follows that

$$(\beta^b_{\mathcal{A}} \otimes \mathbb{E}_{\mathcal{B}})[\vec{\nabla}_{\mathcal{A}} w] = 0$$

which is equivalent to the claimed identity. □

We illustrate this machinery with the problem of additive free convolution, done in two ways.

Example 4.15. Consider $\Psi = (1 - z(X+Y))^{-1} = \sum_{n=0}^{\infty} \bigl(z(X+Y)\bigr)^n$ then $\vec{\delta}_X \Psi = z\Psi \otimes X\Psi$ and the conditional expectation is

$$\mathbb{E}_X[\Psi] = \beta^b_Y(\Psi) + z\beta^b_Y \otimes \mathbb{E}_X[\Psi \otimes X\Psi] = \beta^b_Y(\Psi)(1 + zX\mathbb{E}_X[\Psi]).$$

Thus

$$\mathbb{E}_X[\Psi] = \beta^b_Y(\Psi)\Bigl(1 - z\beta^b_Y(\Psi)X\Bigr)^{-1} = \Bigl(1/\beta^b_Y(\Psi) - zX\Bigr)^{-1}. \tag{4.5}$$

The evaluation of $\beta^b_Y(\Psi)$ will be discussed in Example 5.9.

In order to obtain the usual formulation of free additive subordination one has to consider the resolvent $R = (z - (X+Y))^{-1}$ instead of Ψ.

Example 4.16. Let $R = R(z) = (z - X - Y)^{-1}$, then $\vec{\delta}_X R = R \otimes XR$ and the solution of the recurrence for the conditional expectation

$$\mathbb{E}_X[R] = \beta^b_Y(R) + \beta^b_Y(R)X\mathbb{E}_X[R]$$

is immediately obtained as

$$\mathbb{E}_X[R] = (\beta^b_Y(R)^{-1} - X)^{-1}.$$

Now $\beta^b_Y(R)$ is an analytic function of z (via $R = R(z)$) and its reciprocal $\omega(z) = 1/\beta^b_Y(R)$ is the subordination function from (1.2). For its further evaluation see Example 5.8.

In fact the recurrence (4.3) allows for finding a system of equations for the conditional expectation of resolvents of arbitrary polynomials.

Proposition 4.17. *Let $P \in \mathbb{C}\langle X, Y\rangle$ and suppose $\vec{\delta}_X P = \sum_{i=1}^m u_i \otimes X v_i$, i.e., $\partial_X P = \sum_{i=1}^m u_i \otimes v_i$, and let $\Psi = (1 - zP)^{-1}$. Then we have*

$$\mathbb{E}_X[\Psi] = \beta^b_Y(\Psi) + z\sum_{i=1}^m \beta^b_Y(\Psi u_i) X\, \mathbb{E}_X[v_i \Psi] \tag{4.6}$$

where the conditional expectations on the right hand side satisfy the linear system of equations

$$\begin{bmatrix} \mathbb{E}_X[v_1\Psi] \\ \mathbb{E}_X[v_2\Psi] \\ \vdots \\ \mathbb{E}_X[v_m\Psi] \end{bmatrix} = \begin{bmatrix} \beta^b_Y[v_1\Psi] \\ \beta^b_Y[v_2\Psi] \\ \vdots \\ \beta^b_Y[v_m\Psi] \end{bmatrix} + HX \begin{bmatrix} \mathbb{E}_X[v_1\Psi] \\ \mathbb{E}_X[v_2\Psi] \\ \vdots \\ \mathbb{E}_X[v_m\Psi] \end{bmatrix} \tag{4.7}$$

where $H_{ij} = \beta^b_Y(v_i \Psi u_j)z + \beta^b_Y(u_{ij})$.

Remark 4.18. (1) Let us label each appearance of X in $P(X,Y)$ by consecutive labels $X_1, X_2, \dots$ and similarly for Y we label them as $Y_1, Y_2, \dots$. For example for $XY + YX$ we write $X_1Y_1 + X_2Y_2$. Then one can see that the entry H_{ij} is exactly the functional β^b_Y evaluated in all possible subwords of Ψ which occur between X_i and X_j. More precisely if we consider $(1 - zP)^{-1}$ then if X_i and X_j are in different monomials of P then these monomials are of the form $u_i X_i v_i$ and $u_j X_j v_j$ and all subwords of $(1 - zP)^{-1}$ which are between X_i and X_j are exactly of the form $v_i(1 - zP)^{-1}u_j$ if X_i and X_j are in the same monomial then this monomial is of the form $u_i X_i u_{ij} X_j v_j$. Thus in this case we get the additional term u_{ij} to which we apply β^b_Y.

Observe this very matrix H_Y (with slightly different powers of z) appeared in [14, Theorem 6.1] in the computation of the anti–commutator. However there the goal was to determine the distribution, not the conditional expectation.

(2) At this point it is not clear how to determine the matrix H above. For this it is necessary to evaluate the functional β^b and this will be done in Section 5.
(3) Equation (4.7) is linear and can immediately be turned into a *linearization* (in the sense of [16]) of the rational function $\mathbb{E}_X[v_i\Psi]$. We will develop this systematically in Section 6.

Proof of Proposition 4.17. For the recurrence (4.3) we can use either $\vec{\Delta}_X$ or $\vec{\delta}_X$. The latter has the advantage of being a derivation. In any case

$$\mathbb{E}_X[\Psi] = \beta_Y^b(\Psi) + z(\beta_Y^b \otimes \mathbb{E}_X[(\Psi\otimes 1)(\vec{\delta}_X T)(1\otimes\Psi)] \tag{4.8}$$

$$= \beta_Y^b(\Psi) + z(\beta_Y^b \otimes \mathbb{E}_X[(\Psi\otimes 1)(\vec{\Delta}_X T)(1\otimes\Psi)] \tag{4.9}$$

because the extra terms in (4.8) are annihilated by β_Y^b.

Write

$$\vec{\delta}_X T = \sum_{i=1}^m u_i \otimes X v_i$$

i.e., $\partial_X T = \sum_{i=1}^m u_i \otimes v_i$. Then recurrence (4.3) reads

$$\mathbb{E}_X[\Psi] = \beta_Y^b(\Psi) + z\sum_{i=1}^m \beta_Y^b(\Psi u_i) X\, \mathbb{E}_X[v_i\Psi]$$

and we can finally establish a system of equations for the conditional expectations appearing on the RHS of (4.6). Again we employ (4.3) and get

$$\mathbb{E}_X[v_i\Psi] = \beta_Y^b(v_i\Psi) + \sum_j \beta_Y^b(u_{ij}) X\, \mathbb{E}_X[v_j\Psi] + z\sum_{i=1}^m \beta_Y^b(v_i\Psi u_j) X\, \mathbb{E}_X[v_j\Psi].$$

Define a matrix H as

$$H_{ij} = \beta_Y^b(v_i\Psi u_j) z + \beta_Y^b(u_{ij}).$$

It is straightforward to see that identity (4.7) holds with H_Y defined as above. ☐

5. Calculus for the block cumulant functional β^b

The recurrence from Theorem 4.12 opens a door to the study of conditional expectations and in the case of resolvents the resulting linear system can be solved explicitly to obtain explicit formulas in terms of the functional β^b. Its evaluation is the subject of the present section. However at the time of this writing we lack sufficient understanding of Boolean cumulants required to handle this problem in the general setting of Assumption 4.1. Therefore in the remainder of this paper we will mostly work in the formal setting of polynomial algebras from Remark 4.2, which allows the reduction to univariate Boolean cumulants which are well understood.

Assumption 5.1. We will work in a formal ncps, i.e., the free associative algebra $\mathbb{C}\langle\mathcal{X}\rangle$ over some alphabet $\mathcal{X}$ and some distribution $\mu : \mathbb{C}\langle\mathcal{X}\rangle \to \mathbb{C}$. As usual the augmentation map $\epsilon : \mathbb{C}\langle\mathcal{X}\rangle \to \mathbb{C}$ is the constant coefficient map $\epsilon(P) = P(0)$.

More specifically, we will work in one of the following settings:

(i) $\mathcal{M} = \mathbb{C}\langle\mathcal{X} \cup \mathcal{Y}\rangle$ where $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ and $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ are free from each other with respect to μ and so are $\mathcal{A} = \mathbb{C}\langle\mathcal{X}\rangle$ and $\mathcal{B} = \mathbb{C}\langle\mathcal{Y}\rangle$. In this case we will denote the corresponding conditional expectations, derivations, and functionals by $\mathbb{E}_\mathcal{X}$, $\vec{\Delta}_\mathcal{X}$, $\vec{\delta}_\mathcal{X}$, $\beta^b_\mathcal{X}$ etc.

(ii) $\mathcal{M} = \big(\mathbb{C}\langle X_1, X_2, \ldots, X_n\rangle, \mu\big)$ and

$$\mathcal{A} = \mathbb{C}\langle X_i \mid i \in I\rangle \qquad \mathcal{B} = \mathbb{C}\langle X_j \mid j \in J\rangle.$$

where $I\dot{\cup}J = [n]$ is a partition and we assume in addition that *all* variables $X_1, X_2, \ldots, X_n$ are free from each other with respect to the functional μ. In this case we will denote the corresponding conditional expectations, derivations, and functionals by In this case we will denote conditional expectations by $\mathbb{E}_I$, $\vec{\Delta}_I$, β^b_I etc.

The typical case is

$$\mathcal{A} = \mathbb{C}\langle X_1, X_2, \ldots, X_k\rangle \qquad \mathcal{B} = \mathbb{C}\langle X_{k+1}, X_{k+2}, \ldots, X_n\rangle.$$

The following functionals provide the key to evaluate the functionals β^b. They operate by fully splitting monomials into their factors.

Definition 5.2. On the formal ncps $(\mathbb{C}\langle\mathcal{X}\rangle, \mu)$ we define the *fully factored Boolean cumulant* β^δ by prescribing their values on monomials

$$\beta^\delta(1) = 1,$$
$$\beta^\delta(X_{i_1}X_{i_2}\cdots X_{i_k}) = \beta_k(X_{i_1}, X_{i_2}, \ldots, X_{i_k}).$$

We can decompose the functional β^δ along free subsets of variables similar to the functional β^b:

For a subset $\mathcal{Y} \subseteq \mathcal{X}$ of the variables we define linear functionals $\beta^\delta_\mathcal{Y}$ on monomials as follows:

$$\beta^\delta_\mathcal{Y}(1) = 1,$$
$$\beta^\delta_\mathcal{Y}(X_{i_1}X_{i_2}\cdots X_{i_k}) = \begin{cases} \beta_k(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) & \text{if } X_{i_1}, X_{i_k} \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

Then $\beta^\delta = \beta^\delta_\mathcal{X}$ and if a set of variables $\mathcal{Y}$ is the disjoint union of mutually free (with respect to μ) subsets $\mathcal{Y}_j$ then thanks to property (CAC) we have

$$\beta^\delta_\mathcal{Y} = \epsilon + \sum(\beta^\delta_{\mathcal{Y}_j} - \epsilon).$$

Next we state Lemma 3.10 in terms of the functional β^δ, which will turn out to be useful in the proof of the next theorem.

Lemma 5.3. *Assume the setting from Assumption 5.1* (i), *i.e.,* $\mathcal{M} = \mathbb{C}\langle\mathcal{X} \cup \mathcal{Y}\rangle$ *with* $\mathcal{X}$ *and* $\mathcal{Y}$ *mutually free. Then for any letters* $X_{i_j} \in \mathcal{X}$ *and any elements* $V_j \in \mathbb{C}\langle\mathcal{Y}\rangle$ *we have*

$$\beta^\delta_\mathcal{X}(X_{i_0}V_1X_{i_1}V_2\cdots V_kX_{i_k}) = \beta_{2k+1}(X_{i_0}, V_1, X_{i_1}, V_2, \ldots, V_k, X_{i_k}).$$

We emphasize that it is not required that V_j have vanishing constant term.

For the next theorem we enhance our toolbox with yet another derivation, which implements a kind of elementary unshuffle coproduct.

Definition 5.4. For a subset of variables $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_k\} \subseteq \mathcal{X}$ define the first order unshuffle operator

$$L^*_{\mathcal{Y}} \odot \partial_{\mathcal{Y}} = \sum L_{Y_j} \otimes \partial_{Y_j} : \mathbb{C}\langle\mathcal{X}\rangle \to \mathbb{C}\langle\mathcal{Y}\rangle \otimes \mathbb{C}\langle\mathcal{X}\rangle \otimes \mathbb{C}\langle\mathcal{X}\rangle$$
$$W \mapsto \sum Y_j \otimes \partial_{Y_j} W$$

Its iterations are defined on the left leg

$$(L^*_{\mathcal{Y}} \odot \partial_{\mathcal{Y}})^2 W = \sum_{i,j} Y_i Y_j \otimes (\partial_{Y_i} \otimes \mathrm{id}) \partial_{Y_j} W$$
$$(L^*_{\mathcal{Y}} \odot \partial_{\mathcal{Y}})^p W = \sum_{i_1, i_2, \dots, i_p} Y_{i_1} Y_{i_2} \cdots Y_{i_p} \otimes (\partial_{Y_{i_1}} \otimes \mathrm{id}^{\otimes p-1})(\partial_{Y_{i_2}} \otimes \mathrm{id}^{\otimes p-2}) \cdots \partial_{Y_{i_p}} W$$

In the theorem below we summarize the relations between the functionals β^b and β^δ which will allow to evaluate these functionals on some non–commutative power series.

Theorem 5.5. *Assume the setting from Assumption 5.1* (i), *i.e.,* $\mathcal{M} = \mathbb{C}\langle\mathcal{X} \cup \mathcal{Y}\rangle$ *with* $\mathcal{X}$ *and* $\mathcal{Y}$ *mutually free.*

(i) For any element $P \in \mathbb{C}\langle\mathcal{X} \cup \mathcal{Y}\rangle$ *we have*

$$\begin{aligned} \beta^b_{\mathcal{X}}(P) &= \epsilon(P) + (\beta^\delta_{\mathcal{X}} \otimes \beta^b_{\mathcal{X}})(\overleftarrow{\delta}_{\mathcal{X}} P) \\ &= \epsilon(P) + (\beta^b_{\mathcal{X}} \otimes \beta^\delta_{\mathcal{X}})(\overrightarrow{\delta}_{\mathcal{X}} P). \end{aligned} \tag{5.1}$$

(ii) For any element $P \in \mathbb{C}\langle\mathcal{X} \cup \mathcal{Y}\rangle$ *we have*

(5.2)
$$\beta^\delta_{\mathcal{X}}(P) = \epsilon(P) + \sum_{k=1}^{\infty} \beta^\delta \otimes \left[\epsilon \otimes \left(\beta^b_{\mathcal{Y}}\right)^{\otimes(k-1)} \otimes \epsilon\right] \left((L^*_{\mathcal{X}} \odot \partial_{\mathcal{X}})^k(P)\right)$$
$$= \epsilon(P) + \sum_{k=1}^{\infty} \sum_{i_1, i_2, \dots, i_k} \beta_k(X_{i_1}, X_{i_2}, \dots, X_{i_k}) \left[\epsilon \otimes \left(\beta^b_{\mathcal{Y}}\right)^{\otimes(k-1)} \otimes \epsilon\right] \left(\partial_{X_{i_1}} \partial_{X_{i_2}} \cdots \partial_{X_{i_k}} P\right)$$

In particular, when $\mathcal{X} = \{X\}$ *consists of a single variable, then*

$$\beta^\delta_X(P) = \epsilon(P) + \sum_{k=1}^{\infty} \beta_k(X) \left[\epsilon \otimes \left(\beta^b_{\mathcal{Y}}\right)^{\otimes(k-1)} \otimes \epsilon\right] \left(\partial^k_X P\right) \tag{5.3}$$

Remark 5.6. (1) It is straightforward to extend by linearity the functionals $\beta^b_{\mathcal{X}}$ and $\beta^\delta_{\mathcal{X}}$ and Theorem 5.5 to formal power series. One the one hand, for elements of the algebra $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ of formal power series in non–commuting variables $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$, provided the resulting series converges; on the other hand, to the algebra $\mathbb{C}\langle\mathcal{X}\rangle((z))$ of formal power series with non–commuting coefficients.
(2) Formula (5.2) can also be stated in terms of the half-shuffle coproduct $\Delta_{\prec}$ of [13], but this will be dealt with elsewhere.

Proof of Theorem 5.5. (i) Essentially formula (5.1) is a reformulation of Corollary 2.7 in terms of the functionals and derivations which we introduced above, after observing that the extra terms vanish as a consequence of Property (CAC).

We will only prove the first of the two identities (5.1) as the proof of second equation is essentially the same. By linearity it suffices to consider monomials $w \in \mathbb{C}\langle\mathcal{X} \cup \mathcal{Y}\rangle$ and in fact only words w of type $\mathcal{X}$–$\mathcal{X}$, any word of different type being annihilated on both sides of the equation by definition of $\beta^b_{\mathcal{X}}$. So suppose $w = u_1 v_1 u_2 \cdots v_k u_{k+1}$ is a free factorization into words $u_i \in \mathcal{X}^+$ and $v_i \in \mathcal{Y}^+$, then $\beta^b_{\mathcal{X}}(w) = \beta_{2k+1}(u_1, v_1, u_2, \dots, v_k, u_{k+1})$. On the other hand

$$\overleftarrow{\delta}_{\mathcal{X}} w = \sum_{i=1}^{k+1} u_1 v_1 u_2 \cdots v_{i-1} \cdot \overleftarrow{\delta}_{\mathcal{X}}(u_i) \cdot v_i u_{i+1} \cdots u_{m+1}. \tag{5.4}$$

This produces the same splittings of the factors from $\mathcal{X}$ in the $\mathcal{X}$–$\mathcal{Y}$ factorization as in (2.9), except for two kinds configurations:

(1) Formula (2.9) contains splittings of factors of w coming from both $\mathcal{X}^+$ and $\mathcal{Y}^+$. The latter are absent from (5.4) because whenever a factor $v_i = v_i' v_i''$ is split, at least one of the factors of the arising term $\beta^\delta_{\mathcal{X}}(u_1 v_1 u_2 \cdots v_i')\beta^b_{\mathcal{X}}(v_i'' u_{i+1} \cdots u_{m+1})$ vanishes by definition of $\beta^\delta_{\mathcal{X}}$ and $\beta^b_{\mathcal{X}}$.

(2) In the derivative $\overleftarrow{\delta}_{\mathcal{X}} w$ there are extra terms of type $u_1 v_1 u_2 \cdots u_k \otimes v_k u_{k+1} v_{k+1} \cdots u_{m+1}$ which do not appear in Corollary 2.9, but the application of $\beta^\delta_{\mathcal{X}} \otimes \beta^b_{\mathcal{X}}$ creates the factor $\beta^b_{\mathcal{X}}(v_k \cdots u_{m+1}) = 0$.

(ii) For the proof of (5.2) first observe that both sides of this equation are 1 for the empty word and that both sides are zero unless w is of type $\mathcal{X}$–$\mathcal{X}$. Thus fix a monomial $w \in (\mathcal{X} \cup \mathcal{Y})^+$ of the latter type. We write it as $w = X_{i_1} v_1 X_{i_1} v_2 \cdots v_k X_{i_{k+1}}$ with $v_j \in \mathcal{Y}^*$, i.e., we allow some $v_j = 1$. However the latter will be considered later and we assume first that $v_j \neq 1$ for $i = 1, 2, \dots, k$. Comparing (5.2) with formula (3.2) observe that according to (3.2) we have a sum over all choices of variables from $\mathcal{X}$, with the restriction that both X_{i_1} and $X_{i_{k+1}}$ are always selected. In formula (5.2), all choices of p X's are given by the p-th derivative, the variables annihilated by the free derivative are put to the outer block, moreover application of $\epsilon \otimes \left(\beta^b_{\mathcal{Y}}\right)^{\otimes(k-1)} \otimes \epsilon$ evaluates to zero unless both X_{i_1} and X_{i_k} are annihilated by the derivative. Further from (5.2) we get a product of cumulants of type $\beta_r(v_p, X_{i_{p+1}}, \dots, X_{i_q}, v_q)$, which are extracted between the two X's moved to the outer block and this is exactly equal to $\beta^b_{\mathcal{Y}}(v_p X_{i_{p+1}} \cdots X_{i_q} v_q)$.

It remains to consider the case where some $v_q = 1$. Assume that it comes from a pocket determined by X_{i_p} and X_{i_r} which are both chosen to the outer block, i.e., $p \leq q < r$. The existence of such p and q is guaranteed as we always choose the first and the last X to the outer block. We will consider the contribution of this pocket in both formulas (3.2) and (5.2). There are three possible cases depending on how v_i is placed between j-th and k-th X_1:

- $p = q$ and $r = p + 1$ then the pocket created by these two elements contains only v_q and formula (3.2) gives $\beta_1(v_q) = 1$ while from (5.2) we get $\beta^b_{\mathcal{Y}}(1) = 1$.
- $p = q$ and $r > p + 1$, then the pocket contains $v_p X_{i_{p+1}} \cdots v_{r-1}$ and (3.2) gives precisely $\beta_{2(k-i)-1}(v_p, X_{i_{p+1}}, v_{p+1}, X_{i_{p+2}}, \dots, v_{r-1}) = 0$ because the first argument is 1 and (2.10) applies. On the other hand (5.2) we get $\beta^b_{\mathcal{Y}}(v_p X_{i_{p+1}} v_{p+1} X_{i_{p+1}} \cdots v_{r-1}) = \beta^b_{\mathcal{Y}}(X_{i_{p+1}} v_{p+1} X_{i_{p+1}} \cdots v_{r-1}) = 0$ by definition of the functional $\beta^b_{\mathcal{Y}}$. The case $p < q = r - 1$ is treated similarly.
- Having eliminated all such units, it remains to consider the case $p < q < r - 1$. By (3.2) the contribution of the pocket containing v_q equals

$$\begin{aligned}
&\beta_{2(k-i)-1}(v_p, X_{i_{p+1}}, v_{p+1}, X_{i_{p+2}}, \dots, X_{i_q}, 1, X_{i_{q+1}}, \dots, X_{i_{r-1}}, v_{r-1}) \\
&\qquad = \beta_{2(k-i)-1}(v_p, X_{i_{p+1}}, v_{p+1}, X_{i_{p+2}}, \dots, X_{i_q}, X_{i_{q+1}}, \dots, X_{i_{r-1}}, v_{r-1}) \\
&\qquad\qquad = \beta_{2(k-i)-1}(v_p, X_{i_{p+1}}, v_{p+1}, X_{i_{p+2}}, \dots, X_{i_q} X_{i_{q+1}}, \dots, X_{i_{r-1}}, v_{r-1})
\end{aligned}$$

  because of (2.12) and Lemma 3.10.
  Assuming that $v_p, v_{p+1}, \dots, v_{r-1} \neq 1$, then $\beta^b_{\mathcal{Y}}(v_p X_{i_{p+1}} v_{p+1} \cdots X_{i_q} v_q X_{i_{q+1}} \cdots X_{i_{r-1}} v_{r-1})$ evaluates to the same value. If there are $v_l = 1$ for $p < l < r - 1$ and $l \neq q$ we repeat the same argument. Observe that with the previous steps we have already made sure that $v_p, v_{r-1} \neq 1$.

□

Let us illustrate this with some examples.

Example 5.7. A direct calculation using VNRP shows that

$$\beta^\delta_X(XYX) = \beta_3(X,Y,X) = \beta_2(X)\beta_1(Y).$$

Let us calculate the derivatives of $\partial_X(XYX) = 1\otimes YX + XY\otimes 1$ of course both corresponding terms vanish after application of β^b_Y and thus $n=1$ does not contribute (it never will, effectively the sum starts from n=2). The second derivative gives $D_2(XYX) = 1\otimes Y\otimes 1$, and hence from the formula in the theorem we also get

$$\beta^\delta_X(XYX) = \beta_3(X,Y,X) = \beta_2(X)\beta_1(Y).$$

Example 5.8 (The additive subordination function). In Example 4.16 we concluded that the additive subordination function for $R=(z-X-Y)^{-1}$ is $1/\beta^b_Y(R)$. Now $\overleftarrow{\delta}_Y R = RY\otimes R$ and the recurrence (5.1) yields

$$\begin{aligned}\beta^b_Y(R) &= \epsilon(R) + (\beta^\delta_Y\otimes\beta^b_Y)(\overleftarrow{\delta}_Y R)\\ &= \frac{1}{z} + \beta^\delta_Y(RY)\beta^b_Y(R)\end{aligned}$$

and thus the subordination function is

$$\omega(z) = 1/\beta^b_Y(R) = z - z\beta^\delta_Y(RY)$$

cf. [23, Corollary 3.7].

Example 5.9. In anticipation of matrix valued formulas arising in Section 6 we continue Example 4.15 and consider the simple example where $n=2$ and the power series $\Psi = (1-z(X+Y))^{-1} = \sum_{n=0}^\infty \big(z(X+Y)\big)^n \in \mathbb{C}\langle X,Y\rangle((z))$. Since we work only with two variables we will write $\beta^b_X, \beta^\delta_X$ etc. which should not lead to any confusion.

Clearly we have

$$\overrightarrow{\delta}_X(\Psi) = 1 + z\Psi\otimes X\Psi, \qquad \overrightarrow{\delta}_Y(\Psi) = 1 + z\Psi\otimes Y\Psi.$$

Thus from (5.1) we obtain

$$\beta^b_X(\Psi) = 1 + z\beta^b_X(\Psi)\beta^\delta_X(X\Psi), \qquad \beta^b_Y(\Psi) = 1 + z\beta^b_Y(\Psi)\beta^\delta_Y(Y\Psi).$$

Hence we obtain

$$\beta^b_X(\Psi) = \Big(1 - z\beta^\delta_X(X\Psi)\Big)^{-1}, \qquad \beta^b_Y(\Psi) = \Big(1 - z\beta^\delta_Y(Y\Psi)\Big)^{-1}.$$

Comparing with (4.5) we conclude that

$$\mathbb{E}_X[\Psi] = (1 - z\beta^\delta_Y(Y\Psi) - zX)^{-1}.$$

Moreover observe that

$$\partial^n_X(X\Psi) = z^{n-1}1\otimes\Psi^{\otimes n} + z^n X\Psi\otimes\Psi^{\otimes n}, \qquad \partial^n_Y(Y\Psi) = z^{n-1}1\otimes\Psi^{\otimes n} + z^n Y\Psi\otimes\Psi^{\otimes n}.$$

Thus equation (5.3) gives

$$\begin{aligned}\beta^\delta_X(X\Psi) &= \sum_{n=1}^\infty \beta_n(X)\beta^b_Y(\Psi)^{n-1}z^{n-1} = \widetilde{\eta}_X(z\beta^b_Y(\Psi)),\\ \beta^\delta_Y(Y\Psi) &= \sum_{n=1}^\infty \beta_n(Y)\beta^b_X(\Psi)^{n-1}z^{n-1} = \widetilde{\eta}_Y(z\beta^b_X(\Psi)).\end{aligned}$$

Finally we obtain the following system of equations

$$\beta^\delta_X(X\Psi) = \widetilde{\eta}_X\left(z\left(1 - z\beta^\delta_Y(Y\Psi)\right)^{-1}\right), \quad \beta^\delta_Y(Y\Psi) = \widetilde{\eta}_Y\left(z\left(1 - z\beta^\delta_X(X\Psi)\right)^{-1}\right).$$

We shall see below that this system of equations has unique power series solutions $\beta^\delta_X(X\Psi)$ and $\beta^\delta_Y(Y\Psi)$ analytic at 0 and in fact yields the fixed point equation for subordination function for free additive convolution. Thus this system determines the function needed in Example 4.15.

6. Linearization and conditional expectations

The procedure presented in Proposition 4.17 can be systematized using linearizations of resolvents from the very beginning. For the reader not familiar with rational series and linearizations the basic facts are collected in Appendix B. Here we will adapt and amplify all previously defined operations to the level of matrices and then lift Example 5.9 to the matrix-valued setting.

6.1. **Amplifications of expectations and cumulants.**

Most concepts considered in the present paper can be generalized to the operator valued case. Rather than do this in the general case we restrict the discussion to amplifications to tensor products. In fact the matrix valued case would suffice for later applications to linearizations, however the proofs are conceptually simpler in the language of tensor products.

Notation 6.1. Let $(\mathcal{A},\varphi)$ be an ncps and $\mathcal{C}$ a unital algebra. We shall consider $\mathcal{C}\otimes\mathcal{A}$ as a $\mathcal{C}$-bimodule with action $c_1\cdot(c\otimes a)\cdot c_2 = c_1cc_2\otimes a$.

In the case where $\mathcal{C}=M_N(\mathbb{C})$ is a matrix algebra we will denote the elements by $\sum C_i u_i$ where for a matrix $C\in M_N(\mathbb{C})$ and $u\in\mathcal{A}$ we denote by $Cu=uC\in M_N(\mathcal{A})$ the matrix with entries

$$(Cu)_{ij} = c_{ij}u.$$

The following lemma is easily verified on elementary tensors.

Lemma 6.2. *Let $(\mathcal{A},\varphi)$ be a ncps and $\mathcal{C}$ a unital algebra. and denote by $\varphi_{\mathcal{C}}=\mathrm{id}_{\mathcal{C}}\otimes\varphi:\mathcal{C}\otimes\mathcal{A}\to\mathcal{C}$ the amplification of φ.*

(i) $\varphi_{\mathcal{C}}$ is a $\mathcal{C}$-bimodule map:

$$\varphi_{\mathcal{C}}(c_1\cdot u\cdot c_2) = c_1\varphi_{\mathcal{C}}(u)c_2$$

for all $c_1,c_2\in\mathcal{C}$ and $u\in\mathcal{C}\otimes\mathcal{A}$.

(ii) Let $\mathcal{B}\subseteq\mathcal{A}$ be a subalgebra and $\mathbb{E}_{\mathcal{B}}:\mathcal{A}\to\mathcal{B}$ be a conditional expectation for φ. Then its amplification $\mathrm{id}_{\mathcal{C}}\otimes\mathbb{E}_{\mathcal{B}}:\mathcal{C}\otimes\mathcal{A}\to\mathcal{C}\otimes\mathcal{B}$ is a $\mathcal{C}\otimes\mathcal{B}$-bimodule map and moreover it is a conditional expectation for the amplification $\varphi_{\mathcal{C}}$ in the sense that

$$\varphi_{\mathcal{C}}((\mathrm{id}_{\mathcal{C}}\otimes\mathbb{E}_{\mathcal{B}})[u]v) = \varphi_{\mathcal{C}}(uv)$$

for any $u\in\mathcal{C}\otimes\mathcal{A}$ and any $v\in\mathcal{C}\otimes\mathcal{B}$.

(iii) The $\mathcal{C}$-valued Boolean cumulants defined by the amplification of the recurrence (2.2)

$$\varphi_{\mathcal{C}}(u_1u_2\cdots u_n) = \sum_{k=1}^{n}\beta_k^{\mathcal{C}}(u_1,u_2,\dots,u_k)\,\varphi_{\mathcal{C}}(u_{k+1}u_{k+2}\cdots u_n)$$

are given by

$$\beta_n^{\mathcal{C}}(c_1\otimes a_1, c_2\otimes a_2,\dots,c_n\otimes a_n) = c_1c_2\cdots c_n\,\beta_n(a_1,a_2,\dots,a_n).$$

In the case where $\mathcal{C}=M_N(\mathbb{C})$ is a matrix algebra we can apply the usual identification of $M_n(\mathbb{C})\otimes\mathcal{A}$ with $M_N(\mathcal{A})$ and reformulate the lemma in terms of matrix operations as follows.

Corollary 6.3. *Let $(\mathcal{A},\varphi)$ be a ncps.*

(i) The amplification $\varphi^{(N)}:M_N(\mathcal{A})\to M_N(\mathbb{C})$ to the matrix algebra is the entry-wise application $\varphi^{(N)}([a_{ij}])=[\varphi(a_{ij})]$. These maps form a family of matrix bimodule maps:

$$\varphi^{(k)}(U\cdot A\cdot V) = U\varphi^{(N)}(A)V$$

holds for any matrix $A\in M_N(\mathcal{A})$ and any scalar matrices $U\in M_{k\times N}(\mathbb{C})$ and $V\in M_{N\times k}(\mathbb{C})$, $k\in\mathbb{N}$.

(ii) Let $\mathcal{B}\subseteq\mathcal{A}$ be a subalgebra and $\mathbb{E}_{\mathcal{B}}:\mathcal{A}\to\mathcal{B}$ be a conditional expectation for φ. Then the entry-wise application maps $\mathbb{E}_{\mathcal{B}}^{(N)}\big[[a_{ij}]_{ij}\big]=\big[\mathbb{E}_{\mathcal{B}}[a_{ij}]\big]_{ij}$ form a family of $M_N(\mathcal{B})$-bimodule maps:

$$\mathbb{E}_{\mathcal{B}}^{(k)}[U\cdot A\cdot V] = U\cdot\mathbb{E}_{\mathcal{B}}^{(N)}[A]\cdot V \tag{6.1}$$

holds for any matrix $A \in M_N(\mathcal{A})$ *and any scalar matrices* $U \in M_{k\times N}(\mathbb{C})$ *and* $V \in M_{N\times k}(\mathbb{C})$*. Moreover it is a conditional expectation for the map* $\varphi^{(N)}$ *in the sense that* $\varphi^{(N)}(\mathbb{E}_{\mathcal{B}}^{(N)}(A)B) = \varphi^{(N)}(AB)$ *for any* $A \in M_N(\mathcal{A})$ *and any* $B \in M_N(\mathcal{B})$*.*

(iii) The entries of the $M_N(\mathbb{C})$*-valued Boolean cumulants defined by the amplification of the recurrence* (2.2)

$$\varphi^{(N)}(A_1A_2\cdots A_n) = \sum_{k=1}^{n} \beta_k^{(N)}(A_1, A_2, \ldots, A_k)\,\varphi^{(N)}(A_{k+1}A_{k+2}\cdots A_n)$$

are given by

$$\beta_n^{(N)}(A_1, A_2, \ldots, A_n)_{ij} = \sum_{i_1,i_2,\ldots,i_{n-1}} \beta_n(a_{ii_1}^{(1)}, a_{i_1i_2}^{(2)}, \ldots, a_{i_{n-1}j}^{(n)}).$$

Notation 6.4. The corresponding matrix valued versions of the functionals β^b and β^δ are defined analogously as

$$\beta_{\mathcal{A}}^{b(N)}(A)_{ij} = \beta_{\mathcal{A}}^{b}(a_{ij}) \qquad\qquad \beta_{\mathcal{A}}^{\delta(N)}(A)_{ij} = \beta_{\mathcal{A}}^{\delta}(a_{ij}),$$

whenever these are defined (Assumption 4.1, resp. Assumption 5.1).

6.2. Amplifications of derivations.

Lemma 6.5. *Let* $\mathcal{A}$ *and* $\mathcal{C}$ *be algebras,* $D : \mathcal{A} \to \mathfrak{M}$ *be a derivation into an* $\mathcal{A}$*-bimodule* $\mathfrak{M}$*. Then* $D^{(\mathcal{C})} = \mathrm{id}_{\mathcal{C}} \otimes D : \mathcal{C}\otimes\mathcal{A} \to \mathcal{C}\otimes\mathfrak{M}$ *is a derivation, where* $\mathcal{C}\otimes\mathfrak{M}$ *is an* $\mathcal{C}\otimes\mathcal{A}$*-module with action*

$$(c_1 \otimes a_1)\cdot(c\otimes\mathfrak{m})\cdot(c_2\otimes a_2) = c_1cc_2 \otimes (a_1\cdot\mathfrak{m}\cdot a_2)$$

Proof. It suffices to verify the Leibniz rule (2.15) for $D^{(\mathcal{C})}$ on elementary tensors:

$$\begin{aligned} D^{(\mathcal{C})}\Big((c_1\otimes a_1)(c_2\otimes a_2)\Big) &= c_1c_2\otimes D(a_1a_2) \\ &= c_1c_2\otimes(D(a_1)\cdot a_2) + c_1c_2\otimes(a_1\cdot D(a_2)) \\ &= D^{(\mathcal{C})}(c_1\otimes a_1)\cdot(c_2\otimes a_2) + (c_1\otimes a_1)\cdot D^{(\mathcal{C})}(c_2\otimes a_2) \end{aligned}$$

□

This is true in particular for $\mathcal{C} = M_N(\mathbb{C})$ (cf. [27, Section 3]).

Corollary 6.6. *Let* $\mathfrak{M}$ *be an* $\mathcal{A}$*-bimodule and* $D : \mathcal{A}\to\mathfrak{M}$ *be a derivation. Then* $M_N(\mathfrak{M})$ *is an* $M_N(\mathcal{A})$*-bimodule with action*

$$[a'_{ij}]_{ij}\cdot[\mathfrak{m}_{ij}]_{ij}\cdot[a''_{ij}]_{ij} = \Big[\sum_{k,l} a'_{ik}\cdot\mathfrak{m}_{kl}\cdot a''_{lj}\Big]_{ij}$$

and the matrix amplification $D^{(N)} : M_N(\mathcal{A}) \to M_N(\mathfrak{M})$*, i.e., entry-wise application*

$$D^{(N)}([a_{ij}]) = [D(a_{ij})]$$

satisfies the Leibniz rule (2.15)*.*

Example 6.7. Let $D : \mathcal{A}\to\mathfrak{M}$ be a derivation where $\mathfrak{M} = \mathcal{A}\otimes\mathcal{A}$ is the bimodule with action

$$a'\cdot(a_1\otimes a_2)\cdot a'' = a'a_1\otimes a_2a'' = (a'\otimes 1)(a_1\otimes a_2)(1\otimes a'').$$

Then using Notation 2.9 the Leibniz rule for the product of two matrices $A_1, A_2 \in M_N(\mathcal{A})$ reads

$$D^{(N)}(A_1A_2) = D^{(N)}(A_1)(1_{\mathcal{A}}\odot A_2) + (A_1\odot 1_{\mathcal{A}})D^{(N)}(A_2) \in M_N(\mathcal{A}\otimes\mathcal{A}),$$

i.e.,

$$D^{(N)}(A_1A_2)_{ij} = \sum_k D(a_{ik}^{(1)})(1_{\mathcal{A}}\otimes a_{kj}^{(2)}) + (a_{ik}^{(1)}\otimes 1_{\mathcal{A}})D(a_{kj}^{2}).$$

In particular, for a resolvent $\mathbf{\Psi} = (I_N - zA)^{-1}$ the derivation results in

$$D^{(N)}\mathbf{\Psi} = z(\mathbf{\Psi} \odot 1)D^{(N)}(A)(1 \odot \mathbf{\Psi})$$

and in the case of an elementary tensor $A = C \otimes a$ with $D(a) = \sum u_i \otimes v_i$,

$$D^{(N)}\mathbf{\Psi} = z \sum \mathbf{\Psi} u_i C \odot v_i \mathbf{\Psi}.$$

6.3. **Computing conditional expectations via linearizations.** In the following we will omit the superscripts from $\mathbb{E}_{\mathcal{B}}^{(N)}$ etc. and write $\mathbb{E}_{\mathcal{B}}$ etc. whenever the context is unambiguous. Now the bimodule property (6.1) allows us to rewrite the conditional expectation of a linearization

$$(1 - z^m P)^{-1} = u^t (I - zL)^{-1} v$$

as

$$\mathbb{E}_{\mathcal{B}}\Big[(1 - z^m P)^{-1}\Big] = u^t \mathbb{E}_{\mathcal{B}}\Big[(I - zL)^{-1}\Big] v \tag{6.2}$$

and the problem is reduced to the computation of conditional expectations of matrix pencils. We shall see that this accomplished by repeating the computation from Example 1.6 with matrix coefficients.

It is immediate to verify the amplification of the recurrence (4.3), namely

$$(\mathrm{id}_{\mathcal{C}} \otimes \mathbb{E}_{\mathcal{B}})[c \otimes P] = (\mathrm{id}_{\mathcal{C}} \otimes \beta_{\mathcal{A}}^{b})(P) + (\mathrm{id}_{\mathcal{C}} \otimes \beta_{\mathcal{A}}^{b} \otimes \mathbb{E}_{\mathcal{B}})[\mathrm{id}_{\mathcal{C}} \otimes \vec{\Delta}_{\mathcal{B}}(P)]$$

etc. The matrix analog of Theorem 4.12 in terms of the amplifications of $\mathbb{E}_{\mathcal{B}}$, $\beta_{\mathcal{A}}^{b}$, $\vec{\Delta}_{\mathcal{B}}$ and $\vec{\nabla}_{\mathcal{B}}$ from Lemmas 6.2 and 6.5 reads as follows.

Proposition 6.8. *Let $M \in M_N(\mathcal{M})$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}$ as in Assumption 4.1. Then*

$$\begin{aligned} \mathbb{E}_{\mathcal{B}}^{(N)}[M] &= \beta_{\mathcal{A}}^{b(N)}(M) + (\beta_{\mathcal{A}}^{b} \otimes \mathbb{E}_{\mathcal{B}})^{(N)}[\vec{\Delta}_{\mathcal{B}}^{(N)}(M)] \\ &= \beta_{\mathcal{A}}^{b(N)}(M) + (\beta_{\mathcal{A}}^{b} \otimes \mathbb{E}_{\mathcal{B}})^{(N)}[\vec{\nabla}_{\mathcal{B}}^{(N)}(M)] \\ &= \beta_{\mathcal{A}}^{b(N)}(M) + (\mathbb{E}_{\mathcal{B}} \otimes \beta_{\mathcal{A}}^{b})^{(N)}[\overleftarrow{\Delta}_{\mathcal{B}}^{(N)}(M)] \end{aligned}$$

In particular, for a linear matrix pencil $L = \sum C_i' a_i + C_j'' b_j \in M_N(\mathcal{M})$ the conditional expectation of the resolvent $\mathbf{\Psi} = \Big(I_N - z(\sum C_i' a_i + \sum C_j'' b_j)\Big)^{-1}$ is

$$\mathbb{E}_{\mathcal{B}}^{(N)}[\mathbf{\Psi}] = \Big(I_N - zH_{\mathcal{A}} \sum C_j'' b_j\Big)^{-1} H_{\mathcal{A}}, \tag{6.3}$$

where $H_{\mathcal{A}} = \beta_{\mathcal{A}}^{b(N)}(\mathbf{\Psi}) \in M_N(\mathbb{C})$.

Formula (6.3) is a matrix valued version of Example 4.15 and the proof runs along the same lines.

Next we switch to Assumption 5.1 and lift (4.4) and Theorem 5.5 to the matrix level.

Proposition 6.9. *Let $\mathcal{M} = M_N(\mathbb{C}\langle \mathcal{X} \cup \mathcal{Y} \rangle)$ and $\mathcal{X}, \mathcal{Y}$ free with respect to $\mu : \mathcal{M} \to \mathbb{C}$ as in Assumption 5.1. Then*

$$\mathbb{E}_{\mathcal{Y}}^{(N)}[M] = \beta_{\mathcal{X}}^{b(N)}(M) + (\beta_{\mathcal{X}}^{b} \otimes \mathbb{E}_{\mathcal{Y}})^{(N)}[\vec{\delta}_{\mathcal{Y}}^{(N)}(M)]$$

Next, identity (5.1) trivially lifts to tensor products

$$\begin{aligned} (\mathrm{id}_{\mathcal{C}} \otimes \beta_{\mathcal{X}}^{b})(c \otimes P) &= c\epsilon(P) + c \cdot (\beta_{\mathcal{X}}^{\delta} \otimes \beta_{\mathcal{X}}^{b})(\overleftarrow{\delta}_{\mathcal{X}} P) \\ &= c\epsilon(P) + c \cdot (\beta_{\mathcal{X}}^{b} \otimes \beta_{\mathcal{X}}^{\delta})(\vec{\delta}_{\mathcal{X}} P) \end{aligned}$$

which immediately translates to the case of matrices as follows.

Proposition 6.10. *Let $M \in M_N(\mathcal{M})$, then with Assumption 5.1 we have*

$$\begin{aligned}\beta_{\mathcal{X}}^{b(N)}(M) &= \epsilon^{(N)}(M) + (\beta_{\mathcal{X}}^{\delta} \otimes \beta_{\mathcal{X}}^{b})^{(N)}(\overleftarrow{\delta}_{\mathcal{X}}^{(N)} M) \\ &= \epsilon^{(N)}(M) + (\beta_{\mathcal{X}}^{b} \otimes \beta_{\mathcal{X}}^{\delta})^{(N)}(\overrightarrow{\delta}_{\mathcal{X}}^{(N)} M)\end{aligned}$$

In particular, for a linear matrix pencil $L = \sum C_i' X_i + C_j'' Y_j \in M_N(\mathcal{M})$ the block cumulant functional at $\mathbf{\Psi} = \bigl(I_N - z(\sum C_i' X_i + \sum C_j'' Y_j)\bigr)^{-1}$ is

$$\begin{aligned}\beta_{\mathcal{X}}^{b(N)}(\mathbf{\Psi}) &= \Bigl(I_N - z \sum \beta_{\mathcal{X}}^{\delta(N)}(\mathbf{\Psi} X_i) C_i\Bigr)^{-1} = \Bigl(I_N - z\beta_{\mathcal{X}}^{\delta(N)}(\mathbf{\Psi} L)\Bigr)^{-1} \\ &= \Bigl(I_N - z \sum C_i \beta_{\mathcal{X}}^{\delta(N)}(X_i \mathbf{\Psi})\Bigr)^{-1} = \Bigl(I_N - z\beta_{\mathcal{X}}^{\delta(N)}(L\mathbf{\Psi})\Bigr)^{-1}.\end{aligned} \tag{6.4}$$

Thus in order to conclude the computation of the conditional expectation (6.2) it remains to evaluate $\beta_{\mathcal{X}}^{\delta(N)}(\mathbf{\Psi} X_i)$ for all i. This can be done if we assume that all variables are free with the help of (5.3) which we now lift to the matrix level.

Proposition 6.11. *Let $M \in M_N(\mathbb{C}\langle X_1, X_2, \dots, X_n\rangle)$, then*

$$\beta_{X_1}^{\delta(N)}(M) = \epsilon^{(N)}(M) + \sum_{k=1}^{\infty} \beta_k(X_1) \left[\epsilon \otimes \left(\beta_{\mathcal{B}}^{b}\right)^{\otimes(k-1)} \otimes \epsilon\right]^{(N)} \left(\partial_{X_1}^{k\,(N)}(M)\right). \tag{6.5}$$

We can now subsume the essence of the previous calculations as follows. Theorem 1.3 follows by evaluating formula (6.7) in a specific algebra $\mathcal{A}$.

Theorem 6.12. *Let $\mu : \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle \to \mathbb{C}$ be a distribution such that $X_1, X_2, \dots, X_n$ are free. In the following for an index set $I \subseteq \{1, 2, \dots, n\}$ the conditional expectation onto the subalgebra generated by the subset $\{X_i\}_{i\in I}$ will be denoted by $\mathbb{E}_I$. Suppose that the resolvent of a given polynomial $P = P(X_1, X_2, \dots, X_n) \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$ of degree m has linearization*

$$\Psi = (1 - z^m P)^{-1} = u^t \left(I_N - z(C_1 X_1 + C_2 X_2 + \cdots + C_n X_n)\right)^{-1} v \tag{6.6}$$

with $C_1, C_2, \dots, C_n \in M_N(\mathbb{C})$.

Then the conditional expectations of Ψ are

$$\begin{aligned}\mathbb{E}_I[(1 - z^m P)^{-1}] &= u^t\Bigl(I_N - zH_J\Bigl(\sum_{i\in I} C_i X_i\Bigr)\Bigr)^{-1} H_J v \\ &= u^t\Bigl(I_N - z\Bigl(\sum_{i\in I} C_i X_i - \sum_{j\in J} C_j F_j\Bigr)\Bigr)^{-1} v\end{aligned} \tag{6.7}$$

where $J = [n] \setminus I$ is the complement and $H_J = \Bigl(I_N - z\sum_{j\in J} C_j F_j\Bigr)^{-1}$ and the matrices $F_i = \beta_{X_i}^{\delta(N)}(X_i \mathbf{\Psi}) \in M_N(\mathbb{C})$, $i = 1, 2, \dots, n$ are the unique solution of the system of matrix equations

$$F_i = \widetilde{\eta}_{X_i}\Bigl(z\Bigl(I_N - z\sum_{j\neq i} C_j F_j\Bigr)^{-1} C_i\Bigr) \quad i = 1, 2, \dots, n \tag{6.8}$$

which is analytic at $z = 0$. In particular,

$$\varphi((1 - z^m P)^{-1}) = u^t\Bigl(I_N - z\Bigl(\sum_{j=1}^{n} C_j F_j\Bigr)\Bigr)^{-1} v$$

Proof. Let

$$\mathbf{\Psi} = \bigl(I_N - z(C_1 X_1 + C_2 X_2 + \cdots + C_n X_n)\bigr)^{-1}$$

be the matrix of the linearization $\Psi = u^t \mathbf{\Psi} v$ from (6.6). In the following we abbreviate the functionals

$$\beta_I^b = \sum_{i\in I} \beta_{X_i}^b \qquad\qquad \beta_I^\delta = \sum_{i\in I} \beta_{X_i}^\delta.$$

Then from (6.3) with $\mathcal{B}$ the algebra generated by $\{X_i\}_{i\in I}$ and $\mathcal{A}$ the algebra generated by $\{X_j\}_{j\in J}$ we obtain

$$\mathbb{E}_I[\Psi] = u^t\Big(I_N - zH_J \sum_{i\in I} C_i X_i\Big)^{-1} H_J v$$

where

$$H_J = \beta_J^{b(N)}(\mathbf{\Psi}) = \Big(I_N - z\sum_{j\in J} C_j \beta_J^{\delta(N)}(X_j\mathbf{\Psi})\Big)^{-1}.$$

by (6.4).

Next we apply Proposition 6.11 in order to establish equations for $F_i = \beta_{X_i}^{\delta(N)}(X_i\mathbf{\Psi})$, similar to Example 5.9. First observe that

$$\partial_{X_i}\mathbf{\Psi} = z(\mathbf{\Psi}\odot 1)C_i(1\odot\mathbf{\Psi}) = z\mathbf{\Psi}C_i\odot\mathbf{\Psi}$$

thus

$$\partial_{X_i}X_i\mathbf{\Psi} = 1\odot\mathbf{\Psi} + zX_i\mathbf{\Psi}C_i\odot\mathbf{\Psi}$$

and applying iterated derivatives to the left leg

$$\partial_{X_i}^k X_i\mathbf{\Psi} = z^{k-1}1\odot(\mathbf{\Psi}C_i)^{\odot k-1}\odot\mathbf{\Psi} + z^k X_i(\mathbf{\Psi}C_i)^{\odot k}\odot\mathbf{\Psi}$$

and (6.5) becomes

$$\begin{aligned}\beta_{X_i}^{\delta(N)}(X_i\mathbf{\Psi}) &= \sum_{k=1}^{\infty}\beta_k(X_i)z^{k-1}\beta_{[n]\setminus i}^{b(N)}(\mathbf{\Psi}C_i)^{k-1}\\ &= \tilde{\eta}_{X_i}(zH_{[n]\setminus i}C_i).\end{aligned}$$

Uniqueness of the matrices F_i follows from the iteration in Lemma 6.13 below. □

We have seen that the matrices $F_i(z) = \beta_{X_i}^{\delta(N)}(X_i\mathbf{\Psi}(z))$ satisfy the fixed point equation (6.8) and it remains to show that the latter has a unique solution analytic at 0. To this end we expand $F_i(z)$ into a power series

$$F_i(z) = \beta_{X_i}^{\delta(N)}(X_i\mathbf{\Psi}(z)) = \sum_{k=0}^{\infty}\beta_{X_i}^{\delta(N)}(X_iL^k)z^{km} \tag{6.9}$$

and show that iterating the fixed point equation produces a series whose coefficients converge to those of the series (6.9).

Lemma 6.13. *Start with constant matrices*

$$F_i^{(0)}(z) = \beta_1(X_i)I_N \qquad i=1,2,\dots,n$$

and iterate

$$F_i^{(r+1)}(z) = \tilde{\eta}_i\Big(z\Big(I_N - \sum_{j\neq i} C_jF_j^{(r)}(z)\Big)^{-1}C_i\Big) \qquad i=1,2,\dots,n$$

Then for all $r\in\mathbb{N}_0$ *we have*

$$F_i^{(r)}(z) - F_i(z) = \mathcal{O}(z^{r+1}) \qquad i=1,2,\dots,n$$

Proof. We proceed by induction. First observe that if $F_i^{(r)}(z) - F_i(z) = \mathcal{O}(z^{r+1})$ for $i = 1,2,\dots,n$ then

$$\Big(I_N - \sum_{j\neq i} C_jF_j^{(r)}(z)\Big)^{-1}C_i - \Big(I_N - \sum_{j\neq i} C_jF_j(z)\Big)^{-1}C_j = \mathcal{O}(z^{r+1})$$

as well and thus

$$\begin{aligned}
F_i^{(r+1)}(z) - F_i(z) &= \sum_{k=0}^{r+1} \beta_{k+1}(X_i) z^k \Bigl(\Bigl(I_N - \sum_{j\neq i} C_j F_j^{(r)}(z)\Bigr)^{-1} C_j\Bigr)^k + \mathcal{O}(z^{r+2}) \\
&\qquad - \sum_{k=0}^{r+1} \beta_{k+1}(X_i) z^k \Bigl(\Bigl(I_N - \sum_{j\neq i} C_j F_j(z)\Bigr)^{-1} C_X\Bigr)^k + \mathcal{O}(z^{r+2}) \\
&= \mathcal{O}(z^{r+2})
\end{aligned}$$

□

Remark 6.14. When it comes to the practical solution of the system (6.8) we are faced with a large number of variables. It can be reduced by observing that the solutions F_i enter the final expression (6.7) for the conditional expectation only in the products $C_i F_i$ and we can advantage of the fact that the coefficient matrices C_i usually are sparse. Indeed let P_i be the projection onto $\ker C_i$ and $Q_i = I - P_i$, i.e., $C_i P_i = 0$ and $C_i Q_i = C_i$. Then in the expression (6.7) we can replace the matrices F_i with the matrices $\tilde{F}_i = Q_i F_i$ and the latter satisfy the slightly modified system

$$\tilde{F}_i = Q_i \widetilde{\eta}_{X_i}\Bigl(z\Bigl(I_N - z\sum_{j\neq i} C_j F_j\Bigr)^{-1} C_i\Bigr) \quad i = 1, 2, \dots, n. \tag{6.10}$$

It follows that $\tilde{F}_i = Q_i \tilde{F}_i Q_i$ can be obtained by iterating the fixed point equation (6.10) with starting point $\tilde{F}_i^{(0)} = \beta_1(X_i) Q_i$.

7. Examples

In specific calculations it is always clear which N we need to take in $\mathbb{E}^{(N)}$, thus we will write $\mathbb{E}$ for $\mathbb{E}^{(N)}$ and similarly we will write β^b, β^δ etc. for matricial versions all other mappings.

7.1. **The product of free random variables.**

We start with the example $T = XY$ which can be solved directly without invoking linearizations.

The derivations are $\overrightarrow{\delta}_X \Psi = z\Psi \otimes XY\Psi$ and $\overrightarrow{\delta}_Y \Psi = z\Psi X \otimes Y\Psi$, respectively. First notice that from the resolvent identities

$$\Psi = 1 + zXY\Psi = 1 + z\Psi XY \tag{7.1}$$

we immediately infer

$$\beta_X^b(\Psi) = \beta_Y^b(\Psi) = 1 \tag{7.2}$$

and thus

$$\mathbb{E}_X[\Psi] = \beta_Y^b(\Psi) + z\beta_Y^b \otimes \mathbb{E}_X[\Psi \otimes XY\Psi] = 1 + zX\,\mathbb{E}_X[Y\Psi]; \tag{7.3}$$

further

$$\mathbb{E}_X[Y\Psi] = \beta_Y^b(Y\Psi)(1 + zX\,\mathbb{E}_X[Y\Psi])$$

and thus

$$\mathbb{E}_X[Y\Psi] = \beta_Y^b(Y\Psi)\Big(1 - z\beta_Y^b(Y\Psi)X\Big)^{-1}. \tag{7.4}$$

Putting together (7.3) and (7.4) we find

$$\mathbb{E}_X[\Psi] = \Big(1 - z\beta_Y^b(Y\Psi)X\Big)^{-1}$$

and the subordination equation

$$M(z) = M_X(z\beta_Y^b(Y\Psi)).$$

We have thus identified the first subordination function

$$\omega_1(z) = z\beta_Y^b(Y\Psi).$$

The conditional expectation with respect to Y is simpler to obtain, but the result is the same:

$$\mathbb{E}_Y[\Psi] = 1 + z\beta_X^b(\Psi X)Y\,\mathbb{E}_Y[\Psi]$$

and thus

$$\mathbb{E}_Y[\Psi] = \Big(1 - z\beta_X^b(\Psi X)Y\Big)^{-1}$$
$$M(z) = M_Y(z\beta_X^b(\Psi X)).$$

The second subordination function is

$$\omega_2(z) = z\beta_X^b(\Psi X).$$

To find relations between these functions we will resort to Theorem 5.5. It is immediate to see combinatorially that $\beta_Y^b(Y\Psi) = \beta_Y^\delta(Y\Psi)$ and $\beta_X^b(\Psi X) = \beta_X^\delta(\Psi X)$ but let us verify this with the recurrence from Proposition 5.5. Indeed,

$$\begin{aligned}\beta_Y^b(Y\Psi) &= \beta_Y^\delta \otimes \beta_Y^b(Y \otimes \Psi + zY\Psi XY \otimes \Psi)\\ &= \beta_Y^\delta(Y(1 + z\Psi XY))\beta_Y^b(\Psi) = \beta_Y^\delta(Y\Psi)\end{aligned}$$

because of the identities (7.1) and (7.2). Similarly

$$\beta_X^b(\Psi X) = \beta_X^\delta \otimes \beta_X^b(z\Psi X \otimes Y\Psi X + \Psi X \otimes 1) = \beta_X^\delta(\Psi X).$$

Thus we have

$$\omega_1(z) = z\beta_Y^\delta(Y\Psi) \qquad \omega_2(z) = z\beta_X^\delta(\Psi X).$$

7.2. **Conditional expectations of free commutators and anti–commutators.** As an application of Theorem 1.3 we show that for symmetric and free X, Y the conditional expectation of $(1-z^2P(X,Y))^{-1}$ onto one of the variables coincide for commutator and anti–commutator.

First we consider the anti–commutator $XY+YX$ in free variables.

The linearization matrix described in Remark B.7 is given by

$$L = \begin{bmatrix} 0 & 0 & Y & 0 \\ X & 0 & 0 & Y \\ X & 0 & 0 & Y \\ 0 & X & 0 & 0 \end{bmatrix} = C_X X + C_Y Y$$

where

$$C_X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad C_Y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have

$$(1-z^2(XY+YX))^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} (1-zL)^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

It follows that

$$\mathbb{E}_X[(1-z^2(XY+YX))^{-1}] = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \mathbb{E}_X\left[(1-zL)^{-1}\right] \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Next

$$\begin{aligned} \mathbb{E}_X\left[(1-zL)^{-1}\right] &= (1-zH_XC_XX)^{-1}H_X \\ \mathbb{E}_Y\left[(1-zL)^{-1}\right] &= (1-zH_YC_YY)^{-1}H_Y, \end{aligned}$$

where the the matrices H_X and H_Y they satisfy the following system of equations from Theorem 6.12 and Remark 6.14:

$$\begin{cases} H_X = (I - zC_YF_Y)^{-1} \\ H_Y = (I - zC_XF_X)^{-1} \\ F_X = Q_1\widetilde{\eta}_X(zH_XC_X) \\ F_Y = Q_2\widetilde{\eta}_Y(zH_YC_Y) \end{cases} \tag{7.5}$$

where

$$F_X = \beta_Y^\delta\left(\operatorname{diag}(X,X,Y,Y)(1-zL)^{-1}\right), \qquad F_Y = \beta_X^\delta\left(\operatorname{diag}(X,X,Y,Y)(1-zL)^{-1}\right),$$
$$Q_1 = \operatorname{diag}(1,1,0,0), \qquad Q_2 = \operatorname{diag}(0,0,1,1).$$

Since X and Y are symmetric, all odd cumulants vanish and hence

$$zF_X = \begin{bmatrix} 0 & f_{x,12} & 0 & 0 \\ f_{x,21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad zF_Y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{y,12} \\ 0 & 0 & f_{y,21} & 0 \end{bmatrix}.$$

Substituting F_X and F_Y into the first half of the system (7.5) we obtain

$$H_X = \begin{bmatrix} 1 & 0 & 0 & f_{y,12} \\ 0 & 1 & \frac{f_{y,21}}{1-f_{y,21}} & 0 \\ 0 & 0 & \frac{1}{1-f_{y,21}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H_Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-f_{x,12}} & 0 & 0 \\ 0 & \frac{f_{x,12}}{1-f_{x,12}} & 1 & 0 \\ f_{x,21} & 0 & 0 & 1 \end{bmatrix}. \tag{7.6}$$

By assumption X and Y have symmetric distribution and thus

$$\widetilde{\eta}_X(z) = \sum_{n=1}^{\infty} \beta_{2n}(X) z^{2n-1} \qquad \widetilde{\eta}_Y(z) = \sum_{n=1}^{\infty} \beta_{2n}(Y) z^{2n-1}.$$

We need to understand how odd powers of $H_X C_X$ and and $H_Y C_Y$ behave.

First let us note

$$H_X C_X = \begin{bmatrix} 0 & f_{y,12} & 0 & 0 \\ \frac{1}{1-f_{y,21}} & 0 & 0 & 0 \\ \frac{1}{1-f_{y,21}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad (H_X C_X)^2 = \begin{bmatrix} \frac{f_{y,12}}{1-f_{y,21}} & 0 & 0 & 0 \\ 0 & \frac{f_{y,12}}{1-f_{y,21}} & 0 & 0 \\ 0 & \frac{f_{y,12}}{1-f_{y,21}} & 0 & 0 \\ \frac{1}{1-f_{y,21}} & 0 & 0 & 0 \end{bmatrix}.$$

This immediately implies that

$$(H_X C_X)^{2n-1} = \left(\frac{f_{y,12}}{1 - f_{y,21}} \right)^{n-1} H_X C_X$$

Similarly we obtain

$$(H_Y C_Y)^{2n-1} = \left(\frac{f_{x,21}}{1 - f_{x,12}} \right)^{n-1} H_Y C_Y.$$

This gives

$$\begin{aligned} \widetilde{\eta}_X(zH_X C_X) &= \sum_{n=1}^{\infty} \beta_{2n}(X) \left(zH_X C_X\right)^{2n-1} = \frac{1 - f_{y,21}}{f_{y,12}} H_X C_X \sum_{n=1}^{\infty} \beta_{2n}(X) z^{2n} \left(\frac{f_{y,12}}{1 - f_{y,21}} \right)^n \\ &= \eta_X \left(z\sqrt{\frac{f_{y,12}}{1 - f_{y,21}}} \right) \frac{1 - f_{y,21}}{f_{y,12}} H_X C_X, \\ \widetilde{\eta}_Y(zH_Y C_Y) &= \eta_Y \left(z\sqrt{\frac{f_{x,21}}{1 - f_{x,12}}} \right) \frac{1 - f_{x,12}}{f_{x,21}} H_Y C_Y. \end{aligned}$$

Substituting this into the second half of the system (7.5) results in two matrix equations with exactly two non-zero entries which we can rewrite as

$$\begin{cases} f_{x,12} = (1 - f_{y,21})\, \eta_X \left(z\sqrt{\frac{f_{y,12}}{1 - f_{y,21}}} \right) & f_{y,12} = \frac{1}{f_{x,21}}\, \eta_Y \left(z\sqrt{\frac{f_{x,21}}{1 - f_{x,12}}} \right) \\ f_{x,21} = \frac{1}{f_{y,12}}\, \eta_X \left(z\sqrt{\frac{f_{y,12}}{1 - f_{y,21}}} \right) & f_{y,21} = (1 - f_{x,12})\, \eta_Y \left(z\sqrt{\frac{f_{x,21}}{1 - f_{x,12}}} \right) \end{cases} \tag{7.7}$$

Since

$$\mathbb{E}_X[(1 - z^2(XY + YX))^{-1}] = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} (I - zH_X C_X X)^{-1} H_X \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbb{E}_Y[(1-z^2(XY+YX))^{-1}] = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}(I - zH_YC_YY)^{-1}H_Y \begin{bmatrix}0\\1\\1\\0\end{bmatrix}.$$

A direct calculation with H_X and H_Y as in (7.6) gives

$$\begin{aligned}\mathbb{E}_X[(1-z^2(XY+YX))^{-1}] &= \frac{1}{1-f_{y,21}-f_{y,12}X^2z^2}\\ \mathbb{E}_Y[(1-z^2(XY+YX))^{-1}] &= \frac{1}{1-f_{x,12}-f_{x,21}Y^2z^2}.\end{aligned} \tag{7.8}$$

Next we repeat the previous steps with the commutator, which are very similar. The linearization matrix has the form

$$L = \begin{bmatrix} 0 & 0 & Y & 0\\ iX & 0 & 0 & -iY\\ iX & 0 & 0 & -iY\\ 0 & X & 0 & 0\end{bmatrix} = C_XX + C_YY.$$

Again the system of equations reduces to just four equations which after immediate simplifications boil down to the following:

$$\begin{cases} f_{x,12} = (f_{y,21}-i)\,\eta_X\left(z\sqrt{\dfrac{f_{y,12}}{f_{y,21}-i}}\right) & f_{y,12} = \dfrac{1}{f_{x,21}}\,\eta_Y\left(z\sqrt{\dfrac{f_{x,21}}{i+f_{x,12}}}\right)\\ f_{x,21} = \dfrac{1}{f_{y,12}}\,\eta_X\left(z\sqrt{\dfrac{f_{y,12}}{f_{y,21}-i}}\right) & f_{y,21} = (i+f_{x,12})\,\eta_Y\left(z\sqrt{\dfrac{f_{x,21}}{i+f_{x,12}}}\right)\end{cases} \tag{7.9}$$

And the conditional expectation in this case is equal to

$$\begin{aligned}\mathbb{E}_X[(1-z^2i(XY-YX))^{-1}] &= \frac{1}{1+if_{y,21}-if_{y,12}X^2z^2},\\ \mathbb{E}_Y[(1-z^2i(XY-YX))^{-1}] &= \frac{1}{1-if_{x,12}+if_{x,21}Y^2z^2}.\end{aligned} \tag{7.10}$$

We are now ready to prove the following proposition.

Proposition 7.1. *Assume that X, Y are free and symmetric, then*

$$\begin{aligned}\mathbb{E}_X[(1-z^2i(XY-YX))^{-1}] &= \mathbb{E}_X[(1-z^2(XY+YX))^{-1}]\\ \mathbb{E}_Y[(1-z^2i(XY-YX))^{-1}] &= \mathbb{E}_Y[(1-z^2(XY+YX))^{-1}].\end{aligned}$$

Proof. Let us rewrite (7.8) and (7.10) as

$$\begin{aligned}\mathbb{E}_X[(1-z^2(XY+YX))^{-1}] &= \frac{1}{1-f^{ac}_{y,21}-f^{ac}_{y,12}X^2z^2}\\ \mathbb{E}_X[(1-z^2i(XY-YX))^{-1}] &= \frac{1}{1+if^{c}_{y,21}-if^{c}_{y,12}X^2z^2}\\ \mathbb{E}_Y[(1-z^2(XY+YX))^{-1}] &= \frac{1}{1-f^{ac}_{x,12}-f^{ac}_{x,21}Y^2z^2}\\ \mathbb{E}_Y[(1-z^2i(XY-YX))^{-1}] &= \frac{1}{1-if^{c}_{x,12}+if^{c}_{x,21}Y^2z^2}.\end{aligned}$$

Thus if

$$\begin{cases} f^{ac}_{y,12} = if^{c}_{y,12} & f^{ac}_{y,21} = -if^{c}_{y,21}\\ f^{ac}_{x,12} = if^{c}_{x,12} & f^{ac}_{x,21} = -if^{c}_{x,21}\end{cases} \tag{7.11}$$

then the desired equality holds.

It is straightforward to verify that the substitution of equations (7.11) into the system of equations (7.7) for the anti–commutator results in the system (7.9) for commutator. Let us verify this for the first of four equations

$$f^{ac}_{x,12} = (1 - f^{ac}_{y,21})\,\eta_X\left(z\sqrt{\frac{f^{ac}_{y,12}}{1 - f^{ac}_{y,21}}}\right).$$

Substitution results in

$$if^{c}_{x,12} = (1 + if^{c}_{y,21})\,\eta_X\left(z\sqrt{\frac{if^{c}_{y,12}}{1 + if^{c}_{y,21}}}\right),$$

which is equivalent to

$$f^{c}_{x,12} = (f^{c}_{y,21} - i)\,\eta_X\left(z\sqrt{\frac{f^{c}_{y,12}}{f^{c}_{y,21} - i}}\right)$$

and the above is exactly the first equation in (7.9). □

7.3. **A Lie polynomial.** We consider next the conditional expectation of the Lie polynomial $X + i(XY - YX)$. First we show the general form of the subordination, and next we show that for a special choice of the distributions of X, Y we can make a further step and obtain explicit formulas for the functions present in the conditional expectation. The explicit example is as follows, we take X having semicircular distribution with mean zero and variance one, while Y has distribution $\frac{1}{2}(\delta_{-1} + \delta_1)$. In this explicit case we can use the conditional expectation in order to find the distribution of $X + i(XY - YX)$, we obtain the following corollary.

Corollary 7.2. *For a, b free, a being semicircle element of variance 1 and b being symmetric Bernoulli element, the element $a + i(ab - ba)$ has semicircle distribution with variance 3. Moreover elements a and $i(ab + ba)$ are not free, while $i(ab + ba)$ has semicircle distribution with variance 2.*

Note that ab and ba are free [15, Lemma 3.4], but a and $c = i(ab - ba)$ are not: indeed a quick calculation shows that $\kappa_4(a, a, c, c) = 1$. Yet because of some nontrivial cancellations $\kappa_n(a + c) = 0$ for all $n \neq 2$ and thus $a + c$ has the same distribution as the sum of free copies of a and c. This is another example of a pair of semicircular elements which are not free independent, yet any real linear combination has semicircular distribution [8].

Let us describe the linearization for this example we have $u^t = (1, 0, 0)$, $v^t = (1, 0, 0)$ and $L = C_X X + C_Y Y$ where

$$C_X = \begin{bmatrix} z & 0 & i \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad C_Y = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The complementary kernel projections are

$$Q_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q_Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now the compressed matrices $\widetilde{F}_X = Q_X F_X Q_X$ and $\widetilde{F}_Y = Q_Y F_Y Q_Y$ have the form

$$Q_X F_X Q_X = \begin{bmatrix} fx_{11} & 0 & fx_{13} \\ 0 & 0 & 0 \\ fx_{31} & 0 & fx_{33} \end{bmatrix} \qquad Q_Y F_Y Q_Y = \begin{bmatrix} fy_{11} & fy_{12} & 0 \\ fy_{21} & fy_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the conditional expectations are of the form

$$\mathbb{E}_X\left[(1-z^2P)^{-1}\right] = -\frac{i}{i - fy_{21}z - z^2(i - fy_{11} + fy_{22})X + z^3 fy_{12}X^2},$$
$$\mathbb{E}_Y\left[(1-z^2P)^{-1}\right] = \frac{i}{i - ifx_{11}z^2 + fx_{31}z - z^2\left(fx_{11} - fx_{33} + ifx_{13}z\right)Y - z^3 fx_{13}Y^2}.$$

If we evaluate the above for $X = a$ where a has a semicircle distribution and $Y = b$ where b has Bernoulli distribution then we have that $z\widetilde{\eta}_X(z)^2 - \widetilde{\eta}_X(z) + z = 0$ and $\widetilde{\eta}_Y(z) = z$. Thus we solve the following system of equations

$$\begin{cases} \widetilde{F}_Y - zQ_Y(I - zC_X\widetilde{F}_X)^{-1}C_Y = 0, \\ Q_X(I - zC_Y\widetilde{F}_Y)^{-1}C_X\widetilde{F}_X^2 + zQ_X(I - zC_Y\widetilde{F}_Y)^{-1}C_X - \widetilde{F}_X = 0. \end{cases}$$

Since all formal power series in z considered here are convergent in some neighbourhood of zero, we obtain explicit solutions

$$\mathbb{E}_b\left[(1-z^2P(a,b))^{-1}\right] = \frac{2}{\alpha+1},$$
$$\mathbb{E}_a\left[(1-z^2P(a,b))^{-1}\right] = \frac{3\left(4z^4+1\right)}{z^4\left(3a^2(\alpha-2)+\alpha+14\right) - 24az^6 - 3a\alpha z^2 + \alpha + 2}.$$

where $\alpha = \sqrt{1-12z^4}$.

Since the conditional expectation on b is constant we immediately get the distribution of $a + i(ab - ba)$ as we have

$$\varphi\left((1 - zP(a,b))^{-1}\right) = \frac{2}{\sqrt{1-12z^2}+1}$$

and we can get the moments of $a + i(ab - ba)$ which is A005159 entry at OEIS, we see that $\varphi\left((a + i(ab - ba)^{2n})\right) = 3^n C_n$, where C_n is the n-th Catalan number and Corollary 7.2 follows. We show a histogram of eigenvalues of a matrix approximation together with the graph of the density see Figure 1.

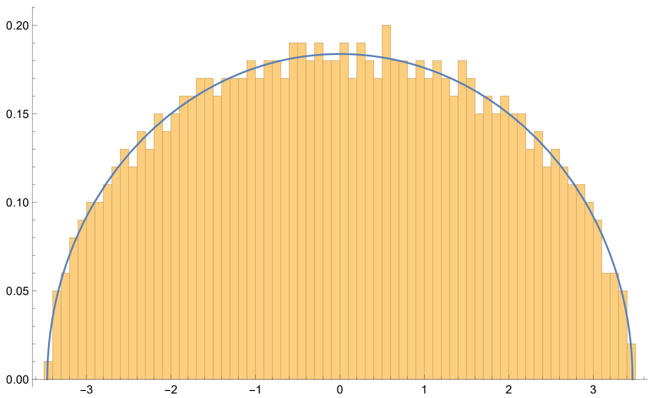

FIGURE 1. Density of $a + i(ab - ba)$ in the case a semicircle and b Bernoulli together with matrix approximation.

7.4. **An example with three variables.** Next we consider the example of product $XZYZX$. The linearization matrix for $(1 - z^5 XZYZX)$ is given by

$$\begin{bmatrix} 0 & X & 0 & 0 & 0 \\ 0 & 0 & Z & 0 & 0 \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & 0 & Z \\ X & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to state an explicit result let us assume that a, b, c are free a, b have distribution $1/2\left(\delta_{-1}+\delta_1\right)$ and c has distribution $1/2\left(\delta_0+\delta_2\right)$. The distribution of $acbca$ is the same as free multiplicative convolution of c and b, however the conditional expectation on a, b is a non-trivial problem. We omit the lengthy calculation wich results in the expression

$$\begin{aligned}&\mathbb{E}_{a,b}\left[(1-z^5acbca)^{-1}\right]\\ &\quad=1-a\left(\left(2z^5\left(8z^{10}+\gamma\right)-\left(b\left(\gamma-4\right)z^5+4z^{10}+2\right)^{-1}\left(-8z^{10}+\gamma\right)\right)^{-1}\right.\\ &\qquad\qquad\left.\left(b\left(\gamma-4\right)z^5+4z^{10}+2\right)\left(\left(-8z^{10}+\gamma\right)a+4az^{10}\right)+a\right)\end{aligned}$$

where $\gamma=\sqrt{1-16z^{10}}+1$ for the conditional expectation.

7.5. **A rational example.** Non–commutative rational functions have linearizations just like polynomials, (see, e.g., [16] for excellent discussion about relations between linearizations and questions in free probability) and consequently our method is not restricted to polynomials but also allows to compute conditional expectations and distributions of non–commutative rational functions in free random variables. Let us illustrate this with the concrete example of a rational function $r(a, b) = a(I-a-b)^{-1}a$ which will show that nevertheless certain technical issues arise. First we must put conditions on the distributions of a and b in order for $I-a-b$ to be invertible. Next we compute the conditional expectation of $(1-zr(a, b))^{-1}$ onto a and b and then integrate the latter in order to determine the distribution of $a(I-a-b)^{-1}a$. Using the method described in Section B we obtain the linearization

$$\left(1-zX(1-X-Y)^{-1}X\right)^{-1}=u^t(I_2-L)^{-1}v$$

where $L=C_XX+C_YY$ with

$$C_X=\begin{bmatrix}0&z\\1&1\end{bmatrix}\quad C_Y=\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

and $u=v=(1,0)^t$.

Let us record some observations here. In order to be able to use iteration from Lemma 6.13 we need to start with correct constant terms $F_X=\beta_1(X)I$ and $F_Y=\beta_1(Y)I$, however there is a catch: The constant term in $F_X(z)=\beta^{\delta}_X(X(1-L)^{-1})$ is not equal to $\beta_1(X)$, as there are more entries which do not depend on z. Therefore in order to be able to distinguish the correct solution, which comes from Lemma 6.13 we introduce an extra parameter $s\in\mathbb{C}$ and we consider $F_X(s,z)=\beta^{\delta}_X(X(1-sL)^{-1})$, Then our iteration gives the unique solution to the system of equations for matrices F_X, F_Y whose entries are power series in s. When we want to evaluate our conditional expectation in $X=a$ and $Y=b$, of some free variables $a, b\in\mathcal{M}$, we have to make sure that for μ being their joint distribution functional $\beta^{\delta}_X\left(X(1-sL)^{-1}\right)$ is analytic at $s=1$. The following rough estimate will ensure this.

Denote by W the set of monomials with coefficient 1 in $\mathbb{C}\langle X, Y\rangle$, i.e., $W=\{X, Y, XY,\ldots\}$. For $w=Z_1\cdots Z_n\in W$ we denote $C_w=C_{Z_1}\cdots C_{Z_n}$. We will also use notation $|w|$ for the number of letters in w and $|w|_X, |w|_Y$ for the number of appearances of X and Y respectively in w, of course $|w|=|w|_X+|w|_Y$. We have

$$\beta^{\delta}_X\left(X(1-sL)^{-1}\right)=\sum_{w\in W}s^{|w|}\beta^{\delta}_X(Xw)C_w.$$

A standard estimate for Boolean cumulant gives $|\beta^{\delta}_X(Xw)|\leq 2^{|w|+1}||a||^{|w|_x+1}||b||^{|w|_y}$. Moreover it is easy to verify that $||C_X||^2\leq 2+|z|^2$ and $||C_Y||=1$. Thus we have

$$|\beta^{\delta}_X\left(X(1-sL)^{-1}\right)|\leq\sum_{w\in W}s^{|w|}|\beta^{\delta}_X(Xw)|\cdot||C_w||$$

$$\leq \sum_{w \in W} s^{|w|} 2^{|w|+1} ||a||^{|w|_x+1} ||b||^{|w|_Y} ||C_X||^{|w|_X+1} \cdot ||C_Y||^{|w|_Y}$$
$$= 2||a|| \left(1 - 2s(||a|| \cdot ||C_X|| + ||b|| \cdot ||C_Y||)\right)^{-1}.$$

The last equality holds whenever $2s(||a|| \cdot ||C_X|| + ||b|| \cdot ||C_Y|| < 1$ and a sufficient condition for this inequality is $2s(||a||\sqrt{2+|z|^2} + ||b||) < 1$. Since we need to evaluate $\beta_X^\delta \left(X(1-sL)^{-1}\right)$ at $s = 1$, our method applies for $||a||, ||b||$ and z such that $2(||a||\sqrt{2+|z|^2} + ||b||) < 1$.

We will apply this to a, b having Bernoulli distributions $\frac{1}{2}(\delta_{-\alpha} + \delta_{-\alpha})$, where in order to have a non-empty range of z we need to assume $\alpha < \frac{\sqrt{2}-1}{2}$. So let us consider the example $\alpha = 1/8$.

Before evaluation in $X = a$ and $Y = b$, taking $s = 1$ we obtain

$$\mathbb{E}_X \left[\left(1 - zX(1-X-Y)^{-1}X\right)^{-1} \right] = \frac{1 - fy_{22} - X}{1 - fy_{22} - X - Xz(fy_{21} + X)}$$

Solving the system of equations for F_X and F_Y, and next evaluating at the resulting expression in $X = a$, $Y = b$ as described above we get

$$\mathbb{E}_a \left[\left(1 - za(1-a-b)^{-1}a\right)^{-1} \right] = \frac{1 - 2a + \alpha^2 z + \sqrt{(\alpha^2 z - 1)^2 - 4\alpha^2}}{1 - 2a(az+1) + \sqrt{\alpha^4 z^2 - 2\alpha^2(z+2) + 1} + \alpha^2 z}.$$

Integrating with respect to the distribution of a we get the moment transform

$$M_{a(1-a-b)^{-1}a}(z) = 1 + \frac{\alpha^2 z}{\sqrt{(\alpha^2 z - 1)^2 - 4\alpha^2}}.$$

Taking $\alpha = 1/8$ we can calculate the density and compare it with the matrix approximation shown in Figure 2.

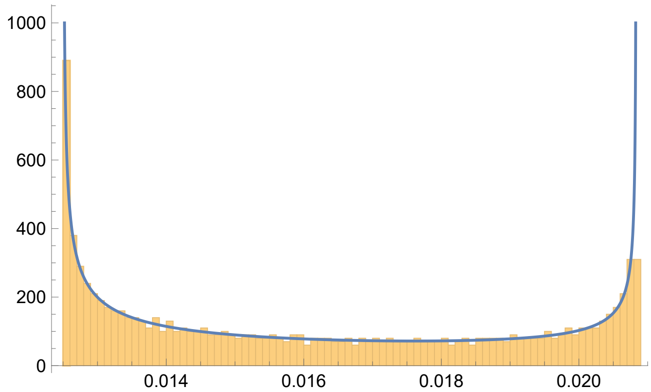

FIGURE 2. Density of $a(1-a-b)^{-1}a$ for a, b with distribution $0.5(\delta_{-1/8} + \delta_{1/8})$. together with matrix approximation.

Appendix A. An algebraic approach to Boolean cumulants

In this first appendix we present purely algebraic proofs of basic facts about Boolean cumulants and implement a formal calculus on the double tensor product.

A.1. **Algebraic proofs of Lemma 2.6 and Corollary 2.7.**

Proof of Lemma 2.6. Induction on n. We compute $\varphi(a_1a_2\cdots a_n)$ in two ways. First consider a_pa_{p+1} as one factor and apply recurrence (2.2), splitting the sum into two at the entry a_pa_{p+1}.

$$\begin{aligned}\text{(A.1)}\quad &\varphi(a_1a_2\cdots(a_pa_{p+1})a_{p+2}\cdots a_n)\\ &= \sum_{k=1}^{p-1}\beta_k(a_1,a_2,\dots,a_k)\,\varphi(a_{k+1}a_{k+2}\cdots a_n)\\ &\quad+\sum_{k=p+1}^{n-1}\beta_{k-1}(a_1,a_2,\dots,a_pa_{p+1},a_{p+2},\dots,a_k)\,\varphi(a_{k+1}a_{k+2}\cdots a_n)\\ &\quad+\beta_{n-1}(a_1,a_2,\dots,a_pa_{p+1},a_{p+2},\dots,a_n)\end{aligned}$$

The left hand side is unchanged if we consider a_p and a_{p+1} as separate factors. We apply recurrence (2.2), (artificially) splitting the sum at the entry a_p.

$$\begin{aligned}\text{(A.2)}\quad &\varphi(a_1a_2\cdots a_pa_{p+1}a_{p+2}\cdots a_n)\\ &= \sum_{k=1}^{p-1}\beta_k(a_1,a_2,\dots,a_k)\,\varphi(a_{k+1}a_{k+2}\cdots a_n)+\beta_p(a_1,a_2,\dots,a_p)\,\varphi(a_{p+1}a_{p+2}\cdots a_n)\\ &\quad+\sum_{k=p+1}^{n-1}\beta_k(a_1,a_2,\dots,a_k)\,\varphi(a_{k+1}a_{k+2}\cdots a_n)+\beta_n(a_1,a_2,\dots,a_n)\end{aligned}$$

The first part is the same in both expressions. By induction hypothesis, the second part of (A.1) can be replaced by

$$\begin{aligned}&\sum_{k=p+1}^{n-1}\beta_{k-1}(a_1,a_2,\dots,a_pa_{p+1},a_{p+2},\dots,a_k)\,\varphi(a_{k+1}a_{k+2}\cdots a_n)\\ &\quad=\sum_{k=p+1}^{n-1}(\beta_p(a_1,a_2,\dots,a_p)\beta_{k-p}(a_{p+1},a_{p+2},\dots,a_k)+\beta_k(a_1,a_2,\dots,a_k))\,\varphi(a_{k+1}a_{k+2}\cdots a_n).\end{aligned}$$

On the other hand, applying recurrence (2.2) to the second part of (A.2) we obtain

$$\begin{aligned}&\beta_p(a_1,a_2,\dots,a_p)\,\varphi(a_{p+1}a_{p+2}\cdots a_n)\\ &\quad=\beta_p(a_1,a_2,\dots,a_p)\sum_{k=p+1}^{n-1}\beta_{k-p}(a_{p+1},a_{p+2},\dots,a_k)\,\varphi(a_{k+1}a_{k+2}\cdots a_n)\\ &\qquad+\beta_p(a_1,a_2,\dots,a_p)\beta_{n-p}(a_{p+1},a_{p+2},\dots,a_n)\end{aligned}$$

Finally canceling equal terms from (A.1) and (A.2) we arrive at (2.8). □

Proof of Corollary 2.7. By induction on the number r of multiplication signs. For $r=0$ there is nothing to prove; for $r=1$ this is Lemma 2.6.

Assume that the formula holds up to $r-1$ multiplication signs and consider a Boolean cumulant of the form (2.9) with r multiplication signs, i.e., $r=n-m-1$. Now apply (2.8) to remove the first multiplication

$$\begin{aligned}&\beta_{m+1}(a_1a_2\cdots a_{d_1},a_{d_1+1}a_{d_1+2}\cdots a_{d_2},\dots,a_{d_m+1}a_{d_m+2}\cdots a_n)\\ &\quad=\beta_1(a_1)\beta_{m+1}(a_2\cdots a_{d_1},a_{d_1+1}a_{d_1+2}\cdots a_{d_2},\dots,a_{d_m+1}a_{d_m+2}\cdots a_n)\end{aligned}$$

$$+ \beta_{m+2}(a_1, a_2 \cdots a_{d_1}, a_{d_1+1}a_{d_1+2} \cdots a_{d_2}, \dots, a_{d_m+1}a_{d_m+2} \cdots a_n).$$

In both terms there are now $r-1$ multiplications and the induction hypothesis can be applied to obtain (2.9). □

A.2. **Boolean cumulants via tensor algebras.** For a vector space V denote by $\mathcal{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ its *tensor algebra* and $\overline{\mathcal{T}}(V) = \bigoplus_{n>0} V^{\otimes n}$ its augmentation ideal, i.e., $\overline{\mathcal{T}}(V) = \ker \epsilon$, where ϵ is the projection onto $\mathbb{C} = V^{\otimes 0}$ called the *counit*. In order to distinguish different levels in the hierarchy of tensors we will denote the multiplication in $\mathcal{T}(V)$ by the symbol $\circledcirc$ and thus elementary tensors are written $a_1 \circledcirc a_2 \circledcirc \cdots \circledcirc a_n$. The "outer" tensor on the double tensor algebra $\mathcal{T}(\overline{\mathcal{T}}(V))$ is denoted by the usual symbol $\otimes$, i.e., any element of $\mathcal{T}(\overline{\mathcal{T}}(V))$ is a sum of simple tensors of the form

$$(a_1 \circledcirc a_2 \circledcirc \cdots \circledcirc a_k) \otimes (b_1 \circledcirc b_2 \circledcirc \cdots \circledcirc b_l) \otimes \cdots \otimes (c_1 \circledcirc c_2 \circledcirc \cdots \circledcirc c_m).$$

The tensor algebra has the following fundamental extension property [26, Prop. 1.1.2]:

Lemma A.1. *Let V be a vector space and M a $\mathcal{T}(V)$-bimodule. Then any linear map $f : V \to M$ can be extended to a derivation $\mathcal{T}(f) : \mathcal{T}(V) \to M$ by setting*

$$\mathcal{T}(f)(v_1 \circledcirc v_2 \circledcirc \cdots \circledcirc v_n) = \sum_{k=1}^{n} v_1 \circledcirc v_2 \circledcirc \cdots \circledcirc v_{k-1} \circledcirc f(v_k) \circledcirc v_{k+1} \circledcirc v_{k+2} \circledcirc \cdots \circledcirc v_n$$

In the following the underlying vector space will always be an algebra $V = \mathcal{A}$, whose multiplication will be denoted as usual by ab or $a \cdot b$.

For example, let $\mathcal{A}$ be the free algebra and $\bar{\Delta}_{\mathrm{d}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \subseteq \overline{\mathcal{T}}(\mathcal{A}) \otimes \overline{\mathcal{T}}(\mathcal{A})$ the reduced deconcatenation coproduct

$$\bar{\Delta}_{\mathrm{d}}(a_1 a_2 \cdots a_n) = \sum_{k=1}^{n-1} a_1 a_2 \cdots a_k \otimes a_{k+1} \cdots a_n;$$

let further

$$\tilde{\Delta}_{\mathrm{d}}(a_1 a_2 \cdots a_n) = a_1 \circledcirc a_2 \circledcirc \cdots \circledcirc a_n \in \mathcal{T}(\mathcal{A})$$

the full deconcatenation, i.e., $\tilde{\Delta}_{\mathrm{d}}(w_1 w_2) = \tilde{\Delta}_{\mathrm{d}}(w_1 \circledcirc w_2)$. Then the derivation from Lemma A.1 is

$$\mathcal{T}(\bar{\Delta}_{\mathrm{d}})(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \sum_{k=1}^{n-1} ((w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_{k-1}) \otimes 1) \circledcirc \bar{\Delta}_{\mathrm{d}}(w_k) \circledcirc (1 \otimes (w_{k+1} \circledcirc \cdots \circledcirc w_n))$$

We will denote this derivation by $\bar{\Delta}_{\mathrm{d}}$ as well.

The moment and cumulant functionals are defined on $\overline{\mathcal{T}}(\mathcal{A})$ via

$$\varphi(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \varphi(w_1 w_2 \cdots w_n)$$
$$\beta(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \beta_n(w_1, w_2, \dots, w_n)$$

and the recurrence (2.2) can be reformulated with the second level deconcatenation operator $\bar{\Delta}_{\mathrm{d}}^{\circledcirc} : \overline{\mathcal{T}}(A) \to \overline{\mathcal{T}}(A) \otimes \overline{\mathcal{T}}(A)$ defined by

$$\bar{\Delta}_{\mathrm{d}}^{\circledcirc}(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \sum_{k=1}^{n-1} w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_k \otimes w_{k+1} \circledcirc w_{k+2} \circledcirc \cdots \circledcirc w_n$$

as follows:

$$\varphi(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \beta(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) + (\beta \otimes \varphi)\, \bar{\Delta}_{\mathrm{d}}^{\circledcirc}(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n)$$

On the other hand, if we define the full Boolean cumulant $\beta^{\delta}(w) = \beta(\tilde{\Delta}_{\mathrm{d}}(w))$, i.e.

$$\beta^{\delta}(a_1 a_2 \cdots a_n) = \beta(a_1 \circledcirc a_2 \circledcirc \cdots \circledcirc a_n) = \beta_n(a_1, a_2, \dots, a_n)$$

and extend it to $\overline{\mathcal{T}}(\mathcal{A})$ via

$$\beta^{\delta}(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \beta^{\delta}(w_1 w_2 \cdots w_n)$$

$$= \beta(\tilde{\Delta}_{\mathrm{d}}(w_1) \circledcirc \tilde{\Delta}_{\mathrm{d}}(w_2) \circledcirc \cdots \circledcirc \tilde{\Delta}_{\mathrm{d}}(w_n))$$

then the full Boolean cumulants satisfy the corresponding recurrence

$$\varphi(w) = \beta^\delta(w) + (\beta^\delta \otimes \varphi)\, \bar{\Delta}_{\mathrm{d}}(w)$$

which we can generalize to $\overline{\mathcal{T}}(\mathcal{A})$. In order to formulate it, we introduce some further notation.

Definition A.2. An interval partition is a partition whose blocks are intervals. The interval partitions $\mathrm{Int}(\{1,2,\dots,n\})$ are thus in bijection with compositions $\mathcal{C}(n) = \{(k_1,k_2,\dots,k_m) \mid k_1+k_2+\cdots k_m = n\}$. The bijection maps an interval partition $\pi \in \mathrm{Int}(n)$ to the sequence of block lengths (in their natural order). We denote this bijection by $C : \mathrm{Int}(n) \to \mathcal{C}(n)$. A composition $\lambda = (k_1,k_2,\dots,k_m) \in \mathcal{C}(n)$ is uniquely determined by its partial sums, called *descents*

$$\mathcal{D}(\lambda) = \{k_1, k_1+k_2, \dots, k_1+k_2+\cdots+k_{m-1}\} \subseteq \{1,2,\dots,n-1\}$$

and thus the compositions of order n are in bijection with the Boolean lattice of order $n-1$, The descents of an interval partition are the descents of the corresponding composition $\mathcal{D}(\pi) = \mathcal{D}(C(\pi))$. This bijection is a poset anti-isomorphism and $\pi \leq \rho$ if and only if $\mathcal{D}(\rho) \supseteq \mathcal{D}(\pi)$. In particular, $\mathcal{D}(\pi \vee \rho) = \mathcal{D}(\pi) \cap \mathcal{D}(\rho)$.

Proposition A.3.

$$\beta(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = \beta^\delta(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) + (\beta^\delta \otimes \beta) \circ \bar{\Delta}_{\mathrm{d}}(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n)$$

Proof. Let $k_i = \ell(w_i)$ be the lengths of the words and $\rho = \{I_1, I_2, \dots, I_m\} \in \mathrm{Int}(n)$ the induced interval partition, i.e., $\left|I_j\right| = k_j$ and $C(\rho) = (k_1,k_2,\dots,k_m)$ with descent set $\mathcal{D}(\rho) = \{r_1, r_2, \dots, r_{m-1}\}$ where $r_0 = d_0(\rho) = 0$, $r_1 = d_1(\rho) = k_1$, $r_2 = d_2(\rho) = k_1+k_2$, etc. Then if $w_i = a_{r_{i-1}+1} a_{r_{i-1}+2} \cdots a_{r_i}$ the product formula (2.7) can be rewritten in terms of descents as

$$\begin{aligned}
\beta_m(a_1 a_2 \cdots a_{r_1}, a_{r_1+1} \cdots a_{r_2}, \dots, a_{r_{m-1}+1} \cdots a_n) \\
&= \sum_{\substack{\pi \in \mathrm{Int}(n) \\ \mathcal{D}(\pi) \cap \mathcal{D}(\rho) = \emptyset}} \beta_\pi(a_1, \dots, a_n) \\
&= \beta_n(a_1, a_2, \dots, a_n) + \sum_{s=1}^{m} \sum_{\substack{\pi \in \mathrm{Int}(n) \setminus \{1_n\} \\ \mathcal{D}(\pi) \cap \mathcal{D}(\rho) = \emptyset \\ d_1(\pi) \in I_s}} \beta_\pi(a_1, \dots, a_n)
\end{aligned}$$

where we split off the term corresponding to $\pi = 1_n$ (i.e., $\mathcal{D}(\pi) = \emptyset$) and regroup the remaining terms according to the location of the first descent $d_1(\pi)$. Fix $1 \leq s \leq m$ and assume that $d = d_1(\pi) \in I_s$. Then the first block of π is $B_1 = \{1, 2, \dots, d_1\}$ and $\pi = \{B_1\} \cup \pi'$ with $\pi' = \pi|_{[d+1,n]} \in \mathrm{Int}[d+1,n]$ and $\mathcal{D}(\pi') \cap \mathcal{D}(\rho') = \emptyset$ where $\rho' = \rho|_{[d+1,n]}$. Now by assumption $r_{s-1} < d < r_s$ and thus

$$\begin{aligned}
\sum_{\substack{\pi \in \mathrm{Int}(n) \setminus \{1_n\} \\ \mathcal{D}(\pi) \cap \mathcal{D}(\rho) = \emptyset \\ d_1(\pi) \in I_s}} \beta_\pi(a_1, \dots, a_n) \\
&= \sum_{d=r_{s-1}+1}^{r_s-1} \beta_d(a_1, a_2, \dots, a_d) \sum_{\substack{\pi' \in \mathrm{Int}([d+1,n]) \\ \mathcal{D}(\pi') \cap \mathcal{D}(\rho') = \emptyset}} \beta_{\pi'}(a_{d+1}, a_{d+2}, \dots, a_n) \\
&= \sum_{d=r_{s-1}+1}^{r_s-1} \beta_d(a_1, a_2, \dots, a_d)\, \beta_{m-s+1}(a_{d+1} a_{d+2} \cdots a_{r_s}, w_{s+1}, w_{s+2}, \dots, w_n) \\
&= (\beta^\delta \otimes \beta)((w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_{s-1} \otimes 1)(\bar{\Delta}_{\mathrm{d}}\, w_s)(1 \otimes (w_{s+1} \circledcirc w_{s+2} \circledcirc \cdots \circledcirc w_m)))
\end{aligned}$$

□

Remark A.4. Although freeness implies property (CAC) (Lemma 3.5) and thus $\beta^\delta(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n) = 0$ if w_1 begins with X and w_n ends in Y, it is important to keep in mind that this is not necessarily true for $\beta(w_1 \circledcirc w_2 \circledcirc \cdots \circledcirc w_n)$.

Example A.5 (Cumulants of products). Let us apply Proposition A.3 to

$$\Psi^\circledcirc = \sum_{n=0}^\infty (XY)^{\circledcirc n} z^{2n}$$

to compute the cumulant generating function

$$B(z) = \beta(\Psi^\circledcirc) = \sum_{n=0}^\infty \beta_n(XY) z^{2n}.$$

From this we can then obtain the moment generating function

$$M(z) = \frac{1}{1 - B(z)}.$$

Applying the Leibniz rule to the identity $\Psi^\circledcirc \circledcirc (1 - z^2 XY) = 1$ we obtain (2.17)

$$\bar{\Delta}_{\mathrm{d}}(\Psi^\circledcirc) = (\Psi^\circledcirc \circledcirc X) \otimes (Y \circledcirc \Psi^\circledcirc) z^2.$$

Now by Proposition A.3 we have

$$\beta(\Psi^\circledcirc) = \beta^\delta(\Psi^\circledcirc) + \beta^\delta(\Psi^\circledcirc \circledcirc X)\beta(Y \circledcirc \Psi^\circledcirc) z^2$$

and similarly

$$\begin{aligned} \bar{\Delta}_{\mathrm{d}}(Y \circledcirc \Psi^\circledcirc) &= (Y \otimes 1) \circledcirc \bar{\Delta}_{\mathrm{d}}(\Psi^\circledcirc) \\ &= (Y \circledcirc \Psi^\circledcirc \circledcirc X) \otimes (Y \circledcirc \Psi^\circledcirc) z^2 \\ \beta(Y \circledcirc \Psi^\circledcirc) &= \beta^\delta(Y \circledcirc \Psi^\circledcirc) + \beta^\delta(Y \circledcirc \Psi^\circledcirc \circledcirc X)\beta(Y \circledcirc \Psi^\circledcirc) z^2. \end{aligned}$$

Consequently,

$$\begin{aligned} \beta(Y \circledcirc \Psi^\circledcirc) &= \frac{\beta^\delta(Y \circledcirc \Psi^\circledcirc)}{1 - \beta^\delta(y \circledcirc \Psi^\circledcirc \circledcirc X) z^2} \\ \beta(\Psi^\circledcirc) &= \beta^\delta(\Psi^\circledcirc) + \frac{\beta^\delta(\Psi^\circledcirc \circledcirc X)\, \beta^\delta(Y \circledcirc \Psi^\circledcirc) z^2}{1 - \beta^\delta(Y \circledcirc \Psi^\circledcirc \circledcirc X) z^2} \end{aligned}$$

where

$$\begin{aligned} \beta^\delta(\Psi^\circledcirc \circledcirc X) &= \sum_{n=0}^\infty \beta_{2n+1}(X, Y, X, Y, \dots, X) z^{2n} \\ \beta^\delta(Y \circledcirc \Psi^\circledcirc) &= \sum_{n=0}^\infty \beta_{2n+1}(Y, X, Y, X, \dots, Y) z^{2n} \end{aligned}$$

If we assume x and y free then we infer from the resolvent identity

$$\Psi = 1 + z^2 XY\Psi$$

that $\beta^\delta(\Psi^\circledcirc) = 1$ and $\beta^\delta(Y \circledcirc \Psi^\circledcirc \circledcirc X) = 0$ and thus

$$\beta(\Psi^\circledcirc) - 1 = \beta^\delta(\Psi^\circledcirc \circledcirc X)\, \beta^\delta(Y \circledcirc \Psi^\circledcirc)$$

Thus

$$B(z) = \beta(\Psi^\circledcirc) = 1 + z\beta^\delta_X(\Psi X)\beta^\delta_Y(Y\Psi)$$

and finally

$$zB(z) = z + \omega_1(z)\, \omega_2(z)$$

which reproduces [3, Theorem 3.2 (3)] and [11, (10)].

Appendix B. Rational series and linearizations

The non–commutative free field and in particular non–commutative rational functions have seen many applications in free probability recently [16]. The main tool for their study are linearizations. For technical reasons we restrict our study to regular rational functions, i.e., rational functions which can be represented as formal power series. These form a subalgebra of $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ and can be characterized in several ways [6].

Definition B.1. For a series $S \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ we denote the extraction of coefficients by $\langle S, w\rangle$, i.e.,

$$S = \sum_{w\in\mathcal{X}^*} \langle S, w\rangle\, w.$$

A series $S \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ is called *proper* if it has no constant term, i.e., $\langle S, 1\rangle = 0$. Proper series have a *quasi-inverse*

$$S^+ = \sum_{k=1}^{\infty} S^k \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle.$$

$Q = S^+$ is the unique solution of the equation

$$Q = S + QS.$$

The algebra of rational series is the smallest subalgebra of $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ which contains all quasi-inverses of its proper elements. This implies that for any element S without constant term the element $1 - S$ is invertible and its inverse is given by the geometric series

$$(1 - S)^{-1} = \sum_{k=0}^{\infty} S^k \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$$

and therefore rational.

A series $S \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ is called *recognizable* if there exists a matrix representation of the free monoid, i.e., a multiplicative map $\rho : \mathcal{X}^* \to M_n(\mathbb{C})$, and vectors $u, v \in \mathbb{C}^n$ such that the coefficients of S are given by

$$\langle S, w\rangle = u^t \rho(w) v.$$

Here the representation ρ is uniquely determined by the matrices $M_x = \rho(x)$ for $x \in \mathcal{X}$. Indeed, $\rho(x_1x_2\cdots x_k) = M_{x_1}M_{x_2}\cdots M_{x_k}$ and thus a recognizable series has a *linearization*

$$S = u^t\Big(\sum_{w\in\mathcal{X}^*} \rho(w)\, w\Big)v = u^t\Big(1 - \sum_{x\in\mathcal{X}} M_x x\Big)^{-1} v.$$

Conversely, any series which is given by a linearization is recognizable.

Definition B.2. For a letter $x \in \mathcal{X}$ denote by L_x the left annihilation operator on $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$, i.e., for a word $w \in \mathcal{X}^*$ set

$$L_x w = \begin{cases} w' & \text{if } w = xw' \\ 0 & \text{otherwise} \end{cases}$$

and extend this operator linearly to $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$. For a word $w = x_1x_2\cdots x_n \in \mathcal{X}^*$ we denote by $L_w = L_{x_n}L_{x_{n-1}}\cdots L_{x_1}$ the composition of the annihilation operators.

A subspace $U \subseteq \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ is called *stable* if it is invariant under L_x for all $x \in \mathcal{X}$.

Remark B.3. With this notation the coefficients of a series are given by

$$\langle S, w\rangle = \langle L_w S, 1\rangle \tag{B.1}$$

One of the main results of the theory of rational series is the following

Theorem B.4 ([6])**.** *For a series $S \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ the following are equivalent,*

(i) S is rational.
(ii) S is recognizable.
(iii) S has a linearization.

(iv) There is a finite dimensional stable subspace of $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ containing S.

The last statement gives rise to an algorithm key to compute linearizations of rational series.

Algorithm B.5. Assume our series S is contained in a finite dimensional stable subspace spanned by a basis $(S_1, S_2, \ldots, S_N) \subseteq \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$, say $S = \sum u_i S_i$.
1. Since the subspace is stable, for any $X \in \mathcal{X}$ we can express
$$L_x S_i = \sum \alpha_{ij}(x) S_j.$$
2. Collect these coefficients in a matrix $A_x = \left[\alpha_{ij}(x)\right]$.
3. Then for a word $w \in \mathcal{X}^*$ after iteration we obtain
$$L_w S_i = \sum_j (M_w)_{ij} S_j.$$
4. From (B.1) we infer $\langle S_i, w\rangle = \langle L_w S, 1\rangle = (M_w v)_i$.
5. Finally $\langle S, w\rangle = u^t M_w v$ for every $w \in \mathcal{X}^*$, i.e.,
$$S = u^t \Big(I_N - \sum_{x\in\mathcal{X}} M_x x\Big)^{-1} v.$$

To illustrate this let us start with polynomials which are clearly rational. Let $w \in \mathcal{X}^*$ be a word. A word $w'' \in \mathcal{X}^*$ is a *right suffix* of w if there exists a word $w' \in \mathcal{X}^*$ such that $w = w'w''$. Equivalently, w'' can be obtained from w by applying the left annihilation operator $L_{w'}$.

Let now $P \in \mathbb{C}\langle\mathcal{X}\rangle$ be a polynomial. If we successively apply annihilation operators L_X, $X \in \mathcal{X}$ we obtain elements contained in the span of all right suffixes of the monomials occurring in P (including the empty word 1). Clearly the span of P and all its right suffixes is stable and thus can be used to obtain a linearization for P.

From this basis we can immediately construct a basis of a stable subspace containing $\Psi = (1 - z^m P)^{-1}$ and construct a linearization of the latter, where $m = \deg(P)$, i.e., the length of the longest monomial in P. Indeed let $\{S_1 = P, S_2, \ldots, S_r = 1\}$ be a basis consisting of P and the right suffixes of P spanning a stable invariant subspace for P. Now we use the resolvent identity
$$\Psi = (1 - z^m P)^{-1} = 1 + z^m P\Psi$$
to obtain $L_x\Psi = z^m (L_x P)\Psi$ and thus $\{\Psi, z^{m_2} S_2\Psi, \ldots, z^{m_{r-1}} S_{r-1}\Psi\}$ spans a stable subspace containing Ψ, where $m_i = \deg S_i$. Moreover $\langle S_i\Psi, 1\rangle = 0$ for $i = 2, \ldots, r-1$ and thus we can set $u = v = e_1$. The inclusion of the factor z^{m_i} ensures that our linearization has the form $\Psi = u^t(I - zL)^{-1}v$, where the entries of L are polynomials in z and thus has a formal power series expansion around zero, which is essential in the Section 6.

Example B.6. Consider $P(X, Y) = X + XYX$, then we are looking for a linearization of $\Psi = (1 - z^3 P)^{-1}$. We have $\Psi = 1 + z^3 P\Psi$ and thus we have
$$S_1 = \Psi, L_X\Psi = z^3\Psi + z^3 YX\Psi = z^3 S_1 + zS_2,$$
where $S_2 = z^2 YX\Psi$. Of course $L_Y S_1 = 0$. Next $L_X S_2 = 0$ and $L_Y S_2 = z^2 X\Psi = zS_3$. Finally $L_X S_3 = z\Psi = zS_1$ and $L_Y S_3 = 0$. Hence we obtain $\Psi = u^t(1 - zL)^{-1}v$ with
$$zL = \begin{bmatrix} z^3 & z & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} Y.$$

Remark B.7. Another way to obtain a linearization of $\Psi = (1 - z^m P)^{-1}$ for a polynomial P is to directly follow the multiplication structure. Let us describe this with an example, occasionally we will use this linearization instead of the one described above.

Consider the non–commutative polynomial $XYX + YXY$ and the corresponding polynomial with numbered variables$X_1Y_1X_2 + Y_2X_3Y_3$. When we look at powers of this polynomial, we see that:

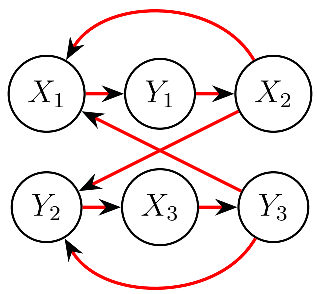

Figure 3. Multiplication for $X_1Y_1X_2 + Y_2X_3Y_3$.

- X_1 is always followed by Y_1,
- X_2 is followed by X_1 or Y_2,
- Y_1 is followed by X_1 or Y_2,
- Y_2 is always followed by X_2.

This can be represented graphically as an automaton, see Figure 3.

Let us name all variables in the considered polynomial $Z_1, Z_2, \ldots, Z_n$, where n is the total number of variables in the polynomial. The polynomial $X_1Y_1X_2 + Y_2X_3Y_3$ we have $Z_1Z_2Z_3 + Z_4Z_5Z_6$, we will consider a square matrix in which each row and column corresponds to one of the variables, more precisely we set $L_{i,j} = Z_j$ if Z_j follows Z_i, otherwise we set $L_{i,j} = 0$. We obtain the following linearization matrix (we show the labels of rows and columns)

$$L = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ Y_1 \\ Y_2 \\ Y_3 \end{array} \begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & Y_1 & Y_2 & Y_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & Y_1 & 0 & 0 \\ X_1 & 0 & 0 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_3 \\ 0 & X_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_3 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & Y_2 & 0 \end{bmatrix} \end{array}$$

In order to get a linearization of $(1 - z^3(X_1Y_1X_2 + Y_2X_3Y_3))^{-1}$ we look at $(1 - zL)^{-1}$, it is clear that every term in expansion of $(1 - z^3(X_1Y_1X_2 + Y_2X_3Y_3))^{-1}$ starts with X_1 or Y_2, hence $u^t = (0, 1, 0, 0, 0, 0)$ or equivalently $u^t = (0, 0, 0, 0, 0, 1)$. On the other hand each term in $(1 - z^3(X_1Y_1X_2 + Y_2X_3Y_3)^{-1}$ ends in X_2 or Y_3, hence we have to terminate with the second or sixth column of L, thus $v^t = (0, 1, 0, 0, 0, 1)$. Clearly we have the equality of formal power series

$$(1 - z^3(X_1Y_1X_2 + Y_2X_3Y_3))^{-1} = u^t(1 - zL)^{-1}v.$$

References

[1] D. Avitzour. Free products of C^*-algebras. *Trans. Amer. Math. Soc.*, 271(2):423–435, 1982.
[2] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev. *Rings with generalized identities*, volume 196 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1996.
[3] S. T. Belinschi and H. Bercovici. A new approach to subordination results in free probability. *J. Anal. Math.*, 101:357–365, 2007.
[4] S. T. Belinschi, T. Mai, and R. Speicher. Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem. *J. Reine Angew. Math.*, 732:21–53, 2017.
[5] S. T. Belinschi and A. Nica. η-series and a Boolean Bercovici-Pata bijection for bounded k-tuples. *Adv. Math.*, 217(1):1–41, 2008.
[6] J. Berstel and C. Reutenauer. *Noncommutative rational series with applications*, volume 137 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2011.
[7] P. Biane. Processes with free increments. *Math. Z.*, 227(1):143–174, 1998.
[8] A. Bose, A. Dey, and W. Ejsmont. Characterization of non-commutative free Gaussian variables. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15(2):1241–1255, 2018.

[9] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Hermann, Paris, 1970.
[10] G. Cébron. Free convolution operators and free Hall transform. *J. Funct. Anal.*, 265(11):2645–2708, 2013.
[11] G. P. Chistyakov and F. Götze. The arithmetic of distributions in free probability theory. *Cent. Eur. J. Math.*, 9(5):997–1050, 2011.
[12] P. Doubilet, G.-C. Rota, and R. Stanley. On the foundations of combinatorial theory. VI. The idea of generating function. pages 267–318, 1972.
[13] K. Ebrahimi-Fard and F. Patras. Cumulants, free cumulants and half-shuffles. *Proc. A.*, 471(2176):20140843, 18, 2015.
[14] M. Fevrier, M. Mastnak, A. Nica, and K. Szpojankowski. Using Boolean cumulants to study multiplication and anticommutators of free random variables. *Trans. Amer. Math. Soc.*, 373:7167–7205, 2020.
[15] U. Haagerup and F. Larsen. Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras. *J. Funct. Anal.*, 176(2):331–367, 2000.
[16] J. W. Helton, T. Mai, and R. Speicher. Applications of realizations (aka linearizations) to free probability. *J. Funct. Anal.*, 274(1):1–79, 2018.
[17] T. W. Hungerford. The free product of algebras. *Illinois J. Math.*, 12:312–324, 1968.
[18] D. Jekel and W. Liu. An operad of non-commutative independences defined by trees, 2020.
[19] S. A. Joni and G.-C. Rota. Coalgebras and bialgebras in combinatorics. In *Umbral calculus and Hopf algebras (Norman, Okla., 1978),*, Contemp. Math., 6,, pages 1–47,. ,, 1982.
[20] B. Krawczyk and R. Speicher. Combinatorics of free cumulants. *J. Combin. Theory Ser. A*, 90:267–292, 2000.
[21] F. Lehner. Free cumulants and enumeration of connected partitions. *European J. Combin.*, 23(8):1025–1031, 2002.
[22] F. Lehner. Cumulants in noncommutative probability theory. I. Noncommutative exchangeability systems. *Math. Z.*, 248(1):67–100, 2004. arXiv:math/0210441.
[23] F. Lehner and K. Szpojankowski. Boolean cumulants and subordination in free probability. *Random Matrices Theory Appl.*, 10(4):Paper No. 2150036, 46, 2021.
[24] V. P. Leonov and A. N. Shiryaev. On a method of calculation of semi-invariants. *Theor. Prob. Appl.*, 4:319–328, 1959.
[25] J.-L. Loday and M. Ronco. On the structure of cofree Hopf algebras. *J. Reine Angew. Math.*, 592:123–155, 2006.
[26] J.-L. Loday and B. Vallette. *Algebraic operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.
[27] T. Mai, R. Speicher, and S. Yin. The free field: zero divisors, atiyah property and realizations via unbounded operators, 2018.
[28] J. A. Mingo and R. Speicher. *Free probability and random matrices*, volume 35 of *Fields Institute Monographs*. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017.
[29] N. Muraki. The five independences as quasi-universal products. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(1):113–134, 2002.
[30] A. Nica and R. Speicher. Commutators of free random variables. *Duke Math. J.*, 92(3):553–592, 1998.
[31] A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.
[32] D. Perales. On the anti-commutator of two free random variables, 2021. arXiv:2101.09444.
[33] M. Popa. A new proof for the multiplicative property of the Boolean cumulants with applications to the operator-valued case. *Colloq. Math.*, 117(1):81–93, 2009.
[34] T. P. Speed. Cumulants and partition lattices. *Austral. J. Statist.*, 25:378–388, 1983.
[35] R. Speicher. Multiplicative functions on the lattice of noncrossing partitions and free convolution. *Math. Ann.*, 298(4):611–628, 1994.
[36] R. Speicher and R. Woroudi. Boolean convolution. In *Free probability theory (Waterloo, ON, 1995)*, volume 12 of *Fields Inst. Commun.*, pages 267–279. Amer. Math. Soc., Providence, RI, 1997.
[37] K. Szpojankowski and J. Wesołowski. Dual Lukacs regressions for non-commutative variables. *J. Funct. Anal.*, 266(1):36–54, 2014.
[38] K. Szpojankowski and J. Wesołowski. Conditional expectations through Boolean cumulants and subordination – towards a better understanding of the lukacs property in free probability. *ALEA, Lat. Am. J. Probab. Math. Stat*, 17:253–272, 2020.
[39] M. Takesaki. *Theory of operator algebras I.* Springer-Verlag, New York-Heidelberg,,, 1979.
[40] R. H. Terwiel. Projection operator method applied to stochastic linear differential equations. *Physica*, 74:248–265, 1974.
[41] V. Vasilchuk. On the asymptotic distribution of the commutator and anticommutator of random matrices. *J. Math. Phys.*, 44(4):1882–1908, 2003.

[42] D. Voiculescu. Symmetries of some reduced free product C^*-algebras. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 556–588. Springer, Berlin, 1985.

[43] D. Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. I. *Comm. Math. Phys.*, 155(1):71–92, 1993.

[44] D. Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. V. Noncommutative Hilbert transforms. *Invent. Math.*, 132(1):189–227, 1998.

[45] D. Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. VI. Liberation and mutual free information. *Adv. Math.*, 146(2):101–166, 1999.

[46] D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992.

[47] W. von Waldenfels. An approach to the theory of pressure broadening of spectral lines. In *Probability and information theory, II*, pages 19–69. Lecture Notes in Math., Vol. 296. 1973.

[48] W. Woess. Nearest neighbour random walks on free products of discrete groups. *Boll. Un. Mat. Ital. B (6)*, 5(3):961–982, 1986.

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