## ON MATRICES IN FINITE FREE POSITION

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ABSTRACT. We study pairs (A, B) of square matrices that are in additive (resp. multiplicative) finite free position, that is, the characteristic polynomial  $\chi_{A+B}(x)$  (resp.  $\chi_{AB}(x)$ ) equals the additive finite free convolution  $\chi_A(x) \boxplus \chi_B(x)$  (resp. the multiplicative finite free convolution  $\chi_A(x) \boxtimes \chi_B(x)$ ), which equals the expected characteristic polynomial  $\mathbb{E}_U [\chi_{A+U^*BU}(x)]$  (resp.  $\mathbb{E}_U [\chi_{AU^*BU}(x)]$ ) over the set of unitary matrices U. We examine the lattice of algebraic varieties of matrices consisting of finite free complementary pairs with respect to the additive (resp. multiplicative) convolution. We show that these pairs include the diagonal matrices vs. the principally balanced matrices, the upper (lower) triangular matrices with a single eigenvalue, and the scalar matrices vs. the set of all square matrices.

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### 1. INTRODUCTION

Finite free probability was introduced by Marcus, Spielman and Srivastava [10] and Marcus [9] in connection with the solution of the Kadison-Singer conjecture. They introduced additive and multiplicative convolutions of characteristic polynomials of matrices over the real and complex numbers. In fact, these convolutions were already studied in the 1920s by Walsh [18] and Szegő [16], who proved among other results that under certain conditions they preserve real rootedness. In [10] explicit formulas for the convolutions were introduced, showing that they express the expectations of the characteristic polynomials of unitarily (orthogonally) invariant random matrices, thus linking them to free probability. That is, when A and B are normal (symmetric) matrices then the additive convolution of their characteristic polynomials equals the expectation of the characteristic polynomials of  $A + U^*BU$ , where the expectation is over randomly distributed unitary (orthogonal) matrices U. A similar result holds for normal matrices with respect to multiplicative convolution.

The analogues of these convolutions in the setting of tropical algebra were studied in [12].

In [9], Marcus developed further the findings of [10] to reach a new theory of "finite free probability", a finite version of Free Probability that was introduced by Voiculescu [17], with many analogous properties. The approach used by Marcus was mainly analytic, through several transforms that are the analogues of the transforms applied in free probability. An alternative combinatorial approach via set partitions was chosen by Arizmendi et al. [2, 1] who established an analogue of Speicher's combinatorial approach to free probability [14].

In the present paper we study pairs of matrices A, B which are in finite free position (FFP), that is, such that the characteristic polynomial of the sum satisfies the identity  $\chi_{A+B}(x) = \chi_A \boxplus \chi_B$ , the additive finite free convolution of  $\chi_A$  and  $\chi_B$ , which further equals the expected characteristic polynomial  $\mathbb{E}_U[\chi_{A+U^*BU}(x)]$ over the set of unitary matrices U. We consider the analogous question in the multiplicative case.

The paper is organized as follows.

In Section 2 we recall the definition of additive finite free convolution and FFP.

In Section 3 we study in detail the simplest case of  $2 \times 2$  matrices in additive FFP.

In Section 4 we change the point of view from single matrices to the lattice of varieties of matrices  $\mathcal{V}$  and  $\mathcal{W}$ , which are maximal with respect to the property that each matrix in  $\mathcal{V}$  is in additive FFP with each matrix in  $\mathcal{W}$ . We say that such families form a finite free complementary pair. We give an upper bound for the rank of each such variety.

In Section 5 we focus on specific complementary pairs: diagonal and principally balanced matrices (matrices with the property that for each i the values of all principal minors of size i coincide); upper triangular and upper triangular matrices with constant diagonal; scalar matrices and the set of all square matrices.

In Section 6, we examine moments and cumulants of matrices A and B that are in additive FFP and show that the moments of A + B can be expressed in terms of the moments of A and B, and compute explicit formulas for the first four moments. We also show that a matrix A that is in additive FFP with itself must have a single eigenvalue. In Section 7 we extend our analysis to matrices the multiplicative FFP. In particular, we show that the three complementary pairs of varieties mentioned above form complementary pairs also in the multiplicative case. In general, however, the pairs of matrices that are in additive FFP and those in multiplicative FFP are not the same.

# 2. MATRICES IN ADDITIVE FINITE FREE POSITION

Additive convolution of polynomials can be defined as follows [10].

**Definition 2.1.** Let  $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ ,  $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$  be two polynomials of degree *n* over  $\mathbb{C}$ . The **additive convolution** of p(x) and q(x), denoted  $p(x) \boxplus q(x)$ , is

(2.1)  

$$p(x) \boxplus q(x) := \sum_{k=0}^{n} \left( \sum_{i+j=k} \frac{\binom{n-i}{j}}{\binom{n}{j}} a_i b_j \right) x^{n-k}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \frac{1}{(n-k)!} \left( \sum_{i+j=k} (n-i)! (n-j)! a_i b_j \right) x^{n-k}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x) q^{(n-k)}(0) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(0) q^{(n-k)}(x),$$

where we denote by  $p^{(k)}$  the k-th derivative of p and  $q^{(n-k)}$  is the (n-k)-th derivative of q.

**Example 2.2.** The neutral element with respect to the operation of additive convolution on polynomials of degree n is  $x^n$ : when p(x) is of degree n and  $q(x) = x^n$  then

$$p(x) \boxplus q(x) = \frac{1}{n!} \sum_{k=0}^{n} p^{(k)}(x) q^{(n-k)}(0) = \frac{1}{n!} p^{(0)}(x) q^{(n)}(0) = \frac{p(x)n!}{n!} = p(x)$$

and similarly,  $x^n \boxplus p(x) = p(x)$ .

We denote by  $\mathcal{M}_n$  the set of  $n \times n$  matrices over  $\mathbb{C}$ . Given a matrix  $A \in \mathcal{M}_n$ , its characteristic polynomial is  $\chi_A(x) = \det(xI - A)$ .

Following is one of the main results of [10]. Recall that a signed permutation matrix is a square matrix with each row and each column having exactly one non-zero entry which is either 1 or -1.

**Theorem 2.3.** [10] Let  $A, B \in \mathcal{M}_n$  be normal  $(A, B \in \mathbb{R}^{n \times n}$  be symmetric) matrices. Then

(2.2) 
$$\chi_A(x) \boxplus \chi_B(x) = \int_{\mathcal{U}(n)} \chi_{A+U^*BU}(x) \, dU = \frac{1}{2^n n!} \sum_{P \in \mathcal{P}^{\pm}(n)} \chi_{A+P^TBP}(x),$$

where the expectation is taken over the set of unitary (orthogonal) matrices  $\mathcal{U}(n)$  or the signed permutation matrices  $\mathcal{P}^{\pm}(n)$ .

In the present paper we study pairs of matrices A and B which satisfy equality (2.2) without taking the expected values of conjugations of B.

**Definition 2.4.** The matrices  $A, B \in \mathcal{M}_n$  are in additive finite free position (or in additive FFP), if

$$\chi_{A+B}(x) = \chi_A(x) \boxplus \chi_B(x).$$

Two families  $\mathcal{A}, \mathcal{B} \subseteq \mathcal{M}_n$  are in additive finite free position (in additive FFP) if  $\chi_{A+B}(x) = \chi_A(x) \boxplus \chi_B(x)$  for every  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ .

## Remarks 2.5.

- (1) As shown in [9], the property of being in additive FFP can be expressed in terms of mixed discriminants [3].
- (2) We notice that  $\chi_A(x) \boxplus \chi_B(x)$  only depends on the characteristic polynomials of the matrices and not on the matrices themselves.
- (3) This is in general not the case for  $\chi_{A+B}(x)$ , we can only assert that for any pair of matrices  $A, B \in \mathcal{M}_n$ , the two leading monomials of  $\chi_{A+B}(x)$  and  $\chi_A(x) \boxplus \chi_B(x)$  coincide and are equal to  $x^n$  and  $-\operatorname{Tr}(A+B)x^{n-1}$ , respectively.
- (4) If A and B are in additive FFP then so are  $PAP^{-1}$  and  $PBP^{-1}$ , for any regular matrix P.

## 3. Matrices of dimension $2 \times 2$

Let us start with an explicit analysis of  $2 \times 2$  matrices.

**Proposition 3.1.** The matrices  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{2 \times 2}$  are in additive FFP if and only if  $(a_{11} - a_{22})(b_{22} - b_{11}) = 2(a_{12}b_{21} + a_{21}b_{12})$ .

*Proof.* Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then

$$\chi_{A+B}(x) = x^2 - (a_{11} + a_{22} + b_{11} + b_{22})x + (a_{11} + b_{11})(a_{22} + b_{22}) - (a_{12} + b_{12})(a_{21} + b_{21})$$

and

$$\chi_A(x) \boxplus \chi_B(x) = x^2 - (a_{11} + a_{22} + b_{11} + b_{22})x + (a_{11}a_{22} - a_{12}a_{21} + b_{11}b_{22} - b_{12}b_{21}) + \frac{1}{2}(a_{11} + a_{22})(b_{11} + b_{22})x$$

We see that both  $\chi_A(x) \boxplus \chi_B(x)$  and  $\chi_A(x) \boxplus \chi_B(x)$  are monic polynomials with the same coefficient of x: negative the sum of the traces (in fact, this holds for any two  $n \times n$  matrices, cf. Remark 2.5). By comparing the free terms of both polynomials we get that A and B are in additive FFP if and only if

(3.1) 
$$(a_{11} - a_{22})(b_{22} - b_{11}) = 2(a_{12}b_{21} + a_{21}b_{12}).$$

Since the property of being in additive FFP is invariant under conjugating both A and B with the same matrix, we can assume that A is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}.$$

Then, condition (3.1) for A and B to be in additive FFP becomes:

$$(3.2) (a_{11} - a_{22})(b_{22} - b_{11}) = 2a_{12}b_{21}.$$

By (3.2), A is in additive FFP with itself if and only if

$$a_{11} = a_{22}$$

that is, A has a single eigenvalue. In fact, by Theorem 6.3, in general, an  $n \times n$  complex matrix is in additive FFP with itself if and only if it has a single eigenvalue.

When A (or B) is a scalar matrix then both sides of (3.1) are zero, so that A and B are in additive FFP. Moreover, when A and B are in additive FFP then it is easy to see that equality (3.1) holds also for  $\lambda_1 A + \lambda_2 I$  and  $\mu_1 B + \mu_2 I$ , for every  $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ . In fact, we have the following more general result. We will see later that it does not hold in higher dimensions, see Remark 4.4.

**Proposition 3.2.** Let  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{2 \times 2}$  be in additive FFP. Then p(A) and q(B) are in additive FFP, for any polynomials p(x) and q(x). Moreover, if A is regular then  $A^{-1}$  and q(B) are in additive FFP.

*Proof.* We infer from the Cayley-Hamilton theorem that every holomorphic function f(A) can be represented as a linear function of the form  $a_0I + a_1A$  and the claim follows from Proposition 4.3.

#### 4. The lattice of additive finite free varieties

In this section we extend the discussion from single matrices to sets of matrices in FFP. To this end we consider the set  $\mathcal{V} \subseteq \mathcal{M}_n$  of all matrices that are in additive FFP with every matrix A in a given set  $\mathcal{A}$ . In fact this set  $\mathcal{V}$  forms an algebraic variety (algebraic set).

For general  $n \times n$  matrices  $A = (A_{ij})$  and  $B = (B_{ij})$ , let

$$\chi_{A+B}(x) = \sum_{k=0}^{n} c_k(A_{11}, A_{12}, \dots, A_{nn}, B_{11}, B_{12}, \dots, B_{nn}) x^{n-k}$$

and

$$\chi_A(x) \boxplus \chi_B(x) = \sum_{k=0}^n d_k(A_{11}, A_{12}, \dots, A_{nn}, B_{11}, B_{12}, \dots, B_{nn}) x^{n-k},$$

where  $c_k(A_{11}, \ldots, A_{nn}, B_{11}, \ldots, B_{nn})$ ,  $d_k(A_{11}, \ldots, A_{nn}, B_{11}, \ldots, B_{nn})$  are multilinear homogeneous polynomials in the variables  $A_{11}, \ldots, A_{nn}, B_{11}, \ldots, B_{nn}$  of degree k. By definition 2.1 of the additive convolution, it is clear that a coefficient of  $x^{n-k}$  in  $\chi_A(x) \boxplus \chi_B(x)$  has as summands the coefficient  $a_k$  of  $\chi_A(x)$ , as well as the coefficient  $b_k$  of  $\chi_B(x)$ . These are also summands of  $\chi_{A+B}(x)$ . Moreover, the coefficients of of  $x^n$  and of  $x^{n-1}$  (1 and  $-\operatorname{Tr}(A+B)$ , respectively) are the same in  $\chi_A(x) \boxplus \chi_B(x)$ and in  $\chi_{A+B}(x)$ , That is, two specific matrices  $A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}_n$  are in additive FFP if and only if

$$c_k(a_{11},\ldots,a_{nn},b_{11},\ldots,b_{nn}) = d_k(a_{11},\ldots,a_{nn},b_{11},\ldots,b_{nn}),$$

for k = 2, ..., n.

Suppose now that  $A = (a_{ij}) \in \mathcal{M}_n$  is a specific matrix. Then the matrices B that are in additive FFP with A are the zero locus of the following set of polynomials over  $\mathbb{C}$ :

$$p_{A,k}(B_{11},\ldots,B_{nn}) := c_k(a_{11},\ldots,a_{nn},B_{11},\ldots,B_{nn}) - d_k(a_{11},\ldots,a_{nn},B_{11},\ldots,B_{nn}),$$

for k = 2, ..., n. We associate with A the ideal  $I_A$  generated by these polynomials:

$$I_A := \langle p_{A,2}(B_{11},\ldots,B_{nn}),\ldots,p_{A,n}(B_{11},\ldots,B_{nn}) \rangle.$$

The affine variety of  $I_A$  is

$$\mathcal{V}(I_A) := \{ B = (b_{ij}) \in \mathcal{M}_n :$$
  
$$p_{A,2}(b_{11}, \dots, b_{nn}) = \dots = p_{A,n}(b_{11}, \dots, b_{nn}) = 0 \},$$

and this is the set of matrices  $B \in \mathcal{M}_n$  that are in additive FFP with A:

$$\mathcal{V}(I_A) = \{ B \in \mathcal{M}_n : \chi_{A+B}(x) = \chi_A(x) \boxplus \chi_B(x) \}.$$

We denote by  $\{A\}^{\boxplus}$  the variety  $\mathcal{V}(I_A)$ . More generally, we define the following.

**Definition 4.1.** Given a set  $\mathcal{A} \subseteq \mathcal{M}_n$ , the additive finite free complement or additive finite free variety  $\mathcal{A}^{\boxplus}$  is the set

$$\mathcal{A}^{\boxplus} := \mathcal{V}(I_{\mathcal{A}}) = \{ B \in \mathcal{M}_n : B \in \mathcal{V}(I_A), \text{ for all } A \in \mathcal{A} \} = \bigcap_{A \in \mathcal{A}} \mathcal{V}(I_A),$$

which is the affine variety of the ideal  $I_{\mathcal{A}}$  generated by the set of polynomials  $p_{A,k}(B_{11},\ldots,B_{nn})$ , for  $A \in \mathcal{A}$  and  $k = 2,\ldots,n$ .

**Definition 4.2.** When  $\mathcal{V}^{\boxplus} = \mathcal{W}$  and  $\mathcal{W}^{\boxplus} = \mathcal{V}$ , we say that the varieties  $\mathcal{V}, \mathcal{W}$  form an additive finite free complementary pair (an additive complementary pair, for short).

Note that when  $\mathcal{W} = \mathcal{V}^{\boxplus}$  then  $\mathcal{W} = \mathcal{W}^{\boxplus \boxplus}$ .

We show next that every additive finite free variety contains the scalar matrices.

**Proposition 4.3.** If A and B are in additive FFP then so are  $A + \lambda I$  and B, for every  $\lambda \in \mathbb{C}$ .

Proof. Let  $A, B \in \mathcal{M}_n$ . Then,  $\chi_{A+\lambda I}(x) = \det(xI - (A + \lambda I)) = \det((x - \lambda)I - A) = \chi_A(x - \lambda)$ . Similarly,  $\chi_{A+\lambda I+B}(x) = \chi_{A+B}(x - \lambda)$ . By assumption  $\chi_A(x) \boxplus \chi_B(x) = \chi_{A+B}(x)$  and it follows that

$$\chi_{A+\lambda I}(x) \boxplus \chi_B(x) = \frac{1}{n!} \sum_{k=0}^n \chi_{A+\lambda I}^{(k)}(x) \chi_B^{(n-k)}(0)$$
$$= \frac{1}{n!} \sum_{k=0}^n \chi_A^{(k)}(x-\lambda) \chi_B^{(n-k)}(0)$$
$$= \chi_{A+B}(x-\lambda) = \chi_{A+\lambda I+B}(x).$$

**Remark 4.4.** Proposition 3.2 raises the question whether other algebraic operations preserve additive FFP also in higher dimensions. The following example shows that the obvious candidates fail: The matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 13 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

are in additive FFP but the following pairs are not:

- (i)  $(\lambda A, B)$ , unless  $\lambda \in \{0, 1\}$ ;
- (ii)  $(A^2, B);$

(iii)  $(A^{-1}, B)$ .

**Definition 4.5.** Let  $\mathcal{A} \subseteq \mathcal{M}_n$  and let  $\mathcal{V} = \mathcal{A}^{\boxplus \boxplus}$ . Then we say that  $\mathcal{A}$  is a **generating set** of  $\mathcal{V}$ , written  $\mathcal{V} = \langle \mathcal{A} \rangle$ . We define the **rank** of  $\mathcal{V}$  to be the minimal cardinality of a generating set of  $\mathcal{V}$ .

**Proposition 4.6.** Every additive finite free variety  $\mathcal{V} \subseteq \mathcal{M}_n$  is of finite rank. An upper bound for the rank is  $\sum_{k=1}^{n-1} k! \binom{n}{k}^2$ .

Proof. Let  $\mathcal{A}$  be a generating set of  $\mathcal{V}$ , that is,  $\mathcal{V} = \mathcal{A}^{\boxplus\boxplus}$ , and let  $I_{\mathcal{A}}$  be the corresponding ideal. That is,  $I_{\mathcal{A}}$  is the sum of the ideals  $I_{\mathcal{A}}$ , for  $\mathcal{A} \in \mathcal{A}$ , where  $I_{\mathcal{A}}$  is generated by the finitely many polynomials  $p_{\mathcal{A},k}(B_{11},\ldots,B_{nn})$ . Since  $I_{\mathcal{A}}$  is an ideal in a Noetherian ring, it is finitely generated. It follows that there exists a finite set  $\mathcal{A}' \subseteq \mathcal{A}$ , such that  $I_{\mathcal{A}} = I_{\mathcal{A}'}$ . But then  $\mathcal{V} = \mathcal{A}^{\boxplus\boxplus} = \mathcal{A}'^{\boxplus\boxplus}$ , that is  $\mathcal{V} = \langle \mathcal{A}' \rangle$ .

Each polynomial  $p_{A,k}(B_{11}, \ldots, B_{nn})$ , for  $k = 2, \ldots, n$ , is of degree at most k - 1, since the part that is of degree k does not involve the elements of A and is the same in  $c_k$  and  $d_k$ , and therefore vanishes in  $p_{A,k}$ . Similarly, the part of degree 0 comes just from A and also vanishes in  $p_{A,k}$ . The monomials in  $p_{A,k}(B_{11}, \ldots, B_{nn})$  are then of degree  $1 \le d \le n - 1$  and of the form  $B_{i_1j_1}B_{i_2j_2}\cdots B_{i_lj_l}$ , where  $\{i_1,\ldots,i_l\}$ and  $\{j_1,\ldots,j_l\}$  are subsets of size l of  $\{1,\ldots,n\}$  (by the way the determinant is defined). It follows that the generators of the ideal  $I_A$  span linearly a subspace of a vector space of dimension

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{n!}{(n-k)!} = \sum_{k=1}^{n-1} k! \binom{n}{k}^2,$$

which gives an upper bound on the number of generators of  $I_{\mathcal{A}}$  and, thus, on the rank of  $\mathcal{A}$ .

**Remark 4.7.** The examples given in this paper suggest that the varieties are probably of rank much smaller than the bound given in Proposition 4.6, more likely of order  $\mathcal{O}(n^2)$ , e.g., the Krull dimension of the coordinate ring  $\mathbb{C}[B_{11}, \ldots, B_{nn}]/I_A$ , possibly even satisfying rank $(\mathcal{V})$  + rank $(\mathcal{W}) = n^2 - 1$ , for varieties  $\mathcal{V}$  and  $\mathcal{W}$  that are complementary.

The set of additive finite free varieties forms a **lattice**  $(\mathbb{L}_n, \vee, \wedge, \mathcal{I}_n, \mathcal{M}_n)$  under set inclusion, where:

- The set  $\perp = S_n$  of scalar matrices is the bottom element of  $\mathbb{L}$  and  $\top = \mathcal{M}_n$  is the top element.
- For every  $\mathcal{V}, \mathcal{W}$  in  $\mathbb{L}_n$ , their **join**  $\mathcal{V} \lor \mathcal{W} := \langle \mathcal{V} \cup \mathcal{W} \rangle$  and their **meet**  $\mathcal{V} \land \mathcal{W} := \mathcal{V} \cap \mathcal{W}$  are in  $\mathbb{L}_n$ .

Some of the properties of the lattice  $\mathbb{L}_n$  that are easy to verify are:

- (1)  $\mathbb{L}_n$  is associative, commutative and distributive with respect to  $\vee$  and  $\wedge$ .
- (2)  $\mathcal{V} \wedge \bot = \bot$  and  $\mathcal{V} \vee \bot = \mathcal{V}$ , for every  $\mathcal{V} \in \mathbb{L}_n$ .
- (3)  $\mathcal{V} \wedge \top = \mathcal{V}$  and  $\mathcal{V} \vee \top = \top$ , for every  $\mathcal{V} \in \mathbb{L}_n$ .
- (4)  $(\mathcal{V} \lor \mathcal{W})^{\boxplus} = \mathcal{V}^{\boxplus} \land \mathcal{W}^{\boxplus}$  and  $(\mathcal{V} \land \mathcal{W})^{\boxplus} = \mathcal{V}^{\boxplus} \lor \mathcal{W}^{\boxplus}$ , for every  $\mathcal{V}, \mathcal{W} \in \mathbb{L}_n$  (De Morgan laws).

- **Remarks 4.8.** (1) The complement of a variety in  $\mathbb{L}_n$  does not obey the standard definition of a complement in a lattice, that is, it does not hold in general that  $\mathcal{V} \vee \mathcal{V}^{\boxplus} = \top$  and  $\mathcal{V} \wedge \mathcal{V}^{\boxplus} = \bot$ .
  - (2) Since every additive finite free variety contains the scalar matrices, we could have defined the lattice in the quotient space  $\mathbb{M}_n := \mathcal{M}_n / \mathcal{S}_n$ .

Let us look at the special case of  $2 \times 2$  matrices.

**Proposition 4.9.** Let the varieties  $\mathcal{V}, \mathcal{W} \subseteq \mathbb{C}^{2 \times 2}$  form an additive complementary pair. Then

- (1)  $\mathcal{V}, \mathcal{W}$  are subspaces of  $\mathbb{C}^{2 \times 2}$ ;
- (2)  $\operatorname{rank}(\mathcal{V}) + \operatorname{rank}(\mathcal{W}) = 3.$
- *Proof.* (1) By (3.1), the elements of the matrices  $A \in \mathcal{V}$ , that are in additive FFP with a matrix  $B = (b_{ij}) \in \mathcal{W}$ , are the solution set of the linear homogeneous equation

$$(4.1) (b_{22} - b_{11})A_{11} - (b_{22} - b_{11})A_{22} - 2a_{21}A_{12} - 2b_{12}A_{21} = 0$$

and thus form a subspace of  $\mathbb{C}^{2\times 2}$ . It follows that  $\mathcal{V}$ , and similarly  $\mathcal{W}$ , are subspaces of  $\mathbb{C}^{2\times 2}$ .

(2) For each matrix  $B \in \mathcal{W}$ , we obtain a linear equation of the form (4.1). Hence, it is sufficient to find the matrices A that are in additive FFP with respect to a linear basis of the subspace  $\mathcal{W}$ . One of the elements of  $\mathcal{W}$  (written as a row vector) is  $(b_{11}, b_{22}, b_{12}, b_{21}) = (1, 1, 0, 0)$ , representing the identity matrix. Thus, by reduction through this row, we may assume that the other basis elements have zero in their first entry. When  $(b_{11}, b_{22}, b_{12}, b_{21}) = (1, 1, 0, 0)$ then Equation (4.1) vanishes, and so, we get that  $\mathcal{V}$  is the set of solutions of an homogeneous system of linear equations of the form

$$(4.2) c_1 A_{11} - c_1 A_{22} + c_2 A_{12} + c_3 A_{21} = 0.$$

The number of these equations is  $\dim(\mathcal{W}) - 1$  (since the identity matrix does not contribute an equation). Moreover, since these equations are independent, we cannot take a smaller number of equations in order to define  $\mathcal{V}$ . A basis for  $\mathcal{V}$  (a basis for the solution of the system of equations) consists of  $4 - (\dim(\mathcal{W}) - 1) = 5 - \dim(\mathcal{W})$  elements. It follows from the above discussion that  $\operatorname{rank}(\mathcal{V}) = \dim(\mathcal{V}) - 1$ ,  $\operatorname{rank}(\mathcal{W}) = \dim(\mathcal{W}) - 1$  (after omitting the identity matrix), and

$$\operatorname{rank}(\mathcal{V}) + \operatorname{rank}(\mathcal{W}) = 5 - 2 = 3.$$

#### 5. Additive finite free complementary pairs

In this section we present pairs of varieties  $(\mathcal{V}, \mathcal{W}) \subseteq \mathcal{M}_n \times \mathcal{M}_n$  that form additive complementary pairs. These are:  $(\mathcal{D}_n, \mathcal{B}_n)$ , the set of diagonal matrices and the set of principally balanced matrices;  $(\hat{\mathcal{R}}_n, \mathcal{R}_n)$  (resp.  $(\hat{\mathcal{L}}_n, \mathcal{L}_n)$ ), the set of upper (resp. lower) triangular matrices with constant diagonal and the set of upper (resp. lower) triangular matrices;  $(\mathcal{S}_n, \mathcal{M}_n)$ , the set of scalar matrices and the set of all  $n \times n$ matrices.

We will make frequent use of the following lemma. We denote by  $E_{kl} \in \mathcal{M}_n$  the matrix with 1 at position (k, l) and 0 elsewhere.

**Lemma 5.1.** Let  $A = (a_{ij}) \in \mathcal{M}_n$  be a matrix having an off-diagonal non-zero entry  $a_{lk}$ . Then  $\chi_{A+E_{kl}}(x) \neq \chi_A(x) = \chi_A(x) \boxplus \chi_{E_{kl}}(x)$ .

*Proof.* Let us look at the coefficient of  $x^{n-2}$  in  $\chi_{A+E_{kl}}(x)$  and in  $\chi_A(x)$ . The only term that is affected by the addition of  $E_{kl}$  to A is the principal minor with indices k, l. The difference is

$$(a_{kk}a_{ll} - (a_{kl} + 1)a_{lk}) - (a_{kk}a_{ll} - a_{kl}a_{lk}) = -a_{lk} \neq 0.$$

It follows that  $\chi_{A+E_{kl}}(x) \neq \chi_A(x)$ . On the other hand,  $\chi_{E_{kl}}(x) = x^n$  and by Example 2.2,  $\chi_A(x) \boxplus \chi_{E_{kl}}(x) = \chi_A(x)$ .

5.1. Diagonal, principally balanced and symmetric matrices. Consider first diagonalizable matrices. Because of invariance under conjugation (Remark 4 (4)) it we may assume that one of the matrices is diagonal. It turns out that the following concept is crucial.

**Definition 5.2.** A matrix  $B \in \mathcal{M}_n$  is called **principally balanced** (or has the **symmetrized principal minors** property [6]) if, for every  $i, 1 \leq i \leq n$ , the values of all principal minors of B of order i coincide. We denote by  $\mathcal{B}_n \subseteq \mathcal{M}_n$  the family of  $n \times n$  matrices which are principally balanced.

This is a special case of the principal minor assignment problem [4, 11] going back to Stouffer [15].

It it easy to see that the family of principally balanced matrices is invariant under conjugation by permutation matrices and diagonal matrices. Clearly, all  $1 \times 1$ matrices are principally balanced, as well as all  $2 \times 2$  matrices with a constant diagonal and all  $n \times n$  triangular matrices with a constant diagonal. Here are examples of specific principally balanced matrices.

**Example 5.3.** The following  $3 \times 3$  matrix is principally balanced: all principal minors of order 1 (the diagonal entries) equal 1 and all principal minors of order 2 equal -11:

$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 1 & -12 \\ 4 & -1 & 1 \end{bmatrix}.$$

**Example 5.4.** Let A be the  $n \times n$  matrix with entries  $a_{ij} = \frac{i}{j}$ , i = 1, ..., n:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \cdots & \frac{1}{n} \\ 2 & 1 & \frac{2}{3} \cdots & \frac{2}{n} \\ 3 & \frac{3}{2} & 1 \cdots & \frac{3}{n} \\ \vdots & & \ddots & \vdots \\ n & \frac{n}{2} & \frac{n}{3} \cdots & 1 \end{bmatrix}.$$

The rank of A is 1, so all minors (not just principal) of order  $k \ge 2$  equal 0, while the principal minors of order 1 equal 1. Thus, A is principally balanced.

More generally, let  $A_i$  denote the *i*-th row of A, i = 1, ..., n. When all entries of  $A_1$  are non-zero and  $A_i = \frac{a_{11}}{a_{1i}}A_1$ , i = 1, ..., n, then A is principally balanced with all principal minors of order 1 equal  $a_{11}$  and all minors of order  $k \ge 2$  equal 0. In fact, it is easy to show that for any positive integer k,  $A^k = (a_{11}n)^{k-1}A$  and so, for any polynomial p(x) with p(0) = 0, p(A) is a scalar multiple of A and thus of the same form as A (just that the principal minors of order 1 are multiplied by  $\lambda$ ) and, in

particular, is principally balanced. Moreover, p(A) is principally balanced for every polynomial P since the set of principally balanced matrices is closed under addition of a scalar matrix by Proposition 4.3 and Theorem 5.7.

**Lemma 5.5.** Let  $B \in \mathcal{B}_n$  be principally balanced and let  $m_i$ , for i = 0, ..., n, be its  $i \times i$  principal minor, where  $m_0$  is defined to be 1. Then  $\chi_B(x) = \sum_{i=0}^n (-1)^i {n \choose i} m_i x^{n-i}$ .

*Proof.* For each i, the coefficient of  $x^{n-i}$  in  $\chi_B(x)$  is  $(-1)^i$  times the sum of all principal minors of order *i*. There are  $\binom{n}{i}$  such principal minors and each one equals  $m_i$ . 

**Remark 5.6.** The family of principally balanced matrices can be equivalently defined by the property of having equal "cycle sums" of the same order:  $c_I = c_J$ , for all  $I, J \subseteq [n]$  with |I| = |J|, where

$$c_I = \sum_{\substack{I = \{i_1, \dots, i_k\}\\i_1 = \min I}} a_{i_1, i_2} \cdots a_{i_{k-1}, i_k} a_{i_k, i_1},$$

are the cycle sums of a matrix  $A = (a_{ij})$ , see [6]. Indeed the authors prove that principal minors and cycle sums are related by Möbius inversion on the lattice of set partitions.

This can be interpreted as follows. Let  $B \in \mathcal{B}_n$  be a principally balanced matrix with principal minors  $m_k$  and cycle sums  $c_k$ . Consider  $\tilde{m}_k = (-1)^k m_k$  as moments of a formal random variable (or "umbra" in the sense of Rota [13]) Z, then

$$\chi_B(x) = \mathbb{E}(x+Z)^i$$

and the corresponding classical cumulants of Z are the numbers  $\tilde{c}_k = (-1)^k c_k$ .

We denote by [n] the set  $\{1, \ldots, n\}$ . When  $M \in \mathcal{M}_n$  and  $S \subseteq [n]$  with |S| = k, we denote by  $M_S$  the  $k \times k$  principal submatrix of M obtained after removing the rows and columns with indices in [n] - S.

**Theorem 5.7.** Let  $\mathcal{D}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  diagonal matrices and let  $\mathcal{B}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  principally balanced matrices. Then  $\mathcal{D}_n$  and  $\mathcal{B}_n$ form an additive complementary pair.

*Proof.* The theorem clearly holds for n = 1, so let us assume that n > 2. First, we show that for arbitrary  $D \in \mathcal{D}_n$  and  $B \in \mathcal{B}_n$ , the equality  $\chi_{D+B}(x) = \chi_D(x) \boxplus \chi_B(x)$ holds. Let  $D = \text{diag}(d_1, \ldots, d_n)$ , hence

$$\chi_D(x) = \sum_{i=0}^n c_i x^{n-i} = \prod_{i=1}^n (x - d_i)$$
  
=  $\sum_{i=0}^n \left( (-1)^i \sum_{J \subseteq [n], |J|=i} \det(D_J) \right) x^{n-i}$   
=  $\sum_{i=0}^n \left( (-1)^i \sum_{J \subseteq [n], |J|=i} d_J \right) x^{n-i},$ 

where  $d_J = d_{j_1} \cdots d_{j_i}$  for  $J = \{j_1, \dots, j_i\}$ . For  $B \in \mathcal{B}_n$ , we have

(5.1)

(5.2) 
$$\chi_B(x) = \sum_{i=0}^n b_i x^{n-i} = \sum_{i=0}^n (-1)^i \binom{n}{i} m_i x^{n-i},$$

where  $m_i$ , i = 0, ..., n, is the value of a principal minor of B of order i.

The characteristic polynomial of B + D is

(5.3) 
$$\chi_{D+B}(x) = \sum_{i=0}^{n} \left( (-1)^{i} \sum_{I \subseteq [n], |I|=i} \det((D+B)_{I}) \right) x^{n-i}$$
$$= \sum_{i=0}^{n} \left( (-1)^{i} \sum_{I \subseteq [n], |I|=i} \sum_{j=0}^{i} \sum_{J \subseteq I, |J|=j} m_{i-j} d_{J} \right) x^{n-i}.$$

We observe that when we compute  $det((D+B)_I)$ , with |I|=i, in (5.3), then we sum over the products of  $d_J$ , which is a principal minor of order  $j \leq i$  of D, and a principal minor of order i - j of B. But all principal minors of order i - j of B equal  $m_{i-j}$ . It follows that for each J, the term  $m_{i-j}d_J$  appears  $\binom{n-j}{i-j}$  times in (5.3), the number of principal minors of order i - j of B with indices of the submatrices outside of J, while the submatrix  $D_J$  of D is kept fixed. Hence

(5.4) 
$$\chi_{D+B}(x) = \sum_{i=0}^{n} \left( (-1)^{i} \sum_{j=0}^{i} \sum_{J \subseteq [n], |J|=j} \binom{n-j}{i-j} m_{i-j} d_{J} \right) x^{n-i}$$
$$= \sum_{i=0}^{n} \left( (-1)^{i} \sum_{j=0}^{i} \binom{n-j}{i-j} m_{i-j} \sum_{J \subseteq [n], |J|=j} d_{J} \right) x^{n-i}.$$

Next, we compute the additive convolution of  $\chi_D(x)$  and  $\chi_B(x)$ . By (2.1),

(5.5) 
$$\chi_D(x) \boxplus \chi_B(x) = \sum_{i=0}^n f_i x^{n-i} = \sum_{i=0}^n c_i x^{n-i} \boxplus \sum_{i=0}^n b_i x^{n-i}$$
  
(5.6)  $= \sum_{i=0}^n \left( \sum_{j=0}^i \frac{(n-j)! (n-i+j)!}{n! (n-i)!} c_j b_{i-j} \right) x^{n-i}.$ 

Substituting for  $c_j$  the appropriate coefficient from (5.1) and for  $b_{i-j}$  the appropriate coefficient from (5.2), we get

$$f_{i} = \sum_{j=0}^{i} \frac{(n-j)! (n-i+j)!}{n! (n-i)!} (-1)^{j} \sum_{J \subseteq [n], |J|=j} d_{J} (-1)^{i-j} {n \choose i-j} m_{i-j}$$
$$= (-1)^{i} \sum_{j=0}^{i} \frac{(n-j)! (n-i+j)! n!}{n! (n-i)! (n-i+j)! (i-j)!} m_{i-j} \sum_{J \subseteq [n], |J|=j} d_{J}$$
$$(5.7) \qquad = (-1)^{i} \sum_{j=0}^{i} {n-j \choose i-j} m_{i-j} \sum_{J \subseteq [n], |J|=j} d_{J}.$$

We got in (5.7) the same coefficient of  $x^{n-i}$  as in (5.4), hence  $\chi_{D+B}(x) = \chi_D(x) \boxplus \chi_B(x)$ . It follows that  $\mathcal{B}_n \subseteq \mathcal{D}_n^{\boxplus}$  and  $\mathcal{D}_n \subseteq \mathcal{B}_n^{\boxplus}$ . We show now that  $\mathcal{D}_n^{\boxplus} \subseteq \mathcal{B}_n$ . Let  $B = (b_{ij}) \in \mathcal{D}_n^{\boxplus}$ . Let  $K \subseteq [n], |K| = k$ ,

0 < k < n, and let  $D(\lambda, K) = (d_{ij})$  be the parametric diagonal matrix with entries

$$d_{ij} = \begin{cases} \lambda & \text{for } i = j \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluating  $\chi_{D(\lambda,K)+B}(x)$  at x=0 gives

$$\chi_{D(\lambda,K)+B}(0) = (-1)^n \det(D(\lambda,K)+B) = (-1)^n \det(B_{[n]-K})\lambda^k + q(\lambda),$$

where  $q(\lambda)$  is a polynomial in  $\lambda$  of degree less than k. It follows that

(5.8) 
$$\lim_{\lambda \to \infty} \frac{\chi_{D(\lambda,K)+B}(0)}{\lambda^k} = (-1)^n \det(B_{[n]-K})$$

Let us now compute this limit for  $\chi_{D(\lambda,K)}(x) \boxplus \chi_B(x)$ . The characteristic polynomial of  $D(\lambda, K)$  is

$$\chi_{D(\lambda,K)}(x) = \sum_{i=0}^{n} a_i x^{n-i} = x^{n-k} (x-\lambda)^k = \sum_{i=0}^{n} \binom{k}{i} (-\lambda)^i x^{n-i},$$

where  $\binom{k}{i} = 0$  when i > k. That is,

$$a_i = \begin{cases} (-1)^i \binom{k}{i} \lambda^i & \text{for } 0 \le i \le k, \\ 0 & \text{for } i > k. \end{cases}$$

The characteristic polynomial of B is

$$\chi_B(x) = \sum_{i=0}^n b_i x^{n-i} = \sum_{i=0}^n \left( (-1)^i \sum_{I \subseteq [n], |I|=i} \det(B_I) \right) x^{n-i}.$$

That is,  $(-1)^i b_i$  is the sum of the  $\binom{n}{i}$  principal minors of order *i*. By (2.1),

$$(\chi_{D(\lambda,K)} \boxplus \chi_B)(0) = \sum_{i=0}^n \frac{(n-i)! \, i!}{n!} a_i b_{n-i}$$
  
=  $\sum_{i=0}^n \left( \frac{(-1)^i \binom{k}{i} (-1)^{n-i}}{\binom{n}{i}} \sum_{I \subseteq [n], |I|=n-i} \det(B_I) \right) \lambda^i,$ 

a polynomial of degree at most k in  $\lambda$ . It follows that

$$(\chi_{D(\lambda,K)} \boxplus \chi_B)(0) = \left(\frac{(-1)^n}{\binom{n}{k}} \sum_{I \subseteq [n], |I|=n-k} \det(B_I)\right) \lambda^k + p(\lambda),$$

where  $p(\lambda)$  is a polynomial in  $\lambda$  of degree less than k. Then,

(5.9) 
$$\lim_{\lambda \to \infty} \frac{(\chi_{D(\lambda,K)} \boxplus \chi_B)(0)}{\lambda^k} = \frac{(-1)^n}{\binom{n}{k}} \sum_{I \subseteq [n], |I|=n-k} \det(B_I).$$

Comparing (5.8) with (5.9), we get that

$$\det(B_{[n]-K}) = \frac{1}{\binom{n}{k}} \sum_{I \subseteq [n], |I|=n-k} \det(B_I),$$

that is, for every  $1 \leq k \leq n$ , every principal minor of B of order n - k equals the mean value of the principal minors of order n - k. In other words, all principal minors of the same order are equal. It follows that B is principally balanced and  $\mathcal{D}_n^{\boxplus} \subseteq \mathcal{B}_n$ . Together with the inclusion  $\mathcal{B}_n \subseteq \mathcal{D}_n^{\boxplus}$  we have

$${\mathcal D}_n{}^{\boxplus}=\mathcal{B}_n$$

It remains to show that the additive free complement of  $\mathcal{B}_n$  is  $\mathcal{D}_n$  and not a larger family. Suppose that A has an off-diagonal entry  $a_{kl} \neq 0$ ,  $k \neq l$  Then the matrix

 $E_{kl}$  from Lemma 5.1 satisfies that  $\chi_{A+E_{kl}}(x) \neq \chi_A(x) \boxplus \chi_{E_{kl}}(x)$ . On the other hand all principal minors of  $E_{kl}$  vanish and hence  $E_{kl} \in \mathcal{B}_n$ . We showed that  $A \notin \mathcal{B}_n^{\boxplus}$ and consequently  $\mathcal{B}_n^{\boxplus} \subseteq \mathcal{D}_n$ . Together with the inclusion in the other direction, we have

$$\mathcal{B}_n^{\oplus} = \mathcal{D}_n$$

and the proof is complete.

**Corollary 5.8.** Let  $\mathcal{A}_n \subseteq \mathbb{R}^{n \times n}$  be a maximal family of commuting diagonalizable matrices and let  $\mathcal{B}_n$  be the family of principally balanced matrices.

- (1) There exists a non-singular matrix P, such that  $\mathcal{A}_n$  and  $P^{-1}\mathcal{B}_n P$  form an additive complementary pair.
- (2) If  $\mathcal{A}_n$  is a maximal family of commuting symmetric matrices then there exists an orthogonal matrix U, such that  $\mathcal{A}_n$  and  $U^T \mathcal{B}_n U$  form an additive complementary pair.

*Proof.* Since the matrices in  $\mathcal{A}_n$  commute with each other, they are simultaneously diagonalizable ([5], Theorem 1.3.21). That is, there exists a non-singular matrix P, such that for each  $A \in \mathcal{A}_n$  the conjugated matrix  $D = PAP^{-1}$  is diagonal. By the maximality of  $\mathcal{A}_n$  and the non-singularity of P,  $\mathcal{A}_n = P^{-1}\mathcal{D}_n P$  is a subspace of dimension n. By Theorem 5.7, for every  $A \in \mathcal{A}_n$  and  $B \in \mathcal{B}_n$ ,

$$\chi_{PAP^{-1}+B}(x) = \chi_{PAP^{-1}}(x) \boxplus \chi_B(x)$$

and the subsets  $P\mathcal{A}_n P^{-1}$  and  $\mathcal{B}_n$  form an additive complementary pair. By similarity transformation, the same holds for  $\mathcal{A}_n$  and  $P^{-1}\mathcal{B}_n P$ .

If the matrices in  $\mathcal{A}_n$  are symmetric then commuting is equivalent to simultaneous orthogonal diagonalization ([5, Theorem 4.5.15]).

5.2. Scalar matrices. Another additive complementary pair is the following.

**Theorem 5.9.** Let  $S_n \subseteq M_n$  be the family of  $n \times n$  scalar matrices. Then  $S_n$  and  $M_n$  form an additive complementary pair.

*Proof.* Let  $A \in \mathcal{M}_n$  be an arbitrary matrix. Then the zero matrix is in finite free position with A because  $\chi_0(x) = x^n$  is the neutral element for additive finite free convolution (Example 2.2). It follows from Proposition 4.3 that any scalar matrix is in finite free position with A as well. It follows that

$$\mathcal{S}_n \subseteq \mathcal{M}_n^{\boxplus}$$

For the converse, by Theorem 5.7, since  $\mathcal{M}_n \supseteq \mathcal{D}_n$  and  $\mathcal{M}_n \supseteq \mathcal{B}_n$ ,

$$\mathcal{M}_n^{\ \boxplus} \subseteq \mathcal{D}_n^{\ \boxplus} \cap \mathcal{B}_n^{\ \boxplus} = \mathcal{B}_n \cap \mathcal{D}_n = \mathcal{S}_n.$$

By both inclusions, we conclude that

$$\mathcal{M}_n^{\ \boxplus} = \mathcal{S}_n$$

Clearly, since  $\mathcal{S}_n^{\boxplus} = \mathcal{M}_n^{\boxplus \boxplus} \supseteq \mathcal{M}_n$ , we get that

$$\mathcal{S}_n^{\ \boxplus} = \mathcal{M}_n$$

It follows that  $\mathcal{S}_n$  and  $\mathcal{M}_n$  form an additive complementary pair.

5.3. Triangular matrices. In this subsection we exhibit an additive complementary pair within the ring of upper triangular matrices.

**Theorem 5.10.** Let  $\mathcal{R}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  upper triangular matrices and let  $\widehat{\mathcal{R}}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  upper triangular matrices with constant diagonal. Then  $\mathcal{R}_n$  and  $\widehat{\mathcal{R}}_n$  form an additive complementary pair.

Analogously, the lower triangular matrices  $\mathcal{L}_n$  and  $\widehat{\mathcal{L}}_n$  form an additive complementary pair.

Proof. Let  $T \in \mathcal{R}_n$  and  $C \in \widehat{\mathcal{R}}_n$ . Then  $\chi_T(x), \chi_C(x)$  and  $\chi_{T+C}(x)$  do not change if we replace them by their diagonals, i.e., if we set all off-diagonal entries to zero. Then T becomes diagonal and C becomes a scalar matrix and thus principally balanced. By Theorem 5.7,  $\chi_{T+C}(x) = \chi_T(x) \boxplus \chi_C(x)$ . This shows that  $\widehat{\mathcal{R}}_n \subseteq \mathcal{R}_n^{\boxplus}$ and  $\mathcal{R}_n \subseteq \widehat{\mathcal{R}}_n^{\boxplus}$ .

Next, we show that  $\widehat{\mathcal{R}}_n^{\boxplus} \subseteq \mathcal{R}_n$ . Suppose that  $A = (a_{ij}) \notin \mathcal{R}_n$ , i.e.,  $a_{lk} \neq 0$  for some (l,k) with l > k. Then the matrix  $E_{kl}$  from Lemma 5.1 is in  $\widehat{\mathcal{R}}_n$  but  $\chi_{A+E_{kl}}(x) \neq \chi_A(x) \boxplus \chi_{E_{kl}}(x)$ . It follows that  $A \notin \widehat{\mathcal{R}}_n^{\boxplus}$  and consequently that  $\widehat{\mathcal{R}}_n^{\boxplus} \subseteq \mathcal{R}_n$ . It remains to show that  $\mathcal{R}_n^{\boxplus} \subseteq \widehat{\mathcal{R}}_n$ . Let  $C \in \mathcal{R}_n^{\boxplus}$ , then the argument of the

It remains to show that  $\mathcal{R}_n^{\oplus} \subseteq \widehat{\mathcal{R}}_n$ . Let  $C \in \mathcal{R}_n^{\oplus}$ , then the argument of the previous paragraph shows that C is upper triangular. Since  $\mathcal{D}_n \subseteq \mathcal{R}_n$ , it follows from Theorem 5.7 that C is principally balanced and thus  $C \in \mathcal{R}_n \cap \mathcal{B}_n = \widehat{\mathcal{R}}_n$ .

**Corollary 5.11.** If  $A \in \mathcal{M}_n$  has a single eigenvalue and  $B \in \mathcal{M}_n$  commutes with A then, for any polynomials p(x) and q(x), p(A) and q(B) are in additive FFP.

*Proof.* Since A and B commute, they are simultaneously triangularizable [5, Theorem 2.3.3], that is, for some regular matrix P,  $P^TAP$  and  $P^TBP$  are upper triangular, with  $P^TAP$  having a constant diagonal. The same holds for p(A) and q(B), so, by Theorem 5.10, p(A) and q(B) are in additive FFP.

**Remark 5.12.** Unlike diagonalization, commuting matrices are simultaneously triangularizable, but the converse is not true in general.

5.4. Conclusion. We conclude this section with the following observations. Denote  $\widehat{\mathcal{V}} = \mathcal{V} \cap \mathcal{B}_n$  for any subset  $\mathcal{V} \subseteq \mathcal{M}_n$ . In particular,  $\mathcal{B}_n = \widehat{\mathcal{M}}_n$ .

- (1) This notation is compatible with Theorem 5.10 by the observation that a triangular matrix is principally balanced if and only if it has constant diagonal.
- (2) In particular, scalar matrices are exactly the principally balanced diagonal matrices,  $\widehat{\mathcal{D}}_n = \mathcal{S}_n$ .
- (3) The examples found so far can be summarized in the following "recipe":
  - 1. Pick a pair  $(\mathcal{V}_1, \mathcal{V}_2)$  among  $(\mathcal{D}_n, \mathcal{M}_n), (\mathcal{R}_n, \mathcal{R}_n)$  or  $(\mathcal{L}_n, \mathcal{L}_n)$ .
  - 2. Restrict one of the components to its principally balanced subset  $\mathcal{V}_i$ .
  - 6. Moments of sums of matrices that are in additive FFP

In [2] Arizmendi and Perales introduced **cumulants** for the additive finite free convolution as the coefficients of a truncated R-transform, by showing that they satisfy the axiomatization of cumulants as defined by Lehner in [7]. Asymptotically, the finite free cumulants converge to free cumulants. Finite free cumulants are

additive with respect to finite free convolution [2], which implies that for matrices A and B in additive FFP we have

(6.1) 
$$\kappa_i(A+B) = \kappa_i(A) + \kappa_i(B),$$

where  $\kappa_i(A)$  is the *i*-th order additive finite free cumulant of (the characteristic polynomial of) A. A similar equality holds for independent as well as for free independent random variables. However, unlike what happens in probability and in free probability, where mixed cumulants of independent (free) random variables vanish, this is not the case for finite free cumulants.

Cumulants are tightly related to moments, with concrete moment-cumulant formulas, which allows passing from one representation to the other. Such formulas were introduced in [2] in the finite free setting. This means that one can obtain formulas for the moments of A + B in terms of the moments of A and those of Bwhen A and B are in finite free position, as we show below. Conversely, when Aand B satisfy these moment formulas, for k = 2, ..., n, then they are in additive free position.

The first **moment** (or normalized trace) of  $A \in \mathcal{M}_n$  is defined to be

$$m_1(A) := \operatorname{tr}(A) = \frac{1}{n} \operatorname{Tr}(A) = \frac{1}{n} \sum_{i=1}^n \lambda_i$$

the arithmetic mean of the eigenvalues  $\lambda_i$  of A. In general, the k-th moment of A,  $k \geq 1$ , is

$$m_k(A) := \operatorname{tr}(A^k) = \frac{1}{n} \sum_{i=1}^n \lambda_i^k.$$

We make use of the following coefficient-moment formula of Lewin [8] that is based on Newton's identities. Let  $\chi_A(x) = \sum_{k=0}^n a_k x^{n-k}$ . Then,  $a_0 = 1$ , and for  $k = 1, \ldots, n$ ,

(6.2) 
$$a_k = \sum_{\pi} \prod_{i=1}^t \frac{(-n \cdot m_{r_i}(A))^{s_i}}{r_i^{s_i} s_i!}$$

where the summation is over all partitions  $\pi$  of k of the form

$$k = \underbrace{r_1 + \dots + r_1}_{s_1 \text{ summands}} + \underbrace{r_2 + \dots + r_2}_{s_2 \text{ summands}} + \dots + \underbrace{r_t + \dots + r_t}_{s_t \text{ summands}} = \sum_{i=1}^t s_i r_i,$$

with  $0 < r_1 < r_2 < \cdots < r_t$ , and  $r_i = r_i(\pi), s_i = s_i(\pi), t = t(\pi)$ .

**Theorem 6.1.** Let  $A, B \in \mathcal{M}_n$  be in additive FFP. Then the moments of A + B can be expressed in terms of the moments of A and the moments of B.

*Proof.* The proof is by induction. For r = 1, we have

$$m_1(A+B) = m_1(A) + m_1(B)$$

by linearity of the trace operator. Suppose that the statement holds for  $1 \le i < r$ . Let  $\chi_{A+B}(x) = \chi_A(x) \boxplus \chi_B(x) = \sum_{i=0}^n c_i x^{n-i}$ . By Newton's identity,

$$-rc_r = \sum_{i=0}^{r-1} c_i \operatorname{Tr}(A+B)^{r-i},$$

thus,

(6.3) 
$$m_r(A+B) = -\frac{r}{n}c_r - \sum_{i=1}^{r-1} c_i m_{r-i}(A+B).$$

Since A and B are in additive FFP, we have, for  $1 \le i \le r$ ,

$$c_i = \sum_{l+j=i} \frac{\binom{n-l}{j}}{\binom{n}{j}} a_l b_j,$$

and by (6.2), the coefficients  $a_l$  and  $b_j$  can be written in terms of the moments of A, respectively B. It remains to express the moments  $m_{r-i}(A+B)$  on the right hand side of (6.3) through the moments of A and B, which is assumed by the induction hypothesis.

Equivalently, we can obtain a complex formula for  $m_r(A + B)$  through the formulas in [2]. By Equation (4.5) in [2],  $m_r(A + B)$  can be written in terms of the cumulants  $\kappa_1(A + B), \ldots, \kappa_r(A + B)$ . Then, as shown in [2], Proposition 3.6, finite free cumulants are additive with respect to polynomial convolution, so that for each  $i, \kappa_i(A + B) = \kappa_i(A) + \kappa_i(B)$ . Finally, by Equation (4.4) in [2], each  $\kappa_i(A)$  can be written as an expression in the moments  $m_1(A), \ldots, m_i(A)$ , and similarly for  $\kappa_i(B)$ .

**Examples 6.2.** We obtain simple formulas for the first, second and third moments of A+B when  $A, B \in \mathcal{M}_n$  are in additive FFP. For the higher moments the formulas become more complex and, unlike the case  $n \leq 3$ , they depend on the dimension of the matrices. Here are the first four moments.

$$\begin{split} m_1(A+B) &= m_1(A) + m_1(B); \\ m_2(A+B) &= m_2(A) + 2m_1(A)m_1(B) + m_2(B); \\ m_3(A+B) &= m_3(A) + 3m_2(A)m_1(B) + 3m_1(A)m_2(B) + m_3(B); \\ m_4(A+B) &= m_4(A) + 4m_3(A)m_1(B) + \frac{2n}{n-1}m_2(A)m_1^2(B) \\ &+ \frac{4n-6}{n-1}m_2(A)m_2(B) - \frac{2n}{n-1}m_1^2(A)m_1^2(B) \\ &+ \frac{2n}{n-1}m_1^2(A)m_2(B) + 4m_1(A)m_3(B) + m_4(B). \end{split}$$

We demonstrate the above formula for the second moment.

$$\begin{split} m_2(A+B) &= -\frac{2}{n}c_2 - c_1m_1(A+B) \\ &= -\frac{2}{n}\Big(a_0b_2 + \frac{n-1}{n}a_1b_1 + a_2b_0\Big) - \Big((a_0b_1 + a_1b_0)m_1(A+B)\Big) \\ &= -\frac{2}{n}\Big((\frac{n^2}{2}m_1^2(B) - \frac{n}{2}m_2(B)) + \frac{n-1}{n}n^2m_1(A)m_1(B) \\ &+ (\frac{n^2}{2}m_1^2(A) - \frac{n}{2}m_2(A))\Big) \\ &+ \Big(n(m_1(A) + m_1(B))(m_1(A) + m_1(B))\Big) \\ &= -nm_1^2(B) + m_2(B) - 2(n-1)m_1(A)m_1(B) - nm_1^2(A) + m_2(A) \\ &+ nm_1^2(B) + 2nm_1(A)m_1(B) + nm_1^2(A) \\ &= m_2(A) + 2m_1(A)m_1(B) + m_2(B). \end{split}$$

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By the linearity of the moment and the fact that Tr(AB) = Tr(BA), it follows from the formulas for the second and third moments that when A and B are in FFP then:

$$m_1(AB) = m_1(A)m_1(B);$$
  
 $m_1(AB^2) + m_1(A^2B) = m_1(A)m_2(B) + m_2(A)m_1(B)$ 

Let us now examine the case when A is in additive FFP with itself. We denote by  $\mathcal{P}(j)$  the set of partitions of the set  $[j] = \{1, \ldots, j\}$ .  $\mathcal{P}(j)$  forms a lattice with a least element  $0_j = \{\{1\}, \{2\}, \ldots, \{j\}\}$  and an upper element  $1_j = \{\{1, 2, \ldots, j\}\}$ . When  $\pi = \{V_1, \ldots, V_r\} \in \mathcal{P}(j)$  is a partition of [j] with r blocks then we set  $|\pi| = r$ and denote by  $\kappa_{\pi}(A)$  the product of the cumulants

$$\kappa_{\pi}(A) := \kappa_{|V_1|}(A) \kappa_{|V_2|}(A) \cdots \kappa_{|V_r|}(A).$$

The Möbius function  $\mu(0_i, \pi)$  is

$$\mu(0_j,\pi) = (-1)^{j-|\pi|} (2!)^{r_3} (3!)^{r_4} \cdots ((j-1)!)^{r_j},$$

where  $r_i$  is the number of blocks of  $\pi$  of size *i*.

**Theorem 6.3.** Let  $A \in \mathcal{M}_n$ . Then A is in additive FFP with itself if and only if A has a single eigenvalue.

*Proof.* When A has a single eigenvalue then it is in additive FFP with itself by Corollary 5.11.

Suppose now that A is in additive FFP with itself. By (6.1),  $\kappa_j(2A) = 2\kappa_j(A)$ , for  $j \ge 1$ . On the other hand, by the definition of a cumulant [2],  $\kappa_j(2A) = 2^j \kappa_j(A)$ . It follows that

$$\kappa_j(A) = 0, \quad \text{for } j \ge 2.$$

It is shown in [2] that for  $j \ge 1$ ,

(6.4) 
$$m_j(A) = \frac{(-1)^{j-1}}{n^{j+1}(j-1)!} \sum_{\pi \in \mathcal{P}(j)} n^{|\pi|} \mu(0_j, \pi) \kappa_\pi(A) \sum_{\rho: \rho \lor \pi = 1_j} n^{|\rho|} \mu(0_j, \rho).$$

Since  $\kappa_j(A) = 0$ , for  $j \ge 2$ , then in the first summation in (6.4), all summands vanish, except for the bottom partition  $\pi = 0_j$ , and then the second summation is over the set containing just the top partition  $\rho = 1_j$ , resulting in

(6.5)  

$$m_{j}(A) = \frac{(-1)^{j-1}}{n^{j+1}(j-1)!} n^{|0_{j}|} \mu(0_{j}, 0_{j}) \kappa_{0_{j}}(A) n^{|1_{j}|} \mu(0_{j}, 1_{j})$$

$$= \frac{(-1)^{j-1}}{n^{j+1}(j-1)!} n^{j} \kappa_{1}^{j}(A) n^{1}(-1)^{j-1}(j-1)!$$

$$= m_{1}^{j}(A).$$

The last equation follows from  $\kappa_1(A) = m_1(A) = tr(A)$ .

The equations (6.5), for j = 2, ..., n, imply that A has a single eigenvalue.

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## 7. Matrices in multiplicative finite free position

In addition to additive convolution of polynomials, the notion of multiplicative convolution was introduced in [10]). We examine here matrices that are in multiplicative free position and obtain results that are similar to those in the additive case.

**Definition 7.1.** Let  $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$ ,  $q(x) = \sum_{i=0}^{n} b_i x^{n-i}$  be two polynomials of degree *n* over  $\mathbb{C}$ . The **multiplicative convolution** of p(x) and q(x), denoted  $p(x) \boxtimes q(x)$ , is

(7.1) 
$$p(x) \boxtimes q(x) := \sum_{k=0}^{n} \frac{(-1)^{k}}{\binom{n}{k}} a_{k} b_{k} x^{n-k}.$$

The following theorem is the analogue of Theorem 2.3 in the multiplicative case.

**Theorem 7.2.** [10] Let  $A, B \in \mathcal{M}_n$  be normal  $(A, B \in \mathbb{R}^{n \times n}$  be symmetric) matrices. Then

(7.2) 
$$\chi_A(x) \boxtimes \chi_B(x) = \int_{\mathcal{U}(n)} \chi_{AU^*BU}(x) \, dU = \frac{1}{2^n n!} \sum_{P \in \mathcal{P}^{\pm}(n)} \chi_{AP^TBP}(x),$$

where the expectation is taken over the set of unitary (orthogonal) matrices  $\mathcal{U}(n)$  or the signed permutation matrices  $\mathcal{P}^{\pm}(n)$ .

**Definition 7.3.** The matrices  $A, B \in \mathcal{M}_n$  are in multiplicative finite free position (or in multiplicative FFP) if

$$\chi_{AB}(x) = \chi_A(x) \boxtimes \chi_B(x).$$

The families  $\mathcal{E}_n, \mathcal{F}_n \subseteq \mathcal{M}_n$  are in multiplicative finite free position (in multiplicative FFP) if  $\chi_{AB}(x) = \chi_A(x) \boxtimes \chi_B(x)$  for every  $A \in \mathcal{E}_n$  and  $B \in \mathcal{F}_n$ .

**Remarks 7.4.** (1) As shown in [9], the property of being in multiplicative FFP can be expressed in terms of mixed discriminants.

- (2) As in the additive case,  $\chi_A(x) \boxtimes \chi_B(x)$  depends on the characteristic polynomials of the matrices and not on the matrices themselves, which is not the case for  $\chi_{AB}(x)$ . However, if A and B are in multiplicative FFP then so are  $PAP^{-1}$  and  $PBP^{-1}$ , for any regular matrix P.
- (3) For any two matrices  $A, B \in \mathcal{M}_n$ , the leading monomial and the free coefficient of  $\chi_{AB}(x)$  and  $\chi_A(x) \boxtimes \chi_B(x)$  are the same:  $x^n$  and  $(-1)^n \det(AB) = (-1)^n \det(A) \det(B)$ .

7.1. Matrices of dimension  $2 \times 2$ . For matrices  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{2 \times 2}$ , we have

$$\chi_{AB}(x) = x^2 - (a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22})x + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22}$$

and

$$\chi_A(x) \boxtimes \chi_B(x) = x^2 - \frac{1}{2}(a_{11} + a_{22})(b_{11} + b_{22})x + (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}).$$

By comparing the coefficients of the powers of x, we get the following result.

**Proposition 7.5.** The matrices  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{2 \times 2}$  are in multiplicative *FFP* if and only if

$$(a_{11} - a_{22})(b_{22} - b_{11}) = 2(a_{12}b_{21} + a_{21}b_{12}).$$

This is the same condition as in (3.1), although here the identical coefficients are of powers 2 and 0 of x, whereas in the additive case the identical coefficients are of powers 2 an 1 of x.

It follows, that also in the multiplicative case, a  $2 \times 2$  matrix A is in FFP with itself if and only if it has a single eigenvalue. The analogue of Proposition 3.2 is the following proposition.

**Proposition 7.6.** Let  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{C}^{2 \times 2}$  be in multiplicative FFP. Then p(A) and q(B) are in multiplicative FFP, for any polynomials p(x) and q(x). Moreover, if A is regular then  $A^{-1}$  and q(B) are in multiplicative FFP.

7.2. Matrices of dimension  $3 \times 3$ . Let us look at the following example.

Example 7.7. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then  $\chi_A(x) = x^3 - x^2$ ,  $\chi_B(x) = x^3 - 3x^2 + 2x$ ,  $\chi_{A+B}(x) = x^3 - 4x^2 + 4x + 1$  and  $\chi_{AB}(x) = x^3 - x^2$ . The additive and multiplicative convolutions are  $\chi_A(x) \boxplus \chi_B(x) = x^3 - 4x^2 + 4x - \frac{2}{3}$  and  $\chi_A(x) \boxtimes \chi_B(x) = x^3 - x^2$ . We see that unlike the situation in dimension  $2 \times 2$ , here the matrices A and B are in multiplicative FFP but not in additive FFP.

Let us now consider algebraic operations.

## Example 7.8.

- (i) Adding a scalar does not preserve multiplicative FFP: The matrices A and B from example 7.7 are in multiplicative FFP, but the pair (I + A, B) is not.
- (ii) Squaring and inverting do not preserve multiplicative FFP: The matrices

	Γ1	0	0		1	-1	
A =	0	2	0	B =	-1	-2	1
	0	0	3		2	1	1

are in multiplicative FFP, but the pairs  $(A^2, B)$  and  $(A^{-1}, B)$  are not.

7.3. The lattice of multiplicative finite free varieties. The lattice of multiplicative finite free varieties is constructed in a way similar to the additive one.

**Definition 7.9.** Given a family  $S \subseteq M_n$ , its multiplicative finite free complement, denoted  $S^{\boxtimes}$ , is

$$\mathcal{S}^{\boxtimes} = \{ B \in \mathcal{M}_n : \chi_{AB}(x) = \chi_A(x) \boxtimes \chi_B(x), \text{ for all } A \in \mathcal{S} \}.$$

When  $\mathcal{E}_n^{\boxtimes} = \mathcal{F}_n$  and  $\mathcal{F}_n^{\boxtimes} = \mathcal{E}_n$ , we say that  $\mathcal{E}_n, \mathcal{F}_n$  form a **multiplicative finite** free complementary pair (or a multiplicative complementary pair).

**Proposition 7.10.** If A and B are in multiplicative FFP then so are  $\lambda A$  and B, for every  $\lambda \in \mathbb{C}$ .

*Proof.* By definition, in both  $\chi_{\lambda A}(x) \boxtimes \chi_B(x)$  and  $\chi_{\lambda AB}(x)$ , the coefficients of  $x^{n-k}$  are multiplied by  $\lambda^k$  compared to the coefficients of  $\chi_A(x) \boxtimes \chi_B(x)$  and  $\chi_{AB}(x)$ .  $\Box$ 

7.4. Diagonal, principally balanced and symmetric matrices. We show here that the pairs of complementary varieties with respect to the additive convolution, discussed in Section 5, are also complementary pairs in the multiplicative setting.

**Theorem 7.11.** Let  $\mathcal{D}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  diagonal matrices and let  $\mathcal{B}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  principally balanced matrices. Then  $\mathcal{D}_n$  and  $\mathcal{B}_n$  form a multiplicative complementary pair.

*Proof.* The theorem clearly holds for n = 1, so let us assume that  $n \ge 2$ . First, we show that for arbitrary  $D \in \mathcal{D}_n$  and  $B \in \mathcal{B}_n$ , the equality  $\chi_{DB}(x) = \chi_D(x) \boxtimes \chi_B(x)$  holds. Let  $D = \text{diag}(d_1, \ldots, d_n)$ ,

$$\chi_D(x) = \sum_{i=0}^n a_i x^{n-i} = \sum_{i=0}^n \left( (-1)^i \sum_{J \subseteq [n], \, |J|=i} d_J \right) x^{n-i},$$

where  $d_J = d_{j_1} \cdots d_{j_i}$  for  $J = \{j_1, \dots, j_i\}$ . Let also

$$\chi_B(x) = \sum_{i=0}^n b_i x^{n-i} = \sum_{i=0}^n (-1)^i \binom{n}{i} m_i x^{n-i},$$

where  $m_i$ , i = 0, ..., n, is the value of a principal minor of B of order i. Since

$$DB = \begin{bmatrix} d_1 B_1 \\ d_2 B_2 \\ \vdots \\ d_n B_n \end{bmatrix},$$

where  $B_{i}$  denotes the *i*-th row of B, we have

(7.3)

$$\chi_{DB}(x) = \sum_{i=0}^{n} \left( (-1)^{i} \sum_{J \subseteq [n], |J|=i} \det((DB)_{J}) \right) x^{n-i}$$
$$= \sum_{i=0}^{n} \left( (-1)^{i} \sum_{J \subseteq [n], |J|=i} d_{J} \det(B_{J}) \right) x^{n-i}$$
$$= \sum_{i=0}^{n} \left( (-1)^{i} \sum_{J \subseteq [n], |J|=i} d_{J} m_{i} \right) x^{n-i}$$
$$= \sum_{i=0}^{n} a_{i} m_{i} x^{n-i} = \sum_{i=0}^{n} a_{i} \frac{(-1)^{i} b_{i}}{\binom{n}{i}} x^{n-i}.$$

By Definition 7.1, this is exactly  $\chi_D(x) \boxtimes \chi_B(x)$  and it follows that  $\mathcal{B}_n \subseteq \mathcal{D}_n^{\boxtimes}$  and  $\mathcal{D}_n \subseteq \mathcal{B}_n^{\boxtimes}$ 

Next, we show that  $\mathcal{D}_n^{\boxtimes} \subseteq \mathcal{B}_n$ . Let  $B = (b_{ij}) \in \mathcal{D}_n^{\boxtimes}$ . Let  $K \subseteq [n], |K| = k$ , 0 < k < n, and let  $D(\lambda, K) = (d_{ij})$  be the parametric diagonal matrix with entries

$$d_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ \lambda & \text{for } i = j \in K, \\ 1 & \text{for } i = j \notin K. \end{cases}$$

$$\chi_{D(\lambda,K)B}(x) = \det(xI - (D(\lambda,K)B))$$
  
=  $x^n + p_1(\lambda)x^{n-1} + \dots + p_{n-1}(\lambda)x + (-1)^n \det(D(\lambda,K)B),$ 

$$(\chi_{D(\lambda,K)B})^{(n-k)}(0) = (n-k)! p_k(\lambda) = (-1)^k (n-k)! \lambda^k \det(B_K) + q(\lambda),$$

where  $q(\lambda)$  is a polynomial in  $\lambda$  of degree less than k. We have

(7.4) 
$$\lim_{\lambda \to \infty} \frac{(\chi_{D(\lambda,K)B})^{(n-k)}(0)}{(-1^k)(n-k)!\,\lambda^k} = \det(B_K).$$

We now compute the same limit for  $\chi_{D(\lambda,K)}(x) \boxtimes \chi_B(x)$ . The characteristic polynomial of  $D(\lambda, K)(x)$  is

$$\chi_{D(\lambda,K)}(x) = \sum_{i=0}^{n} a_i x^{n-i} = (x-\lambda)^k (x-1)^{n-k}.$$

Then,

(7.5) 
$$a_k = (-1)^k \lambda^k + g(\lambda),$$

where  $g(\lambda)$  is a polynomial in  $\lambda$  of degree less than k. The characteristic polynomial of B is  $\chi_B(x) = \sum_{i=0}^n b_i x^{n-i}$  with

(7.6) 
$$b_k = (-1)^k \sum_{I \subseteq [n], |I|=k} \det(B_I).$$

By (7.1),

$$(\chi_{D(\lambda,K)} \boxtimes \chi_B)^{(n-k)}(0) = \left(\sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} a_k b_k x^{n-k}\right)^{(n-k)}(0)$$
$$= \frac{(-1)^k (n-k)!}{\binom{n}{k}} a_k b_k$$

Substituting for  $a_k$  and  $b_k$  the expressions in (7.5) and (7.6), we get

$$= \frac{(-1)^k (n-k)!}{\binom{n}{k}} ((-1)^k \lambda^k + g(\lambda)) (-1)^k \sum_{I \subseteq [n], |I|=k} \det(B_I).$$

Computing the limit as in (7.4) gives

(7.7) 
$$\lim_{\lambda \to \infty} \frac{(\chi_{D(\lambda,K)} \boxtimes \chi_B)^{(n-k)}(0)}{(-1^k)(n-k)! \,\lambda^k} = \frac{1}{\binom{n}{k}} \sum_{I \subseteq [n], |I|=k} \det(B_I).$$

Comparing (7.4) with (7.7), we get that

$$\det(B_K) = \frac{1}{\binom{n}{k}} \sum_{I \subseteq [n], |I|=k} \det(B_I),$$

and as in Theorem 5.7, it follows that B is principally balanced and therefore  $\mathcal{D}_n^{\boxtimes} \subseteq \mathcal{B}_n$ . Together with the inclusion in the opposite direction, we have  $\mathcal{D}_n^{\boxtimes} = \mathcal{B}_n$ .

Finally, we show that  $\mathcal{B}_n^{\boxtimes} \subseteq \mathcal{D}_n$ . So, assume by contradiction that  $A = (a_{ij}) \in \mathcal{B}_n^{\boxtimes}$  and  $A \notin \mathcal{D}_n$ . It follows that for some  $(j_0, i_0), j_0 \neq i_0, a_{j_0i_0} = a \neq 0$ . Let  $B = (b_{ij})$  be the matrix  $B = I_n + E_n(i_0j_0)$ :

$$b_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 1 & \text{for } (i, j) = (i_0, j_0), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $B \in \mathcal{B}_n$ . Let the characteristic polynomial of A be  $\chi_A(x) = \sum_{i=0}^n a_i x^{n-i}$ and that of B:

$$\chi_B(x) = \sum_{i=0}^n b_i x^{n-i} = (x-1)^n = \sum_{i=0}^n \binom{n}{i} (-i)^i x^{n-i}.$$

Since Tr(AB) = Tr(A) + a, the characteristic polynomial of AB is

(7.8) 
$$\chi_{AB}(x) = x^n + (a_1 - a)x^{n-1} + q(x),$$

where q(x) is a polynomial in x of degree less than n-1.

The multiplicative convolution of  $\chi_A(x)$  and  $\chi_B(x)$  is

(7.9) 
$$\chi_A(x) \boxtimes \chi_B(x) = \sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} a_k b_k x^{n-k}$$

(7.10) 
$$= \sum_{k=0}^{n} \frac{(-1)^{k}}{\binom{n}{k}} a_{k} \binom{n}{k} (-1)^{k} x^{n-k}$$

(7.11) 
$$= \sum_{k=0}^{n} a_k x^{n-k} = \chi_A(x).$$

We see that the coefficients of  $x^{n-1}$  in (7.8) and in (7.9) are different from each other, implying that  $\chi_{AB}(x) \neq \chi_A(x) \boxtimes \chi_B(x)$ . It follows that  $\mathcal{B}_n^{\boxtimes} \subseteq \mathcal{D}_n$  and with the inclusion in the opposite direction, we have  $\mathcal{B}_n^{\boxtimes} = \mathcal{D}_n$ .

The next corollary is analogues to Corollary 5.8 and is proved in a similar way.

**Corollary 7.12.** Let  $\mathcal{A}_n \subseteq \mathcal{M}_n$  be a maximal family of commuting diagonalizable matrices and let  $\mathcal{B}_n$  be the family of principally balanced matrices. Then there exists a non-singular matrix P, such that  $\mathcal{A}_n$  and  $P^{-1}\mathcal{B}_n P$  form a multiplicative complementary pair.

If  $\mathcal{A}_n$  is a maximal family of commuting symmetric matrices then there exists an orthogonal matrix U, such that  $\mathcal{A}_n$  and  $U^T \mathcal{B}_n U$  form a multiplicative complementary pair.

Similarly to Theorem 2.3 we have the following theorem of [10] with respect to the multiplicative convolution.

**Theorem 7.13.** Let  $A, B \in \mathbb{R}^{n \times n}$  with A being symmetric. Then

(7.12) 
$$\chi_A(x) \boxtimes \chi_B(x) = \mathbb{E}_U [\chi_{AU^T BU}(x)],$$

where the expectation is taken over the set of orthogonal matrices U.

7.5. **Triangular matrices.** As with the additive case, the upper triangular matrices and the upper triangular matrices with constant diagonal form a multiplicative complementary pair, and similarly for lower triangular matrices.

**Theorem 7.14.** Let  $\mathcal{R}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  upper triangular matrices and let  $\widehat{\mathcal{R}}_n \subseteq \mathcal{M}_n$  be the family of  $n \times n$  upper triangular matrices with constant diagonal. Then  $\mathcal{R}_n$  and  $\widehat{\mathcal{R}}_n$  form a multiplicative complementary pair.

*Proof.* The lines of proof are analogous to those of Theorem 5.10. We only mention that when  $A = (a_{ij}) \notin \mathcal{R}_n$  with  $a_{j_0i_0} = a \neq 0$ , for some  $j_0 > i_0$ , then we define  $B = (b_{ij}) \in \widehat{\mathcal{R}}_n$  to be the matrix  $B = I_n + E_n(i_0j_0)$  as in Theorem 7.11. Then  $\chi_A(x) \boxtimes \chi_B(x) = \chi_A(x)$  whereas  $\chi_{AB}(x) \neq \chi_A(x)$ .

7.6. Scalar matrices. Also for scalar matrices the multiplicative convolution behaves analogously to the additive convolution.

**Theorem 7.15.** Let  $S_n \subseteq M_n$  be the family of  $n \times n$  scalar matrices. Then  $S_n$  and  $M_n$  form a multiplicative complementary pair.

*Proof.* First, we show that when  $A = (a_{ij}) \in \mathcal{M}_n$  and  $C = \text{diag}(c, \ldots, c)$  then  $\chi_{AC}(x) = \chi_A(x) \boxtimes \chi_C(x)$ . As before, let

$$\chi_A(x) = \sum_{i=0}^n a_i x^{n-i},$$

and

$$\chi_C(x) = \sum_{i=0}^n c_i x^{n-i} = \sum_{i=0}^n (-1)^i \binom{n}{i} c^i x^{n-i}$$

Then

$$\chi_{AC}(x) = \sum_{k=0}^{n} a_k c^k x^{n-k}$$

and

$$\chi_A(x) \boxtimes \chi_C(x) = \sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} a_k c_k x^{n-k}$$
$$= \sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} a_k (-1)^k \binom{n}{k} c^k x^{n-k}$$
$$= \sum_{k=0}^n a_k c^k x^{n-k} = \chi_{AC}(x).$$

The rest of the proof is similar to that of Theorem 5.9.

7.7. Matrices in multiplicative FFP with themselves. The following result is similar to what happens in the additive case.

**Proposition 7.16.** Let  $A \in \mathcal{M}_n$ . Then A is in multiplicative FFP with itself if and only if A has a single eigenvalue.

*Proof.* By similarity transformation, we can assume, without loss of generality, that A is in upper triangular form with its eigenvalues  $\lambda_1, \lambda_2, \ldots, \lambda_n$  on the diagonal. Let  $\chi_A(x) = \sum_{k=0}^n a_k x^{n-k}$ , then

$$a_k = (-1)^k \sum_{\substack{I = \{i_1, i_2, \dots, i_k\}\\1 \le i_1 < i_2 < \dots < i_k \le n}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

Hence

(7.13) 
$$\chi_{A^2}(x) = \sum_{k=0}^n (-1)^k \sum_{\substack{I=\{i_1,i_2,\dots,i_k\}\\1 \le i_1 < i_2 < \dots < i_k \le n}} \lambda_{i_1}^2 \lambda_{i_2}^2 \cdots \lambda_{i_k}^2 x^{n-k}.$$

On the other hand,

(7.14) 
$$\chi_A(x) \boxtimes \chi_A(x) = \sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} a_k^2 x^{n-k} = \sum_{k=0}^n \frac{(-1)^k}{\binom{n}{k}} \Big(\sum_{\substack{I=\{i_1,i_2,\dots,i_k\}\\1\le i_1< i_2<\dots< i_k\le n}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \Big)^2 x^{n-k}.$$

By comparing the corresponding coefficients in (7.13) and (7.14), for k = 1, ..., n-1, we get that  $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ .

In the other direction, when all eigenvalues of A are the same then A is similar to an upper triangular matrix with a constant diagonal and by Theorem 7.14, A is in multiplicative FFP with itself.

7.8. Moments of products of matrices that are in multiplicative FFP. Moments and cumulants for the multiplicative convolution of polynomials were handled by Arizmendi et al. [1]. Among others, they derived a formula for the finite free cumulants of the multiplicative convolution of polynomials p and q in terms of the finite free cumulants of p and of q. They also were able to express the moments of the empirical root distribution of  $p \boxtimes q$  in terms of the finite free cumulants of p and the moments of q. As shown in [1], finite free multiplicative convolution converges to the free multiplicative convolution.

As in the additive case, when A and B are in multiplicative FFP, we can compare the coefficients of  $\chi_A(x) \boxtimes \chi_B(x)$  with the corresponding coefficients of  $\chi_{AB}(x)$  and derive formulas for the moments of AB in terms of those of A and B. Next, we derive these formulas for the first and second moments.

**Proposition 7.17.** Let  $A, B \in \mathcal{M}_n$  be in multiplicative FFP. Then

(1) 
$$m_1(AB) = m_1(A)m_1(B).$$
  
(2)  $m_2(AB) = \frac{n}{n-1}[m_2(A)m_1^2(B) + m_1^2(A)m_2(B) - m_1^2(A)m_1^2(B)] - \frac{1}{n-1}m_2(A)m_2(B)$ 

*Proof.* (1) Let  $\chi_A(x) = \sum_{k=0}^n a_k x^{n-k}$  and  $\chi_B(x) = \sum_{k=0}^n b_k x^{n-k}$ . The coefficient of  $x^{n-1}$  in  $\chi_A(x) \boxtimes \chi_B(x)$  is

(7.15) 
$$-\frac{1}{n}a_1b_1 = -\frac{1}{n}\operatorname{Tr}(A)\operatorname{Tr}(B) = -n \cdot m_1(A)m_1(B).$$

whereas the coefficient of  $x^{n-1}$  in  $\chi_{AB}(x)$  is

(7.16) 
$$-\operatorname{Tr}(AB) = -n \cdot m_1(AB).$$

The result then follows.

(2) The coefficient of  $x^{n-2}$  in  $\chi_A(x) \boxtimes \chi_B(x)$  is  $-\frac{1}{\binom{n}{2}}a_2b_2$ , which equals, by (6.2),  $2 \qquad \left(\operatorname{Tr}^2(A) \qquad \operatorname{Tr}(A^2)\right) \left(\operatorname{Tr}^2(B) \qquad \operatorname{Tr}(B^2)\right)$ 

$$-\frac{2}{n(n-1)}\left(\frac{\Pi(A)}{2} - \frac{\Pi(A)}{2}\right)\left(\frac{\Pi(D)}{2} - \frac{\Pi(D)}{2}\right)$$

In terms of moments it equals

(7.17) 
$$\frac{n}{2(n-1)} [n^2 \cdot m_1^2(A)m_1^2(B) + m_2(A)m_2(B) - n \cdot m_1^2(A)m_2(B) - n \cdot m_2(A)m_1^2(B)].$$

The coefficient of  $x^{n-2}$  in  $\chi_{AB}(x)$  is

$$\frac{\operatorname{Tr}^2(AB)}{2} - \frac{\operatorname{Tr}(AB)^2}{2}$$

In terms of moments and by formula (1) it equals

(7.18) 
$$\frac{n^2}{2} \cdot m_1^2(A)m_1^2(B) - \frac{n}{2} \cdot m_2(AB).$$

Formula (2) is then obtained by the fact that (7.17) and (7.18) are equal for matrices in multiplicative FFP.  $\hfill \Box$ 

Comparing other coefficients of powers of x do not result in simple and nice formulas for higher moments of AB.

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