# MULTIPLICATIVE AND SEMI-MULTIPLICATIVE FUNCTIONS ON NON-CROSSING PARTITIONS, AND RELATIONS TO CUMULANTS 

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#### Abstract

We consider the group $(\mathcal{G}, *)$ of unitized multiplicative functions in the incidence algebra of non-crossing partitions, where "*" denotes the convolution operation. We introduce a larger group ( $\widetilde{\mathcal{G}}, *$ ) of unitized functions from the same incidence algebra, which satisfy a weaker semi-multiplicativity condition. The natural action of $\tilde{\mathcal{G}}$ on sequences of multilinear functionals of a non-commutative probability space captures the combinatorics of transitions between moments and some brands of cumulants that are studied in the non-commutative probability literature. We use the framework of $\widetilde{\mathcal{G}}$ in order to explain why the multiplication of free random variables can be very nicely described in terms of Boolean cumulants and more generally in terms of $t$-Boolean cumulants, a oneparameter interpolation between free and Boolean cumulants arising from work of Bożejko and Wysoczanski.

It is known that the group $\mathcal{G}$ can be naturally identified as the group of characters of the Hopf algebra Sym of symmetric functions. We show that $\tilde{\mathcal{G}}$ can also be identified as group of characters of a Hopf algebra $\mathcal{T}$, which is an incidence Hopf algebra in the sense of Schmitt. Moreover, the inclusion $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$ turns out to be the dual of a natural bialgebra homomorphism from $\mathcal{T}$ onto Sym.


## 1. Introduction

### 1.1. The group $\mathcal{G}$ of unitized multiplicative functions on $N C(n)$ 's.

The idea of studying the convolution of multiplicative functions defined on the set of all intervals of a "coherent" collection of lattices $\left(\mathcal{L}_{n}\right)_{n=1}^{\infty}$ goes back to the 1960 's work of Rota and collaborators, e.g. in [10]. The phenomenon which prompts this study is that, in a number of important examples: for every $\pi \leq \sigma$ in an $\mathcal{L}_{n}$, the sublattice $[\pi, \sigma]:=\left\{\rho \in \mathcal{L}_{n} \mid\right.$ $\pi \leq \rho \leq \sigma\}$ of $\mathcal{L}_{n}$ is canonically isomorphic to a direct product,

$$
\begin{equation*}
[\pi, \sigma] \approx \mathcal{L}_{1}^{p_{1}} \times \cdots \times \mathcal{L}_{n}^{p_{n}}, \quad \text { with } p_{1}, \ldots, p_{n} \geq 0 . \tag{1.1}
\end{equation*}
$$

A function $f: \sqcup_{n=1}^{\infty}\left\{(\pi, \sigma) \mid \pi, \sigma \in \mathcal{L}_{n}, \pi \leq \sigma\right\} \rightarrow \mathbb{C}$ is declared to be multiplicative when there exists a sequence of complex numbers $\left(\alpha_{n}\right)_{n=1}^{\infty}$ such that, for $\pi, \sigma$ and non-negative integers $p_{1}, \ldots, p_{n}$ as in (1.1), one has $f(\pi, \sigma):=\alpha_{1}^{p_{1}} \cdots \alpha_{n}^{p_{n}}$.

In the present paper we are interested in the case when $\mathcal{L}_{n}$ is the lattice $N C(n)$ of noncrossing partitions of $\{1, \ldots, n\}$, endowed with the partial order by reverse refinement. In the 1990's it was found by Speicher [29] that, when considered in connection to the $N C(n)$ 's, the convolution of multiplicative functions plays an essential role in the combinatorial development of free probability. For the purposes of the present paper it is convenient to focus on the set $\mathcal{G}$ consisting of multiplicative functions on the $N C(n)$ 's where the sequence $\left(\alpha_{n}\right)_{n=1}^{\infty}$ defining the function has $\alpha_{1}=1$. Then $\mathcal{G}$ is a group under convolution. While this is a self-standing structure, which can be considered without any knowledge of what is a

[^0]non-commutative probability space, it nevertheless turns out that the group operation of $\mathcal{G}$ encapsulates the combinatorics of the multiplication of free random variables; for a detailed presentation of how this goes, we refer to Lectures 14 and 18 of the monograph [24].

### 1.2. The group $\widetilde{\mathcal{G}}$ of unitized semi-multiplicative functions on $N C(n)$ 's.

 In the case of the lattices $N C(n)$, the canonical isomorphism indicated in (1.1) is obtained by combining two kinds of lattice isomorphisms, as follows.First kind of isomorphism: one observes that for every $\pi \leq \sigma$ in some $N C(n)$, the interval $[\pi, \sigma] \subseteq N C(n)$ is canonically isomorphic to a direct product of intervals of the form $\left[\theta, 1_{k}\right]$, with $\theta \in N C(k)$ for some $1 \leq k \leq n$, and where $1_{k}$ is the maximal element of $N C(k)$, i.e. it is the partition of $\{1, \ldots, k\}$ into a single block.

Second kind of isomorphism: for every $k \geq 1$ and $\theta \in N C(k)$ one finds $\left[\theta, 1_{k}\right]$ to be canonically isomorphic to a direct product $N C(1)^{q_{1}} \times \cdots \times N C(k)^{q_{k}}$, with $q_{1}, \ldots, q_{k} \geq 0$.

These two kinds of isomorphisms will be reviewed precisely as soon as the notation is set for them, cf. Remark 2.3 below. But we signal right now that our main point is this:

> It is worth studying convolution for functions
> $g: \sqcup_{n=1}^{\infty}\{(\pi, \sigma) \mid \pi, \sigma \in N C(n), \pi \leq \sigma\} \rightarrow \mathbb{C}$ which
> are only required to be multiplicative with respect
> to the first kind of isomorphism mentioned above.

We will use the term semi-multiplicative for a function $g$ as in (1.2), and we will denote

$$
\widetilde{\mathcal{G}}=\left\{g: \sqcup_{n=1}^{\infty}\{(\pi, \sigma) \mid \pi \leq \sigma \text { in } N C(n)\} \rightarrow \mathbb{C} \left\lvert\, \begin{array}{l}
g \text { is semi-multiplicative and }  \tag{1.3}\\
g(\pi, \pi)=1, \forall \pi \in \sqcup_{n=1}^{\infty} N C(n)
\end{array}\right.\right\}
$$

It turns out that $\widetilde{\mathcal{G}}$ is a group under convolution. This group and some of its subgroups (in particular the subgroup $\mathcal{G}$ from Section 1.1) are the main players in the considerations of the present paper. The benefits that come from studying $\widetilde{\mathcal{G}}$ are presented in the next subsections of this Introduction.

### 1.3. Relations of $\widetilde{\mathcal{G}}$ with moments and with some brands of cumulants.

Consider now the framework of a non-commutative probability space $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ and $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ is a linear functional such that the algebra unit is mapped to one $\left(\varphi\left(1_{\mathcal{A}}\right)=1\right)$, and look at

$$
\mathfrak{M}_{\mathcal{A}}:=\left\{\underline{\psi} \mid \underline{\psi}=\left(\psi_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}, \text { where } \psi_{n} \text { is an } n \text {-linear functional }\right\} .
$$

In $\mathfrak{M}_{\mathcal{A}}$ we have a special element $\underline{\varphi}=\left(\varphi_{n}\right)_{n=1}^{\infty}$ called family of moment functionals of $(\mathcal{A}, \varphi)$, where $\varphi_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}$ is defined by putting $\varphi_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right):=\varphi\left(a_{1} a_{2} \cdots a_{n}\right)$ for all $n \geq 1$ and $a_{1}, \ldots, a_{n} \in \mathcal{A}$. Then in $\mathfrak{M}_{\mathcal{A}}$ there also are several families of cumulant functionals which relate to $\varphi$ via summation formulas over non-crossing partitions, and receive constant attention in the research literature on non-commutative probability: free cumulants, Boolean cumulants, monotone cumulants (see e.g. (1). In this paper we also devote some attention to a continuous interpolation between Boolean and free cumulants, which we refer to as $t$-Boolean cumulants, and are arising from the work of Bożejko and Wysoczanski [7 - the case $t=0$ gives Boolean cumulants and the case $t=1$ gives free cumulants.

The group $\widetilde{\mathcal{G}}$ has a natural action on $\mathfrak{M}_{\mathcal{A}}$, which is discussed in detail in Section 6 of the paper. This action captures the transitions between moment functionals and the brands of cumulants mentioned above, and as a consequence it also captures the formulas for direct transitions between two such brands of cumulants. We mention that the study of direct
transitions between different brands of cumulants goes back to the work of Lehner 20, and was thoroughly pursued in [1]. The benefit of using the group $\widetilde{\mathcal{G}}$ is that it offers an efficient framework for streamlining calculations related to various moment-cumulant and inter-cumulant formulas.

For full disclosure, we reiterate here a fact implicitly present in the above discussion, namely that this paper only addresses brands of cumulants which live within the world of non-crossing partitions. It is an interesting direction of future research to clarify how some of the considerations of the paper can be adjusted to the setting of full lattices of partitions of sets $\{1, \ldots, n\}$ (where crossings are allowed). The examination of this direction has been started in Chapters 5-7 of the thesis [25], and promises to extend the results of the present paper to a setting which will also include the "classical" cumulants commonly used in the probability literature.

Returning to the group $\widetilde{\mathcal{G}}$, our next point is that it is possible to identify precisely some notions of what it means for a function $h \in \widetilde{\mathcal{G}}$ to be of cumulant-to-moment type, and what it means for a $g \in \widetilde{\mathcal{G}}$ to be of cumulant-to-cumulant type. This is done in Section 7 of the paper. Denoting

$$
\begin{gathered}
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}=\{h \in \widetilde{\mathcal{G}} \mid h \text { is of cumulant-to-moment type }\} \text { and } \\
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}=\{g \in \widetilde{\mathcal{G}} \mid g \text { is of cumulant-to-cumulant type }\}
\end{gathered}
$$

we prove in Section 8 that $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ is a subgroup of $(\widetilde{\mathcal{G}}, *)$, while $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ is a right coset of $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. The latter statement means that we have

$$
\begin{equation*}
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}=\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}} * h:=\left\{g * h \mid g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}\right\}, \tag{1.4}
\end{equation*}
$$

for no matter what $h \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ we choose to fix. An easy choice is to fix the $h$ which is identically equal to 1 ; this is indeed a function in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$, and encodes the transition from free cumulants to moment functionals. However, as pointed out in Section 8.2 of the paper, it seems to be more advantageous (both for writing proofs and for finding applications) if in (1.4) we use a different choice for $h$, and pick the function which encodes the transition from Boolean cumulants to moments.

### 1.4. The 1-parameter subgroup $\left\{u_{q} \mid q \in \mathbb{R}\right\}$ of $\widetilde{\mathcal{G}}_{c-c}$.

The method we use for proving (1.4) draws attention to the subgroup of $\widetilde{\mathcal{G}}$ generated by the function which encodes transition between free cumulants and Boolean cumulants. In the notation system used throughout the paper, the latter function is denoted as $g_{\mathrm{fc}-\mathrm{bc}}$. The subgroup $\left\{g_{\mathrm{fc}-\mathrm{bc}}^{p} \mid p \in \mathbb{Z}\right\} \subseteq \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ can be naturally incorporated into a continuous 1 parameter subgroup of $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$, which we denote as $\left\{u_{q} \mid q \in \mathbb{R}\right\}$ (thus $u_{q}=g_{\mathrm{fc}-\mathrm{bc}}^{q}$ for $q \in \mathbb{Z}$ ). Working with the $u_{q}$ 's nicely streamlines the various formulas involving $t$-Boolean cumulants, and in particular gives an easy way (cf. Corollary 9.5 below) to write the transition formula between $s$-Boolean cumulants and $t$-Boolean cumulants for distinct values $s, t \in \mathbb{R}$.

In Section 10 we prove that every $u_{q}$ belongs to the normalizer of the subgroup $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$ from Section 1.1:

$$
\begin{equation*}
(q \in \mathbb{R}, f \in \mathcal{G}) \Rightarrow u_{q}^{-1} * f * u_{q} \in \mathcal{G} . \tag{1.5}
\end{equation*}
$$

This is a non-trivial fact, as the $u_{q}$ 's are coming from $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$, and there is no obvious direct connection between $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ and $\mathcal{G}$ - it is, in any case, easy to check that the intersection $\mathcal{G} \cap \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ only contains the unit $e$ of $\widetilde{\mathcal{G}}$, while the intersection of $\mathcal{G}$ with the coset $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ only contains the function $h$ which is constantly equal to 1 .
1.5. Multiplication of free random variables, in terms of $t$-Boolean cumulants. The result obtained in (1.5) can be used in order to give a neat explanation of the intriguing fact that the multiplication of freely independent random variables is nicely described in terms of Boolean cumulants (who aren't a priori meant to be related to free probability).

We find it convenient to place the discussion in the more general framework of $t$-Boolean cumulants. So let us consider a non-commutative probability space $(\mathcal{A}, \varphi)$, let $x, y$ be two freely independent elements of $\mathcal{A}$, and let $t$ be a parameter with values in $\mathbb{R}$. What happens is that the formula describing the $t$-Boolean cumulants of the product $x y$ in terms of the separate $t$-Boolean cumulants of $x$ and of $y$ is one and the same, no matter what value of $t$ we are using. More precisely: denoting the family of $t$-Boolean cumulants as $\underline{\beta}^{(t)}=\left(\beta_{n}^{(t)}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$, the formula for $t$-Boolean cumulants of $x y$ says that:

$$
\begin{equation*}
\beta_{n}^{(t)}(x y, \ldots, x y)=\sum_{\pi \in N C(n)} \beta_{\pi}^{(t)}(x, \ldots, x) \cdot \beta_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y), \quad \forall n \geq 1 \tag{1.6}
\end{equation*}
$$

Equation (1.6) contains some notation that has to be clarified (such as what is the multilinear functional $\beta_{\pi}^{(t)}: \mathcal{A}^{n} \rightarrow \mathbb{C}$ associated to a partition $\pi \in N C(n)$, and the fact that every $\pi \in N C(n)$ has a complement $\operatorname{Kr}(\pi) \in N C(n))$. All the necessary notation will be reviewed in the body of the paper; the reason for giving the formula (1.6) at this point is so that we can explain our way of proving it.

Our approach can be summarized as follows. For every $t \in \mathbb{R}$, consider the statement:

$$
\text { (Statement t) } \quad\left\{\begin{array}{c}
\text { The formula (1.6) holds true for this } t \\
\text { and for any freely independent elements } x, y \text { in } \\
\text { some non-commutative probability space }(\mathcal{A}, \varphi)
\end{array}\right\} .
$$

The action by conjugation of the $u_{q}$ 's on multiplicative functions allows us to prove the following fact:

Fact. If there exists a $t_{o} \in \mathbb{R}$ for which (Statement $t_{o}$ ) is true, then it follows that (Statement $t$ ) is true for all $t \in \mathbb{R}$.
But it has been known since the 1990's that (Statement $t_{o}$ ) is true for $t_{o}=1$ - this is the very basic description of multiplication of free random variables in terms of free cumulants, cf. [24, Theorem 14.4]. The above "Fact" then assures us that (Statement $t$ ) is indeed true for all $t$; in particular, at $t=0$ we retrieve the result (first found in [2] via a direct combinatorial analysis) about how multiplication of free random variables is described in terms of Boolean cumulants.

### 1.6. Hopf algebra aspects.

A significant fact about the group $\mathcal{G}$ from Section 1.1, observed in [22], is that it can be naturally identified as the group of characters of the Hopf algebra Sym of symmetric functions. When combined with the log map for characters of Sym, this identification retrieves the celebrated $S$-transform of Voiculescu [32], which is the most efficient tool for computing distributions of products of free random variables.

In analogy to that, we present the construction of a Hopf algebra $\mathcal{T}$, done in such a way that the character group $\mathbb{X}(\mathcal{T})$ is naturally isomorphic to $\widetilde{\mathcal{G}}$. $\mathcal{T}$ can be identified as an incidence Hopf algebra, cf. [27, 12], and is also closely related to one of the Hopf algebras studied in the recent paper [11]. Moreover, we find that the inclusion of groups $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$ is precisely (in view of the canonical isomorphisms $\mathcal{G} \approx \mathbb{X}(\mathrm{Sym})$ and $\widetilde{\mathcal{G}} \approx \mathbb{X}(\mathcal{T})$ ) the dual $\Psi^{*}: \mathbb{X}(\mathrm{Sym}) \rightarrow \mathbb{X}(\mathcal{T})$ of a natural bialgebra homomorphism $\Psi: \mathcal{T} \rightarrow$ Sym provided by the Kreweras complementation map.

A promising feature of $\mathcal{T}$ is that its antipode map can, in principle, serve as a universal tool for inversion in formulas that relate moments to cumulants, or relate different brands of cumulants living in the $N C(n)$ framework. In Section 13 of the paper we examine the antipode of $\mathcal{T}$ and in particular we identify (Theorem 13.13) a cancellation-free formula for how the antipode works, described in terms of a suitable notion of "efficient chains" in the lattices $N C(n)$.

### 1.7. Organising of the paper.

Following to the present Introduction, the sections of the paper can be roughly divided into four parts.

- In the first part, Sections 2-5, we establish some basic relevant facts concerning the group $(\widetilde{\mathcal{G}}, *)$. More precisely: after setting some background and notation in Section 2, we introduce $\widetilde{\mathcal{G}}$ in Section 3. Then in Section 4 (Theorem (4.3) we prove that $\widetilde{\mathcal{G}}$ is indeed a group under convolution. The review of the smaller group $\mathcal{G}$ and some discussion around the inclusion $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$ appears in Section 5 .
- In the second part, Sections $6-8$, we demonstrate the relevance of $\widetilde{\mathcal{G}}$ to the study of noncommutative cumulants. This comes into the picture via a natural action which a function $g \in \widetilde{\mathcal{G}}$ has on sequences of multilinear functionals on a non-commutative probability space. This action is presented in Section 6. Then in Section 7 we look at specific examples of cumulants and, based on them, we identify what it means for $g \in \widetilde{\mathcal{G}}$ to encode transitions of "moment-to-cumulant" type or of "cumulant-to-cumulant" type.

In Section 8 we prove (Propositions 8.2 and 8.4) that, as announced in the above subsection 1.3: the set $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ of functions of cumulant-to-cumulant type is a subgroup of $\widetilde{\mathcal{G}}$, and the set $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ of functions of cumulant-to-moment type is a right coset of $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. The method of proof of Proposition 8.4 points to the importance of the function $g_{\mathrm{bc}-\mathrm{m}} \in \widetilde{\mathcal{G}}$ which encodes the transition from Boolean cumulants to moments; as an application, we show (Section 8.3) how this leads to the known "rule of thumb" that Boolean cumulants are the easiest cumulants to relate to, when we start from a moment-cumulant formula given for some other brand of cumulants.

- In the third part, Sections 9-11, we present the results announced in the above subsections 1.4 and 1.5. Section 9 discusses the 1-parameter subgroup $\left\{u_{q} \mid q \in \mathbb{R}\right\} \subseteq \widetilde{\mathcal{G}}_{\text {c-c }}$ and its applications to $t$-Boolean cumulants. In Section 10 we prove (Theorem 10.1) that the $u_{q}$ 's normalize $\mathcal{G}$, and in Section 11 we flesh out the plan outlined in Section 1.5 for how to derive the description of the multiplication of free random variables in terms of $t$-Boolean cumulants.
- Finally the fourth part, Sections 12-14, discusses Hopf algebra aspects of the study of $\widetilde{\mathcal{G}}$. In Section 12 we provide a detailed explicit description of the Hopf algebra $\mathcal{T}$ and we put into evidence the canonical isomorphism $\mathbb{X}(\mathcal{T}) \approx \widetilde{\mathcal{G}}$. Section 13 is devoted to studying the antipode of $\mathcal{T}$, and in Section 14 we present (Theorem 14.6) the bialgebra homomorphism $\Psi: \mathcal{T} \rightarrow$ Sym which dualizes the inclusion of $\mathcal{G}$ into $\widetilde{\mathcal{G}}$.


## 2. Background and notation

### 2.1. Some $\boldsymbol{N C}(\boldsymbol{n})$ terminology.

We will assume the reader is familiar with the lattices of non-crossing partitions $N C(n)$, and we will follow standard notation commonly used in connection to them, as presented for instance in Lectures 9 and 10 of [24]. Here is a quick review of some notational highlights.

Notation 2.1. Let $n$ be a positive integer.
$1^{o}$ The number of blocks of a partition $\pi \in N C(n)$ is denoted as $|\pi|$. One can meaningfully define what it means for two blocks of $\pi$ to be nested inside each other; consequently, one gets a notion of outer block (a block $V \in \pi$ which is not nested into anything else) versus inner block (a block which is not outer). We will use the notation inner $(\pi)$ and outer $(\pi)$ for the numbers of inner respectively outer blocks of $\pi$. We thus have $|\pi|=\operatorname{inner}(\pi)+\operatorname{outer}(\pi)$, with $\operatorname{inner}(\pi) \geq 0$ and outer $(\pi) \geq 1$. Note that

$$
\begin{equation*}
\{\pi \in N C(n) \mid \operatorname{inner}(\pi)=0\}=: \operatorname{Int}(n) \tag{2.1}
\end{equation*}
$$

is the set of all interval partitions of $\{1, \ldots, n\}$; these are the partitions $\pi \in N C(n)$ where every block $V \in \pi$ is an interval of $\{1, \ldots, n\}$.
$2^{o}$ The main partial order we consider on $N C(n)$ is the one given by reverse refinement: for $\pi, \sigma \in N C(n)$ we write " $\pi \leq \sigma$ " to mean that every block of $\sigma$ is a union of blocks of $\pi$. We will also make occasional use of two other partial orders on $N C(n)$, denoted $\ll$ and $\sqsubseteq$, which are reviewed in the next subsection.
$3^{o}$ We denote by $0_{n} \in N C(n)$ the partition with $n$ blocks of cardinality 1 , and by $1_{n} \in$ $N C(n)$ the partition consisting of a single block. These are the minimal respectively maximal element of the partially ordered set (poset) $(N C(n), \leq)$.
$4^{o}$ Every $\pi \in N C(n)$ has a Kreweras complement $\operatorname{Kr}(\pi) \in N C(n)$, and the map
$\mathrm{Kr}: N C(n) \rightarrow N C(n)$ so defined is an anti-automorphism of the poset $(N C(n), \leq)$. For the description of how $\operatorname{Kr}(\pi)$ is constructed and for some of its basic properties, see e.g. pages 147-148 in Lecture 9 of [24]. Occasionally it is useful to consider the more general notion of relative Kreweras complement of $\pi$ in $\sigma$, defined for any $\pi \leq \sigma$ in $N C(n)$, and where the "usual" Kreweras complement corresponds to the special case $\sigma=1_{n}$; see the discussion on pages 288-291 in Lecture 18 of [24].

Since throughout the paper we will work extensively with restrictions of non-crossing partitions, we take a moment to state clearly what is our notation for how this works.

Notation 2.2. (Relabeled-restrictions of partitions.) Let $n \geq 1$, let $\pi$ be an element in $N C(n)$, and consider a set $W=\left\{p_{1}, \ldots, p_{m}\right\} \subseteq\{1, \ldots, n\}$ where $1 \leq m \leq n$ and $p_{1}<$ $\cdots<p_{m}$. We use the notation " $\pi_{W}$ " for the partition of $\{1, \ldots, m\}$ described as follows: for $i, j \in\{1, \ldots, m\}$ we have

$$
\binom{i \text { and } j \text { belong to }}{\text { the same block of } \pi_{W}} \Leftrightarrow\binom{p_{i} \text { and } p_{j} \text { belong to }}{\text { the same block of } \pi} .
$$

It is immediate that the hypothesis of $\pi$ being non-crossing implies that $\pi_{W} \in N C(m)$.

Remark 2.3. We now have the notation set to state precisely what are the two kinds of lattice isomorphisms indicated in Section 1.2 of the Introduction.

First kind of isomorphism: for every $n \geq 1$ and $\pi \leq \sigma$ in $N C(n)$ one has

$$
\begin{equation*}
[\pi, \sigma] \approx \prod_{W \in \sigma}\left[\pi_{W}, 1_{|W|}\right] \tag{2.2}
\end{equation*}
$$

where the relabeled-restriction $\pi_{W} \in N C(|W|)$ is as above, and $1_{|W|} \in N C(|W|)$ is the partition with a single block.

[^1]Second kind of isomorphism: for every $k \geq 1$ and $\theta \in N C(k)$ one has

$$
\begin{equation*}
\left[\theta, 1_{k}\right] \approx\left[0_{k}, \operatorname{Kr}(\theta)\right] \approx \prod_{U \in \operatorname{Kr}(\theta)}\left[0_{|U|}, 1_{|U|}\right]=\prod_{U \in \operatorname{Kr}(\theta)} N C(|U|) . \tag{2.3}
\end{equation*}
$$

It is immediate how these two kinds of isomorphisms work together to yield the fact that for any $n \geq 1$ and $\pi \leq \sigma$ in $N C(n)$ one has a canonical isomorphism

$$
\begin{equation*}
[\pi, \sigma] \approx N C(1)^{p_{1}} \times N C(2)^{p_{2}} \times \cdots \times N C(n)^{p_{n}} \text { for some } p_{1}, \ldots, p_{n} \geq 0 \tag{2.4}
\end{equation*}
$$

For a detailed discussion of all this we refer to [24, pages 148-153 in Lecture 9]. It may be re-assuring to know that, more than being canonical, the exponents $p_{2}, \ldots, p_{n}$ in (2.4) are in fact uniquely determined - cf. [24, Proposition 9.38]. (The exponent $p_{1}$ in (2.4) is not uniquely determined, since $|N C(1)|=1$.)

Notation and Remark 2.4. (Concatenation and irreducibility).
$1^{o}$ Given $n_{1}, n_{2} \geq 1$ and $\pi_{1} \in N C\left(n_{1}\right), \pi_{2} \in N C\left(n_{2}\right)$, we denote by $\pi_{1} \diamond \pi_{2}$ the non-crossing partition in $N C\left(n_{1}+n_{2}\right)$ which is obtained by placing $\pi_{1}$ on the points $1, \ldots, n_{1}$ and $\pi_{2}$ on the points $n_{1}+1, \ldots, n_{1}+n_{2}$.
$2^{o}$ A non-crossing partition $\pi \in N C(n)$ is said to be irreducible when it cannot be written in the form $\pi=\pi_{1} \diamond \pi_{2}$ with $\pi_{1} \in N C\left(n_{1}\right)$ and $\pi_{2} \in N C\left(n_{2}\right)$ for some $n_{1}, n_{2} \geq 1$ with $n_{1}+n_{2}=n$. This condition is easily seen to be equivalent to the fact that the numbers 1 and $n$ belong to the same block of $\pi$, i.e., outer $(\pi)=1$.
$3^{o}$ Every $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ can be written as a concatenation of irreducible partitions. This is best understood by referring to the outer blocks of $\pi$. Indeed, it is straightforward to check that these outer blocks can be listed as $W_{1}, \ldots, W_{k}$, with

$$
\min \left(W_{1}\right)=1, \min \left(W_{2}\right)=1+\max \left(W_{1}\right), \ldots, \min \left(W_{k}\right)=1+\max \left(W_{k-1}\right), \max \left(W_{k}\right)=n .
$$

For every $1 \leq i \leq k$, consider the interval $J_{i}:=\left\{m \in \mathbb{N} \mid \min \left(W_{i}\right) \leq m \leq \max \left(W_{i}\right)\right\}$, which is a union of blocks of $\pi$, and consider the restricted partition $\pi_{J_{i}} \in N C\left(\left|J_{i}\right|\right)$. The concatenation $\pi_{J_{1}} \diamond \cdots \diamond \pi_{J_{k}}$ will then give back the $\pi$ we started with, and every $\pi_{J_{i}}$ is an irreducible partition in $N C\left(\left|J_{i}\right|\right)$.
$4^{o}$ We mention that, in the setting of part $3^{o}$, the interval partition $\theta:=\left\{J_{1}, \ldots, J_{k}\right\}$ is called the interval cover of $\pi$. It is easily checked that this $\theta$ is the smallest upper bound for $\pi \operatorname{in} \operatorname{Int}(n)$, in the sense that one has

$$
(\tau \in \operatorname{Int}(n) \text { and } \tau \geq \pi) \Rightarrow \tau \geq \theta
$$

### 2.2. The partial orders $\ll$ and $\sqsubseteq$ on $N C(n)$.

We will make use of two partial order relations on $N C(n)$ which are coarser than reverse refinement, and are defined as follows.

Notation 2.5. Let $n \in \mathbb{N}$ and $\pi, \sigma \in N C(n)$.
$1^{o}$ We will write " $\pi \ll \sigma$ " to mean that $\pi \leq \sigma$ in the reverse refinement order and that, in addition, the following happens:
$\left\{\begin{array}{l}\text { For every block } W \in \sigma \text { there exists a block } \\ V \in \pi \text { such that } \min (W), \max (W) \in V .\end{array}\right.$
$2^{o}$ We will write " $\pi \sqsubseteq \sigma$ " to mean that $\pi \leq \sigma$ in the reverse refinement order and that, in addition, the following happens:
$\left\{\begin{array}{l}\text { Suppose } W \in \sigma \text { and } i_{1}<i_{2}<i_{3} \text { are elements of } W . \\ \text { Suppose moreover that } i_{1} \text { and } i_{3} \text { belong to the same block } V \in \pi . \\ \text { Then it follows that } i_{2} \in V \text { as well. }\end{array}\right.$

Remark 2.6. Both partial orders $\ll$ and $\sqsubseteq$ have been considered before: $\ll$ was introduced in [2], in connection to the study of the so-called Boolean Bercovici-Pata bijection, while $\sqsubseteq$ was introduced and studied in [19.

We mention that the recent paper [6] generalizes $\ll$ and $\sqsubseteq$ to the setting of Coxeter groups and puts into evidence the fact that these two partial orders are, in a certain sense, dual to each other. A special case of this duality, which we use in Section 10 of the paper, is reviewed in Remark 2.8 below.

Remark 2.7. $1^{o}$ Let $n \in \mathbb{N}$ and $\pi \in N C(n)$, and let us record what happens when in Notation [2.5 we put $\sigma=1_{n}$. We note that:

- " $\pi \ll 1_{n}$ " means that 1 and $n$ are in the same block of $\pi$, i.e. that $\pi$ is irreducible.
- " $\pi \sqsubseteq 1_{n}$ " means precisely that $\pi$ is an interval partition.
$2^{o}$ More generally, let $n \in \mathbb{N}$ and let $\pi, \sigma \in N C(n)$ be such that $\pi \leq \sigma$. One can construe the latter inequality as saying that $\pi$ is obtained out of $\sigma$ by taking, one by one, the blocks of $\sigma$, and by performing a non-crossing partition of each of these blocks. From this perspective: the relation $\pi \ll \sigma$ amounts to the fact that $\pi$ is obtained by performing an irreducible partition of every block of $\sigma$, while the relation $\pi \sqsubseteq \sigma$ amounts to the fact that $\pi$ is obtained by performing an interval partition of every block of $\sigma$.

Remark 2.8. We record here two facts about the order relations $\ll$ and $\sqsubseteq$ that will be used later on in the paper.
$1^{o}$ The Kreweras complementation map Kr on $N C(n)$ provides us with a bijection

$$
\begin{equation*}
\{\pi \in N C(n) \mid \pi \text { is irreducible }\} \ni \tau \mapsto \operatorname{Kr}(\tau) \in\{\sigma \in N C(n) \mid\{n\} \text { is a block of } \sigma\}, \tag{2.7}
\end{equation*}
$$

which is a poset anti-isomorphism when the set on the left-hand side of (2.7) is endowed with the partial order $\ll$, while the set on the right-hand side is endowed with $\sqsubseteq$. For reference, see e.g. [14, Lemma 2.10].
$2^{o}$ For an irreducible partition $\pi \in N C(n)$, the upper ideal $\{\sigma \in N C(n) \mid \sigma \gg \pi\}$ has cardinality $2^{|\pi|-1}$. And more precisely: for $\pi \in N C(n)$ and every $1 \leq k \leq|\pi|$, one has that

$$
\begin{equation*}
\left|\left\{\sigma \in N C(n)|\sigma \gg \pi,|\sigma|=k\} \left\lvert\,=\binom{|\pi|-1}{k-1} .\right.\right.\right. \tag{2.8}
\end{equation*}
$$

For reference, see e.g. [2, Proposition 2.13].

## 3. Definition of $\widetilde{\mathcal{G}}$

### 3.1. Framework of the incidence algebra on non-crossing partitions.

Definition 3.1. We denote

$$
\begin{equation*}
N C^{(2)}:=\sqcup_{n=1}^{\infty}\{(\pi, \sigma) \mid \pi, \sigma \in N C(n), \pi \leq \sigma\} \tag{3.1}
\end{equation*}
$$

The set of functions from $N C^{(2)}$ to $\mathbb{C}$ goes under the name of incidence algebra of noncrossing partitions. This set of functions carries a natural associative operation of convolution, denoted as "*", where for any $f, g: N C^{(2)} \rightarrow \mathbb{C}$ and any $\pi \leq \sigma$ in an $N C(n)$ one puts

$$
\begin{equation*}
f * g(\pi, \sigma)=\sum_{\substack{\rho \in N C(n), \pi \leq \rho \leq \sigma}} f(\pi, \rho) \cdot g(\rho, \sigma) \tag{3.2}
\end{equation*}
$$

In the next remark we collect a few relevant facts concerning the above mentioned convolution operation. The reader is referred to [24, Lecture 10] (cf. pages 155-158 there) for a more detailed presentation. The general framework of incidence algebras comes from work of Rota and collaborators, e.g. in [10]; a detailed presentation of this appears in Chapter 3 of (30].

Remark 3.2. It is easy to verify that the convolution operation "*" defined by (3.2) is associative and unital, where the unit is the function $e: N C^{(2)} \rightarrow \mathbb{C}$ given by

$$
e(\pi, \sigma)= \begin{cases}1, & \text { if } \pi=\sigma  \tag{3.3}\\ 0, & \text { otherwise }\end{cases}
$$

For a function $f: N C^{(2)} \rightarrow \mathbb{C}$ one has (see e.g. [24, Proposition 10.4]) that

$$
\begin{equation*}
\binom{f \text { is invertible }}{\text { with respect to "*" }} \Leftrightarrow\left(f(\pi, \pi) \neq 0, \quad \forall \pi \in \sqcup_{n=1}^{\infty} N C(n)\right) . \tag{3.4}
\end{equation*}
$$

Moreover, if $f$ is invertible with respect to " $*$ ", then upon writing explicitly what it means to have $f * f^{-1}(\pi, \pi)=e(\pi, \pi)=1$, one immediately sees that the inverse $f^{-1}$ satisfies

$$
\begin{equation*}
f^{-1}(\pi, \pi)=\frac{1}{f(\pi, \pi)}, \quad \forall n \geq 1 \text { and } \pi \in N C(n) \tag{3.5}
\end{equation*}
$$

A reader who is matrix-inclined may choose to take the point of view that a function $f: N C^{(2)} \rightarrow \mathbb{C}$ is just an upper triangular matrix with rows and columns indexed by $\sqcup_{n=1}^{\infty} N C(n)$, and where the values $f(\pi, \sigma)$ appear as certain entries of the matrix. Then the operation " $*$ " amounts to matrix multiplication, and the formulas (3.3), (3.4), (3.5) have obvious meanings in that language as well.

Notation and Remark 3.3. (Unitized functions on $N C^{(2)}$.) We denote

$$
\begin{equation*}
\mathcal{F}:=\left\{f: N C^{(2)} \rightarrow \mathbb{C} \mid f(\pi, \pi)=1, \quad \forall \pi \in \sqcup_{n=1}^{\infty} N C(n)\right\} . \tag{3.6}
\end{equation*}
$$

The observations made in (3.4), (3.5) show that every $f \in \mathcal{F}$ is invertible under convolution, where the inverse $f^{-1}$ still belongs to $\mathcal{F}$. It is also immediate that if $f, g \in \mathcal{F}$ then $f * g \in$ $\mathcal{F}$, since for every $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ the formula defining $f * g(\pi, \pi)$ boils down to just $f * g(\pi, \pi)=f(\pi, \pi) \cdot g(\pi, \pi)=1$. Thus $(\mathcal{F}, *)$ is a group.

### 3.2. Unitized semi-multiplicative functions on $N C^{(2)}$.

We now proceed, as promised, to looking at functions on $N C^{(2)}$ which are (only) required to be multiplicative with respect to the first kind of isomorphism indicated in Section 1.2.

Definition 3.4. We will denote by $\widetilde{\mathcal{G}}$ the set of functions $g: N C^{(2)} \rightarrow \mathbb{C}$ which have $g(\pi, \pi)=1$ for all $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ and satisfy the following condition:

$$
\left\{\begin{array}{c}
\text { For every } n \geq 1 \text { and } \pi \leq \sigma \text { in } N C(n) \text { one has the factorization }  \tag{3.7}\\
\qquad g(\pi, \sigma)=\prod_{W \in \sigma} g\left(\pi_{W}, 1_{|W|}\right)
\end{array}\right.
$$

(where $\pi_{W}$ and $1_{|W|}$ are the same as in Equation (2.2) of Remark 2.3). We will refer to the condition (3.7) by calling it semi-multiplicativity, in contrast with the stronger multiplicativity condition from the work of Speicher [29], which also considers the second kind of isomorphism reviewed in Remark 2.3.

From (3.7) it is obvious that a function $g \in \widetilde{\mathcal{G}}$ is completely determined when we know the values $g\left(\pi, 1_{n}\right)$ for all $n \geq 1$ and $\pi \in N C(n)$. It is hence clear that the map indicated in (3.8) below is injective. This map turns out to also be surjective; it thus identifies $\widetilde{\mathcal{G}}$, as a set, with the countable direct product of copies of $\mathbb{C}$ denoted as " $\mathcal{Z}$ " in the next proposition.

Proposition 3.5. Let us denote $\mathcal{Z}:=\left\{\underline{z} \mid \underline{z}: \sqcup_{n=1}^{\infty} N C(n) \backslash\left\{1_{n}\right\} \rightarrow \mathbb{C}\right\}$.
$1^{o}$ One has a bijection $\widetilde{\mathcal{G}} \ni g \mapsto \underline{z} \in \mathcal{Z}$, with $\underline{z}$ obtained out of $g$ by putting

$$
\begin{equation*}
\underline{z}(\pi)=g\left(\pi, 1_{n}\right) \text { for every } n \geq 1 \text { and } \pi \in N C(n) \backslash\left\{1_{n}\right\} \tag{3.8}
\end{equation*}
$$

$2^{o}$ The inverse of the bijection from (3.8) is described as follows. Given $a \underline{z} \in \mathcal{Z}$, we "fill in" values $\underline{z}\left(1_{n}\right)=1$ for all $n \geq 1$, and then define $g: N C^{(2)} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
g(\pi, \sigma):=\prod_{W \in \sigma} \underline{z}\left(\pi_{W}\right), \quad \forall(\pi, \sigma) \in N C^{(2)} \tag{3.9}
\end{equation*}
$$

Then $g \in \widetilde{\mathcal{G}}$, and is sent by the map from (3.8) onto the $\underline{z}$ we started with.
Proof. Let $\underline{z} \in \mathcal{Z}$ be given and let $g: N C^{(2)} \rightarrow \mathbb{C}$ be defined as in (3.9). Then (3.8) is satisfied, because it is the special case " $\sigma=1_{n}$ " of (3.9). Upon combining (3.9) and (3.8), we thus see that $g$ satisfies the factorization condition indicated in (3.7). We have moreover that $g(\pi, \pi)=\prod_{W \in \pi} z\left(1_{|W|}\right)=1, \quad \forall \pi \in \sqcup_{n=1}^{\infty} N C(n)$, so we conclude that $g \in \widetilde{\mathcal{G}}$. Clearly, this $g$ is sent by the map (3.8) into the $\underline{z} \in \mathcal{Z}$ that we started with.

The argument in the preceding paragraph covers at the same time the surjectivity which was left to check in part $1^{\circ}$ of the proposition, and the inverse description in part $2^{\circ}$.

Remark 3.6. Recall that, in parallel with $1_{n} \in N C(n)$, one uses the notation " $0_{n}$ " for the partition in $N C(n)$ which has $n$ blocks of cardinality 1 . We warn the reader that $0_{n}$ and $1_{n}$ do not play symmetric roles in the study of $\widetilde{\mathcal{G}}$. Indeed, it is immediate that if $g \in \widetilde{\mathcal{G}}$ corresponds to a $\underline{z} \in \mathcal{Z}$ in the way described in Proposition 3.5, then we have

$$
\begin{equation*}
g\left(0_{n}, \sigma\right)=\prod_{W \in \sigma} g\left(0_{|W|}, 1_{|W|}\right)=\prod_{W \in \sigma} \underline{z}\left(0_{|W|}\right), \quad \forall n \geq 1 \text { and } \sigma \in N C(n) \tag{3.10}
\end{equation*}
$$

quite different from the Equation (3.8) giving the values $g\left(\pi, 1_{n}\right)$.

## 4. $\widetilde{\mathcal{G}}$ IS A GROUP UNDER CONVOLUTION

In this section we prove that $\widetilde{\mathcal{G}}$ is a subgroup of the convolution group $(\mathcal{F}, *)$ considered in Remark 3.3. We start by observing that the semi-multiplicativity condition (3.7) has an automatic upgrade to a "local" version, shown in the next lemma (where the special case $U=\{1, \ldots, n\}$ retrieves the original definition of semi-multiplicativity).

Lemma 4.1. (Local semi-multiplicativity.)
Let $n \geq 1$ and $\pi, \sigma \in N C(n)$ be such that $\pi \leq \sigma$. Let $U$ be a non-empty subset of $\{1, \ldots, n\}$ which is a union of blocks of $\sigma$. For every $g \in \widetilde{\mathcal{G}}$ one has:

$$
\begin{equation*}
g\left(\pi_{U}, \sigma_{U}\right)=\prod_{\substack{W \in \sigma, W \subseteq U}} g\left(\pi_{W}, 1_{|W|}\right) \tag{4.1}
\end{equation*}
$$

where the relabeled-restrictions ( $\pi_{U}$ and such) are in the sense of Notation 2.2.
Proof. We write explicitly $U=W_{1} \cup \cdots \cup W_{k}$ with blocks $W_{1}, \ldots, W_{k} \in \sigma$, and for every $1 \leq i \leq k$ we write $W_{i}=W_{i, 1} \cup \cdots \cup W_{i, p_{i}}$ with blocks $W_{i, 1}, \ldots, W_{i, p_{i}} \in \pi$. It follows that the partitions $\pi_{U}, \sigma_{U} \in N C(|U|)$ are of the form

$$
\sigma_{U}=\left\{T_{1}, \ldots, T_{k}\right\} \text { and } \pi_{U}=\left\{T_{1,1}, \ldots, T_{1, p_{1}}, \ldots, T_{k, 1}, \ldots, T_{k, p_{k}}\right\}
$$

with $T_{i}=T_{i, 1} \cup \cdots \cup T_{i, p_{i}} \subseteq\{1, \ldots,|U|\}$ for every $1 \leq i \leq k$. We leave it as an exercise to the reader to follow the necessary relabeled-restrictions and verify that for every $1 \leq i \leq k$ we have

$$
\begin{equation*}
\left.\left(\pi_{U}\right)_{T_{i}}=\pi_{W_{i}} \text { (equality of partitions in } N C\left(n_{i}\right), \text { with } n_{i}=\left|W_{i}\right|=\left|T_{i}\right|\right) \tag{4.2}
\end{equation*}
$$

When applied to the partitions $\pi_{U} \leq \sigma_{U}$ in $N C(|U|)$, the original semi-multiplicativity condition from (3.7) says that

$$
g\left(\pi_{U}, \sigma_{U}\right)=\prod_{i=1}^{k} g\left(\left(\pi_{U}\right)_{T_{i}}, 1_{\left|T_{i}\right|}\right)
$$

On the right-hand side of the latter equation we replace $\left(\pi_{U}\right)_{T_{i}}$ by $\pi_{W_{i}}$ and $1_{\left|T_{i}\right|}$ by $1_{\left|W_{i}\right|}$, and the required formula (4.1) follows.

As an application of local semi-multiplicativity, we get the following fact.
Lemma 4.2. Let $n \geq 1$, let $\pi, \rho, \sigma \in N C(n)$ with $\pi \leq \rho \leq \sigma$, and let $g \in \widetilde{\mathcal{G}}$. One has

$$
\begin{equation*}
g(\pi, \rho)=\prod_{U \in \sigma} g\left(\pi_{U}, \rho_{U}\right) \tag{4.3}
\end{equation*}
$$

Proof. Every block $U \in \sigma$ is a union of blocks of $\rho$, hence Lemma 4.1 can be invoked in connection to this $U$ and the partitions $\pi \leq \rho$ to infer that

$$
\begin{equation*}
g\left(\pi_{U}, \rho_{U}\right)=\prod_{\substack{W \in \rho, W \subseteq U}} g\left(\pi_{W}, 1_{|W|}\right) \tag{4.4}
\end{equation*}
$$

We thus find that

$$
\begin{aligned}
\prod_{U \in \sigma} g\left(\pi_{U}, \rho_{U}\right) & =\prod_{U \in \sigma}\left(\prod_{\substack{W \in \rho, W \subseteq U}} g\left(\pi_{W}, 1_{|W|}\right)\right)(\text { by (4.4) }) \\
& \left.=\prod_{W \in \rho} g\left(\pi_{W}, 1_{|W|}\right)\right) \\
& =g(\pi, \rho) \quad \text { (by definition of semi-multiplicativity) }
\end{aligned}
$$

and the required formula (4.3) is obtained.

Theorem 4.3. $\widetilde{\mathcal{G}}$ is a subgroup of $(\mathcal{F}, *)$.

Proof. $1^{o}$ We pick two functions $g_{1}, g_{2} \in \widetilde{\mathcal{G}}$, and we prove that $g_{1} * g_{2}$ is in $\widetilde{\mathcal{G}}$ as well.
The function $g_{1} * g_{2}: N C^{(2)} \rightarrow \mathbb{C}$ can be in any case considered as an element of the larger group $\mathcal{F}$. Proposition 3.5 gives us a function $g \in \widetilde{\mathcal{G}}$ such that

$$
\begin{equation*}
g\left(\pi, 1_{n}\right)=g_{1} * g_{2}\left(\pi, 1_{n}\right) \text { for every } n \geq 1 \text { and } \pi \in N C(n) \tag{4.5}
\end{equation*}
$$

We will prove that, for this $g$, we actually have

$$
\begin{equation*}
g(\pi, \sigma)=g_{1} * g_{2}(\pi, \sigma) \text { for every } n \geq 1 \text { and } \pi \leq \sigma \text { in } N C(n) . \tag{4.6}
\end{equation*}
$$

This will imply in particular that $g_{1} * g_{2}=g \in \widetilde{\mathcal{G}}$, as required.
So let us fix an $n \geq 1$ and some $\pi \leq \sigma$ in $N C(n)$, for which we will verify that (4.6) holds. We write explicitly $\sigma=\left\{W_{1}, \ldots, W_{k}\right\}$, and we calculate as follows:

$$
\begin{gathered}
g(\pi, \sigma)=\prod_{i=1}^{k} g\left(\pi_{W_{i}}, 1_{\left|W_{i}\right|}\right) \quad \text { (by semi-multiplicativity) } \\
=\prod_{i=1}^{k} g_{1} * g_{2}\left(\pi_{W_{i}}, 1_{\left|W_{i}\right|}\right) \quad(\text { by }(\underline{4.5 \mid)}) \\
=\prod_{i=1}^{k}\left(\sum_{\substack{\rho_{i} \in N C\left(\left|W_{i}\right|\right), \rho_{i} \geq \pi_{W_{i}}}} g_{1}\left(\pi_{W_{i}}, \rho_{i}\right) \cdot g_{2}\left(\rho_{i}, 1_{\left|W_{i}\right|}\right)\right)(\text { by the def. of "*") } \\
=\begin{array}{c}
\sum_{\substack{ \\
\rho_{1} \geq \pi_{W_{1}} \in N C\left(\left|W_{1}\right|\right), \ldots \\
\ldots, \rho_{k} \geq \pi_{W_{k}} \in N C\left(\left|W_{k}\right|\right)}}\left(\prod_{i=1}^{k} g_{1}\left(\pi_{W_{i}}, \rho_{i}\right)\right) \cdot\left(\prod_{i=1}^{k} g_{2}\left(\rho_{i}, 1_{\left|W_{i}\right|}\right)\right)
\end{array}
\end{gathered}
$$

where the latter equality is obtained by expanding the product from the preceding line.
But now, one has a natural order-preserving bijection

$$
\left\{\begin{array}{rll}
\{\rho \in N C(n) \mid \rho \leq \sigma\} & \longrightarrow & N C\left(\left|W_{1}\right|\right) \times \cdots \times N C\left(\left|W_{k}\right|\right),  \tag{4.8}\\
\rho & \mapsto & \left(\rho_{W_{1}}, \ldots, \rho_{W_{k}}\right),
\end{array}\right.
$$

where the relabeled-restrictions $\rho_{W_{1}}, \ldots \rho_{W_{k}}$ are as described in Notation [2.2. We observe that the bijection from (4.8) sends the set $\{\rho \in N C(n) \mid \pi \leq \rho \leq \sigma\}$ onto

$$
\left\{\left(\rho_{1}, \ldots, \rho_{k}\right) \in N C\left(\left|W_{1}\right|\right) \times \cdots \times N C\left(\left|W_{k}\right|\right) \mid \rho_{1} \geq \pi_{W_{1}}, \ldots, \rho_{k} \geq \pi_{W_{k}}\right\}
$$

Consequently, the latter bijection can be used in order to perform a "change of variable" in the summation from (4.7), and turn it into a summation over the set $\{\rho \in N C(n) \mid \pi \leq$ $\rho \leq \sigma\}$. When we perform this change of variable we arrive to the formula

$$
\begin{equation*}
g(\pi, \sigma)=\sum_{\substack{\rho \in N C(n), \pi \leq \rho \leq \sigma}}\left(\prod_{i=1}^{k} g_{1}\left(\pi_{W_{i}}, \rho_{W_{i}}\right)\right) \cdot\left(\prod_{i=1}^{k} g_{2}\left(\rho_{W_{i}}, 1_{\left|W_{i}\right|}\right)\right) . \tag{4.9}
\end{equation*}
$$

At this point we recognize the products on the right-hand side of (4.9) as

$$
\begin{cases}\prod_{i=1}^{k} g_{1}\left(\pi_{W_{i}}, \rho_{W_{i}}\right)=g_{1}(\pi, \rho) & \left(\text { by Lemma 4.2 for } g_{1}\right), \text { and }  \tag{4.10}\\ \prod_{i=1}^{k} g_{2}\left(\rho_{W_{i}}, 1_{\left|W_{i}\right|}\right)=g_{2}(\rho, \sigma) & \text { (by plain semi-multiplicativity for } \left.g_{2}\right)\end{cases}
$$

Upon substituting (4.10) into (4.9), we arrive to

$$
g(\pi, \sigma)=\sum_{\substack{\rho \in N C(n), \pi \leq \rho \leq \sigma}} g_{1}(\pi, \rho) \cdot g_{2}(\rho, \sigma)=g_{1} * g_{2}(\pi, \sigma), \text { as required in (4.6). }
$$

$2^{o}$ We pick a $g \in \widetilde{\mathcal{G}}$ and we prove that $g^{-1}$ (inverse under convolution) is in $\widetilde{\mathcal{G}}$ as well. The inverse $g^{-1}$ of $g$ can be in any case considered in the larger group $\mathcal{F}$. Our task here is to prove that $g^{-1}$ belongs in fact to $\widetilde{\mathcal{G}}$.

For every $n \geq 1$ we define a family of complex numbers $\{\underline{z}(\pi) \mid \pi \in N C(n)\}$ in the way described as follows. We first put $\underline{z}\left(1_{n}\right)=1$, then for $\pi \in N C(n) \backslash\left\{1_{n}\right\}$ we proceed by induction on the number $|\pi|$ of blocks of $\pi$ and put

$$
\begin{equation*}
\underline{z}(\pi):=-\sum_{\substack{\sigma \in N C(n), \sigma \geq \pi, \sigma \neq \pi}} g(\pi, \sigma) \underline{z}(\sigma) . \tag{4.11}
\end{equation*}
$$

Note that all the values $\underline{z}(\sigma)$ invoked on the right-hand side of (4.11) can indeed be used in this inductive definition, since the conditions $\sigma \geq \pi, \sigma \neq \pi$ imply that $|\sigma|<|\pi|$.

Proposition 3.5 gives us a function $h \in \widetilde{\mathcal{G}}$ such that $h\left(\pi, 1_{n}\right)=\underline{z}(\pi)$ for every $n \geq 1$ and $\pi \in N C(n)$. It is immediate that, with $h$ so defined, Equation (4.11) can be read as saying that

$$
\begin{equation*}
g * h\left(\pi, 1_{n}\right)=0, \text { for every } n \geq 1 \text { and } \pi \in N C(n) \backslash\left\{1_{n}\right\} . \tag{4.12}
\end{equation*}
$$

Now, in view of part $1^{o}$ of the present proof, we have that $g * h \in \widetilde{\mathcal{G}}$. Equation (4.12) states that $g * h$ agrees with the unit $e$ of $\widetilde{\mathcal{G}}$ on all couples $\left(\pi, 1_{n}\right)$ with $n \geq 1$ and $\pi \in N C(n) \backslash\left\{1_{n}\right\}$. Since an element of $\widetilde{\mathcal{G}}$ is uniquely determined by its values on such couples $\left(\pi, 1_{n}\right)$, we conclude that $g * h=e$.

Upon reading the equality $g * h=e$ in the larger group $\mathcal{F}$, we see that $h=g^{-1}$. Hence $g^{-1}=h \in \widetilde{\mathcal{G}}$, as we had to prove.

## 5. Multiplicative vs SEmi-multiplicative: the inclusion $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$

In this section we briefly review the situation when a function $g \in \widetilde{\mathcal{G}}$ is also required to respect the second kind of isomorphism reviewed in Remark 2.3, and is thus a multiplicative function on non-crossing partitions in the sense considered by Speicher [29]. It is easily seen that in order to upgrade to this situation, it suffices to require $g$ to be well-behaved with respect to the isomorphism $\left[\theta, 1_{k}\right] \approx\left[0_{k}, \operatorname{Kr}(\theta)\right]$ mentioned at the beginning of the line in (2.3). We can therefore go with the following concise definition.

Definition 5.1. Consider the group of semi-multiplicative functions $\widetilde{\mathcal{G}}$ discussed in Sections 3 and 4. A function $g \in \widetilde{\mathcal{G}}$ will be said to be multiplicative when it has the property that

$$
\begin{equation*}
g\left(\pi, 1_{n}\right)=g\left(0_{n}, \operatorname{Kr}(\pi)\right), \quad \forall n \geq 1 \text { and } \pi \in N C(n), \tag{5.1}
\end{equation*}
$$

where Kr is the Kreweras complementation map on $N C(n)$. We will denote

$$
\begin{equation*}
\mathcal{G}:=\{g \in \widetilde{\mathcal{G}} \mid g \text { satisfies the condition (5.1) }\} . \tag{5.2}
\end{equation*}
$$

Remark 5.2. Let $g$ be a function in $\mathcal{G}$ and let us denote

$$
\begin{equation*}
g\left(0_{n}, 1_{n}\right)=: \lambda_{n}, \quad n \geq 1 . \tag{5.3}
\end{equation*}
$$

Upon combining (5.1) with the formula for $g\left(0_{n}, \sigma\right)$ that had been recorded in Remark 3.6, we find that for every $n \geq 1$ and $\pi \in N C(n)$ we have

$$
\begin{equation*}
g\left(\pi, 1_{n}\right)=\prod_{W \in \operatorname{Kr}(\pi)} \lambda_{|W|} . \tag{5.4}
\end{equation*}
$$

This generalizes to

$$
\begin{equation*}
g(\pi, \sigma)=\prod_{W \in \operatorname{Kr}_{\sigma}(\pi)} \lambda_{|W|}, \quad \forall n \geq 1 \text { and } \pi \leq \sigma \text { in } N C(n), \tag{5.5}
\end{equation*}
$$

where $\operatorname{Kr}_{\sigma}(\pi)$ is the relative Kreweras complement of $\pi$ in $\sigma$. Indeed, Equation (5.5) follows easily from (5.4) when we invoke the semi-multiplicativity factorization (3.7) and then take into account that $\mathrm{Kr}_{\sigma}(\pi)$ is obtained by performing in parallel Kreweras complementation on all the restricted partitions $\pi_{W}$, with $W$ running among the blocks of $\sigma$.

In connection to the above, we have the following statement.
Proposition 5.3. Let $\left(\lambda_{n}\right)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, with $\lambda_{1}=1$. There exists a multiplicative function $g \in \mathcal{G}$, uniquely determined, such that $g\left(0_{n}, 1_{n}\right)=\lambda_{n}$ for all $n \geq 1$.

The uniqueness part of Proposition 5.3 is clearly implied by the formula (5.5). For the existence part, one defines $g$ by using the formula (55.5), and then proves (via a discussion very similar to the one on pages 164 -167 of [24, Lecture 10]) that $g \in \mathcal{G}$.

Remark 5.4. It turns out that $\mathcal{G}$ is in fact a subgroup of $\widetilde{\mathcal{G}}$. For the proof of this fact we refer to [24, Theorem 18.11]. Due to some basic symmetry properties enjoyed by the Kreweras complementation map it turns out, moreover, that $\mathcal{G}$ (unlike $\widetilde{\mathcal{G}}$ ) is a commutative group - see [24, Corollary 17.10].

## 6. The action of $\widetilde{\mathcal{G}}$ on Sequences of multilinear functionals

The relevance of the group $\widetilde{\mathcal{G}}$ for non-commutative probability considerations stems from a natural action that this group has on certain sequences of multilinear functionals. In order to describe this action, it is convenient to introduce the following notation.

Notation 6.1. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. We denote

$$
\begin{equation*}
\mathfrak{M}_{\mathcal{A}}:=\left\{\underline{\psi} \mid \underline{\psi}=\left(\psi_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}, \text { where } \psi_{n} \text { is an } n \text {-linear functional }\right\} . \tag{6.1}
\end{equation*}
$$

Remark and Notation 6.2. $1^{o}$ In Notation 6.1 we did not need to assume that $\mathcal{A}$ is an algebra, or that it comes endowed with an expectation functional $\varphi: \mathcal{A} \rightarrow \mathbb{C}$. If that would be the case, and we would thus be dealing with a non-commutative probability space $(\mathcal{A}, \varphi)$, then the set $\mathfrak{M}_{\mathcal{A}}$ would get to have a special element $\underline{\varphi}=\left(\varphi_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$ where

$$
\begin{equation*}
\varphi_{n}\left(x_{1}, \ldots, x_{n}\right):=\varphi\left(x_{1} \cdots x_{n}\right), \quad \forall x_{1}, \ldots, x_{n} \in \mathcal{A} . \tag{6.2}
\end{equation*}
$$

Such a $\underline{\varphi}$ is called "family of moment functionals" of $(\mathcal{A}, \varphi)$.
$2^{o}$ Given a $\underline{\psi}=\left(\psi_{n}\right)_{n=1}^{\infty}$ as in (6.1), there is a standard way of enlarging $\underline{\psi}$ by adding to it some multilinear functionals indexed by non-crossing partitions. More precisely: for any
$n \geq 1$ and $\pi \in N C(n)$, it is customary to denote as $\psi_{\pi}: \mathcal{A}^{n} \rightarrow \mathbb{C}$ the multilinear functional which acts by

$$
\begin{equation*}
\psi_{\pi}\left(x_{1}, \ldots, x_{n}\right)=\prod_{V \in \pi} \psi_{|V|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right), \quad x_{1}, \ldots, x_{n} \in \mathcal{A} \tag{6.3}
\end{equation*}
$$

[A concrete example: if we have, say, $n=5$ and $\pi=\{\{1,2,5\},\{3,4\}\} \in N C(5)$, then the formula defining $\psi_{\pi}$ becomes $\psi_{\pi}\left(x_{1}, \ldots, x_{5}\right):=\psi_{3}\left(x_{1}, x_{2}, x_{5}\right) \cdot \psi_{2}\left(x_{3}, x_{4}\right)$.]
The convention for how to enlarge $\underline{\psi}$ is useful when we introduce the following notation.
Notation 6.3. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. For every $\underline{\psi}=\left(\psi_{n}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ and $g \in \widetilde{\mathcal{G}}$, we denote by " $\underline{\psi} \cdot g$ " the element $\underline{\theta}=\left(\theta_{n}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined by putting

$$
\begin{equation*}
\theta_{n}=\sum_{\pi \in N C(n)} g\left(\pi, 1_{n}\right) \psi_{\pi}, \quad \forall n \geq 1 \tag{6.4}
\end{equation*}
$$

The right-hand side of (6.4) has a linear combination done in the vector space of $n$-linear functionals from $\mathcal{A}^{n}$ to $\mathbb{C}$, where the $\psi_{\pi}$ are as defined in Notation [6.2, 2 .

We will prove that the map introduced in Notation 6.3 is a group action. It is convenient to first record an extension of the formula used to define $\underline{\psi} \cdot g$.

Lemma 6.4. Let $\underline{\psi}, \underline{\theta} \in \mathfrak{M}_{\mathcal{A}}$ and $g \in \widetilde{\mathcal{G}}$ be such that $\underline{\theta}=\underline{\psi} \cdot g$. Consider the extended families of multilinear functionals $\left\{\psi_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty} N C(n)\right\}$ an $\bar{d}\left\{\theta_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty} N C(n)\right\}$ that are obtained out of $\underline{\psi}$ and $\underline{\theta}$, respectively, in the way indicated in Notation 6.2.2. Then for every $n \geq 1$ and $\sigma \bar{\in} N C(\bar{n})$ one has

$$
\begin{equation*}
\theta_{\sigma}=\sum_{\substack{\pi \in N C(n), \pi \leq \sigma}} g(\pi, \sigma) \psi_{\pi} \tag{6.5}
\end{equation*}
$$

Proof. Let us write explicitly $\sigma=\left\{W_{1}, \ldots, W_{k}\right\}$. Then for every $x_{1}, \ldots, x_{n} \in \mathcal{A}$ we have

$$
\begin{aligned}
\theta_{\sigma}\left(x_{1}, \ldots, x_{n}\right) & =\prod_{j=1}^{k} \theta_{\left|W_{j}\right|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid W_{j}\right) \text { (by the definition of } \theta_{\sigma} \text { ) } \\
& =\prod_{j=1}^{k}\left(\sum_{\pi_{j} \in N C\left(\left|W_{j}\right|\right)} g\left(\pi_{j}, 1_{\left|W_{j}\right|}\right) \psi_{\pi_{j}}\left(\left(x_{1}, \ldots, x_{n}\right) \mid W_{j}\right)\right) \text { (by Eqn. (6.4)). }
\end{aligned}
$$

Upon expanding the latter product of $k$ factors, we find $\theta_{\sigma}\left(x_{1}, \ldots, x_{n}\right)$ to be equal to

$$
\begin{equation*}
\sum_{\substack{\pi_{1} \in N C\left(\left|W_{1}\right|\right), \ldots \\ \ldots, \pi_{k} \in N C\left(\left|W_{k}\right|\right)}}\left(\prod_{j=1}^{k} g\left(\pi_{j}, 1_{\left|W_{j}\right|}\right)\right) \cdot\left(\prod_{j=1}^{k} \psi_{\pi_{j}}\left(\left(x_{1}, \ldots, x_{n}\right) \mid W_{j}\right)\right) \tag{6.6}
\end{equation*}
$$

But now, one has a natural bijection

$$
\left\{\begin{array}{rll}
\{\pi \in N C(n) \mid \pi \leq \sigma\} & \longrightarrow & N C\left(\left|W_{1}\right|\right) \times \cdots \times N C\left(\left|W_{k}\right|\right),  \tag{6.7}\\
\pi & \mapsto & \left(\pi_{W_{1}}, \ldots, \pi_{W_{k}}\right),
\end{array}\right.
$$

where the partitions $\pi_{W_{j}} \in N C\left(\left|W_{j}\right|\right)$ are relabeled-restrictions of $\pi$ (cf. Notation [2.2). When we use this bijection in order to perform a change of variables in the summation from (6.6), the semi-multiplicativity property of $g$ assures us that the product $\prod_{j=1}^{k} g\left(\pi_{j}, 1_{\left|W_{j}\right|}\right)$ is converted into just " $g(\pi, \sigma)$ ". On the other hand, it is easily checked that the said change of variable transforms $\prod_{j=1}^{k} \psi_{\pi_{j}}\left(\left(x_{1}, \ldots, x_{n}\right) \mid W_{j}\right)$ into " $\psi_{\pi}\left(x_{1}, \ldots, x_{n}\right)$ ". Hence
our computation of what is $\theta_{\sigma}\left(x_{1}, \ldots, x_{n}\right)$ has lead to $\sum_{\pi \leq \sigma} g(\pi, \sigma) \cdot \psi_{\pi}\left(x_{1}, \ldots, x_{n}\right)$, as required.

Proposition 6.5. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. The formula 6.4) from Notation 6.3 defines an action of the group $\widetilde{\mathcal{G}}$ on the set $\mathfrak{M}_{\mathcal{A}}$. That is, one has

$$
\begin{equation*}
\left(\underline{\psi} \cdot g_{1}\right) \cdot g_{2}=\underline{\psi} \cdot\left(g_{1} * g_{2}\right), \quad \forall \underline{\psi} \in \mathfrak{M}_{\mathcal{A}} \text { and } g_{1}, g_{2} \in \widetilde{\mathcal{G}} \tag{6.8}
\end{equation*}
$$

Proof. We denote $\underline{\psi} \cdot g_{1}=: \underline{\theta}=\left(\theta_{n}\right)_{n=1}^{\infty}$ and $\left(\underline{\psi} \cdot g_{1}\right) \cdot g_{2}=: \underline{\eta}=\left(\eta_{n}\right)_{n=1}^{\infty}$. Our goal for the proof is to verify that $\underline{\eta}=\underline{\psi} \cdot\left(g_{1} * g_{2}\right)$, i.e. that we have

$$
\begin{equation*}
\eta_{n}=\sum_{\pi \in N C(n)} g_{1} * g_{2}\left(\pi, 1_{n}\right) \psi_{\pi}, \quad \forall n \geq 1 \tag{6.9}
\end{equation*}
$$

where $\left\{\psi_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty} N C(n)\right\}$ is the extension of $\underline{\psi}$. We thus fix an $n \geq 1$ for which we will verify that (6.9) holds. We write the formula given for $\eta_{n}$ by the relation $\underline{\eta}=\underline{\theta} \cdot g_{2}$ and then we invoke Lemma 6.4 in connection to the relation $\underline{\theta}=\underline{\psi} \cdot g_{1}$, to find that:

$$
\eta_{n}=\sum_{\sigma \in N C(n)} g_{2}\left(\sigma, 1_{n}\right) \theta_{\sigma}=\sum_{\sigma \in N C(n)} g_{2}\left(\sigma, 1_{n}\right)\left(\sum_{\substack{\pi \in N C(n) \\ \pi \leq \sigma}} g_{1}(\pi, \sigma) \psi_{\pi}\right)
$$

Changing the order of summation in the latter double sum then leads to:

$$
\begin{equation*}
\eta_{n}=\sum_{\pi \in N C(n)}\left(\sum_{\substack{\sigma \in N C(n) \\ \pi \leq \sigma}} g_{1}(\pi, \sigma) g_{2}\left(\sigma, 1_{n}\right)\right) \psi_{\pi} \tag{6.10}
\end{equation*}
$$

The interior sum in (6.10) is equal to $g_{1} * g_{2}\left(\pi, 1_{n}\right)$, and we have thus obtained the required Equation (6.9).

Remark 6.6. Throughout this section we have considered, for the sake of simplicity, only multilinear functionals with values in $\mathbb{C}$. We invite the reader to take a moment to observe that the whole discussion could have been pursued, without any change, in the framework where we consider multilinear functionals with values in a unital commutative algebra $\mathcal{C}$ over $\mathbb{C}$. Indeed, suppose we have fixed such a $\mathcal{C}$. Then Notation 6.1 is adjusted by putting

$$
\mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}:=\left\{\underline{\underline{\psi}} \left\lvert\, \begin{array}{l}
\frac{\psi}{}=\left(\psi_{n}: \mathcal{A}^{n} \rightarrow \mathcal{C}\right)_{n=1}^{\infty}, \text { where every } \psi_{n}  \tag{6.11}\\
\text { is a } \mathbb{C} \text {-multilinear functional }
\end{array}\right.\right\}
$$

Given $\underline{\psi} \in \mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}$ and $g \in \widetilde{\mathcal{G}}$, we define what is $\underline{\psi} \cdot g \in \mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}$ by the very same formula as in (6.4) of Notation 6.3. The proof of Proposition 6.5 goes through without any changes, to show that in this way we obtain a right group action of $\widetilde{\mathcal{G}}$ on $\mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}$.

In the rest of the paper we will stick everywhere to the basic case when $\mathcal{C}=\mathbb{C}$, with only one exception: Section 7.4 will have an occurrence of the case where $\mathcal{C}$ is the Grassmann algebra $\mathbb{G}:=\{\alpha+\varepsilon \beta \mid \alpha, \beta \in \mathbb{C}\}$, with multiplication defined by

$$
\left(\alpha_{1}+\varepsilon \beta_{1}\right) \cdot\left(\alpha_{2}+\varepsilon \beta_{2}\right)=\alpha_{1} \alpha_{2}+\varepsilon\left(\alpha_{1} \beta_{2}+\alpha_{2} \beta_{1}\right), \quad \text { for } \alpha_{1}, \beta_{1}, \alpha_{2}, \beta_{2} \in \mathbb{C}
$$

## 7. Cumulant-To-moment type, and Cumulant-To-Cumulant type

There are several brands of cumulants which live naturally in the universe of non-crossing partitions, and are commonly used in the non-commutative probability literature. Each such brand of cumulants has its own "moment-cumulant" summation formula, and there also exist useful summation formulas that connect different brands of cumulants. The action of $\widetilde{\mathcal{G}}$ on sequences of multilinear functionals that was observed in Section 6 offers an efficient way to do calculations related to these moment-cumulant and inter-cumulant formulas. In connection to that, we next put into evidence: a factorization property which seems to always be fulfilled when one considers functions $g \in \widetilde{\mathcal{G}}$ involved in moment-cumulant formulas; and a vanishing property fulfilled by functions $g \in \widetilde{\mathcal{G}}$ which are involved in intercumulant formulas. Both these properties are phrased in connection to the operation " $\diamond$ " of concatenation of non-crossing partitions, and to the notion of irreducibility with respect to concatenation, as reviewed in Notation 2.4.

Definition 7.1. $1^{o}$ A function $g \in \widetilde{\mathcal{G}}$ will be said to be of cumulant-to-moment type when it has the property that

$$
\begin{equation*}
g\left(\pi_{1} \diamond \pi_{2}, 1_{n_{1}+n_{2}}\right)=g\left(\pi_{1}, 1_{n_{1}}\right) \cdot g\left(\pi_{2}, 1_{n_{2}}\right), \tag{7.1}
\end{equation*}
$$

holding for all $n_{1}, n_{2} \geq 1$ and $\pi_{1} \in N C\left(n_{1}\right), \pi_{2} \in N C\left(n_{2}\right)$. We denote

$$
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}:=\{g \in \widetilde{\mathcal{G}} \mid g \text { is of cumulant-to-moment type }\} .
$$

$2^{o}$ A function $g \in \widetilde{\mathcal{G}}$ will be said to be of cumulant-to-cumulant type when it satisfies

$$
\begin{equation*}
g\left(\pi_{1} \diamond \pi_{2}, 1_{n_{1}+n_{2}}\right)=0, \quad \forall n_{1}, n_{2} \geq 1 \text { and } \pi_{1} \in N C\left(n_{1}\right), \pi_{2} \in N C\left(n_{2}\right) . \tag{7.2}
\end{equation*}
$$

We denote

$$
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}:=\{g \in \widetilde{\mathcal{G}} \mid g \text { is of cumulant-to-cumulant type }\} .
$$

Remark 7.2. $1^{o}$ A function $g \in \widetilde{\mathcal{G}_{c-c}}$ is completely determined when we know its values $g\left(\pi, 1_{n}\right)$ with $\pi \in N C(n)$ irreducible. Indeed, the condition on $g$ stated in (7.2) just says that if $\pi \in N C(n)$ is not irreducible, then $g\left(\pi, 1_{n}\right)=0$. So we know the values $g\left(\pi, 1_{n}\right)$ for all $n \geq 1$ and $\pi \in N C(n)$, which determines $g$ (cf. Proposition 3.5).
$2^{o}$ Consider now a function $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$. An easy induction shows that for every $k \geq 1$, $n_{1}, \ldots, n_{k} \geq 1$ and $\pi_{1} \in N C\left(n_{1}\right), \ldots, \pi_{k} \in N C\left(n_{k}\right)$, one has:

$$
\begin{equation*}
g\left(\pi_{1} \diamond \cdots \diamond \pi_{k}, 1_{n_{1}+\cdots+n_{k}}\right)=\prod_{j=1}^{k} g\left(\pi_{j}, 1_{n_{j}}\right) . \tag{7.3}
\end{equation*}
$$

Since every non-crossing partition can be written as a concatenation of irreducible ones, we conclude that our $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ can be completely reconstructed if we know its values $g\left(\pi, 1_{n}\right)$ with $\pi \in N C(n)$ irreducible - indeed, Equation (7.3) then tells us what is $g\left(\pi, 1_{n}\right)$ for all $n \geq 1$ and $\pi \in N C(n)$, and Proposition 3.5 can be applied.

In the next section we will examine $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ and $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ from the group structure point of view, within the group $(\widetilde{\mathcal{G}}, *)$. For now we only want to show, by example, what is the rationale for the terms "cumulant-to-moment" and "cumulant-to-cumulant" used in Definition 7.1, This is an opportunity to review a few salient examples of cumulants, and to display some of the functions in $\widetilde{\mathcal{G}}$ which encode transition formulas from these cumulants to moments, or encode transition formulas between two different brands of cumulants.

### 7.1. Free and Boolean cumulants.

Throughout this subsection we fix a non-commutative probability space $(\mathcal{A}, \varphi)$, we look at

$$
\mathfrak{M}_{\mathcal{A}}:=\left\{\underline{\psi} \mid \underline{\psi}=\left(\psi_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}, \text { where } \psi_{n} \text { is an } n \text {-linear functional }\right\}
$$

and we consider the family of moment functionals $\underline{\varphi}=\left(\varphi_{n}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ which was introduced in Notation 6.2, 1.

Definition and Remark 7.3. The family of free cumulant functionals of $(\mathcal{A}, \varphi)$ is the family $\underline{\kappa}=\left(\kappa_{n}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in$ $\mathcal{A}$ one has:

$$
\begin{equation*}
\varphi\left(x_{1} \cdots x_{n}\right)=\sum_{\pi \in N C(n)} \prod_{V \in \pi} \kappa_{|V|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right) \tag{7.4}
\end{equation*}
$$

This requirement can be re-phrased as follows: let $g_{\mathrm{fc}-\mathrm{m}}: N C^{(2)} \rightarrow \mathbb{C}$ be $2^{2}$ defined by

$$
\begin{equation*}
g_{\mathrm{fc}-\mathrm{m}}(\pi, \sigma)=1, \quad \forall n \geq 1 \text { and } \pi \leq \sigma \text { in } N C(n) \tag{7.5}
\end{equation*}
$$

It is immediate that $g_{\mathrm{fc}-\mathrm{m}} \in \widetilde{\mathcal{G}}$ and that it fulfills the factorization condition (7.1), hence $g_{\mathrm{fc}-\mathrm{m}} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$. The "moment-cumulant" formula (7.4) can be read as an instance of the group action from Section 6, it just says that

$$
\begin{equation*}
\underline{\varphi}=\underline{\kappa} \cdot g_{\mathrm{fc}-\mathrm{m}} . \tag{7.6}
\end{equation*}
$$

Indeed, (7.4) asks for the equality of $n$-linear functionals $\varphi_{n}=\sum_{\pi \in N C(n)} \kappa_{\pi}$, holding for every $n \geq 1$, and with $\kappa_{\pi}$ 's defined as in Notation 6.2.2; but the latter equality is the same as (7.6).

We next repeat the same moment-cumulant formulation in connection to Boolean cumulants, where we now refer to interval partitions.

Definition and Remark 7.4. The family of Boolean cumulant functionals of $(\mathcal{A}, \varphi)$ is the family $\underline{\beta}=\left(\beta_{n}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{A}$ one has:

$$
\begin{equation*}
\varphi\left(x_{1} \cdots x_{n}\right)=\sum_{\pi \in \operatorname{Int}(n)} \prod_{J \in \pi} \beta_{|J|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid J\right) \tag{7.7}
\end{equation*}
$$

Now, consider the function $g_{\mathrm{bc}-\mathrm{m}} \in \widetilde{\mathcal{G}}$ defined via the requirement that for every $n \geq 1$ and $\pi \in N C(n)$ we have

$$
g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)= \begin{cases}1, & \text { if } \pi \in \operatorname{Int}(n)  \tag{7.8}\\ 0, & \text { otherwise }\end{cases}
$$

Such a function does indeed exist and is unique, as guaranteed by Proposition 3.5. We see moreover that $g_{\mathrm{bc}-\mathrm{m}}$ is a function of cumulant-to-moment type: indeed, given any $n_{1}, n_{2} \geq 1$ and $\pi \in N C\left(n_{1}\right), \pi_{2} \in N C\left(n_{2}\right)$, it is immediate that

$$
g_{\mathrm{bc}-\mathrm{m}}\left(\pi_{1} \diamond \pi_{2}, 1_{n_{1}+n_{2}}\right)=g_{\mathrm{bc}-\mathrm{m}}\left(\pi_{1}, 1_{n_{1}}\right) \cdot g_{\mathrm{bc}-\mathrm{m}}\left(\pi_{2}, 1_{n_{2}}\right)=\left\{\begin{array}{rc}
1, & \text { if both } \pi_{1} \text { and } \pi_{2} \text { are } \\
0, & \text { interval partitions } \\
0,
\end{array}\right.
$$

[^2]Hence $g_{\mathrm{bc}-\mathrm{m}} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ and (exactly as we did for free cumulants in Remark 7.3) we see that the moment-cumulant formula (7.7) amounts to just:

$$
\begin{equation*}
\underline{\varphi}=\underline{\beta} \cdot g_{\mathrm{bc}-\mathrm{m}} . \tag{7.9}
\end{equation*}
$$

Remark 7.5. It was convenient to introduce the function $g_{\mathrm{bc}-\mathrm{m}}$ by just postulating its values $g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)$, and then by invoking Proposition 3.5, It is not hard to actually write down the formula for the values taken by $g_{\mathrm{bc}-\mathrm{m}}$ on general couples in $N C^{(2)}$; this is found by using the semi-multiplicativity property, and comes out (immediate verification) as

$$
g_{\mathrm{bc}-\mathrm{m}}(\pi, \sigma)=\left\{\begin{array}{ll}
1, & \text { if } \pi \sqsubseteq \sigma,  \tag{7.10}\\
0, & \text { otherwise. }
\end{array}\right\}, \quad \forall n \geq 1
$$

where $\sqsubseteq$ is one of the partial order relations reviewed in Section 2.2.

### 7.2. An interpolation between free and Boolean: $\boldsymbol{t}$-Boolean cumulants.

In this subsection we continue to use the notation from Section 7.1, where $\varphi \in \mathfrak{M}_{\mathcal{A}}$ is the family of moment functionals of the non-commutative probability space $(\mathcal{A}, \varphi)$, and $\underline{\kappa}, \underline{\beta} \in \mathfrak{M}_{\mathcal{A}}$ are the families of free and respectively Boolean cumulants of the same space. Our goal for the subsection is to review a 1-parameter interpolation between $\underline{\beta}$ and $\underline{\kappa}$, arising from the work of Bożejko and Wysoczanski [7], and defined in the way described as follows.

Definition 7.6. Let $t \in \mathbb{R}$ be a parameter. We will use the name $t$-Boolean cumulant functionals of $(\mathcal{A}, \varphi)$ to refer to the sequence of multilinear functionals $\beta^{(t)}=\left(\beta_{n}^{(t)}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{A}$ one has:

$$
\begin{equation*}
\varphi\left(x_{1} \cdots x_{n}\right)=\sum_{\pi \in N C(n)} t^{\operatorname{inner}(\pi)} \prod_{V \in \pi} \beta_{|V|}^{(t)}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right) . \tag{7.11}
\end{equation*}
$$

Recall that $\operatorname{inner}(\pi)$ is our notation for the number of inner blocks of $\pi \in N C(n)$.

Remark 7.7. It is clear that for $t=1$ one gets $\underline{\beta}^{(1)}=\underline{\kappa}$. On the other hand, for $t=0$ one gets that $\underline{\beta}^{(0)}=\underline{\beta}$, because in this case the right-hand side of Equation (7.11) reduces to a sum over $\overline{\operatorname{In}} \mathrm{t}(n) \overline{\text { (cf. (2.1) }}$ in the review of background).

Notation and Remark 7.8. For every $t \in \mathbb{R}$, let $g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ be the function in $\widetilde{\mathcal{G}}$ defined via the requirement that

$$
\begin{equation*}
g_{\mathrm{bc}-\mathrm{m}}^{(t)}\left(\pi, 1_{n}\right):=t^{\operatorname{inner}(\pi)}, \text { for all } n \geq 1 \text { and } \pi \in N C(n) \tag{7.12}
\end{equation*}
$$

As an immediate consequence of the obvious fact that

$$
\operatorname{inner}\left(\pi_{1} \diamond \pi_{2}\right)=\operatorname{inner}\left(\pi_{1}\right)+\operatorname{inner}\left(\pi_{2}\right), \quad \forall \pi_{1}, \pi_{2} \in \sqcup_{n=1}^{\infty} N C(n),
$$

one has that $g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ is a function of cumulant-to-moment type. The formula (7.11) used to define the $t$-Boolean cumulant functionals can be concisely re-written in the form

$$
\begin{equation*}
\underline{\varphi}=\underline{\beta}^{(t)} \cdot g_{\mathrm{bc}-\mathrm{m}}^{(t)} . \tag{7.13}
\end{equation*}
$$

This is a common generalization of the formulas (7.4) and (7.7) observed for free and for Boolean cumulants - the latter formulas are obtained by setting the parameter to $t=1$ and to $t=0$, respectively.

Remark 7.9. When using the action of the group $\widetilde{\mathcal{G}}$, one sees very clearly how to combine moment-cumulant formulas for two different brands of cumulants in order to get a direct connection between the cumulants themselves. We illustrate how this works when we want to go from $s$-Boolean cumulants to $t$-Boolean cumulants for two distinct parameters $s, t \in \mathbb{R}$. We have $\underline{\beta}^{(t)} \cdot g_{\mathrm{bc}-\mathrm{m}}^{(t)}=\underline{\varphi}=\underline{\beta}^{(s)} \cdot g_{\mathrm{bc}-\mathrm{m}}^{(s)}$, hence:

$$
\begin{equation*}
\underline{\beta}^{(t)}=\underline{\varphi} \cdot\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1}=\left(\underline{\beta}^{(s)} \cdot g_{\mathrm{bc}-\mathrm{m}}^{(s)}\right) \cdot\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1}=\underline{\beta}^{(s)} \cdot\left(g_{\mathrm{bc}-\mathrm{m}}^{(s)} *\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1}\right) . \tag{7.14}
\end{equation*}
$$

In short: the transition from $s$-Boolean cumulants to $t$-Boolean cumulants is encoded by the function $g_{\mathrm{bc}-\mathrm{m}}^{(s)} *\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. The values of this function turn out to have a nice explicit description (cf. Remark 9.2.1 and Corollary 9.5 below), where in particular we find that for $n \geq 1$ and $\pi \in N C(n)$ we have:

$$
g_{\mathrm{bc}-\mathrm{m}}^{(s)} *\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1}\left(\pi, 1_{n}\right)= \begin{cases}(s-t)^{|\pi|-1}, & \text { if } \pi \text { is irreducible, }  \tag{7.15}\\ 0, & \text { otherwise. }\end{cases}
$$

Hence, when spelled out explicitly, the transition formula (7.14) says this: for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{A}$ we have

$$
\begin{equation*}
\beta_{n}^{(t)}\left(x_{1}, \ldots, x_{n}\right)=\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}}(s-t)^{\operatorname{inner}(\pi)} \prod_{V \in \pi} \beta_{|V|}^{(s)}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right) . \tag{7.16}
\end{equation*}
$$

In the special case when $s=1$ and $t=0$, Equation (7.16) becomes the transition formula from free cumulants to Boolean cumulants, which is well-known since the work of Lehner [20]. When swapping the role of the parameters and putting $s=0$ and $t=1$, one finds the inverse transition formula which writes free cumulants in terms of Boolean cumulants, and is also well-known (cf. [2, Proposition 3.9], [1, Section 4]).

### 7.3. Monotone cumulants.

We continue to use the framework and notation of the subsections 7.1 and 7.2. Another family of cumulant functionals associated to $(\mathcal{A}, \varphi)$ that gets constant attention in the research literature on non-commutative probability is the family of monotone cumulant functionals which were introduced in [16], based on the notion of monotone ordering of a partition $\pi \in N C(n)$. The latter notion is defined as a bijection $\ell: \pi \rightarrow\{1, \ldots,|\pi|\}$ (or in other words: a total ordering of the blocks of $\pi$ ) which has the property that

$$
\left\{\begin{array}{l}
\text { If } V, W \in \pi \text { are such that } V \text { is nested inside } W  \tag{7.17}\\
\text { then it follows that } \ell(V) \geq \ell(W) .
\end{array}\right.
$$

With this notion in hand, one then proceeds as follows.
Definition 7.10. The family $\underline{\rho}=\left(\rho_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$ of monotone cumulants of $(\mathcal{A}, \varphi)$ is defined via the requirement that for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{A}$ one has

$$
\begin{equation*}
\varphi\left(x_{1} \cdots x_{n}\right)=\sum_{\pi \in N C(n)} \frac{\# \text { of monotone orderings of } \pi}{|\pi|!} \cdot \prod_{V \in \pi} \rho_{|V|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right) . \tag{7.18}
\end{equation*}
$$

Notation and Remark 7.11. In order to re-phrase the preceding definition in terms of the action of $\widetilde{\mathcal{G}}$ on $\mathfrak{M}_{\mathcal{A}}$, we let $g_{\mathrm{mc}-\mathrm{m}}$ be the function in $\widetilde{\mathcal{G}}$ defined via the requirement that

$$
\begin{equation*}
g_{\mathrm{mc}-\mathrm{m}}\left(\pi, 1_{n}\right)=\frac{\# \text { of monotone orderings of } \pi}{|\pi|!}, \quad \forall n \geq 1 \text { and } \pi \in N C(n) . \tag{7.19}
\end{equation*}
$$

An elementary counting argument (presented for instance in [1, Proposition 3.3]) shows that $g_{\mathrm{mc}-\mathrm{m}}$ satisfies the factorization condition stated in (7.1), and is therefore of cumulant-tomoment type. The formula (7.18) from the preceding definition gets to be re-phrased as

$$
\begin{equation*}
\underline{\varphi}=\underline{\rho} \cdot g_{\mathrm{mc}-\mathrm{m}}, \tag{7.20}
\end{equation*}
$$

in close analogy to how the definitions of $\underline{\kappa}, \underline{\beta}, \underline{\beta}^{(t)}$ were re-phrased in the preceding subsections.

### 7.4. Infinitesimal cumulants.

There exists an "infinitesimal" extension of the notion of non-commutative probability space, which has been considered primarily for the purpose of pinning down an infinitesimal version of the notion of free independence for non-commutative random variables (cf. [4], and the follow-up in [28] relating this topic to random matrix theory). An infinitesimal non-commutative probability space is a triple $\left(\mathcal{A}, \varphi, \varphi^{\prime}\right)$ where $(\mathcal{A}, \varphi)$ is a non-commutative probability space in the usual sense and one also has a second linear functional $\varphi^{\prime}: \mathcal{A} \rightarrow \mathbb{C}$ such that $\varphi^{\prime}\left(1_{\mathcal{A}}\right)=0$. To such a space one associates:

- a sequence of free infinitesimal cumulant functionals $\underline{\kappa}^{\prime}=\left(\kappa_{n}^{\prime}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$;
- a sequence of Boolean infinitesimal cumulant functionals $\underline{\beta}^{\prime}=\left(\beta_{n}^{\prime}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$;
- a sequence of monotone infinitesimal cumulant functionals $\underline{\rho}^{\prime}=\left(\rho_{n}^{\prime}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$.

The infinitesimal free cumulants $\underline{\kappa}^{\prime}$ were introduced in [13], while $\underline{\beta}^{\prime}, \underline{\rho}^{\prime}$ were introduced in [17] (see also the detailed study of all these notions appearing in the recent paper [8]).

We note that $\underline{\kappa}^{\prime}, \underline{\beta}^{\prime}, \underline{\rho}^{\prime}$ belong to $\mathfrak{M}_{\mathcal{A}}$, the set bearing the action of $\widetilde{\mathcal{G}}$ from Section 6 . The definitions of these infinitesimal cumulants can be described in terms of a variation of this action of $\widetilde{\mathcal{G}}$, going now on $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$. The occurrence of $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$ comes from the fact that the summations over lattices $N C(n)$ used to describe infinitesimal cumulants have terms which depend on both the linear functionals $\varphi, \varphi^{\prime}$ considered on $\mathcal{A}$. The details are as follows.

Notation 7.12. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$ and let $\mathfrak{M}_{\mathcal{A}}$ be the set of sequences of multilinear functionals introduced in Notation 6.1. Suppose we are given a couple $\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) \in \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$, where $\underline{\psi}^{(1)}=\left(\psi_{n}^{(1)}\right)_{n=1}^{\infty}$ and $\underline{\psi}^{(2)}=\left(\psi_{n}^{(2)}\right)_{n=1}^{\infty}$, and suppose we are also given a function $g \in \widetilde{\mathcal{G}}$. We then denote

$$
\begin{equation*}
\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) \stackrel{\text { inf }}{\odot} g:=\left(\underline{\theta}^{(1)}, \underline{\theta}^{(2)}\right) \in \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}, \tag{7.21}
\end{equation*}
$$

where $\underline{\theta}^{(1)}=\underline{\psi}^{(1)} \cdot g$ (exactly as in Notation 6.3) and $\underline{\theta}^{(2)}=\left(\theta_{n}^{(2)}\right)_{n=1}^{\infty}$ is defined by the requirement that for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{A}$ we have

$$
\begin{gather*}
\theta_{n}^{(2)}\left(x_{1}, \ldots, x_{n}\right)=  \tag{7.22}\\
\sum_{\pi \in N C(n)} g\left(\pi, 1_{n}\right) \cdot \sum_{W \in \pi}\left(\psi_{|W|}^{(2)}\left(\left(x_{1}, \ldots, x_{n}\right) \mid W\right) \cdot \prod_{\substack{V \in \pi, V \neq W}} \psi_{|V|}^{(1)}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right)\right) .
\end{gather*}
$$

Remark 7.13. Let $\mathcal{A}$ be as in Notation 7.12, What hides behind (7.21) and (7.22) is the fact that we have a canonical identification:

$$
\begin{equation*}
\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \ni\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) \mapsto \widetilde{\underline{\psi}} \in \mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}, \tag{7.23}
\end{equation*}
$$

where $\mathbb{G}$ is the Grassmann algebra and $\mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}$ is as considered in Remark 6.6 at the end of Section 6. That is: given $\underline{\psi}^{(1)}=\left(\psi_{n}^{(1)}\right)_{n=1}^{\infty}$ and $\underline{\psi}^{(2)}=\left(\psi_{n}^{(2)}\right)_{n=1}^{\infty}$ in $\mathfrak{M}_{\mathcal{A}}$, we create a sequence of $\mathbb{C}$-multilinear functionals $\underline{\psi}=\left(\widetilde{\psi}_{n}: \mathcal{A}^{n} \rightarrow \mathbb{G}\right)_{n=1}^{\infty}$ by simply putting

$$
\widetilde{\psi}_{n}\left(x_{1}, \ldots, x_{n}\right)=\psi_{n}^{(1)}\left(x_{1}, \ldots, x_{n}\right)+\varepsilon \psi_{n}^{(2)}\left(x_{1}, \ldots, x_{n}\right), \quad \forall x_{1}, \ldots, x_{n} \in \mathcal{A}
$$

As explained in Remark [6.6, the group $\widetilde{\mathcal{G}}$ acts on the right on $\mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}$. The explicit formula for $\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) \stackrel{\text { inf }}{\odot} g$ shown in the preceding notation is just the conversion of the formula for $\widetilde{\underline{\psi}} \cdot g \in \mathfrak{M}_{A}^{(\mathbb{G})}$, via the identification (7.23).
$\overline{\text { As }}$ a byproduct of the connection with the Grassmann algebra, one also gets an immediate proof of the following fact.
Proposition 7.14. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. Then "" $\odot$ " from Notation 7.12 is an action on the right of the group $\widetilde{\mathcal{G}}$ on $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$. That is, one has

$$
\begin{equation*}
\left(\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) \stackrel{\inf }{\odot} g\right) \stackrel{\inf }{\odot} h=\left(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}\right) \stackrel{\inf }{\odot}(g * h), \quad \forall \underline{\psi}^{(1)}, \underline{\psi}^{(2)} \in \mathfrak{M}_{\mathcal{A}} \text { and } g, h \in \widetilde{\mathcal{G}}, \tag{7.24}
\end{equation*}
$$

where on the right-hand side we use the convolution operation of $\widetilde{\mathcal{G}}$.
Remark 7.15. Consider now an infinitesimal non-commutative probability space $\left(\mathcal{A}, \varphi, \varphi^{\prime}\right)$, and let us spell out how the infinitesimal cumulants $\underline{\kappa}^{\prime}, \underline{\beta}^{\prime}, \underline{\rho}^{\prime} \in \mathfrak{M}_{\mathcal{A}}$ mentioned at the beginning of this subsection are described in terms of the action $\stackrel{i n f}{\odot}$ of the group $\widetilde{\mathcal{G}}$. To that end, let $\underline{\varphi}=\left(\varphi_{n}\right)_{n=1}^{\infty}$ and $\underline{\varphi}^{\prime}=\left(\varphi_{n}^{\prime}\right)_{n=1}^{\infty}$ be the sequences of moment functionals associated to $\varphi$ and to $\varphi^{\prime}$; that is, for every $n \geq 1$ the $n$-linear functionals $\varphi_{n}, \varphi_{n}^{\prime}: \mathcal{A}^{n} \rightarrow \mathbb{C}$ act by

$$
\varphi_{n}\left(x_{1}, \ldots, x_{n}\right)=\varphi\left(x_{1} \cdots x_{n}\right) \text { and } \varphi_{n}^{\prime}\left(x_{1}, \ldots, x_{n}\right)=\varphi^{\prime}\left(x_{1} \cdots x_{n}\right), \quad \forall x_{1}, \ldots, x_{n} \in \mathcal{A}
$$

The infinitesimal cumulants we are interested in are determined by the "moment-cumulant" equations

$$
\begin{equation*}
\left(\underline{\varphi}, \underline{\varphi}^{\prime}\right)=\left(\underline{\kappa}, \underline{\kappa}^{\prime}\right) \stackrel{\inf }{\odot} g_{\mathrm{fc}-\mathrm{m}}=\left(\underline{\beta}, \underline{\beta}^{\prime}\right) \stackrel{\mathrm{inf}}{\odot} g_{\mathrm{bc}-\mathrm{m}}=\left(\underline{\rho}, \underline{\rho}^{\prime}\right) \odot g_{\mathrm{mc}-\mathrm{m}}, \tag{7.25}
\end{equation*}
$$

where $g_{\mathrm{fc}-\mathrm{m}}, g_{\mathrm{bc}-\mathrm{m}}, g_{\mathrm{mc}-\mathrm{m}} \in \widetilde{\mathcal{G}}$ are the functions of cumulant-to-moment type that appeared in the preceding subsections (cf. Equations (7.5), (7.10) and (7.19), respectively). So for instace the sequence of free infinitesimal cumulants $\underline{\kappa}^{\prime}$ is found by picking the second component in the formula

$$
\begin{equation*}
\left(\underline{\kappa}, \underline{\kappa}^{\prime}\right)=\left(\underline{\varphi}, \underline{\varphi}^{\prime}\right) \stackrel{\inf }{\odot} g_{\mathrm{fc}-\mathrm{m}}^{-1} \tag{7.26}
\end{equation*}
$$

We note that the $\underline{\kappa}$ appearing in (7.25), (7.26) is precisely the sequence of free cumulant functionals of $(\mathcal{A}, \varphi)$, as one sees by picking the first component in (7.26) and by taking into account that on the first component of $\stackrel{\text { inf }}{\odot}$ we have the "usual" action of $\widetilde{\mathcal{G}}$ of $\mathfrak{M}_{\mathcal{A}}$.
8. $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ IS A SUBGROUP, AND $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ IS A RIGHT COSET

In this section we follow up on the subsets $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}, \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}} \subseteq \widetilde{\mathcal{G}}$ introduced in Definition 7.1. We will prove that: (i) $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ is a subgroup of $(\widetilde{\mathcal{G}}, *)$, and
(ii) $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ is a right coset of the subgroup $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$.

The statement (ii) means that we can write

$$
\begin{equation*}
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}=\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}} * h=\left\{g * h \mid g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}\right\} \tag{8.1}
\end{equation*}
$$

for no matter what $h \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ we choose to fix. The easiest choice for $h$ is to pick $h(\pi, \sigma)=1$ for all $(\pi, \sigma) \in N C^{(2)}$, that is, let $h$ be the special function $g_{\mathrm{fc}-\mathrm{m}}$ from Definition 7.3, However, as we will see in Section 8.2 below, it may be more advantageous for proofs and applications if we go instead with $h=g_{\mathrm{bc}-\mathrm{m}}$, picked from Definition 7.4.

### 8.1. Proof that $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ is a subgroup of $(\widetilde{\mathcal{G}}, *)$.

We first record a straightforward extension of the vanishing condition postulated in Equation (7.2), in the definition of $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$.

Lemma 8.1. Let $n \geq 1$ and let $\pi \leq \sigma$ be two partitions in $N C(n)$, where:

$$
\left\{\begin{array}{l}
\text { there exists a block } W_{o} \text { of } \sigma \text { such that }  \tag{8.2}\\
\min \left(W_{o}\right) \text { and } \max \left(W_{o}\right) \text { belong to different blocks of } \pi .
\end{array}\right.
$$

Then $g(\pi, \sigma)=0$ for all $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$.
Proof. For any $g \in \widetilde{\mathcal{G}}_{\text {c-c }}$ we write the factorization $g(\pi, \sigma)=\prod_{W \in \sigma} g\left(\pi_{W}, 1_{|W|}\right)$ provided by semi-multiplicativity, and we observe that the factor $g\left(\pi_{W_{o}}, 1_{\left|W_{o}\right|}\right)$ of this factorization is sure to be equal to 0 , since the partition $\pi_{W_{o}} \in N C\left(\left|W_{o}\right|\right)$ is not irreducible.

Proposition 8.2. $1^{o}$ Let $g_{1}, g_{2}$ be in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. Then $g_{1} * g_{2}$ is in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ as well. $2^{o}$ Let $g$ be in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. Then $g^{-1}$ (inverse under convolution) is in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ as well.

Proof. $1^{o}$ Let $n \geq 1$ and let $\pi \in N C(n)$ which is not irreducible, that is, 1 and $n$ belong to distinct blocks of $\pi$. We want to prove that $g_{1} * g_{2}\left(\pi, 1_{n}\right)=0$. We have

$$
g_{1} * g_{2}\left(\pi, 1_{n}\right)=\sum_{\sigma \in N C(n), \sigma \geq \pi} g_{1}(\pi, \sigma) \cdot g_{2}\left(\sigma, 1_{n}\right),
$$

and we will argue that every term of the latter sum is equal to 0 . Indeed, for a $\sigma \in N C(n)$ such that $\sigma \geq \pi$ there are two possible cases.

Case 1: $\sigma$ is not irreducible. In this case $g_{2}\left(\sigma, 1_{n}\right)=0$, and thus $g_{1}(\pi, \sigma) \cdot g_{2}\left(\sigma, 1_{n}\right)=0$.
Case 2: $\sigma$ is irreducible. In this case the numbers 1 and $n$ belong to the same block of $\sigma$, but belong to different blocks of $\pi$. Lemma 8.1] applies, and tells us that $g_{1}(\pi, \sigma)=0$. It thus follows that $g_{1}(\pi, \sigma) \cdot g_{2}\left(\sigma, 1_{n}\right)=0$ in this case as well.
$2^{o}$ We fix an $n \geq 1$, for which we prove that:

$$
g^{-1}\left(\pi, 1_{n}\right)=0 \text { for every } \pi \in N C(n) \text { which is not irreducible. }
$$

What we will do is to prove by induction on $m$, with $1 \leq m \leq n$, that:

$$
\left\{\begin{array}{c}
\pi \in N C(n), \text { not irreducible, }  \tag{8.3}\\
\text { with }|\pi|=m
\end{array}\right\} \Rightarrow g^{-1}\left(\pi, 1_{n}\right)=0
$$

The base-case $m=1$ holds trivially, because the set of partitions indicated in (8.3) is empty in that case (the only partition with $|\pi|=1$ is $\pi=1_{n}$, which is irreducible). In the remaining part of the proof we discuss the induction step: we pick an $m_{o} \in\{2, \ldots, n\}$, we assume that (8.3) is true for all $m<m_{o}$, and we verify that it is also true for $m_{o}$.

So consider a partition $\pi \in N C(n)$ which is not irreducible and has $|\pi|=m_{o}$. We have $g * g^{-1}\left(\pi, 1_{n}\right)=e\left(\pi, 1_{n}\right)=0$, and upon writing explicitly what is $g * g^{-1}\left(\pi, 1_{n}\right)$ we find,
very similar to Equation (4.11) in the proof of Theorem 4.3) that

$$
\begin{equation*}
g^{-1}\left(\pi, 1_{n}\right)=-\sum_{\substack{\sigma \in N C(n) \\ \sigma \geq \pi, \sigma \neq \pi}} g(\pi, \sigma) g^{-1}\left(\sigma, 1_{n}\right) \tag{8.4}
\end{equation*}
$$

In order to arrive to the desired conclusion that $g^{-1}\left(\pi, 1_{n}\right)=0$, we now verify that every term in the sum on the right-hand side of (8.4) is equal to 0 . In reference to the partition $\sigma$ which indexes the terms of that sum, we distinguish two cases.

Case 1: $\sigma$ is not irreducible. Since $|\sigma|<|\pi|=m_{o}$ (as implied by the conditions $\sigma \geq \pi, \sigma \neq \pi)$, the induction hypothesis applies to $\sigma$, and tells us that $g^{-1}\left(\sigma, 1_{n}\right)=0$. Hence $g(\pi, \sigma) g^{-1}\left(\sigma, 1_{n}\right)=0$, as we wanted.

Case 2: $\sigma$ is irreducible. In this case the numbers 1 and $n$ belong to the same block of $\sigma$, but belong to different blocks of $\pi$. Lemma 8.1 tells us that $g(\pi, \sigma)=0$, and we thus get that $g(\pi, \sigma) g^{-1}\left(\sigma, 1_{n}\right)=0$ in this case as well.

### 8.2. Proof that $\widetilde{\mathcal{G}}_{\mathbf{c}-\mathbf{m}}$ is a right coset of $\widetilde{\mathcal{G}}_{\mathbf{c}-\mathbf{c}}$.

The claim about $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ that we want to prove is as stated in Equation (8.1) at the beginning of the section, where on the right-hand side we have to choose a suitable "representative" $h$ picked from $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$. As mentioned immediately following to (8.1), the coset representative we will go with is the function $g_{\mathrm{bc}-\mathrm{m}}$ introduced in Equation (7.8) of the preceding section. In connection to it, we first prove a lemma.

Lemma 8.3. Let $g$ be in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. Then: $1^{o} g * g_{\mathrm{bc}-\mathrm{m}} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$.
$2^{o}$ One has

$$
\begin{equation*}
g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)=g\left(\pi, 1_{n}\right) \text { for every } n \geq 1 \text { and irreducible } \pi \in N C(n) \tag{8.5}
\end{equation*}
$$

Proof. We start by recording the general fact that

$$
\begin{equation*}
g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)=\sum_{\sigma \in \operatorname{Int}(n), \sigma \geq \pi} g(\pi, \sigma), \quad \forall n \geq 1 \text { and } \pi \in N C(n) \tag{8.6}
\end{equation*}
$$

This is obtained directly from the definition of the convolution operation, when we take into account the specifics of what is $g_{\mathrm{bc}-\mathrm{m}}$.

Proof of $1^{o}$. We take a partition $\pi=\pi_{1} \diamond \pi_{2} \in N C(n)$ where $n=n_{1}+n_{2}$ with $n_{1}, n_{2} \geq 1$ and where $\pi_{1} \in N C\left(n_{1}\right), \pi_{2} \in N C\left(n_{2}\right)$. Our goal here is to verify that

$$
\begin{equation*}
g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)=\left(g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi_{1}, 1_{n_{1}}\right)\right) \cdot\left(g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi_{2}, 1_{n_{2}}\right)\right) \tag{8.7}
\end{equation*}
$$

Observe that we clearly have

$$
\{\sigma \in \operatorname{Int}(n) \mid \sigma \geq \pi\} \supseteq\left\{\begin{array}{l|l}
\sigma_{1} \diamond \sigma_{2} & \left.\begin{array}{l}
\sigma_{1} \in \operatorname{Int}\left(n_{1}\right), \sigma_{1} \geq \pi_{1} \\
\sigma_{2} \in \operatorname{Int}\left(n_{2}\right), \sigma_{2} \geq \pi_{2}
\end{array}\right\} \tag{8.8}
\end{array}\right\}
$$

By starting from (8.6), we can thus write

$$
\begin{equation*}
g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)=\left(\sum_{\substack{\sigma_{1} \in \operatorname{Int}\left(n_{1}\right), \sigma_{1} \geq \pi_{1} \\ \sigma_{2} \in \operatorname{Int}\left(n_{2}\right), \sigma_{2} \geq \pi_{2}}} g\left(\pi, \sigma_{1} \diamond \sigma_{2}\right)\right)+\sum_{\sigma \in \mathcal{J}} g(\pi, \sigma), \tag{8.9}
\end{equation*}
$$

where $\mathcal{J}$ denotes the difference of the two sets indicated in (8.8). But note that $\mathcal{J}$ can be described as

$$
\mathcal{J}=\left\{\begin{array}{l|l}
\sigma \in \operatorname{Int}(n) & \begin{array}{l}
\sigma \geq \pi \text { and there exists a block } W_{o} \text { of } \sigma \\
\text { such that } \min \left(W_{o}\right) \leq n_{1}, \max \left(W_{o}\right)>n_{1}
\end{array}
\end{array}\right\} ;
$$

a direct application of Lemma 8.1 then gives that $g(\pi, \sigma)=0$ for every $\sigma \in \mathcal{J}$. So the second sum on the right-hand side of Equation (8.9) is actually equal to 0 . Concerning the first sum appearing there, we observe that its general term can be written as

$$
g\left(\pi, \sigma_{1} \diamond \sigma_{2}\right)=g\left(\pi_{1} \diamond \pi_{2}, \sigma_{1} \diamond \sigma_{2}\right)=g\left(\pi_{1}, \sigma_{1}\right) \cdot g\left(\pi_{2}, \sigma_{2}\right)
$$

with the factorization at the second equality sign coming from the semi-multiplicativity of $g$. When we put these observations together, we find that (8.9) leads to the factorization

$$
\begin{equation*}
g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)=\left(\sum_{\sigma_{1} \in \operatorname{Int}\left(n_{1}\right), \sigma_{1} \geq \pi_{1}} g\left(\pi_{1}, \sigma_{1}\right)\right) \cdot\left(\sum_{\sigma_{2} \in \operatorname{Int}\left(n_{2}\right), \sigma_{2} \geq \pi_{2}} g\left(\pi_{2}, \sigma_{2}\right)\right) . \tag{8.10}
\end{equation*}
$$

Finally, upon specializing the general Equation (8.6) to the case of the partitions $\pi_{1}$ and $\pi_{2}$, we identify the right-hand side of (8.10) as being the product that had been announced on the right-hand side of (8.7). This completes the verification that had to be done in this part of the proof.

Proof of $2^{\circ}$. Let $n \geq 1$ and let $\pi \in N C(n)$ be irreducible. It is immediate that, since 1 and $n$ belong to the same block of $\pi$, the only $\sigma \in \operatorname{Int}(n)$ such that $\sigma \geq \pi$ is $\sigma=1_{n}$. Hence, in this special case, the sum on the right-hand side of (8.6) has only 1 term, which is equal to $g\left(\pi, 1_{n}\right)$. It follows that $g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right)=g\left(\pi, 1_{n}\right)$, as stated in (8.5).

Proposition 8.4. $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}=\left\{g * g_{\mathrm{bc}-\mathrm{m}} \mid g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}\right\}$.
Proof. " $\supseteq$ ": This inclusion is provided by Lemma 8.3.1.
" $\subseteq$ ": Let a function $h \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ be given. We have to prove that $h$ can be written in the form $h=g * g_{\mathrm{bc}-\mathrm{m}}$, with $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$.

Proposition 3.5 assures us that there exists $g \in \widetilde{\mathcal{G}}$, uniquely determined, such that for every $n \geq 1$ and $\pi \in N C(n)$ we have

$$
g\left(\pi, 1_{n}\right)= \begin{cases}h\left(\pi, 1_{n}\right), & \text { if } \pi \text { is irreducible },  \tag{8.11}\\ 0, & \text { otherwise }\end{cases}
$$

Since Equation (8.11) includes the fact that $g\left(\pi, 1_{n}\right)=0$ whenever $\pi$ is not irreducible, we know that $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. Let $\widetilde{h}:=g * g_{\mathrm{bc}-\mathrm{m}}$. Then $\widetilde{h} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$, by Lemma 8.3,1. Moreover, for every $n \geq 1$ and irreducible partition $\pi \in N C(n)$ we have

$$
\begin{aligned}
\widetilde{h}\left(\pi, 1_{n}\right) & =g\left(\pi, 1_{n}\right) \quad(\text { by Lemma 8.3, } 2) \\
& =h\left(\pi, 1_{n}\right) \quad(\text { by }(\text { (8.11) }) .
\end{aligned}
$$

We thus get to have two functions $h$ and $\widetilde{h}$ in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$, such that $h\left(\pi, 1_{n}\right)=\widetilde{h}\left(\pi, 1_{n}\right)$ whenever $\pi \in N C(n)$ is irreducible. As observed in Remark [7.2,2, this implies $h=\widetilde{h}$. In particular we have obtained $h=g * g_{\mathrm{bc}-\mathrm{m}}$ with $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$, and this concludes the proof.

Proposition 8.4 has established the required claim that $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ is a right coset of the subgroup $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}} \subseteq \widetilde{\mathcal{G}}$. It is useful to also record the following fact, which gives a converse to Lemma 8.3.2, and was implicitly included in our method of deriving Proposition 8.4.

Corollary 8.5. Suppose that $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}, h \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ and it holds true that

$$
\begin{equation*}
g\left(\pi, 1_{n}\right)=h\left(\pi, 1_{n}\right), \quad \forall n \geq 1 \text { and irreducible } \pi \in N C(n) . \tag{8.12}
\end{equation*}
$$

Then it follows that $h=g * g_{\mathrm{bc}-\mathrm{m}}$.
Proof. We have $g * g_{\mathrm{bc}-\mathrm{m}} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$, by Proposition 8.4. We are thus required to prove an equality between two functions in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$. To this end, we know (cf. Remark [7.2,2) it is sufficient to verify that the two functions in question, $h$ and $g * g_{\mathrm{bc}-\mathrm{m}}$, agree on every couple $\left(\pi, 1_{n}\right)$ where $n \geq 1$ and $\pi \in N C(n)$ is irreducible. And indeed, for such $\left(\pi, 1_{n}\right)$ we have

$$
\begin{aligned}
g * g_{\mathrm{bc}-\mathrm{m}}\left(\pi, 1_{n}\right) & =g\left(\pi, 1_{n}\right)(\text { by Lemma } 8.3,2) \\
& =h\left(\pi, 1_{n}\right)(\text { by hypothesis }) .
\end{aligned}
$$

### 8.3. An application: why Boolean cumulants are the easiest to connect to.

In this subsection we show how the method of proof used in Proposition 8.4 can be invoked to retrieve a known "rule of thumb", which says that when given a cumulant-to-moment summation formula, it is usually immediate to write down the corresponding cumulant-to(Boolean cumulant) summations: one uses the very same coefficients as in the description of moments, only that the summations are now restricted to non-crossing partitions that are irreducible. The precise statement of this fact goes as follows.

Proposition 8.6. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Suppose we are given a sequence of multilinear functionals $\underline{\lambda}=\left(\lambda_{n}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ and a family of complex coefficients $(c(\pi))_{\pi \in \cup_{n=1}^{\infty} N C(n)}$ such that for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{A}$ we have

$$
\begin{equation*}
\varphi\left(x_{1} \cdots x_{n}\right)=\sum_{\pi \in N C(n)} c(\pi) \prod_{V \in \pi} \lambda_{|V|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right) . \tag{8.13}
\end{equation*}
$$

Suppose moreover that, in relation to the operation " $\checkmark$ " of concatenating non-crossing partitions, the coefficients $c(\pi)$ have the property that

$$
\begin{equation*}
c\left(\pi_{1} \diamond \pi_{2}\right)=c\left(\pi_{1}\right) \cdot c\left(\pi_{2}\right), \quad \forall \pi_{1}, \pi_{2} \in \sqcup_{n=1}^{\infty} N C(n) \tag{8.14}
\end{equation*}
$$

Then: denoting by $\underline{\beta}=\left(\beta_{n}\right)_{n=1}^{\infty}$ the Boolean cumulant functionals of $(\mathcal{A}, \varphi)$, we have

$$
\begin{equation*}
\beta_{n}\left(x_{1}, \ldots, x_{n}\right)=\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}} c(\pi) \prod_{V \in \pi} \lambda_{|V|}\left(\left(x_{1}, \ldots, x_{n}\right) \mid V\right), \tag{8.15}
\end{equation*}
$$

holding for every $n \geq 1$ and every $x_{1}, \ldots, x_{n} \in \mathcal{A}$.
Proof. Let $h$ be the function in $\widetilde{\mathcal{G}}$ which is defined via the requirement that $h\left(\pi, 1_{n}\right)=c(\pi)$, for all $n \geq 1$ and $\pi \in N C(n)$. The factorization hypothesis (8.14) satisfied by the coefficients $c(\pi)$ tells us that $h$ is a function of cumulant-to-moment type. On the other hand: Equation (8.13) can be re-written concisely in terms of $h$, in the form of the relation $\underline{\varphi}=\underline{\lambda} \cdot h$, where $\underline{\varphi}$ is the family of moment functionals of the space $(\mathcal{A}, \varphi)$. We can therefore write that

$$
\begin{aligned}
\underline{\beta} & \left.=\underline{\varphi} \cdot g_{\mathrm{bc}-\mathrm{m}}^{-1} \text { (Boolean cumulants expressed in terms of moments }\right) \\
& =(\underline{\lambda} \cdot h) \cdot g_{\mathrm{bc}-\mathrm{m}}^{-1}=\underline{\lambda} \cdot\left(h * g_{\mathrm{bc}-\mathrm{m}}^{-1}\right)=\underline{\lambda} * g,
\end{aligned}
$$

where we denoted $g:=h * g_{\mathrm{bc}-\mathrm{m}}^{-1}$.
Since $h \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$, Proposition 8.4 implies that $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. Moreover, since $g$ and $h$ are related by the convolution $h=g * g_{\mathrm{bc}-\mathrm{m}}$, Lemma 8.3 , 2 tells us that we have

$$
g\left(\pi, 1_{n}\right)=h\left(\pi, 1_{n}\right), \quad \forall n \geq 1 \text { and irreducible } \pi \in N C(n)
$$

By taking into account that $h\left(\pi, 1_{n}\right)=c(\pi)$, we thus come to the conclusion that for every $n \geq 1$ and $\pi \in N C(n)$ we have

$$
g\left(\pi, 1_{n}\right)= \begin{cases}c(\pi), & \text { if } \pi \text { is irreducible }  \tag{8.16}\\ 0, & \text { otherwise }\end{cases}
$$

Finally, for a fixed $n \geq 1$ we obtain that

$$
\beta_{n}=\sum_{\pi \in N C(n)} g\left(\pi, 1_{n}\right) \lambda_{\pi}=\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}} c(\pi) \lambda_{\pi},
$$

where the first equality is just spelling out the meaning of " $\underline{\beta}=\underline{\lambda} \cdot g$ ", and the second equality makes use of (8.16). The formula for $\beta_{n}$ obtained in this way is precisely the one stated in Equation (8.15).

Here is how the preceding proposition applies to some examples discussed in Section 7. We mention that a rather general result of this kind, going in a framework of cumulant constructions related to trees, appears as Lemma 7.6 of [18].

Example 8.7. (Boolean cumulants in terms of $t$-Boolean cumulants.)
Let $t \in \mathbb{R}$ be a parameter, and in Proposition 8.6 let us make $\underline{\lambda}$ be the family of $t$-Boolean cumulant functionals of $(\mathcal{A}, \varphi): \underline{\lambda}=\underline{\beta}^{(t)}=\left(\beta_{n}^{(t)}\right)_{n=1}^{\infty}$, when the coefficients of interest are $c(\pi)=t^{\operatorname{inner}(\pi)}$ and Equation (8.13) becomes the moment-cumulant formula recorded in Definition 7.6. The factorization condition from (8.14) is holding; this corresponds precisely to the fact that the function $g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ introduced in Notation 7.8 is of cumulant-to-moment type. Thus Proposition 8.6 applies, and yields a formula expressing Boolean cumulants in terms of $t$-Boolean cumulants:

$$
\begin{equation*}
\beta_{n}=\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}} t^{|\pi|-1} \beta_{\pi}^{(t)}, \quad n \geq 1, \tag{8.17}
\end{equation*}
$$

where on the right-hand side we took into account that an irreducible partition $\pi$ has $\operatorname{inner}(\pi)=|\pi|-1$, and therefore has $c(\pi)=t^{|\pi|-1}$. A generalization of this formula appears in Corollary 9.5 of the next section.

Example 8.8. (Boolean cumulants in terms of monotone cumulants.)
In Proposition 8.6 let us make $\underline{\lambda}$ be the family of monotone cumulant functionals of $(\mathcal{A}, \varphi)$ : $\underline{\lambda}=\underline{\gamma}=\left(\gamma_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$, when the coefficients of interest are

$$
c(\pi)=\frac{\# \text { of monotone orderings of } \pi}{|\pi|!}, \pi \in \sqcup_{n=1}^{\infty} N C(n),
$$

and Equation (8.13) becomes the moment-cumulant formula recorded in Definition 7.10, The factorization condition from (8.14) is holding; this corresponds precisely to the fact that the function $g_{\mathrm{mc}-\mathrm{m}}$ introduced in Notation 7.10 is of cumulant-to-moment type. Thus Proposition 8.6 applies, and yields a formula expressing Boolean cumulants in terms of monotone cumulants:

$$
\begin{equation*}
\beta_{n}=\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}} \frac{\# \text { of monotone orderings of } \pi}{|\pi|!} \cdot \gamma_{\pi}, \quad n \geq 1 \tag{8.18}
\end{equation*}
$$

This retrieves Equation (1.6) of [1], which is the beginning of the analysis done in that paper on how to relate monotone cumulants to other brands of cumulants.

Equation (8.18) is equivalent to a formula that gives an explicit description of the function $g_{\mathrm{mc}-\mathrm{bc}}:=g_{\mathrm{mc}-\mathrm{m}} * g_{\mathrm{bc}-\mathrm{m}}^{-1} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$, encoding the transition from monotone cumulants to Boolean cumulants. We mention that it is an interesting and non-trivial issue, addressed in [1] and in the recent paper [9], to describe explicitly the inverse

$$
g_{\mathrm{mc}-\mathrm{bc}}^{-1}=\left(g_{\mathrm{mc}-\mathrm{m}} * g_{\mathrm{bc}-\mathrm{m}}^{-1}\right)^{-1}=g_{\mathrm{bc}-\mathrm{m}} * g_{\mathrm{mc}-\mathrm{m}}^{-1} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}
$$

which encodes the reverse transition from Boolean to monotone cumulants.
9. The 1-Parameter subgroup $\left\{u_{q} \mid q \in \mathbb{R}\right\}$ of $\widetilde{\mathcal{G}}$, and its action on $g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ 'S

The method used for studying the right coset $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$ in Section 8.2 draws attention to a 1-parameter family of functions in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$, defined as follows.

Notation 9.1. For every $q \in \mathbb{R}$, we denote by $u_{q}$ the function in $\widetilde{\mathcal{G}}$ which is determined via the requirement that for all $n \geq 1$ and $\pi \in N C(n)$ we have

$$
u_{q}\left(\pi, 1_{n}\right)= \begin{cases}q^{|\pi|-1}, & \text { if } \pi \text { is irreducible }  \tag{9.1}\\ 0, & \text { otherwise }\end{cases}
$$

Clearly, this is a function of cumulant-to-cumulant type.

Remark 9.2. $1^{o}$ In the case $q=0$, the usual conventions apply to yield that $u_{0}\left(1_{n}, 1_{n}\right)=1$ and $u_{0}\left(\pi, 1_{n}\right)=0$ for every $\pi \neq 1_{n}$ in $N C(n)$. This implies that $u_{0}=e$, the unit of $\widetilde{\mathcal{G}}$.
$2^{o}$ The formula for the values taken by $u_{q}$ on general couples in $N C^{(2)}$ is determined from (9.1) by using the semi-multiplicativity property; we leave it as an easy exercise to the reader to check that for every $n \geq 1$ and $\pi \leq \sigma$ in $N C(n)$ one gets:

$$
u_{q}(\pi, \sigma)= \begin{cases}q^{|\pi|-|\sigma|}, & \text { if } \pi \ll \sigma  \tag{9.2}\\ 0, & \text { otherwise }\end{cases}
$$

where $\ll$ is one of the partial order relations reviewed in Section 2.2.

Proposition 9.3. The $u_{q}$ 's form a 1-parameter subgroup of $\widetilde{\mathcal{G}}$ :

$$
\begin{equation*}
u_{q_{1}} * u_{q_{2}}=u_{q_{1}+q_{2}}, \quad \text { for all } q_{1}, q_{2} \in \mathbb{R} \tag{9.3}
\end{equation*}
$$

Proof. We fix $q_{1}, q_{2} \in \mathbb{R}$ for which we will prove that (9.3) holds. The case when $q_{1}=0$ or $q_{2}=0$ is clear, so we will assume that $q_{1} \neq 0 \neq q_{2}$.

Since both $u_{q_{1}} * u_{q_{2}}$ and $u_{q_{1}+q_{2}}$ are in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$, in order to prove their equality it will suffice (cf. Remark 7.2) to check that

$$
\left\{\begin{array}{l}
u_{q_{1}} * u_{q_{2}}\left(\pi, 1_{n}\right)=u_{q_{1}+q_{2}}\left(\pi, 1_{n}\right)  \tag{9.4}\\
\text { for every } n \geq 1 \text { and every irreducible } \pi \in N C(n) .
\end{array}\right.
$$

For the rest of the proof, we fix an $n \geq 1$ and an irreducible $\pi \in N C(n)$ for which we will verify that Equation (9.4) holds.

The right-hand side of (9.4) is, directly from the definitions, equal to $\left(q_{1}+q_{2}\right)^{|\pi|-1}$. So our job is to verify that the left-hand side of (9.4) is equal to that same quantity.

We compute:

$$
u_{q_{1}} * u_{q_{2}}\left(\pi, 1_{n}\right)=\sum_{\substack{\sigma \in N C(n), \sigma \geq \pi}} u_{q_{1}}(\pi, \sigma) \cdot u_{q_{2}}\left(\sigma, 1_{n}\right)=\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}} q_{1}^{|\pi|-|\sigma|} \cdot q_{2}^{|\sigma|-1},
$$

where at the second equality sign we used (9.2) and also the fact that every $\sigma \geq \pi$ in $N C(n)$ is irreducible, thus has $u_{q_{2}}\left(\sigma, 1_{n}\right)=q_{2}^{|\sigma|-1}$. In the latter summation over $\sigma$ we sort out the terms according to what is $|\sigma|$. As reviewed in Remark [2.8,2, one has

$$
\left|\left\{\sigma \in N C(n)|\sigma \gg \pi,|\sigma|=k\} \left\lvert\,=\binom{|\pi|-1}{k-1}\right., \quad \forall k \in\{1, \ldots,|\pi|\} .\right.\right.
$$

Hence our evaluation of the left-hand side of Equation (9.4) continues as follows:

$$
u_{q_{1}} * u_{q_{2}}\left(\pi, 1_{n}\right)=\sum_{k=1}^{|\pi|}\binom{|\pi|-1}{k-1} q_{1}^{|\pi|-k} q_{2}^{k-1}=\sum_{\ell=0}^{|\pi|-1}\binom{|\pi|-1}{\ell} q_{1}^{(|\pi|-1)-\ell} q_{2}^{\ell}=\left(q_{1}+q_{2}\right)^{|\pi|-1}
$$

which is precisely the value we wanted to obtain.
We now consider the functions of cumulant-to-moment type $g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ introduced in Section 7.2, which encode moment-cumulant formulas for $t$-Boolean cumulants, and we look at how our 1-parameter subgroup of $u_{q}$ 's acts on $g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ 's, by left translations.

## Proposition 9.4. One has

$$
\begin{equation*}
u_{q} * g_{\mathrm{bc}-\mathrm{m}}^{(t)}=g_{\mathrm{bc}-\mathrm{m}}^{(q+t)}, \quad \text { for all } q, t \in \mathbb{R} \tag{9.5}
\end{equation*}
$$

Proof. We first verify the special case of (9.5) when $t=0$. Since $g_{\mathrm{bc}-\mathrm{m}}^{(0)}$ is just the function $g_{\mathrm{bc}-\mathrm{m}}$ from Definition 7.4, this case amounts to checking that for every $q \in \mathbb{R}$ we have

$$
\begin{equation*}
u_{q} * g_{\mathrm{bc}-\mathrm{m}}=g_{\mathrm{bc}-\mathrm{m}}^{(q)} . \tag{9.6}
\end{equation*}
$$

And indeed, let us notice that: $u_{q}$ is in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}, g_{\mathrm{bc}-\mathrm{m}}^{(t)}$ is in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{m}}$, and they are such that

$$
g_{\mathrm{bc}-\mathrm{m}}^{(q)}\left(\pi, 1_{n}\right)=q^{\operatorname{inner}(\pi)}=q^{|\pi|-1}=u_{q}\left(\pi, 1_{n}\right), \quad \forall \pi \in N C(n), \text { irreducible. }
$$

Thus Equation (9.6) does hold, as a special case of Corollary 8.5,
Going to general $q, t \in \mathbb{R}$ we can then write:

$$
\begin{aligned}
u_{q} * g_{\mathrm{bc}-\mathrm{m}}^{(t)} & =u_{q} *\left(u_{t} * g_{\mathrm{bc}-\mathrm{m}}\right)(\text { by (9.6) }) \\
& =\left(u_{q} * u_{t}\right) * g_{\mathrm{bc}-\mathrm{m}} \\
& =u_{q+t} * g_{\mathrm{bc}-\mathrm{m}}^{(\text {by Proposition (9.3) })} \\
& =g_{\mathrm{bc}-\mathrm{m}}^{(q+t)}(\text { by }(9.6)),
\end{aligned}
$$

yielding the required Equation (9.5).

The preceding proposition yields, in particular, the explicit transition formula from $s$ Boolean cumulants to $t$-Boolean cumulants which was anticipated in Remark 7.9 of Section 7.2. Recall that, in the said Remark 7.9, the point that remained to be justified was the validity of Equation (7.15); this is now very easy to fill in.

Corollary 9.5. (A repeat of Equation (7.15).)
Let $s$ and $t$ be real parameters, and consider the functions $g_{\mathrm{bc}-\mathrm{m}}^{(s)}, g_{\mathrm{bc}-\mathrm{m}}^{(t)} \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$. For every $n \geq 1$ and $\pi \in N C(n)$ one has:

$$
g_{\mathrm{bc}-\mathrm{m}}^{(s)} *\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1}\left(\pi, 1_{n}\right)= \begin{cases}(s-t)^{|\pi|-1}, & \text { if } \pi \text { is irreducible, } \\ 0, & \text { otherwise } .\end{cases}
$$

Proof. Proposition 9.4 says that $g_{\mathrm{bc}-\mathrm{m}}^{(s)}=u_{s-t} * g_{\mathrm{bc}-\mathrm{m}}^{(t)}$, which implies that

$$
\begin{equation*}
g_{\mathrm{bc}-\mathrm{m}}^{(s)} *\left(g_{\mathrm{bc}-\mathrm{m}}^{(t)}\right)^{-1}=u_{s-t} . \tag{9.7}
\end{equation*}
$$

We evaluate both sides of (9.7) at $\left(\pi, 1_{n}\right)$, then we refer to the formula for $u_{s-t}\left(\pi, 1_{n}\right)$ which comes from Equation (9.1), and the corollary follows.
10. The action of $\left\{u_{q} \mid q \in \mathbb{R}\right\}$, by conjugation, on multiplicative functions

The goal of the present section is to prove the following result.
Theorem 10.1. Let $\left\{u_{q} \mid q \in \mathbb{R}\right\}$ be as in the preceding section and let $\mathcal{G}$ be the subgroup of $\widetilde{\mathcal{G}}$ which consists of multiplicative functions, as reviewed in Section 5. One has that:

$$
\begin{equation*}
(q \in \mathbb{R} \text { and } f \in \mathcal{G}) \Rightarrow u_{q}^{-1} * f * u_{q} \in \mathcal{G} . \tag{10.1}
\end{equation*}
$$

Remark 10.2. Recall from Remark 5.2 and Proposition 5.3 that a function $f \in \mathcal{G}$ is completely determined by the sequence of complex numbers $\left(\lambda_{n}\right)_{n=1}^{\infty}$, where $\lambda_{n}:=f\left(0_{n}, 1_{n}\right)$, $n \geq 1$. If we accept Theorem 10.1, then it follows that $u_{q}^{-1} * f * u_{q}$ must be determined in a similar way by the sequence of $\theta_{n}$ 's where $\theta_{n}:=u_{q}^{-1} * f * u_{q}\left(0_{n}, 1_{n}\right)$ for $n \geq 1$. It is easy to write down the explicit formula which gives $\theta_{n}$ in terms of $\lambda_{1}, \ldots, \lambda_{n}$ and $q$, this is:

$$
\begin{equation*}
\theta_{n}=\sum_{\substack{\pi \in N C(n) \\ \text { irreducible }}} q^{|\pi|-1} \prod_{V \in \pi} \lambda_{|V|} . \tag{10.2}
\end{equation*}
$$

This gives for instance:

$$
\theta_{1}=\lambda_{1}=1, \theta_{2}=\lambda_{2}, \theta_{3}=\lambda_{3}+q \lambda_{2}, \theta_{4}=\lambda_{4}+2 q \lambda_{3}+q \lambda_{2}^{2}+q^{2} \lambda_{2} .
$$

Verification of (10.2): use the definition of the convolution operation "*" to find that

$$
\begin{equation*}
\theta_{n}=u_{q}^{-1} * f * u_{q}\left(0_{n}, 1_{n}\right)=\sum_{\substack{\sigma, \tau \in N C(n), \sigma \leq \tau}} u_{q}^{-1}\left(0_{n}, \sigma\right) f(\sigma, \tau) u_{q}\left(\tau, 1_{n}\right), \tag{10.3}
\end{equation*}
$$

then notice that $u_{q}^{-1}\left(0_{n}, \sigma\right)=0$ for every $\sigma \neq 0_{n}$ in $N C(n)$ (since $u_{q}^{-1}=u_{-q}$ and we can invoke the formula (9.2)). Thus $\sigma$ in (10.3) is forced to be $0_{n}$, and we continue with

$$
=\sum_{\tau \in N C(n)} f\left(0_{n}, \tau\right) u_{q}\left(\tau, 1_{n}\right),
$$

which yields (10.2) upon replacing $f\left(0_{n}, \tau\right)$ by $\prod_{V \in \tau} \lambda_{|V|}$ and $u_{q}\left(\tau, 1_{n}\right)$ from (9.1).
Deriving the formula (10.2) for $\theta_{n}$ does not, however, substitute for a proof of Theorem 10.1. We still need to evaluate $u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)$ for general $\pi \in N C(n)$, and to suitably express the resulting value as a product of $\theta_{m}$ 's. In order to achieve this we will prove two factorization formulas, presented in Lemmas 10.5 and 10.8 below.

For Lemma 10.5 we will need the following notation.

Notation and Remark 10.3. (Irreducible cover of a non-crossing partition.)
Let $n \geq 1$ and let $\pi$ be in $N C(n)$.
$1^{o}$ It is easy to see that there exists a partition $\bar{\pi}^{\mathrm{irr}} \in N C(n)$, uniquely determined, with the following properties:

$$
\left\{\begin{array}{l}
\text { (i) } \bar{\pi}^{\mathrm{irr}} \text { is irreducible and } \bar{\pi}^{\mathrm{irr}} \geq \pi ;  \tag{10.4}\\
\text { (ii) } \\
\text { If } \sigma \in N C(n) \text { is irreducible and } \sigma \geq \pi \text {, then } \sigma \geq \bar{\pi}^{\mathrm{irr}} .
\end{array}\right.
$$

We will refer to $\bar{\pi}^{\mathrm{irr}}$ as the irreducible cover of $\pi$. For its explicit description we distinguish two cases.

Case 1. $\pi$ is irreducible. Then, clearly, $\bar{\pi}^{\mathrm{irr}}=\pi$.
Case 2. $\pi$ is not irreducible. Then the blocks $V_{\text {left }}, V_{\text {right }} \in \pi$ which contain the numbers 1 respectively $n$ are such that $V_{\text {left }} \neq V_{\text {right }}$. In this case, $\bar{\pi}^{\text {irr }}$ is obtained out of $\pi$ by merging together $V_{\text {left }}$ and $V_{\text {right }}$. (It is easy to see that the said merger is sure to give a partition which is still non-crossing and has all the properties required in (10.4).)
$2^{o}$ In what follows, we will need at some point to deal with the relative Kreweras complement of $\pi$ in $\bar{\pi}^{\mathrm{irr}}$. In the case when $\bar{\pi}^{\mathrm{irr}}=\pi$, the complement $\mathrm{Kr}_{\bar{\pi}_{\mathrm{irr}}}(\pi)$ is just $0_{n}$. In the case when $\bar{\pi}^{\mathrm{irr}} \neq \pi$ (i.e. in the Case 2 indicated above), the complement $\operatorname{Kr}_{\bar{\pi}^{\mathrm{irr}}}(\pi)$ has 1 block with 2 elements and $n-2$ blocks with 1 element. Upon drawing a picture which features the outer blocks of $\pi$, the reader should have no difficulty to check that, in Case 2, the unique 2-element block of $\operatorname{Kr}_{\bar{\pi}^{\operatorname{irr}}}(\pi)$ is of the form $\{m, n\}$, with $m$ described as follows:

$$
\left\{\begin{array}{c}
m=\max \left(V_{\text {left }}\right)=\min \left(W_{\text {right }}\right) \text {, where }  \tag{10.5}\\
V_{\text {left }} \text { is the block of } \pi \text { which contains the number } 1, \text { and } \\
W_{\text {right }} \text { is the block of } \operatorname{Kr}(\pi) \text { which contains the number } n .
\end{array}\right.
$$

$3^{o}$ The drawing of the outer blocks of $\pi$ that was recommended above also reveals that the block $W_{\text {right }} \in \operatorname{Kr}(\pi)$ can be explicitly written in the form

$$
\begin{equation*}
W_{\text {right }}=\left\{\max \left(U_{1}\right), \ldots, \max \left(U_{k}\right)\right\}, \tag{10.6}
\end{equation*}
$$

where $U_{1}, \ldots, U_{k}$ are the outer blocks of $\pi$ (in particular, $U_{1}=V_{\text {left }}$ ). A consequence of (10.6) which will be needed in the sequel is this: if $\sigma \in N C(n)$ is such that $\sigma \gg \pi$ and if $W_{\text {right }}^{\prime}$ denotes the block of $\operatorname{Kr}(\sigma)$ which contains the number $n$, then it follows that $W_{\text {right }}^{\prime}=W_{\text {right }}$. This is because the relation $\gg$ forces $\sigma$ to have the same maximal elements of outer blocks as $\pi$ does, and thus the right-hand side of (10.6) also serves as an explicit description for what is $W_{\text {right }}^{\prime}$.

Notation and Remark 10.4. In Lemma 10.5 we will use three sequences of numbers $\left(\alpha_{n}\right)_{n=1}^{\infty},\left(\widehat{\alpha}_{n}\right)_{n=1}^{\infty}$ and $\left(\widetilde{\alpha}_{n}\right)_{n=1}^{\infty}$, where the $\widehat{\alpha}_{n}$ 's and $\widetilde{\alpha}_{n}$ 's are obtained out of the $\alpha_{n}$ 's via summation formulas, as follows:

$$
\begin{equation*}
\widehat{\alpha}_{n}=\sum_{\pi \in N C(n)} \prod_{V \in \pi} \alpha_{|V|} \quad \text { and } \quad \widetilde{\alpha}_{n}=\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}} \prod_{V \in \pi} \alpha_{|V|}, \quad n \geq 1 . \tag{10.7}
\end{equation*}
$$

One can also write summation formulas which give a direct relation between the two derived sequences $\left(\widehat{\alpha}_{n}\right)_{n=1}^{\infty}$ and $\left(\widetilde{\alpha}_{n}\right)_{n=1}^{\infty}$. For future reference, we record here one such formula (which is not hard to verify via direct calculation) saying that

$$
\begin{equation*}
\widetilde{\alpha}_{n}=\sum_{\rho \in \operatorname{Int}(n)}(-1)^{|\rho|+1} \prod_{J \in \rho} \widehat{\alpha}_{|J|}, \quad \forall n \geq 1 . \tag{10.8}
\end{equation*}
$$

Lemma 10.5. (A factorization formula.) Consider sequences of numbers as in Notation 10.4, and on the other hand let us pick an $n \geq 1$ and a partition $\pi \in N C(n)$. We consider the Kreweras complement $\operatorname{Kr}(\pi)$ and, same as in Remark 10.3, we denote by $W_{\text {right }}$ the block of $\operatorname{Kr}(\pi)$ which contains the number $n$. Then:

$$
\begin{equation*}
\sum_{\substack{\sigma \in N C(n), \sigma \geq \bar{\pi}^{\mathrm{irr}}}}\left(\prod_{U \in \operatorname{Kr}_{\sigma}(\pi)} \alpha_{|U|}\right)=\widetilde{\alpha}_{\left|W_{\mathrm{right}}\right|} \cdot \prod_{\substack{W \in \operatorname{Kr}(\pi), W \not \supset n}} \widehat{\alpha}_{|W|} \cdot \tag{10.9}
\end{equation*}
$$

Proof. On the left-hand side of (10.9) we perform the change of variable " $\tau=\operatorname{Kr}_{\sigma}(\pi)$ ". When $\sigma$ runs in the interval $\left[\bar{\pi}^{\mathrm{irr}}, 1_{n}\right] \subseteq N C(n)$, the relative Kreweras complement $\tau$ runs in the interval $\left[\mathrm{Kr}_{\bar{\pi}^{\text {irr }}}(\pi), \operatorname{Kr}_{1_{n}}(\pi)\right]$, where $\operatorname{Kr}_{1_{n}}(\pi)$ is just $\operatorname{Kr}(\pi)$. For a discussion of this nice behaviour of the partition $\operatorname{Kr}_{\sigma}(\pi)$ viewed as a function of $\sigma$ (and with $\pi$ fixed) see [24, Lemma 18.9].

Let us also recall, from Remark 10.3 , 2 , that the inequality $\tau \geq \mathrm{Kr}_{\pi_{\operatorname{irr}}}(\pi)$ amounts to requesting that $\tau$ connects $m$ with $n$, where $m=\min \left(W_{\text {right }}\right)$. Our processing of the left-hand side of Equation (10.9) has thus taken us to:

$$
\begin{equation*}
\sum_{\tau \in N C(n), \tau \leq \operatorname{Kr}(\pi)} \prod_{U \in \tau} \alpha_{|U|} . \tag{10.10}
\end{equation*}
$$

and $\tau$ connects $m$ with $n$
Now let us write explicitly $\operatorname{Kr}(\pi)=\left\{W_{1}, \ldots, W_{p}\right\}$, with the blocks listed such that $W_{p}=W_{\text {right }}$. A standard decomposition argument shows that a partition $\tau \in N C(n)$ such that $\tau \leq \operatorname{Kr}(\pi)$ is bijectively identified to the tuple

$$
\left(\tau_{W_{1}}, \ldots, \tau_{W_{p}}\right) \in N C\left(\left|W_{1}\right|\right) \times \cdots \times N C\left(\left|W_{p}\right|\right)
$$

which records the relabeled-restrictions of $\tau$ to the blocks $W_{1}, \ldots, W_{p}$. At the level of the tuple $\left(\tau_{W_{1}}, \ldots, \tau_{W_{p}}\right)$, the requirement that " $\tau$ connects $m$ with $n$ " (where $m$ and $n$ are the minimal and maximal elements of the block $W_{\text {right }}=W_{p}$ ) is transformed into the requirement that $\tau_{W_{p}} \in N C\left(\left|W_{p}\right|\right)$ is irreducible. We leave it as a straightforward exercise to the reader to check that, upon performing the change of variable $\tau \leftrightarrow\left(\tau_{W_{1}}, \ldots, \tau_{W_{p}}\right)$ in the summation from (10.10), one gets precisely the product of $p$ separate summations which is indicated on the right-hand side of (10.9).

Notation and Remark 10.6. The second factorization formula that we want to use is presented in Lemma 10.8, We find it convenient to first prove this lemma in a special case, stated separately as Lemma 10.7. In these lemmas we use two sequences of complex numbers, $\left(\gamma_{n}\right)_{n=1}^{\infty}$ and $\left(\widehat{\gamma}_{n}\right)_{n=1}^{\infty}$, with $\gamma_{1}=\widehat{\gamma}_{1} \neq 0$, and where the $\widehat{\gamma}_{n}$ 's are expressed in terms of $\gamma_{n}$ 's by summations over interval partitions, as follows:

$$
\begin{equation*}
\widehat{\gamma}_{n}=\sum_{\rho \in \operatorname{Int}(n)}\left(\prod_{J \in \rho} \gamma_{|J|}\right), \quad \forall n \geq 1 \tag{10.11}
\end{equation*}
$$

Lemma 10.7. Consider the framework of Notation 10.6, and on the other hand consider an $n \geq 1$ and an irreducible partition $\pi \in N C(n)$. Then:

$$
\begin{equation*}
\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}}\left(\prod_{\substack{U \in \operatorname{Kr}(\sigma), U \nexists n}} \gamma_{|U|}\right)=\prod_{\substack{W \in \operatorname{Kr}(\pi), W \not \supset n}} \widehat{\gamma}_{|W|} \tag{10.12}
\end{equation*}
$$

Proof. Due to the hypothesis that $\pi$ is irreducible, the Kreweras complement $\operatorname{Kr}(\pi)$ has a singleton block $\{n\}$. The same is true for any Kreweras complement $\operatorname{Kr}(\sigma)$ with $\sigma \geq \pi$ (since $\sigma$ will have to be irreducible as well). Hence when we multiply the left-hand side of (10.12) by $\gamma_{1}$ and the right-hand side of (10.12) by $\widehat{\gamma}_{1}$, where $\gamma_{1}=\widehat{\gamma}_{1} \neq 0$, we find that (10.12) is equivalent to

$$
\begin{equation*}
\sum_{\substack{\sigma \in N C(n),}}\left(\prod_{U \in \operatorname{Kr}(\sigma)} \gamma_{|U|}\right)=\prod_{W \in \operatorname{Kr}(\pi)} \widehat{\gamma}_{|W|} \tag{10.13}
\end{equation*}
$$

It is thus all right if we prove (10.13) instead of (10.12).
We now recall the poset anti-isomorphism (2.7) given by Kreweras complementation between $\ll$ and $\sqsubseteq$, where $\ll$ is considered on irreducible partitions in $N C(n)$ while $\sqsubseteq$ is considered on non-crossing partitions which have $\{n\}$ as a 1-element block. Since the set $\{\sigma \in N C(n) \mid \sigma \gg \pi\}$ only contains irreducible partitions, we can use (2.7) as a change of variable in the summation on the left-hand side of (10.13), which is thus transformed into

$$
\begin{align*}
& \sum_{\substack{\tau \in N C(n),}}\left(\prod_{U \in \tau} \gamma_{|U|}\right) .  \tag{10.14}\\
& \tau \sqsubset \operatorname{Kr}(\pi)
\end{align*}
$$

From here on we proceed with a variation of the argument that finalized the proof of Lemma 10.5? we list explicitly the blocks of $\operatorname{Kr}(\pi)$ as $W_{1}, \ldots, W_{p}$, and we use the fact that a $\tau \in N C(n)$ with $\tau \leq \operatorname{Kr}(\pi)$ is bijectively identified to the tuple

$$
\left(\tau_{W_{1}}, \ldots, \tau_{W_{p}}\right) \in N C\left(\left|W_{1}\right|\right) \times \cdots \times N C\left(\left|W_{p}\right|\right) .
$$

At the level of the latter tuple, the requirement " $\tau \sqsubseteq \operatorname{Kr}(\pi)$ " amounts to asking that $\tau_{W_{1}}, \ldots, \tau_{W_{p}}$ are interval partitions. Performing the change of variable $\tau \leftrightarrow\left(\tau_{W_{1}}, \ldots, \tau_{W_{p}}\right)$ in the summation from (10.14) thus takes us to a summation over $\operatorname{Int}\left(\left|W_{1}\right|\right) \times \cdots \times \operatorname{Int}\left(\left|W_{p}\right|\right)$. We leave it as a straightforward exercise to the reader to check that the latter summation factors as the product of $p$ separate summations over $\operatorname{Int}\left(\left|W_{1}\right|\right), \ldots, \operatorname{Int}\left(\left|W_{p}\right|\right)$, and that one obtains in this way the product indicated on the right-hand side of (10.13).

Lemma 10.8. The factorization formula stated in Equation (10.12) of Lemma 10.7 holds even if we do not assume the partition $\pi$ to be irreducible.

Proof. Consider the canonical decomposition $\pi=\pi_{1} \diamond \cdots \diamond \pi_{k}$ with $\pi_{1} \in N C\left(n_{1}\right), \ldots, \pi_{k} \in$ $N C\left(n_{k}\right)$ irreducible, as reviewed in Remark [2.4]3. The specifics of the partial order $\ll$ force that we have

$$
\{\sigma \in N C(n) \mid \sigma \gg \pi\}=\left\{\begin{array}{l|r}
\sigma_{1} \diamond \cdots \diamond \sigma_{k} & \left.\begin{array}{r}
\sigma_{1} \gg \pi_{1} \text { in } N C\left(n_{1}\right), \ldots, \\
\sigma_{k} \gg \pi_{k} \text { in } N C\left(n_{k}\right)
\end{array}\right\} . ~ . ~ \tag{10.15}
\end{array} .\right.
$$

For a partition $\sigma=\sigma_{1} \diamond \cdots \diamond \sigma_{k}$ as in (10.15) we note that $\sigma_{1}, \ldots, \sigma_{k}$ are all irreducible, and an examination of the relevant Kreweras complements leads to the formula

$$
\begin{equation*}
\prod_{\substack{U \in \operatorname{Kr}(\sigma), U \not \not n}} \gamma_{|U|}=\prod_{j=1}^{k}\left(\prod_{\substack{U \in \operatorname{Kr}\left(\sigma_{j}\right), U \not \not n_{j}}} \gamma_{|U|}\right) . \tag{10.16}
\end{equation*}
$$

The observations from (10.15), (10.16) and a straightforward conversion of sum into product then imply that we have:

$$
\begin{equation*}
\sum_{\substack{\sigma \in N C(n), \sigma \gg \\ \sum_{U \in \operatorname{Kr}(\sigma),} \\ U \not \supset n}} \gamma_{|U|}=\prod_{j=1}^{k}\left(\prod_{\substack{\sigma_{j} \in N C\left(n_{j}\right), U \in \operatorname{Kr}\left(\sigma_{j}\right), \sigma_{j} \gg \pi_{j}}} \gamma_{|U|}\right) \tag{10.17}
\end{equation*}
$$

But now, Lemma 10.7 can be applied to each of $\pi_{1}, \ldots, \pi_{k}$. When we do this, we find that Equation (10.17) can be continued with

$$
=\prod_{j=1}^{k}\left(\prod_{\substack{W \in \operatorname{Kr}\left(\pi_{j}\right), W \not \supset n_{j}}} \widehat{\gamma}_{|W|}\right)=\prod_{\substack{W \in \operatorname{Kr}(\pi), W \not \supset n}} \widehat{\gamma}_{|W|},
$$

where at the second equality sign we used the counterpart of (10.16) in connection to the numbers $\widehat{\gamma}_{i}$, and for the decomposition $\pi=\pi_{1} \diamond \cdots \diamond \pi_{k}$.
10.9. Proof of Theorem 10.1, We fix a $q \in \mathbb{R}$ and an $f \in \mathcal{G}$ for which we will prove that $u_{q}^{-1} * f * u_{q} \in \mathcal{G}$. The case when $q=0$ is clear, since $u_{0}^{-1} * f * u_{0}=f$, so we assume $q \neq 0$.

Let us denote $\lambda_{n}:=f\left(0_{n}, 1_{n}\right), n \geq 1$, and let $\left(\theta_{n}\right)_{n=1}^{\infty}$ be the sequence of complex numbers obtained out of the $\lambda_{n}$ 's by using the formula (10.2) from Remark 10.2. As anticipated in that remark, we will obtain the desired conclusion about $u_{q}^{-1} * f * u_{q}$ by proving that

$$
\begin{equation*}
u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)=\prod_{W \in \operatorname{Kr}(\pi)} \theta_{|W|}, \quad \forall n \geq 1 \text { and } \pi \in N C(n) \tag{10.18}
\end{equation*}
$$

From now on and until the end of the proof we fix an $n \geq 1$ and a $\pi \in N C(n)$ for which we will verify that (10.18) holds. We divide the argument into several steps.

Step 1. Write explicitly what is $u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)$, as a double sum "over $\sigma$ and $\tau$ ". Similarly to the derivation of Equation (10.3), we start from

$$
\begin{equation*}
u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)=\sum_{\substack{\sigma, \tau \in N C(n) \\ \text { such that } \pi \leq \sigma \leq \tau}} u_{q}^{-1}(\pi, \sigma) f(\sigma, \tau) u_{q}\left(\tau, 1_{n}\right) \tag{10.19}
\end{equation*}
$$

We have

$$
u_{q}^{-1}(\pi, \sigma)=u_{-q}(\pi, \sigma)= \begin{cases}(-q)^{|\pi|-|\sigma|}, & \text { if } \pi \ll \sigma \\ 0, & \text { otherwise }\end{cases}
$$

We plug this into the right-hand side of (10.19), and also replace the values of $f(\sigma, \tau)$ and of $u_{q}\left(\tau, 1_{n}\right)$ by using (5.5) and (9.1), respectively. In this way we arrive to the formula:

$$
\begin{equation*}
u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)=\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}}(-q)^{|\pi|-|\sigma|}\left(\sum_{\substack{\tau \in N C(n), \tau \geq \sigma \\ \text { and } \tau \text { irreducible }}}\left(\prod_{W \in \operatorname{Kr}_{\tau}(\sigma)} \lambda_{|V|}\right) \cdot q^{|\tau|-1}\right) \tag{10.20}
\end{equation*}
$$

In the summation over $\tau$ performed in (10.20), the conditions " $\tau \geq \sigma$ " and " $\tau$ irreducible" are consolidated in the requirement that $\tau \geq \bar{\sigma}^{\text {irr }}$. Let us also re-arrange the factor $q^{|\tau|-1}$
appearing in that summation: we have (cf. [24, Exercise 18.23]) $|\sigma|+\left|\operatorname{Kr}_{\tau}(\sigma)\right|=|\tau|+n$, which implies that $q^{|\tau|-1}=q^{\left|\operatorname{Kr}_{\tau}(\sigma)\right|} \cdot q^{|\sigma|-(n+1)}$. With these changes, we thus arrive to

$$
\begin{equation*}
u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)=\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}} \frac{(-q)^{|\pi|-|\sigma|}}{q^{(n+1)-|\sigma|}} \cdot\left(\sum_{\substack{\tau \in N C(n), \tau \geq \bar{\sigma}^{\mathrm{irr}}}} \prod_{W \in \mathrm{Kr}_{\tau}(\sigma)}\left(q \lambda_{|V|}\right)\right) . \tag{10.21}
\end{equation*}
$$

Step 2. Use the factorization formula from Lemma 10.5
Here we must first clarify what are the input sequences " $\alpha_{k}, \widehat{\alpha}_{k}, \widetilde{\alpha}_{k}$ " that we plan to use in Lemma 10.5. We go as follows: start from the sequence $\left(\lambda_{k}\right)_{k=1}^{\infty}$ which was fixed from the beginning of the proof and put $\alpha_{k}:=q \lambda_{k}$ for every $k \geq 1$; after that, define sequences $\left(\widehat{\alpha}_{k}\right)_{k=1}^{\infty}$ and $\left(\widetilde{\alpha}_{k}\right)_{k=1}^{\infty}$ via the formulas (10.7) given in Notation 10.4,

In view of what are our $\alpha_{k}$ 's, we re-write (10.21) in the form

$$
\left.u_{q}^{-1} * f * u_{q}\left(\pi, 1_{n}\right)=\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}} \frac{(-1)^{|\pi|-|\sigma|} q^{|\pi|-|\sigma|}}{q^{(n+1)-|\sigma|}} \cdot\left(\sum_{\substack{\tau \in N C(n), \tau \geq \bar{\sigma}^{\mathrm{ir}}}} \prod_{W \in \operatorname{Kr}_{\tau}(\sigma)} \alpha_{|W|}\right)\right),
$$

and we invoke Lemma 10.5 in order to continue with

$$
\begin{equation*}
=\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}}(-1)^{|\pi|}(-1)^{|\sigma|} q^{|\pi|-(n+1)} \cdot\left(\widetilde{\alpha}_{\left|W_{\mathrm{right}}\right|} \cdot \prod_{\substack{W \in \operatorname{Kr}(\sigma), W \ngtr n}} \widehat{\alpha}_{|W|}\right) . \tag{10.22}
\end{equation*}
$$

In the expression we arrived to, note that we can pull to the front of the sum the factors $(-1)^{|\pi|}, q^{|\pi|-(n+1)}$ and $\widetilde{\alpha}_{\left|W_{\text {right }}\right|}$. The justification for pulling out the latter factor comes from Remark $10.3,3$ - the block $W_{\text {right }}$ is the same for all the partitions $\sigma$ with $\sigma \gg \pi$. Thus from (10.22) we go on with

$$
\begin{equation*}
=(-1)^{|\pi|} q^{|\pi|-(n+1)} \widetilde{\alpha}_{\left|W_{\mathrm{right}}\right|} \cdot \sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}}(-1)^{|\sigma|} \cdot\left(\prod_{\substack{W \in \operatorname{Kr}(\sigma), W \not \supset n}} \widehat{\alpha}_{|W|}\right) . \tag{10.23}
\end{equation*}
$$

Step 3. Use the factorization formula from Lemma 10.8.
Here we must clarify what are the input sequences " $\gamma_{k}$ and $\widehat{\gamma}_{k}$ " that we plan to use in Lemma 10.8. We go as follows: put $\gamma_{k}=-\widehat{\alpha}_{k}$ for all $k \geq 1$; after that, define the sequence $\left(\widehat{\gamma}_{k}\right)_{k=1}^{\infty}$ via the formula (10.11) indicated in Notation 10.6. Observe that the common value of $\gamma_{1}$ and $\widehat{\gamma}_{1}$ is equal to $-q$ (this is found by backtracking in the definitions: $\gamma_{1}=-\widehat{\alpha}_{1}=$ $-\alpha_{1}=-q \lambda_{1}=-q$ ). Since it is assumed that $q \neq 0$, we are thus in a situation where the hypotheses of Lemma 10.8 are satisfied.

Next observation: in (10.23), the factor $(-1)^{|\sigma|}$ can be written as

$$
(-1)^{n} \cdot(-1)^{n-|\sigma|}=(-1)^{n} \cdot(-1)^{|\operatorname{Kr}(\sigma)|-1}
$$

where at the second equality we use the fact that one always has $|\sigma|+|\operatorname{Kr}(\sigma)|=n+1$. The $(-1)^{|\operatorname{Kr}(\sigma)|-1}$ can be absorbed into the product of $\widehat{\alpha}_{|W|}$ 's (which has $|\operatorname{Kr}(\sigma)|-1$ factors), and therefore (10.23) continues with

$$
=(-1)^{|\pi|} q^{|\pi|-(n+1)} \widetilde{\alpha}_{\left|W_{\text {right }}\right|} \cdot \sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}}(-1)^{n} \cdot\left(\prod_{\substack{W \in \operatorname{Kr}(\sigma), W \not \supset n}}\left(-\widehat{\alpha}_{|W|}\right)\right)
$$

$$
\begin{equation*}
\left.\left.=(-1)^{n-|\pi|} q^{|\pi|-(n+1)} \widetilde{\alpha}_{\left|W_{\text {right }}\right|} \sum_{\substack{ \\ \\ \\ \\ \\\sigma \gg \pi C(n), \prod_{W \in \operatorname{Kr}(\sigma)}}} \gamma_{|W|}\right)\right) \tag{10.24}
\end{equation*}
$$

The sum over $\sigma \gg \pi$ in (10.24) is precisely the one to which Lemma 10.8 applies, and in this way we arrive to the conclusion of Step 3, which is that we have

Step 4. Identify the factors in the product found in (10.25).
It is convenient to re-write the right-hand side of (10.25) in the form

$$
\begin{equation*}
\left(\frac{1}{q} \widetilde{\alpha}_{\left|W_{\text {right }}\right|}\right) \cdot \prod_{\substack{U \in \operatorname{Kr}(\pi), U \not \supset n}}\left(-\frac{1}{q} \widehat{\gamma}_{|U|}\right) \tag{10.26}
\end{equation*}
$$

with the pre-factor $(-1)^{n-|\pi|} q^{|\pi|-(n+1)}$ distributed among the $(n+1)-|\pi|$ blocks of $\operatorname{Kr}(\pi)$.
We are then left to chase through the formulas used in Steps 2 and 3 , and verify that the product over blocks of $\operatorname{Kr}(\pi)$ that appears in (10.26) is the same as the one on the right-hand side of our target Equation (10.18) indicated at the beginning of the proof. It is visible that everything would be in place if we had that:

$$
\begin{equation*}
\widetilde{\alpha}_{k}=q \theta_{k} \quad \text { and } \quad \widehat{\gamma}_{k}=-q \theta_{k}, \quad \forall k \geq 1 \tag{10.27}
\end{equation*}
$$

We will argue that the desirable relations stated in (10.27) are indeed holding.
The first relation (10.27) comes out by direct comparison of the formulas defining $\widetilde{\alpha}_{k}$ and $\theta_{k}$. Indeed, upon replacing $\alpha_{|V|}=q \lambda_{|V|}$ in the formula (10.7) which defines $\widetilde{\alpha}_{k}$, we find that

$$
\widetilde{\alpha}_{k}=\sum_{\substack{\rho \in N C(k), \\ \text { irreducible }}} \prod_{V \in \rho}\left(q \lambda_{|V|}\right)=\sum_{\substack{\rho \in N C(k), \\ \text { irreducible }}} q^{|\rho|} \prod_{V \in \rho} \lambda_{|V|}=q \theta_{k}
$$

where at the third equality $\operatorname{sign}$ we refer to the formula (10.2) for $\theta_{k}$.
For the second relation (10.27) it suffices to check that $\widehat{\gamma}_{k}=-\widetilde{\alpha}_{k}$. We have

$$
\begin{aligned}
\widehat{\gamma}_{k} & =\sum_{\rho \in \operatorname{Int}(k)} \prod_{J \in \rho} \gamma_{|J|} \quad\left(\text { by the definition of } \widehat{\gamma}_{k}, \text { in Equation (10.11) }\right) \\
& =\sum_{\rho \in \operatorname{Int}(k)} \prod_{J \in \rho}\left(-\widehat{\alpha}_{|J|}\right) \quad\left(\text { by the definition of } \gamma_{|J|}\right. \text { in Step 3) } \\
& =\sum_{\rho \in \operatorname{Int}(k)}(-1)^{|\rho|} \prod_{J \in \rho} \widehat{\alpha}_{|J|}=-\widetilde{\alpha}_{k}
\end{aligned}
$$

where the latter equality follows from Equation (10.8) of Remark 10.4.

## 11. An application: multiplication of free Random variables, in terms of $t$-BOOLEAN CUMULANTS

As explained in Section 1.5 of the Introduction, the multiplication of free random variables has a nice description in terms of $t$-Boolean cumulants, by a formula which is actually the
same for all values of $t$. In the present section we show how this fact can be neatly derived by using the 1-parameter subgroup $\left\{u_{q} \mid q \in \mathbb{R}\right\}$ of $\widetilde{\mathcal{G}_{\mathrm{c}}-\mathrm{c}}$.

Notation and Remark 11.1. (Framework, and discussion of what we will prove.)
We fix for the whole section a non-commutative probability space $(\mathcal{A}, \varphi)$ and two unital subalgebras $\mathcal{M}, \mathcal{N} \subseteq \mathcal{A}$ which are freely independent with respect to $\varphi$. For every $t \in \mathbb{R}$ we consider the family $\underline{\beta}^{(t)}=\left(\beta_{n}^{(t)}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$ of $t$-Boolean cumulants of $(\mathcal{A}, \varphi)$; we also consider the standard enlargement of $\underline{\beta}^{(t)}$ to $\left(\beta_{\pi}^{(t)}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n \geq 1, \pi \in N C(n)}$, as discussed in Notation 6.2, 2. It will be convenient to aim for a formula slightly more general than what was announced in Equation (1.6) of the Introduction, and which is stated as follows:

$$
\left\{\begin{array}{c}
\text { One has } \beta_{n}^{(t)}\left(x_{1} y, \ldots, x_{n} y\right)=\sum_{\pi \in N C(n)} \beta_{\pi}^{(t)}\left(x_{1}, \ldots, x_{n}\right) \cdot \beta_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y),  \tag{11.1}\\
\text { holding for every } n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in \mathcal{M}, y \in \mathcal{N} \text { and } t \in \mathbb{R}
\end{array}\right.
$$

Our approach to (11.1) is this: we note that for fixed $y$ and $t$, the family of equalities stated in (11.1) is equivalent to one equation concerning the action of the group $\widetilde{\mathcal{G}}$ on the space $\mathfrak{M}_{\mathcal{M}}$ of sequences of multilinear functionals on $\mathcal{M}$. The latter equation can then be treated by using results from Sections 9 and 10, particularly Theorem 10.1.

In order for the trick of fixing a $y$ to play smoothly into the setting from Sections 9 and 10 , it is good to arrange that $\varphi(y)=1$. We start by pointing out that, without loss of generality, we can make this assumption.

Lemma 11.2. Assume it is true that (11.1) holds whenever $y \in \mathcal{N}$ has $\varphi(y)=1$. Then (11.1) is sure to hold with $y \in \mathcal{N}$ arbitrary.

Proof. We first extend the validity of (11.1) to the case when $\varphi(y) \neq 0$. If $\varphi(y)=\lambda \neq 0$ then for every $t \in \mathbb{R}, n \in \mathbb{N}$ and $x_{1}, \ldots, x_{n} \in \mathcal{M}$ we have

$$
\begin{aligned}
\beta_{n}^{(t)}\left(x_{1} y, \ldots, x_{n} y\right) & =\beta_{n}^{(t)}\left(\left(\lambda x_{1}\right) \cdot\left(\lambda^{-1} y\right), \ldots,\left(\lambda x_{n}\right) \cdot\left(\lambda^{-1} y\right)\right) \\
& =\sum_{\pi \in N C(n)} \beta_{\pi}^{(t)}\left(\lambda x_{1}, \ldots, \lambda x_{n}\right) \cdot \beta_{\operatorname{Kr}(\pi)}^{(t)}\left(\lambda^{-1} y, \ldots, \lambda^{-1} y\right)
\end{aligned}
$$

(by hypothesis, since $\varphi\left(\lambda^{-1} y\right)=1$ )

$$
\begin{aligned}
& =\sum_{\pi \in N C(n)} \lambda^{n} \beta_{\pi}^{(t)}\left(x_{1}, \ldots, x_{n}\right) \cdot \lambda^{-n} \beta_{\operatorname{Kr}(\pi)}^{(t)}(y, \ldots, y) \\
& =\sum_{\pi \in N C(n)} \beta_{\pi}^{(t)}\left(x_{1}, \ldots, x_{n}\right) \cdot \beta_{\operatorname{Kr}(\pi)}^{(t)}(y, \ldots, y), \text { as required. }
\end{aligned}
$$

Now consider a $y \in \mathcal{N}$ with $\varphi(y)=0$. From the fact proved in the preceding paragraph, it follows that: for every $t \in \mathbb{R}, n \in \mathbb{N}$ and $x_{1}, \ldots, x_{n} \in \mathcal{M}$, one has

$$
\begin{gather*}
\beta_{n}^{(t)}\left(x_{1}\left(y+\delta 1_{\mathcal{A}}\right), \ldots, x_{n}\left(y+\delta 1_{\mathcal{A}}\right)\right)  \tag{11.2}\\
=\sum_{\pi \in N C(n)} \beta_{\pi}^{(t)}\left(x_{1}, \ldots, x_{n}\right) \cdot \beta_{\operatorname{Kr}(\pi)}^{(t)}\left(y+\delta 1_{\mathcal{A}}, \ldots, y+\delta 1_{\mathcal{A}}\right), \quad \forall \delta \neq 0 \text { in } \mathbb{C} .
\end{gather*}
$$

It is easy to check that the two sides of (11.2) depend continuously (in fact polynomially) on $\delta$. We can thus make $\delta \rightarrow 0$ in (11.2), to conclude that (11.1) holds for this $y$ as well.

Notation 11.3. $1^{o}$ For the remaining part of this section we fix an element $y \in \mathcal{N}$ with $\varphi(y)=1$, in connection to which we will prove that (11.1) is holding.
$2^{o}$ It is convenient that, by using the $y$ which was fixed, we introduce some sequences of multilinear functionals on $\mathcal{M}$, as follows: for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$, let $\gamma_{n}^{(t, \mathcal{M})}: \mathcal{M}^{n} \rightarrow \mathbb{C}$ be defined by

$$
\begin{equation*}
\gamma_{n}^{(t, \mathcal{M})}\left(x_{1}, \ldots, x_{n}\right)=\beta_{n}^{(t)}\left(x_{1} y, \ldots, x_{n} y\right), \quad \forall x_{1}, \ldots, x_{n} \in \mathcal{M} \tag{11.3}
\end{equation*}
$$

Clearly, we have that $\underline{\gamma}^{(t, \mathcal{M})}:=\left(\gamma_{n}^{(t, \mathcal{M})}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{M}}$, where $\mathfrak{M}_{\mathcal{M}}$ is defined exactly as in Notation 6.1, but by using $\mathcal{M}$ instead of $\mathcal{A}$.

In the same vein, it is convenient that for every $t \in \mathbb{R}$ and $n \geq 1$ we use the notation $\beta_{n}^{(t, \mathcal{M})}: \mathcal{M}^{n} \rightarrow \mathbb{C}$ for the restriction of the multilinear functional $\beta_{n}^{(t)}: \mathcal{A}^{n} \rightarrow \mathbb{C}$ to the subspace $\mathcal{M}^{n}$. Then $\underline{\beta}^{(t, \mathcal{M})}:=\left(\beta_{n}^{(t, \mathcal{M})}\right)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{M}}$, and (as immediately verified) it is the family of $t$-Boolean cumulants of the non-commutative probability space $(\mathcal{M}, \varphi \mid \mathcal{M})$.
$3^{o}$ Recall from Section 5 that every sequence $\left(\alpha_{n}\right)_{n=1}^{\infty}$ of complex numbers, with $\alpha_{1}=1$, defines a multiplicative function $f \in \mathcal{G}$ via the requirement that $f\left(0_{n}, 1_{n}\right)=\alpha_{n}$ for all $n \geq 1$. For every $t \in \mathbb{R}$ we can therefore consider a multiplicative function $f_{t} \in \mathcal{G}$ defined via the requirement that

$$
\begin{equation*}
f_{t}\left(0_{n}, 1_{n}\right)=\beta_{n}^{(t)}(y, \ldots, y), \quad \forall n \geq 1, \tag{11.4}
\end{equation*}
$$

where $y$ is the element of $\mathcal{N}$ fixed in part $1^{o}$ of this notation. Note that when defining $f_{t}$ we use the fact that $\varphi(y)=1$, which ensures that the sequence of numbers proposed on the right-hand side of (11.4) does indeed start with $\beta_{1}^{(t)}(y)=\varphi(y)=1$.

For every $t \in \mathbb{R}$ and for general $\pi \leq \sigma$ in some $N C(n)$, an explicit formula giving $f_{t}(\pi, \sigma)$ is then obtained out of (11.4), in the way reviewed in Remark 5.2, Recall, in particular, that for every $n \geq 1$ and $\pi \in N C(n)$ we have

$$
\begin{equation*}
f_{t}\left(\pi, 1_{n}\right)=\prod_{W \in \operatorname{Kr}(\pi)} f_{t}\left(0_{|W|}, 1_{|W|}\right)=\prod_{W \in \operatorname{Kr}(\pi)} \beta_{|W|}^{(t)}(y, \ldots, y)=\beta_{\operatorname{Kr}(\pi)}^{(t)}(y, \ldots, y) . \tag{11.5}
\end{equation*}
$$

In terms of the notation just introduced, we can give an equivalent form of (11.1), which is stated as follows.

Lemma 11.4. For every $t \in \mathbb{R}$, one has:

$$
\begin{equation*}
\binom{\text { Formula (11.1) holds for our }}{\text { fixed } y \text { and this particular value of } t} \Leftrightarrow\binom{\gamma^{(t, \mathcal{M})}=\beta^{(t, \mathcal{M})} \cdot f_{t}}{\left.\overline{\left(\text { an equality in } \mathfrak{M}_{\mathcal{M}}\right.}\right)} . \tag{11.6}
\end{equation*}
$$

Proof. The equality stated on the right-hand side of the equivalence is spelled out as follows:

$$
\left\{\begin{array}{r}
\gamma_{n}^{(t, \mathcal{M})}\left(x_{1}, \ldots, x_{n}\right)=\sum_{\pi \in N C(n)} f_{t}\left(\pi, 1_{n}\right) \beta_{\pi}^{(t, \mathcal{M})}\left(x_{1}, \ldots, x_{n}\right)  \tag{11.7}\\
\text { holding for every } n \geq 1 \text { and } x_{1}, \ldots, x_{n} \in \mathcal{M}
\end{array}\right.
$$

We leave it as an immediate exercise to the reader to replace the various quantities mentioned in (11.7) by their definition from Notation [11.3, and to verify that what comes out is indeed equivalent to the instance of (11.1) referring to our fixed $y$ and $t$.

We next examine how one can connect two instances of the equation appearing on the right-hand side of the equivalence (11.6), considered for two different values $s, t \in \mathbb{R}$. This is done by using the 1-parameter subgroup $\left\{u_{q} \mid q \in \mathbb{R}\right\}$ from the preceding sections, both in reference to $\beta^{(t, \mathcal{M})}, \gamma^{(t, \mathcal{M})}$ (in Lemma 11.5) and in reference to $f_{t}$ (in Lemma 11.6).

Lemma 11.5. For every $s, t \in \mathbb{R}$ we have

$$
\begin{equation*}
\underline{\beta}^{(t, \mathcal{M})}=\underline{\beta}^{(s, \mathcal{M})} \cdot u_{s-t} \text { and } \underline{\gamma}^{(t, \mathcal{M})}=\underline{\gamma}^{(s, \mathcal{M})} \cdot u_{s-t} \tag{11.8}
\end{equation*}
$$

Proof. The first formula (11.8) is a direct consequence of Corollary 9.5 written in connection to the non-commutative probability space $(\mathcal{M}, \varphi \mid \mathcal{M})$.

The second formula (11.8) also follows from Corollary 9.5. Indeed, for every $n \geq 1$ and $x_{1}, \ldots, x_{n} \in \mathcal{M}$ we can write

$$
\gamma_{n}^{(t, \mathcal{M})}\left(x_{1}, \ldots, x_{n}\right)=\beta_{n}^{(t)}\left(x_{1} y, \ldots, x_{n} y\right)=\sum_{\substack{\pi \in N C(n) \\ \text { irreducible }}}(s-t)^{|\pi|-1} \cdot \beta_{\pi}^{(s)}\left(x_{1} y, \ldots, x_{n} y\right)
$$

where at the second equality sign we use Equation (7.16) from Remark 7.9. An inspection of the definition of the functionals $\beta_{\pi}^{(s)}$ and $\gamma_{\pi}^{(s)}$ shows that in the latter expression we can replace $\beta_{\pi}^{(s)}\left(x_{1} y, \ldots, x_{n} y\right)$ with $\gamma_{\pi}^{(s, \mathcal{M})}\left(x_{1}, \ldots, x_{n}\right)$; hence what we got is

$$
\gamma_{n}^{(t, \mathcal{M})}\left(x_{1}, \ldots, x_{n}\right)=\sum_{\substack{\pi \in N C(n) \\ \text { irreducible }}}(s-t)^{|\pi|-1} \cdot \gamma_{\pi}^{(s)}\left(x_{1}, \ldots, x_{n}\right)
$$

where the right-hand side is indeed the value at $\left(x_{1}, \ldots, x_{n}\right)$ of the $n$-th functional in the family $\underline{\gamma}^{(s, \mathcal{M})} \cdot u_{s-t} \in \mathfrak{M}_{\mathcal{M}}$.

Lemma 11.6. Let $t, q$ be in $\mathbb{R}$. One has that $u_{q}^{-1} * f_{t} * u_{q}=f_{t-q}$.
Proof. We have that $f_{t-q}$ is multiplicative (by definition, cf. Notation 11.3.3) and $u_{q}^{-1} * f_{t} * u_{q}$ is multiplicative as well (due to Theorem10.1); so in order to prove their equality, it suffices to verify that

$$
\begin{equation*}
u_{q}^{-1} * f_{t} * u_{q}\left(0_{n}, 1_{n}\right)=f_{t-q}\left(0_{n}, 1_{n}\right), \quad \forall n \geq 1 \tag{11.9}
\end{equation*}
$$

The right-hand side of (11.9) is, by definition, equal to $\beta_{n}^{(t-q)}(y, \ldots, y)$. For the left-hand side of the same equation we resort to Equation (10.2) of Remark 10.2 , which says that

$$
\begin{equation*}
u_{q}^{-1} * f_{t} * u_{q}\left(0_{n}, 1_{n}\right)=\sum_{\substack{\pi \in N C(n) \\ \text { irreducible }}} q^{|\pi|-1} \prod_{V \in \pi} f_{t}\left(0_{|V|}, 1_{|V|}\right) \tag{11.10}
\end{equation*}
$$

Upon replacing $f_{t}\left(0_{|V|}, 1_{|V|}\right)$ from its definition, the right-hand side of (11.10) becomes

$$
\sum_{\substack{\pi \in N C(n), \\ \text { irreducible }}} q^{|\pi|-1} \prod_{V \in \pi} \beta_{|V|}^{(t)}(y, \ldots, y)
$$

and this is indeed equal to $\beta_{n}^{(t-q)}(y, \ldots, y)$, thanks to Equation (7.16) of Remark [7.9,

Lemma 11.7. Suppose there exists a value $t_{o} \in \mathbb{R}$ for which it is true that $\underline{\gamma}^{\left(t_{o}, \mathcal{M}\right)}=$ $\underline{\beta}^{\left(t_{o}, \mathcal{M}\right)} \cdot f_{t_{o}}$. Then it follows that $\underline{\gamma}^{(t, \mathcal{M})}=\underline{\beta}^{(t, \mathcal{M})} \cdot f_{t}$ for all $t \in \mathbb{R}$.
Proof. Fix a $t \in \mathbb{R}$. We use Lemmas 11.5 and 11.6 , with $q:=t_{o}-t$, to replace $\underline{\beta}^{(t, \mathcal{M})}=$ $\underline{\beta}^{\left(t_{o}, \mathcal{M}\right)} \cdot u_{t_{o}-t}$ and $f_{t}=u_{t_{o}-t}^{-1} * f_{t_{o}} * u_{t_{o}-t}$, and thus get:

$$
\underline{\beta}^{(t, \mathcal{M})} \cdot f_{t}=\left(\underline{\beta}^{\left(t_{o}, \mathcal{M}\right)} \cdot u_{t_{o}-t}\right) \cdot\left(u_{t_{o}-t}^{-1} * f_{t_{o}} * u_{t_{o}-t}\right)=\left(\underline{\beta}^{\left(t_{o}, \mathcal{M}\right)} \cdot f_{t_{o}}\right) \cdot u_{t_{o}-t}
$$

In the latter expression we can replace $\underline{\beta}^{\left(t_{o}, \mathcal{M}\right)} \cdot f_{t_{o}}$ with $\underline{\gamma}^{\left(t_{o}, \mathcal{M}\right)}$ (by hypothesis), then we can invoke Lemma 11.5 to conclude that $\underline{\gamma}^{\left(t_{o}, \mathcal{M}\right)} \cdot u_{t_{o}-t}=\underline{\gamma}^{(t, \mathcal{M})}$. In this way we obtain that $\underline{\beta}^{(t, \mathcal{M})} \cdot f_{t}=\underline{\gamma}^{(t, \mathcal{M})}$, as required.

11．8．Proof of the statement（11．1）．In view of Lemma 11．2，it suffices to prove（11．1） in connection to the element $y \in \mathcal{N}$ with $\varphi(y)=1$ which was fixed since Notation 11．3，

The special case $t_{o}=1$ of（11．1）concerns the description of multiplication of free elements in terms of free cumulants．This is a basic result in the combinatorics of free probability， which is not hard to obtain via a suitable grouping of terms in the moment－cumulant formula for free cumulants，followed by an application of Möbius inversion．For the details of how this goes，see for instance［24，Theorem 14．4］．

We therefore accept the case $t_{o}=1$ in（11．1）．The equivalence noticed in Lemma 11.4 then tells us that that the equality $\underline{\beta}^{\left(t_{o}, \mathcal{M}\right)} \cdot f_{t_{o}}=\underline{\gamma}^{\left(t_{o}, \mathcal{M}\right)}$ holds for $t_{o}=1$ ．This puts us in the position to invoke Lemma 11．7，in order to conclude that the equality $\underline{\beta}^{(t, \mathcal{M})} \cdot f_{t}=\underline{\gamma}^{(t, \mathcal{M})}$ holds for every $t \in \mathbb{R}$ ．Finally，the equivalence noticed in Lemma 11.4 is used again（this time in the direction from right to left）to conclude that（11．1）holds for all values $t \in \mathbb{R}$ ， as required．

Remark 11．9．In the formula（1．6）of the Introduction，the roles played by the elements $x, y \in \mathcal{A}$ were similar to each other．This symmetry was broken when we moved to the more general statement in（11．1），where we continue to work with $(y, \ldots, y) \in \mathcal{A}^{n}$ but we use a tuple $\left(x_{1}, \ldots, x_{n}\right)$ instead of just $(x, \ldots, x)$ ．In connection to that，we mention that （11．1）can be further extended to the following statement：

$$
\left\{\begin{array}{c}
\text { Let } \mathcal{M}, \mathcal{N} \subseteq \mathcal{A} \text { be as in (11.1). One has that }  \tag{11.11}\\
\beta_{n}^{(t)}\left(x_{1} y_{1}, \ldots, x_{n} y_{n}\right)=\sum_{\pi \in N C(n)} \beta_{\pi}^{(t)}\left(x_{1}, \ldots, x_{n}\right) \cdot \beta_{\operatorname{Kr}(\pi)}^{(t)}\left(y_{1}, \ldots, y_{n}\right), \\
\text { holding for every } n \in \mathbb{N}, x_{1}, \ldots, x_{n} \in \mathcal{M}, y_{1}, \ldots, y_{n} \in \mathcal{N} \text { and } t \in \mathbb{R} .
\end{array}\right.
$$

The proof shown above for（11．1）does not cover the more general statement（11．11）， because our handling of the multiplicative function $f_{t}$ makes effective use（e．g．when con－ sidering the functionals $\beta_{|W|}^{(t)}$ in（11．5））of the fact that $y_{1}=\cdots=y_{n}=y$ ．For a reader who is interested to pursue this，we outline below a possible approach to（11．11），which is however straying a bit outside the main body of ideas of the paper，and requires some work around a certain＂$t$－Boolean Bercovici－Pata bijection＂that was introduced in［3］．

Let us quickly review some notation from［24，Lectures 16 and 17］．We consider a sheer algebraic setting，with a＂space of distributions＂defined as

$$
\mathcal{D}_{\mathrm{alg}}(n):=\left\{\mu: \mathbb{C}\left\langle X_{1}, \ldots, X_{n}\right\rangle \rightarrow \mathbb{C} \mid \mu \text { linear, } \mu(1)=1\right\} .
$$

Every $\mu \in \mathcal{D}_{\text {alg }}(n)$ has an $R$－transform $R_{\mu}$ which belongs to the space $\mathbb{C}_{o}\left\langle\left\langle z_{1}, \ldots, z_{n}\right\rangle\right\rangle$ of formal power series without constant coefficient in the non－commuting indeterminates $z_{1}, \ldots, z_{n}$ ．The series $R_{\mu}$ is put together by using the free cumulants of $\mu$ as coefficients （cf．［24，Definition 16．3］）．The multiplication of freely independent $n$－tuples of elements in a non－commutative probability space is encoded by a binary operation $\boxtimes$ on $\mathcal{D}_{\text {alg }}(n)$ ．Then， upon taking $R$－transforms，$\boxtimes$ is turned into a certain binary operation 柬 on power series：

$$
\begin{equation*}
R_{\mu \boxtimes \nu}=R_{\mu} \text { 困 } R_{\nu}, \quad \forall \mu, \nu \in \mathcal{D}_{\mathrm{alg}}(n) . \tag{11.12}
\end{equation*}
$$

Moreover：for $f, g \in \mathbb{C}_{o}\left\langle\left\langle z_{1}, \ldots, z_{n}\right\rangle\right\rangle$ ，the coefficients of $f$ 困 $g$ can be explicitly described in terms of the coefficients of $f$ and of $g$ via a formula which is reminiscent of（11．11）－cf．［24， Definition 17.1 and Proposition 17．2］．

For our discussion here it is relevant that for every $\mu \in \mathcal{D}_{\text {alg }}(n)$ and $t \in \mathbb{R}$ we can define an $\eta^{(t)}$－transform，

$$
\eta_{\mu}^{(t)} \in \mathbb{C}_{o}\left\langle\left\langle z_{1}, \ldots, z_{n}\right\rangle\right\rangle ;
$$

the series $\eta_{\mu}^{(t)}$ is put together by using the $t$－Boolean cumulants of $\mu$ as coefficients．The $R$－transform is retrieved at $t=1, \eta_{\mu}^{(1)}=R_{\mu}$ ．The point of relevance for the proof of（11．11）
is that one can extend Equation（11．12）from the case $t=1$ to the case of a general $t \in \mathbb{R}$ ：

$$
\begin{equation*}
\eta_{\mu \boxtimes \nu}^{(t)}=\eta_{\mu}^{(t)} \text { 困 } \eta_{\nu}^{(t)} \quad \forall t \in \mathbb{R} \text { and } \mu, \nu \in \mathcal{D}_{\text {alg }}(n) . \tag{11.13}
\end{equation*}
$$

Verification that（11．11）follows from（11．13）：consider the setting from（11．11），and let $\mu, \nu \in \mathcal{D}_{\text {alg }}(n)$ be the joint distributions of the tuples $\left(x_{1}, \ldots, x_{n}\right)$ and $\left(y_{1}, \ldots, y_{n}\right)$ ， respectively．The definition of $\boxtimes$ ensures that the joint distribution of $\left(x_{1} y_{1}, \ldots, x_{n} y_{n}\right)$ is $\mu \boxtimes \nu$ ．Thus $\beta_{n}^{(t)}\left(x_{1} y_{1}, \ldots, x_{n} y_{n}\right)$ is retrieved as the coefficient of $z_{1} \cdots z_{n}$ in $\eta_{\mu \boxtimes \nu}^{(t)}$ ，and in view of（11．13）we get that

$$
\begin{equation*}
\beta_{n}^{(t)}\left(x_{1} y_{1}, \ldots, x_{n} y_{n}\right)=\left[\text { Coefficient of } z_{1} \cdots z_{n} \text { in } \eta_{\mu}^{(t)} \text { 团 } \eta_{\nu}^{(t)}\right] . \tag{11.14}
\end{equation*}
$$

From（11．14），the explicit description of how $⿴ 囗 大$ works takes us precisely to the right－hand side of the formula indicated in（11．11）．

Now，the reason for reducing（11．11）to（11．13）is that the latter formula can be studied in connection to a family of bijective maps $\left(\mathbb{B}_{t}: \mathcal{D}_{\text {alg }}(n) \rightarrow \mathcal{D}_{\text {alg }}(n)\right)_{t \in[0, \infty)}$ introduced in［3］．These maps form a semigroup $\left(\mathbb{B}_{s} \circ \mathbb{B}_{t}=\mathbb{B}_{s+t}\right.$ for all $s, t \in[0, \infty)$ ），and have the property that

$$
\begin{equation*}
\mathbb{B}_{t}(\mu \boxtimes \nu)=\mathbb{B}_{t}(\mu) \boxtimes \mathbb{B}_{t}(\nu), \quad \forall t \in[0, \infty) \text { and } \mu, \nu \in \mathcal{D}_{\mathrm{alg}}(n) \tag{11.15}
\end{equation*}
$$

When $t=1$ ，the map $\mathbb{B}_{1}$ is known as＂Boolean Bercovici－Pata bijection＂，and has the following description（the idea of which can be tracked back all the way to［5］）：

$$
\left\{\begin{array}{c}
\text { For every } \mu \in \mathcal{D}_{\text {alg }}(n) \text {, we have that } \mathbb{B}_{1}(\mu) \text { is }  \tag{11.16}\\
\text { the unique distribution } \nu \in \mathcal{D}_{\text {alg }}(n) \text { such that } R_{\nu}=\eta_{\mu}^{(0)} .
\end{array}\right.
$$

A reader interested in pursuing this line of thought should be able to develop the relevant $N C(n)$－combinatorics presented in［3］and obtain the following generalization of（11．16）：

$$
\left\{\begin{array}{c}
\text { For every } t \in[0, \infty) \text { and } \mu \in \mathcal{D}_{\text {alg }}(n) \text {, we have that } \mathbb{B}_{t}(\mu) \text { is }  \tag{11.17}\\
\text { the unique distribution } \nu \in \mathcal{D}_{\mathrm{alg}}(n) \text { such that } \eta_{\nu}^{(t)}=\eta_{\mu}^{(0)} .
\end{array}\right.
$$

By using（11．17）and the machinery of the $\mathbb{B}_{t}$＇s，it is then rather straightforward to upgrade from（11．12）to the case of（11．13）with $t \in[0,1]$ ．Indeed，if $\mu, \nu \in \mathcal{D}_{\text {alg }}(n)$ and $t \in[0,1]$ are given，one puts $\mu^{\prime}:=\mathbb{B}_{1-t}(\mu)$ and $\nu^{\prime}:=\mathbb{B}_{1-t}(\nu)$ ，and processes the equality $R_{\mu^{\prime} \boxtimes \nu^{\prime}}=R_{\mu^{\prime}}$ 图 $R_{\nu^{\prime}}$ into becoming $\eta_{\mu \boxtimes \nu}^{(t)}=\eta_{\mu}^{(t)}$ 团 $\eta_{\nu}^{(t)}$ ．Finally，it is also straightforward to observe that（11．13）can be expressed in the guise of a family of polynomial identities in $t$ ； and if such an identity holds for all $t \in[0,1]$ ，then it must actually hold for all $t \in \mathbb{R}$ ．

Remark 11．10．Upon seeing how things came up in（11．1），one is prompted to ask the analogous question in connection to the other important brand of cumulants mentioned in Section 7，the monotone cumulants．More precisely：let $(\mathcal{A}, \varphi)$ ，the freely independent unital subalgebras $\mathcal{M}, \mathcal{N} \subseteq \mathcal{A}$ and the element $y \in \mathcal{N}$ be the same as above，and consider the sequence of monotone cumulant functionals $\underline{\rho}=\left(\rho_{n}: \mathcal{A}^{n} \rightarrow \mathbb{C}\right)_{n=1}^{\infty}$ ．Is there a nice formula which expresses a monotone cumulant

$$
\rho_{n}\left(x_{1} y, \ldots, x_{n} y\right), \quad \text { with } n \geq 1 \text { and } x_{1}, \ldots, x_{n} \in \mathcal{M}
$$

in terms of the monotone cumulants of the $x_{i}$＇s（on the one hand）and the monotone cumulants of $y$（on the other hand）？It may seem intriguing that low order calculations show an analogy with（11．1）：one has

$$
\left\{\begin{array}{l}
\rho_{n}\left(x_{1} y, \ldots, x_{n} y\right)=\sum_{\pi \in N C(n)} \rho_{\pi}\left(x_{1}, \ldots, x_{n}\right) \cdot \rho_{\mathrm{Kr}(\pi)}(y, \ldots, y),  \tag{11.18}\\
\quad \text { for all } n \leq 4 \text { and } x_{1}, \ldots, x_{n} \in \mathcal{M} .
\end{array}\right.
$$

This turns out to be an accident which no longer holds for $n \geq 5$. In an appendix at the end of the paper we show the output of some computer calculations which check the difference of the two sides of (11.18) for $5 \leq n \leq 8$ and in the case when $x_{1}=\cdots=x_{n}=: x \in \mathcal{M}$. It is probably another low-dimensional accident that the "irregular" terms appearing in this difference don't seem to be that numerous. In any case, it would be interesting to have a theorem establishing an analogue of (11.1) for monotone cumulants; this theorem would then also explain the structure of the irregular terms shown in the appendix.

## 12. Identifying $\widetilde{\mathcal{G}}$ as character group of a Hopf algebra

The machinery of incidence algebras on posets can be re-cast in a way which uses Hopf algebra considerations. More precisely: the main object studied in the present paper, the group $\widetilde{\mathcal{G}}$, will now be identified in a natural way as the group of characters a Hopf algebra $\mathcal{T}$. The construction of $\mathcal{T}$ is quite direct, when one pursues the following guidelines:

- As an algebra: $\mathcal{T}$ should have a universality property which makes it that the characters of $\mathcal{T}$ are parametrized by functions from $\widetilde{\mathcal{G}}$.
- As a coalgebra: the comultiplication of $\mathcal{T}$ has to play into the formula (3.7) which governs the group operation on $\widetilde{\mathcal{G}}$.

The construction of $\mathcal{T}$ works seamlessly due to certain underlying properties of the lattices $N C(n)$. This falls within the framework of "hereditary family of posets" in the sense developed by Schmitt [27, and consequently $\mathcal{T}$ is an incidence Hopf algebra in the sense of that paper. A version of $\mathcal{T}$ has also been recently studied in the paper [11.

### 12.1. Description of $\mathcal{T}$.

This subsection describes the Hopf algebra $\mathcal{T}$ we are interested in, with detailed explicit formulas for the algebra and coalgebra operations. We assume the reader to be familiar with basic notions and facts concerning Hopf algebras with combinatorial flavour, as presented for instance in [15, Section 1B and Chapter 14] or in [21, Chapters I and II].

Notation and Remark 12.1. $1^{\circ}$ We let $\mathcal{T}$ be the commutative algebra of polynomials over $\mathbb{C}$ which uses a countable collection of indeterminates indexed by non-crossing partitions with at least two blocks:

$$
\begin{equation*}
\mathcal{T}:=\mathbb{C}\left[X_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right)\right] \tag{12.1}
\end{equation*}
$$

We also make the convention to denote

$$
\begin{equation*}
X_{1_{n}}:=1_{\mathcal{T}}, \quad \forall n \geq 1, \tag{12.2}
\end{equation*}
$$

and thus get to have elements $X_{\pi} \in \mathcal{T}$ defined for all the partitions in $\sqcup_{n=1}^{\infty} N C(n)$.
$2^{o}$ As an immediate consequence of how notation is set in $1^{o}, \mathcal{T}$ has a universality property described as follows:

$$
\left\{\begin{array}{l}
\text { If } \mathcal{A} \text { is a unital commutative algebra over } \mathbb{C} \text { and we are given }  \tag{12.3}\\
\text { elements }\left\{a_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty} N C(n)\right\} \text { in } \mathcal{A} \text {, with } a_{1_{n}}=1_{\mathcal{A}} \text { for all } n \geq 1, \\
\text { then there exists a unital algebra homomomorphism } \Phi: \mathcal{T} \rightarrow \mathcal{A} \text {, uniquely } \\
\text { determined, such that } \Phi\left(X_{\pi}\right)=a_{\pi} \text { for all } \pi \in \sqcup_{n=1}^{\infty} N C(n) .
\end{array}\right.
$$

$3^{\circ}$ Consider the unital algebra $\mathcal{T} \otimes \mathcal{T}$. The universality property of $\mathcal{T}$ observed in $2^{\circ}$ assures us that there exists a unital algebra homomorphism $\Delta: \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}$, uniquely determined,
such that for every $n \geq 1$ and $\pi \in N C(n)$ we have

$$
\begin{equation*}
\Delta\left(X_{\pi}\right)=\sum_{\sigma \in N C(n), \sigma \geq \pi}\left(\prod_{W \in \sigma} X_{\pi_{W}}\right) \otimes X_{\sigma} \tag{12.4}
\end{equation*}
$$

with the relabeled-restrictions $\pi_{W} \in N C(|W|)$ considered in the sense of Notation 2.2. We will refer to the homomorphism $\Delta$ by calling it "the comultiplication of $\mathcal{T}$ ".
$4^{o}$ The universality property observed in $2^{o}$ also assures us that there exists a unital algebra homomorphism $\epsilon: \mathcal{T} \rightarrow \mathbb{C}$, uniquely determined, such that

$$
\begin{equation*}
\epsilon\left(X_{\pi}\right)=0, \quad \forall \pi \in \sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right) . \tag{12.5}
\end{equation*}
$$

We will refer to $\epsilon$ by calling it "the counit of $\mathcal{T}$ ".
$5^{o}$ We denote $\mathcal{T}_{0}:=\left\{\lambda \cdot 1_{\mathcal{T}} \mid \lambda \in \mathbb{C}\right\}$, and for every $m \geq 1$ we denote

$$
\mathcal{T}_{m}=\operatorname{span}\left(\begin{array}{l|l}
\left.\left.\bigcup_{k=1}^{m}\left\{\begin{array}{ll}
X_{\pi_{1}} \cdots X_{\pi_{k}} & \begin{array}{l}
\pi_{1}, \ldots, \pi_{k} \in \sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right) \\
\text { with }\left|\pi_{1}\right|+\cdots+\left|\pi_{k}\right|=m+k
\end{array}
\end{array}\right\}\right) . . \text {. } \begin{array}{l}
\text { win }
\end{array}\right) . \tag{12.6}
\end{array}\right.
$$

In other words, $\mathcal{T}_{m}$ is the linear span of all monomials "of degree $m$ ", where we declare that every indeterminate $X_{\pi}$ has degree $|\pi|-1$. This gives a direct sum decomposition $\mathcal{T}=\bigoplus_{m=0}^{\infty} \mathcal{T}_{m}$, which we will refer to as "the grading of $\mathcal{T}$ ".

Notation and Remark 12.2. For every $n \geq 1$ and $\pi \leq \sigma$ in $N C(n)$ let us denote

$$
\begin{equation*}
M_{\pi, \sigma}:=\prod_{W \in \sigma} X_{\pi_{W}} \in \mathcal{T} \tag{12.7}
\end{equation*}
$$

Note that for $\sigma=1_{n}$ the monomial $M_{\pi, \sigma}$ consists of only one factor, so we get

$$
M_{\pi, 1_{n}}=X_{\pi}, \quad \forall n \geq 1 \text { and } \pi \in N C(n)
$$

At the other extreme, setting $\sigma=\pi$ makes $M_{\pi, \sigma}$ consist of factors $X_{1_{|V|}}$ with $V$ running among the blocks of $\pi$, and we thus get

$$
M_{\pi, \pi}=1_{\mathcal{T}}, \quad \forall n \geq 1 \text { and } \pi \in N C(n) .
$$

In terms of the monomials $M_{\pi, \sigma}$, the formula (12.4) defining the comultiplication of $\mathcal{T}$ takes the more appealing form

$$
\begin{equation*}
\Delta\left(X_{\pi}\right)=\sum_{\sigma \in N C(n), \sigma \geq \pi} M_{\pi, \sigma} \otimes X_{\sigma}, \text { for all } n \geq 1 \text { and } \pi \in N C(n) . \tag{12.8}
\end{equation*}
$$

It is easy to further extend this, in the way indicated in the next lemma.

Lemma 12.3. Let $n \geq 1$ and let $\pi, \tau \in N C(n)$ be such that $\pi \leq \tau$. Then

$$
\begin{equation*}
\Delta\left(M_{\pi, \tau}\right)=\sum_{\sigma \in N C(n), \pi \leq \sigma \leq \tau} M_{\pi, \sigma} \otimes M_{\sigma, \tau} \tag{12.9}
\end{equation*}
$$

Proof. Let us write explicitly $\tau=\left\{U_{1}, \ldots, U_{k}\right\}$ and let us denote $\pi^{(j)}:=\pi_{U_{j}} \in N C\left(\left|U_{j}\right|\right)$ for $1 \leq j \leq k$. The left-hand side of Equation (12.9) then becomes

$$
\Delta\left(\prod_{j=1}^{k} X_{\pi^{(j)}}\right)=\prod_{j=1}^{k} \Delta\left(X_{\pi^{(j)}}\right)=\prod_{j=1}^{k}\left(\sum_{\substack{\sigma^{(j)} \in N C\left(\left|U_{j}\right|\right), \sigma^{(j)} \geq \pi^{(j)}}} M_{\pi^{(j)}, \sigma^{(j)}} \otimes X_{\sigma^{(j)}}\right) .
$$

Expanding the product over $j$ in the latter expression takes us to

$$
\begin{equation*}
\sum_{\substack{\sigma^{(1)} \in N C\left(\left|U_{1}\right|\right), \ldots, \sigma^{(k)} \in N C\left(\left|U_{k}\right|\right), \sigma^{(1)} \geq \pi^{(1)}, \ldots, \sigma^{(k)} \geq \pi^{(k)}}}\left(\prod_{j=1}^{k} \prod_{V \in \sigma^{(j)}} X_{\pi_{V}^{(j)}}\right) \otimes\left(\prod_{j=1}^{k} X_{\sigma^{(j)}}\right) . \tag{12.10}
\end{equation*}
$$

The index set for the sum in (12.10) can be identified as the bijective image of the set $\{\sigma \in N C(n) \mid \pi \leq \sigma \leq \tau\}$, via the map

$$
\begin{equation*}
\sigma \mapsto\left(\sigma_{U_{1}}, \ldots \sigma_{U_{k}}\right) \tag{12.11}
\end{equation*}
$$

We leave it as an exercise to the patient reader to check that, when the bijection (12.11) is used as a change of variable in the summation from (12.10), what comes out is indeed the right-hand side of the formula (12.9) claimed by the lemma.

Proposition 12.4. When endowed with the structure introduced in Notation 12.1, $\mathcal{T}$ becomes a graded bialgebra.

Proof. The proof of consists of three verifications, pertaining to comultiplication, counit and grading, respectively.
(i) Verification that $\Delta$ is coassociative.

Here we have to check that $(\operatorname{Id} \otimes \Delta) \circ \Delta=(\Delta \otimes \mathrm{Id}) \circ \Delta$. Since both sides of this equality are unital algebra homomorphisms from $\mathcal{T}$ to $\mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T}$, it suffices to check that they agree on every generator $X_{\pi}$ of $\mathcal{T}$. We thus pick an $n \geq 1$ and a $\pi \neq 1_{n}$ in $N C(n)$, and we will verify that both $\operatorname{Id} \otimes \Delta\left(\Delta\left(X_{\pi}\right)\right)$ and $\Delta \otimes \operatorname{Id}\left(\Delta\left(X_{\pi}\right)\right)$ are equal to

$$
\begin{equation*}
\sum_{\substack{\sigma, \tau \in N C(n) \\ \tau \geq \sigma \geq \pi}} M_{\pi, \sigma} \otimes M_{\sigma, \tau} \otimes X_{\tau}(\text { element of } \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T}) \tag{12.12}
\end{equation*}
$$

Indeed, if in the double sum of (12.12) we first sum over $\tau$, then we get

$$
\sum_{\substack{\sigma \in N C(n) \\ \sigma \geq \pi}} M_{\pi, \sigma} \otimes\left(\sum_{\substack{\tau \in N C(n) \\ \tau \geq \sigma}} M_{\sigma, \tau} \otimes X_{\tau}\right)=\sum_{\substack{\sigma \in N C(n) \\ \sigma \geq \pi}} M_{\pi, \sigma} \otimes \Delta\left(X_{\sigma}\right)=\operatorname{Id} \otimes \Delta\left(\Delta\left(X_{\pi}\right)\right) .
$$

While if in (12.12) we first sum over $\sigma$, then we get

$$
\sum_{\substack{\tau \in N C(n), \tau \geq \pi}}\left(\sum_{\substack{\sigma \in N C(n), \pi \leq \sigma \leq \tau}} M_{\pi, \sigma} \otimes M_{\sigma, \tau}\right) \otimes X_{\tau}=\sum_{\substack{\tau \in N C(n), \tau \geq \pi}} \Delta\left(M_{\pi, \tau}\right) \otimes X_{\tau} \quad(\text { by Lemma 12.3) })
$$

and the latter quantity is precisely equal to $\Delta \otimes \operatorname{Id}\left(\Delta\left(X_{\pi}\right)\right)$.
(ii) Verification that $\epsilon$ satisfies the counit property, i.e. that $(\operatorname{Id} \otimes \epsilon) \circ \Delta=\operatorname{Id}=(\epsilon \otimes \mathrm{Id}) \circ \Delta$. Here again it suffices to focus on a generator $X_{\pi}$. Upon chasing through the definitions, we see that what needs to be verified is this: given $n \geq 2$ and $\pi \neq 1_{n}$ in $N C(n)$, check that

$$
\begin{equation*}
\sum_{\sigma \geq \pi} \epsilon\left(X_{\sigma}\right) \cdot \prod_{W \in \sigma} X_{\pi_{W}}=X_{\pi}=\sum_{\sigma \geq \pi} \prod_{W \in \sigma} \epsilon\left(X_{\pi_{W}}\right) \cdot X_{\sigma} . \tag{12.13}
\end{equation*}
$$

And indeed: the first of the two equalities (12.13) holds because the only non-zero contribution to the sum occurs for $\sigma=1_{n}$, when $\prod_{W \in 1_{n}} X_{\pi_{W}}=X_{\pi}$. The second equality (12.13) also holds, with the only non-zero contribution now coming from the term indexed by $\pi$ :

$$
\left(0 \neq \prod_{W \in \sigma} \epsilon\left(X_{\pi_{W}}\right)\right) \Leftrightarrow\left(\pi_{W}=1_{|W|}, \forall W \in \sigma\right) \Leftrightarrow(\sigma=\pi) .
$$

(iii) Verifications related to the grading.

We leave it to the reader to go over the list of conditions that have to be verified here, and confirm that the only non-obvious item on the list is this: given $\pi \in N C(n)$ with $|\pi|=m$ (hence with $X_{\pi} \in \mathcal{T}_{m-1}$ ) for some $2 \leq m \leq n$, one has that $\Delta\left(X_{\pi}\right) \in \bigoplus_{i=0}^{m-1} \mathcal{T}_{i} \otimes \mathcal{T}_{m-1-i}$. In order to verify this fact one checks that, in the sum defining $\Delta\left(X_{\pi}\right)$ in (12.4) one has:

$$
\begin{equation*}
\left(\prod_{W \in \sigma} X_{\pi_{W}}\right) \otimes X_{\sigma} \in \bigoplus_{i=0}^{m-1} \mathcal{T}_{i} \otimes \mathcal{T}_{m-1-i}, \quad \forall \sigma \in N C(n) \text { such that } \sigma \geq \pi \tag{12.14}
\end{equation*}
$$

Indeed, in the tensor indicated in (12.14), the product of generators that appears to the left of the tensor sign has degree equal to

$$
\sum_{W \in \sigma}\left(\left|\pi_{W}\right|-1\right)=\sum_{W \in \sigma}\left|\pi_{W}\right|-\sum_{W \in \sigma} 1=|\pi|-|\sigma|=m-|\sigma| .
$$

Since $X_{\sigma} \in \mathcal{T}_{|\sigma|-1}$, the tensor indicated in (12.14) thus belongs to $\mathcal{T}_{m-|\sigma|} \otimes \mathcal{T}_{|\sigma|-1}$, with $(m-|\sigma|)+(|\sigma|-1)=m-1$, as required.

Remark 12.5. Recall that the space $\mathcal{T}_{0} \subseteq \mathcal{T}$ of homogeneous elements of degree 0 consists precisely of the scalar multiples of the unit $1_{\mathcal{T}}$. One refers to this property of $\mathcal{T}$ by saying that it is connected. As a consequence of being a graded connected bialgebra, $\mathcal{T}$ is sure to be a Hopf algebra; that is, we are guaranteed (cf. [21, Section II.3]) to have a unital algebra homomorphism $S: \mathcal{T} \rightarrow \mathcal{T}$, called antipode of $\mathcal{T}$, which is in a certain sense the convolution inverse to the identity map Id : $\mathcal{T} \rightarrow \mathcal{T}$. A discussion of the antipode of $\mathcal{T}$ is made in the next section of the paper. Right now we only record the fact that, due to these general Hopf algebra considerations, Proposition 12.4 can be restated in the following stronger form.

Theorem 12.6. When endowed with the structure introduced in Notation 12.1, $\mathcal{T}$ becomes a graded connected Hopf algebra.

Remark 12.7. As mentioned at the beginning of the subsection, the Hopf algebra $\mathcal{T}$ can be treated as an incidence Hopf algebra in the sense of Schmitt 27. The present remark gives a brief outline of how this happens.

For every $n \geq 1$ and $\pi \leq \sigma$ in $N C(n)$ let us denote

$$
\begin{equation*}
[\pi, \sigma]=\{\rho \in N C(n) \mid \pi \leq \rho \leq \sigma\} \quad \text { (a sub-poset of }(N C(n), \leq)) \tag{12.15}
\end{equation*}
$$

and let $\mathcal{P}$ denote the collection of all the posets $[\pi, \sigma]$ considered in (12.15). On $\mathcal{P}$ we have a natural operation of multiplication defined by

$$
\begin{equation*}
\left[\pi_{1}, \sigma_{1}\right] \times\left[\pi_{2}, \sigma_{2}\right]=:\left[\pi_{1} \diamond \pi_{2}, \sigma_{1} \diamond \sigma_{2}\right] \tag{12.16}
\end{equation*}
$$

where " $>$ " denotes concatenation (as in Notation (2.4). It turns out that on $\mathcal{P}$ one can introduce an equivalence relation " $\sim$ " which is compatible with the multiplication (12.16) and produces a commutative quotient monoid $\mathcal{P} / \sim$ generated by

$$
\left\{\widehat{\left[\pi, 1_{n}\right]} \mid n \geq 1 \text { and } \pi \in N C(n) \backslash\left\{1_{n}\right\}\right\},
$$

where we use the notation " $\widehat{\pi, \sigma]}$ " for the image of $[\pi, \sigma]$ under the quotient map $\mathcal{P} \rightarrow \mathcal{P} / \sim$. Moreover, the equivalence relation $\sim$ is set in such a way that one gets factorizations

$$
\begin{equation*}
\widehat{[\pi, \sigma]}=\prod_{W \in \sigma}\left[\widehat{\pi_{W}, 1_{|W|}}\right], \text { for every }[\pi, \sigma] \in \mathcal{P} \tag{12.17}
\end{equation*}
$$

When plugged into the general machinery described in [27, Sections 2-4], the monoid algebra $\mathbb{C}[\mathcal{P} / \sim]$ becomes a Hopf algebra, which turns out to be naturally isomorphic (as Hopf
algebras!) to $\mathcal{T}$ from Theorem 12.6 , via the unital algebra homomorphism defined by requiring that

$$
\mathcal{T} \ni X_{\pi} \mapsto \widehat{\left[\pi, 1_{n}\right]} \in \mathbb{C}[\mathcal{P} / \sim], \quad \forall n \geq 1 \text { and } \pi \in N C(n) \backslash\left\{1_{n}\right\}
$$

### 12.2. The isomorphism $\widetilde{\mathcal{G}} \approx \mathbb{X}(\mathcal{T})$.

Remark 12.8. (Review of the $\operatorname{group}(\mathbb{X}(\mathcal{T}), *)$ ). A unital algebra homomorphism from $\mathcal{T}$ to $\mathbb{C}$ is also known under the name of character of $\mathcal{T}$, and it is customary to denote

$$
\begin{equation*}
\mathbb{X}(\mathcal{T}):=\{\chi: \mathcal{T} \rightarrow \mathbb{C} \mid \chi \text { is a character }\} \tag{12.18}
\end{equation*}
$$

The definition of $\mathbb{X}(\mathcal{T})$ only uses the algebra structure on $\mathcal{T}$. But the coalgebra structure is important too, because it allows us to define a convolution operation for characters, via the formula $\chi_{1} * \chi_{2}=\left(\chi_{1} \otimes \chi_{2}\right) \circ \Delta$. That is: given $\chi_{1}, \chi_{2} \in \mathbb{X}(\mathcal{T})$ and $P \in \mathcal{T}$, one considers some concrete writing $\Delta(P)=\sum_{i=1}^{n} P_{i}^{\prime} \otimes P_{i}^{\prime \prime}$, and defines

$$
\begin{equation*}
\chi_{1} * \chi_{2}(P):=\sum_{i=1}^{n} \chi_{1}\left(P_{i}^{\prime}\right) \chi_{2}\left(P_{i}^{\prime \prime}\right) \tag{12.19}
\end{equation*}
$$

It is easily checked that the definition of $\chi_{1} * \chi_{2}$ makes sense, and that in this way one gets an associative operation " $*$ ", called convolution, on $\mathbb{X}(\mathcal{T})$.

It is clear that the counit $\epsilon$ introduced in Notation $12.1,4$ belongs to $\mathbb{X}(\mathcal{T})$. Then the counit verification from (ii) in the proof of Proposition 12.4 shows precisely that $\epsilon$ is the (necessarily unique) unit element of $(\mathbb{X}(\mathcal{T}), *)$. Finally, for every $\chi \in \mathbb{X}(\mathcal{T})$ one can consider the new character $\chi \circ S \in \mathbb{X}(\mathcal{T})$, where $S: \mathcal{T} \rightarrow \mathcal{T}$ is the antipode map, and one can verify (see e.g. [21, Proposition II.4.1]) that $\chi \circ S$ is inverse to $\chi$ with respect to convolution. Hence the overall conclusion is that $(\mathbb{X}(\mathcal{T}), *)$ is a group.

Theorem 12.9. $1^{o}$ For every $g \in \widetilde{\mathcal{G}}$ there exists a character $\chi_{g} \in \mathbb{X}(\mathcal{T})$, uniquely determined, such that

$$
\begin{equation*}
\chi_{g}\left(X_{\pi}\right)=g\left(\pi, 1_{n}\right), \quad \text { for all } n \geq 1 \text { and } \pi \in N C(n) \tag{12.20}
\end{equation*}
$$

$2^{o}$ The $\operatorname{map} \widetilde{\mathcal{G}} \ni g \mapsto \chi_{g} \in \mathbb{X}(\mathcal{T})$ is a group isomorphism, i.e. it is bijective and has

$$
\begin{equation*}
\chi_{g_{1} * g_{2}}=\chi_{g_{1}} * \chi_{g_{2}}, \quad \forall g_{1}, g_{2} \in \widetilde{\mathcal{G}} \tag{12.21}
\end{equation*}
$$

Proof. The universality property noted in (12.3) implies that the characters of $\mathcal{T}$ are in bijective correspondence with families of complex numbers of the form $\{z(\pi) \mid \pi \in$ $\sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right\}$, where the family of numbers corresponding to $\chi \in \mathbb{X}(\mathcal{T})$ is simply obtained by putting $z(\pi)=\chi\left(X_{\pi}\right)$ for all $n \geq 1$ and $\pi \in N C(n) \backslash\left\{1_{n}\right\}$. When considered in conjunction with the Proposition 3.5 about functions in $\widetilde{\mathcal{G}}$, this immediately implies the statement $1^{o}$ of the theorem, and also the fact that the map $\widetilde{\mathcal{G}} \ni g \mapsto \chi_{g} \in \mathbb{X}(\mathcal{T})$ is a bijection.

We are left to check that (12.21) holds. In order to establish the equality of the characters $\chi_{g_{1}} * \chi_{g_{2}}$ and $\chi_{g_{1} * g_{2}}$ it suffices to verify that they agree on every generator $X_{\pi}$ of $\mathcal{T}$. We
thus fix an $n \geq 1$ and a $\pi \in N C(n) \backslash\left\{1_{n}\right\}$, and we compute:

$$
\chi_{g_{1}} * \chi_{g_{2}}\left(X_{\pi}\right)=\sum_{\sigma \geq \pi \text { in } N C(n)} \chi_{g_{1}}\left(\prod_{W \in \sigma} X_{\pi_{W}}\right) \cdot \chi_{g_{2}}\left(X_{\sigma}\right)
$$

(by (12.19), where we use the explicit formula for $\Delta\left(X_{\pi}\right)$ )

$$
=\sum_{\sigma \geq \pi \text { in } N C(n)}\left(\prod_{W \in \sigma} g_{1}\left(\pi_{W}, 1_{|W|}\right)\right) \cdot g_{2}\left(\sigma, 1_{n}\right)
$$

$$
\text { (by formulas defining } \chi_{g_{1}}, \chi_{g_{2}} \text { in terms of } g_{1}, g_{2} \text { ) }
$$

$$
=\sum_{\sigma \geq \pi \text { in } N C(n)} g_{1}(\pi, \sigma) \cdot g_{2}\left(\sigma, 1_{n}\right) \quad \text { (by Eqn.(3.7) in Definition 3.4) }
$$

$$
\left.=g_{1} * g_{2}\left(\pi, 1_{n}\right) \quad \text { (by the definition of convolution in } \widetilde{\mathcal{G}}\right)
$$

$$
=\chi_{g_{1} * g_{2}}\left(X_{\pi}\right) \quad\left(\text { by the formula defining } \chi_{g_{1} * g_{2}}\right)
$$

### 12.3. A discussion of the primitive elements of $\mathcal{T}$.

Remark 12.10. We now consider the set of primitive elements of $\mathcal{T}$,

$$
\operatorname{Prim}(\mathcal{T}):=\left\{P \in \mathcal{T} \mid \Delta(P)=P \otimes 1_{\mathcal{T}}+1_{\mathcal{T}} \otimes P\right\}
$$

The study of primitive elements is of great importance for co-commutative Hopf algebras, due to a fundamental theorem of Milnor-Moore which holds in that framework (see e.g. [15, Section 14.3]). The Hopf algebra $\mathcal{T}$ studied here is not co-commutative (corresponding to the fact that the group $\widetilde{\mathcal{G}}$ is not commutative), thus the role played by $\operatorname{Prim}(\mathcal{T})$ in the study of $\mathcal{T}$ is less significant. But for the sake of completeness, we give below a precise description of how $\operatorname{Prim}(\mathcal{T})$ looks like.

We start by observing that: if $\pi \in N C(n)$ has $|\pi|=2$, then $\{\sigma \in N C(n) \mid \sigma \geq \pi\}=$ $\left\{\pi, 1_{n}\right\}$, hence the sum which defined the comultiplication $\Delta\left(X_{\pi}\right)$ in Equation (12.4) only has two terms. It is moreover immediate that the terms indexed by $\pi$ and by $1_{n}$ in the said sum are $1_{\mathcal{T}} \otimes X_{\pi}$ and respectively $X_{\pi} \otimes 1_{\mathcal{T}}$. It thus follows that $X_{\pi} \in \operatorname{Prim}(\mathcal{T})$ for every $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ with $|\pi|=2$. Since the space $\mathcal{T}_{1} \subseteq \mathcal{T}$ of homogeneous elements of degree 1 (as defined in Notation [12.1,5) is just

$$
\mathcal{T}_{1}=\operatorname{span}\left\{X_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty} N C(n) \text { with }|\pi|=2\right\},
$$

we conclude that $\mathcal{T}_{1} \subseteq \operatorname{Prim}(\mathcal{T})$. The goal of the present subsection is to point out that the opposite inclusion holds as well, and we therefore have:

$$
\begin{equation*}
\operatorname{Prim}(\mathcal{T})=\mathcal{T}_{1} . \tag{12.22}
\end{equation*}
$$

We will prove this equality in Proposition 12.14 below. Towards that goal, we first introduce some notation and prove a couple of lemmas.

Notation 12.11. $1^{o}$ The algebra $\mathcal{T}$ has a linear basis $\mathcal{M}$ consisting of monomials. It consists of elements of the form

$$
\begin{equation*}
M:=X_{\pi_{1}}^{q_{1}} \cdots X_{\pi_{k}}^{q_{k}}, \tag{12.23}
\end{equation*}
$$

where $\left\{\pi_{1}, \ldots, \pi_{k}\right\}$ is a finite subset of $\sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right)$ and $\left(q_{1}, \ldots, q_{k}\right) \in \mathbb{N}^{k}$ is a tuple of multiplicities. We will use the notation $\#(M)$ for the total number of $X_{\pi}$ 's that are multiplied together to give an $M \in \mathcal{M}$; thus the monomial shown in (12.23) has $\#(M)=$
$q_{1}+\cdots+q_{k}$. We make the convention that if the set $\left\{\pi_{1}, \ldots, \pi_{k}\right\}$ considered in (12.23) is empty, i.e. if $k=0$, then the corresponding monomial is $M:=1_{\mathcal{T}}$ with $\#(M)=0$.
$2^{o}$ For every $M \in \mathcal{M}$ we let $\xi_{M}: \mathcal{T} \rightarrow \mathbb{C}$ be the linear functional which acts on $\mathcal{M}$ by the prescription that $\xi_{M}(M)=1$ and $\xi_{M}(N)=0$ for any $N \in \mathcal{M} \backslash\{M\}$.
$3^{o}$ Moving to $\mathcal{T} \otimes \mathcal{T}:$ we have a linear basis $\mathcal{M} \otimes \mathcal{M}:=\left\{M_{1} \otimes M_{2} \mid M_{1}, M_{2} \in \mathcal{M}\right\}$. For every $M_{1}, M_{2} \in \mathcal{M}$ we let $\xi_{M_{1}, M_{2}}^{(2)}: \mathcal{T} \otimes \mathcal{T} \rightarrow \mathbb{C}$ be the linear functional which acts on $\mathcal{M} \otimes \mathcal{M}$ by:
$\xi_{M_{1}, M_{2}}^{(2)}\left(N_{1} \otimes N_{2}\right)=\xi_{M_{1}}\left(N_{1}\right) \cdot \xi_{M_{2}}\left(N_{2}\right)=\left\{\begin{array}{ll}1, & \text { if } N_{1}=M_{1} \text { and } N_{2}=M_{2}, \\ 0, & \text { otherwise }\end{array}\right\}$, for $N_{1}, N_{2} \in \mathcal{M}$.
Lemma 12.12. $1^{\circ}$ Let $M \in \mathcal{M}$ be such that $\#(M) \geq 2$. There exist $M_{1}, M_{2} \in \mathcal{M}$ with $\#\left(M_{i}\right) \geq 1$ for $i=1,2$ and $a \in \mathbb{N}$ such that

$$
\begin{equation*}
\xi_{M_{1}, M_{2}}^{(2)} \circ \Delta=q \xi_{M} \tag{12.24}
\end{equation*}
$$

$2^{o}$ Let $n \in \mathbb{N}$ and let $\pi$ be a partition in $N C(n)$ such that $|\pi| \geq 3$. There exist $M_{1}, M_{2}, M_{3} \in$ $\mathcal{M}$ with $\#\left(M_{1}\right), \#\left(M_{2}\right) \geq 1$ and $\#\left(M_{3}\right) \geq 2$ such that

$$
\begin{equation*}
\xi_{M_{1}, M_{2}}^{(2)} \circ \Delta=\xi_{X_{\pi}}+\xi_{M_{3}} . \tag{12.25}
\end{equation*}
$$

Proof. $1^{o}$ The monomial $M$ has an explicit writing $M=X_{\pi_{1}}^{q_{1}} \cdots X_{\pi_{k}}^{q_{k}}$ with $\pi_{1} \in N C\left(n_{1}\right), \ldots$, $\pi_{k} \in N C\left(n_{k}\right)$ and $q_{1}, \ldots, q_{k} \in \mathbb{N}$, and where $\pi_{1}, \ldots, \pi_{k}$ are arranged such that $n_{1} \geq n_{2} \geq$ $\cdots \geq n_{k}$. For the role of the required $M_{1}, M_{2}$ we pick $M_{2}:=X_{\pi_{k}}$ and we put

$$
M_{1}:=X_{\pi_{1}}^{q_{1}} \cdots X_{\pi_{k-1}}^{q_{k-1}} \cdot X_{\pi_{k}}^{q_{k}-1}
$$

that is, $M_{1}$ is chosen in such a way that $M_{1} X_{\pi_{k}}=M$. Note that this is always possible due to the hypothesis that $\#(M) \geq 2$. We leave it as an exercise to the reader to verify that

$$
\begin{equation*}
\xi_{M_{1}, M_{2}}^{(2)}(\Delta(M))=q_{k} \text { and that } \xi_{M_{1}, M_{2}}^{(2)}(\Delta(N))=0 \text { for every } N \neq M \text { in } \mathcal{M} \tag{12.26}
\end{equation*}
$$

As a hint towards the verification of the latter equality, we mention that it can be obtained by writing $N$ as a product $X_{\rho_{1}} \cdots X_{\rho_{\ell}}$ and then by applying the formula (12.4) to every $\Delta\left(X_{\rho_{j}}\right)$ in the factorization $\Delta(N)=\Delta\left(X_{\rho_{1}}\right) \cdots \Delta\left(X_{\rho_{\ell}}\right)$.

As a consequence of (12.26), it is immediate that the required formula (12.24) is holding in connection to the $M_{1}, M_{2}$ indicated above and where we take $q:=q_{k}$.
$2^{\circ}$ The requirements of this part of the lemma can be fulfilled by putting

$$
M_{1}=X_{\sigma_{1}}, M_{2}=X_{\sigma_{2}} \text { and } M_{3}=X_{\sigma_{1}} X_{\sigma_{2}}
$$

for suitably chosen non-crossing partitions $\sigma_{1}, \sigma_{2}$. A concrete recipe for finding such $\sigma_{1}, \sigma_{2}$ is as follows: we let $\sigma_{2}$ be of the form $\sigma_{2}=\{U, W\} \in N C(n)$ where $U$ is a special block of $\pi$ (to be picked below) and $W=\{1, \ldots, n\} \backslash U$; then we put $\sigma_{1}:=\pi_{W} \in N C(|W|)$, the relabeled-restriction of $\pi$ to $W$.

For an $M_{1}, M_{2}$ defined by a recipe as above, we leave it as an exercise to the reader to examine what are the conditions on a monomial $N \in \mathcal{M}$ which would allow $\xi_{M_{1}, M_{2}}^{(2)}(\Delta(N))$ to be non-zero. The result of the examination is that one has

$$
\xi_{M_{1}, M_{2}}^{(2)}\left(\Delta\left(X_{\pi}\right)\right)=\xi_{M_{1}, M_{2}}^{(2)}\left(\Delta\left(M_{3}\right)\right)=1
$$

and that $\xi_{M_{1}, M_{2}}^{(2)}(\Delta(N))=0$ for all other $N$, with the exception of a stray $N$ that can only exist when $|U|=|W|$. The conclusion we draw is this: if in the construction of $\sigma_{1}, \sigma_{2}$ we can also arrange to have $|U| \neq|W|$, then the desired Equation (12.25) will hold.

It remains to make certain that we can always pick a block $U \in \pi$ such that, with $W:=\{1, \ldots, n\} \backslash U$, we have that $|W| \neq|U|$ and that $\sigma_{2}:=\{U, W\}$ is non-crossing. The
non-crossing property of $\{U, W\}$ is sure to hold when $U$ is an interval block of $\pi$; thus, if $\pi$ has an interval block $J$ with $|J| \neq n / 2$, then we take $U=J$ and we are done. But what if $\pi$ has a unique interval block $J$, with $|J|=n / 2$ ? In that case, a quick examination of the nesting structure for the blocks of $\pi$ will show that $\pi$ also has a unique outer block $H$, and that picking $U=H$ will give all the properties that $\sigma_{2}$ needs to have.

Lemma 12.13. Let $A$ be an element in $\operatorname{Prim}(\mathcal{T})$.
$1^{o}$ One has $\xi_{M}(A)=0$ for every $M \in \mathcal{M}$ with $\#(M) \geq 2$.
$2^{o}$ One has $\xi_{X_{\pi}}(A)=0$ for every $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ with $|\pi| \geq 3$.
Proof. $1^{o}$ Pick an $M \in \mathcal{M}$ with $\#(M) \geq 2$, and let $M_{1}, M_{2} \in \mathcal{M}$ and $q \in \mathbb{N}$ be such that (12.24) holds. Since $\Delta(A)-A \otimes 1_{\mathcal{T}}-1_{\mathcal{T}} \otimes A=0 \in \mathcal{T}$, we can write:

$$
\begin{aligned}
0 & =\xi_{M_{1}, M_{2}}^{(2)}\left(\Delta(A)-A \otimes 1_{\mathcal{T}}-1_{\mathcal{T}} \otimes A\right) \\
& =\left(\xi_{M_{1}, M_{2}}^{(2)} \circ \Delta\right)(A)-\xi_{M_{1}}(A) \cdot \xi_{M_{2}}\left(1_{\mathcal{T}}\right)-\xi_{M_{1}}\left(1_{\mathcal{T}}\right) \cdot \xi_{M_{2}}(A) \\
& =q \xi_{M}(A)-0-0,
\end{aligned}
$$

where at the latter equality sign we took into account (12.24) and the fact that $\xi_{M_{1}}\left(1_{\mathcal{T}}\right)=$ $\xi_{M_{2}}\left(1_{\mathcal{T}}\right)=0$. We thus found that $q \xi_{M}(A)=0$, and the conclusion follows.
$2^{o}$ Pick a $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ with $|\pi| \geq 3$, and let $M_{1}, M_{2}, M_{3} \in \mathcal{M}$ be such that (12.25) holds. By using the same trick as in the proof of part $1^{\circ}$ we find that

$$
\left.0=\xi_{M_{1}, M_{2}}^{(2)}\left(\Delta(A)-A \otimes 1_{\mathcal{T}}-1_{\mathcal{T}} \otimes A\right)=\left(\xi_{M_{1}, M_{2}}^{(2)} \circ \Delta\right)(A)\right)-0-0 .
$$

Since this time our handle on $\xi_{M_{1}, M_{2}}^{(2)} \circ \Delta$ comes from (12.25), we now get:

$$
0=\left(\xi_{X_{\pi}}+\xi_{M_{3}}\right)(A)=\xi_{X_{\pi}}(A)+\xi_{M_{3}}(A)
$$

But $\xi_{M_{3}}(A)=0$, as proved in $1^{o}$ above. We thus conclude that $\xi_{X_{\pi}}(A)=0$, as required.
$\operatorname{Proposition~12.14.~} \operatorname{Prim}(\mathcal{T})=\mathcal{T}_{1}$.
Proof. In view of Remark 12.10, we only have to verify the inclusion " $\subseteq$ ". We thus fix for the whole proof an $A \in \operatorname{Prim}(\mathcal{T})$, for which we want to prove that $A \in \mathcal{T}_{1}$.

We know that $A$ (same as any other element of $\mathcal{T}$ ) can be decomposed as a sum of homogeneous elements. That is: for $m \in \mathbb{N}$ large enough we can write

$$
\begin{equation*}
A=A_{0}+A_{1}+\cdots+A_{m} \text { with } A_{0} \in \mathcal{T}_{0}, \ldots, A_{m} \in \mathcal{T}_{m} \tag{12.27}
\end{equation*}
$$

where the homogeneous spaces $\mathcal{T}_{0}, \ldots, \mathcal{T}_{m} \subseteq \mathcal{T}$ are as defined in Equation (12.6) above.
By using Lemma 12.13 it is however easy to see that we must have $A_{j}=0$ for every $2 \leq j \leq m$. Indeed, it is immediate that if we had $A_{j} \neq 0$ then we would be able to find a monomial $M \in \mathcal{T}_{j}$ such that $\xi_{M}\left(A_{j}\right) \neq 0$. The fact that $M \in \mathcal{T}_{j}$ implies that $\mathcal{T}_{i} \subseteq \operatorname{Ker}\left(\xi_{M}\right)$ for all $i \neq j$ in $\mathbb{N} \cup\{0\}$, which implies in turn that

$$
\xi_{M}(A)=\xi_{M}\left(A_{0}\right)+\cdots+\xi_{M}\left(A_{m}\right)=\xi_{M}\left(A_{j}\right) \neq 0
$$

But on the other hand, the fact that $M \in \mathcal{T}_{j}$ with $j \geq 2$ also implies that either $\#(M) \geq 2$ or that $M$ is of the form $X_{\pi}$ for a partition $\pi$ with $|\pi| \geq 3$; thus Lemma 12.13 asserts that $\xi_{M}(A)=0-$ contradiction!

Hence the decomposition (12.27) has $A_{j}=0$ for every $2 \leq j \leq m$, and since $A_{0} \in \mathcal{T}_{0}=$ $\mathbb{C} 1_{\mathcal{T}}$, we thus get an equality of the form

$$
\begin{equation*}
A=\lambda 1_{\mathcal{T}}+A_{1}, \text { with } \lambda \in \mathbb{C} \text { and } A_{1} \in \mathcal{T}_{1} . \tag{12.28}
\end{equation*}
$$

The comultiplication of the right-hand side of (12.28) is computed to be

$$
\Delta\left(\lambda 1_{\mathcal{T}}+A_{1}\right)=\lambda 1_{\mathcal{T}} \otimes 1_{\mathcal{T}}+\left(A_{1} \otimes 1_{\mathcal{T}}+1_{\mathcal{T}} \otimes A_{1}\right)
$$

(where we took into account that $A_{1} \in \mathcal{T}_{1} \subseteq \operatorname{Prim}(\mathcal{T})$ ). By comparing this against

$$
\Delta(A)=A \otimes 1_{\mathcal{T}}+1_{\mathcal{T}} \otimes A=2 \lambda 1_{\mathcal{T}} \otimes 1_{\mathcal{T}}+\left(A_{1} \otimes 1_{\mathcal{T}}+1_{\mathcal{T}} \otimes A_{1}\right)
$$

we see that $\lambda=0$. Hence $A=A_{1} \in \mathcal{T}_{1}$, as we had to prove.

## 13. A DISCUSSION OF THE ANTIPODE OF $\mathcal{T}$

The antipode of the Hopf algebra $\mathcal{T}$ deserves special attention due to its potential use as a tool for inversion in formulas that relate moments to cumulants, or relate different brands of cumulants living in the $N C(n)$ framework. The issue of performing such inversions is constantly present in the literature on cumulants. Indeed, it is typical that cumulants (of one brand or another) are introduced via some simple formulas which are deemed to express moments in terms of the desired cumulants; these simple formulas then need to be inverted, if one wants to see explicit formulas describing cumulants in terms of moments. In such a situation, the tool that is typically used for inversion is the Möbius function of some underlying poset which luckily turns out to be related to the cumulants in question.

The considerations on the Hopf algebra $\mathcal{T}$ suggest an alternate method which can provide a unified way of treating the inversions of various cumulant-to-moment formulas, and also for doing inversions in cumulant-to-cumulant formulas. For a concrete illustration: consider the framework of Example 8.8, and the question of computing the inverse in $\widetilde{\mathcal{G}}$ for the semimultiplicative function $g_{\mathrm{mc}-\mathrm{bc}}$ which encodes the transition from monotone cumulants to Boolean cumulants. The antipode strategy for this job amounts to looking at the character $\chi_{\mathrm{mc}-\mathrm{bc}} \in \mathbb{X}(\mathcal{T})$ which corresponds to $g_{\mathrm{mc}-\mathrm{bc}}$, and then by performing the required inversion via the formula (cf. [21, Proposition II.4.1])

$$
\chi_{\mathrm{mc}-\mathrm{bc}}^{-1}=\chi_{\mathrm{mc}-\mathrm{bc}} \circ S, \quad \text { where } S: \mathcal{T} \rightarrow \mathcal{T} \text { is the antipode map. }
$$

In this section we make a start towards the study of the antipode of $\mathcal{T}$, with the hope that applications of the kind described above will be obtained in future work.

Remark 13.1. (Review of antipode basics.) Consider the graded bialgebra $\mathcal{T}$, as discussed in Section 12.1. The space of linear operators $L(\mathcal{T})=\{F: \mathcal{T} \rightarrow \mathcal{T} \mid F$ is linear $\}$ carries an associative operation of convolution defined as follows: one puts

$$
\begin{equation*}
F^{\prime} * F^{\prime \prime}=m \circ\left(F^{\prime} \otimes F^{\prime \prime}\right) \circ \Delta, \text { for } F^{\prime}, F^{\prime \prime} \in L(\mathcal{T}) \tag{13.1}
\end{equation*}
$$

where the map " $m$ " indicated on the right-hand side is the multiplication, $m: \mathcal{T} \otimes \mathcal{T} \rightarrow \mathcal{T}$ acting by $m(P \otimes Q)=P Q$ for $P, Q \in \mathcal{T}$. What (13.1) says is that in order to evaluate $F^{\prime} * F^{\prime \prime}$ on an element $P \in \mathcal{T}$ we should pick a writing $\Delta(P)=\sum_{i=1}^{n} P_{i}^{\prime} \otimes P_{i}^{\prime \prime}$ for the comultiplication of $P$, which yields that

$$
\begin{equation*}
F^{\prime} * F^{\prime \prime}(P)=\sum_{i=1}^{n} F^{\prime}\left(P_{i}^{\prime}\right) F^{\prime \prime}\left(P_{i}^{\prime \prime}\right) \in \mathcal{T} \tag{13.2}
\end{equation*}
$$

It is easy to see that the convolution operation "*" on $L(\mathcal{T})$ is well-defined and is associative. Moreover, if we consider the map $\widehat{\epsilon} \in L(\mathcal{T})$ defined by

$$
\begin{equation*}
\widehat{\epsilon}(P)=\epsilon(P) 1_{\mathcal{T}}, \quad \forall P \in \mathcal{T}, \quad \text { where } \epsilon: \mathcal{T} \rightarrow \mathbb{C} \text { is the counit of } \mathcal{T} \tag{13.3}
\end{equation*}
$$

then it is easily verified that $\widehat{\epsilon}$ is the (necessarily unique) unit for the semigroup $(L(\mathcal{T}), *)$.

Now comes the point anticipated in Remark 12.5 of the preceding section, that the identity map Id $\in L(\mathcal{T})$ is sure to be an invertible element of $(L(\mathcal{T}), *)$. This follows from general considerations on graded connected bialgebras - see for instance [21, Corollary II.3.2]. The inverse of $\operatorname{Id}$ in $(L(\mathcal{T}), *)$ is called the antipode of $\mathcal{T}$, denoted by $S$, and the existence of $S$ makes $\mathcal{T}$ be a Hopf algebra, as anticipated in Theorem 12.6.

General bialgebra considerations, which also take into account that $\mathcal{T}$ is commutative, yield the fact that $S: \mathcal{T} \rightarrow \mathcal{T}$ is a unital algebra homomorphism (cf. [21, Proposition I.7.1]). This implies in particular that $S$ is completely determined by how it acts on the generators $X_{\pi}$ of $\mathcal{T}$. In Proposition 13.3 below we state some formulas which allow recursive calculations of values $S\left(X_{\pi}\right)$, and go under the name of Bogoliubov formulas.
Notation 13.2. For $\pi, \sigma \in N C(n)$, we will write " $\pi<\sigma$ " to mean that $\pi \leq \sigma$ (in the sense or reverse refinement) and that $\pi \neq \sigma$.
Proposition 13.3. (Bogoliubov formulas.) For $n \geq 1$ and $\pi \in N C(n) \backslash\left\{1_{n}\right\}$ one has:

$$
\begin{equation*}
S\left(X_{\pi}\right)=-X_{\pi}-\sum_{\substack{\sigma \in N C(n), \pi<\sigma<1_{n}}} M_{\pi, \sigma} S\left(X_{\sigma}\right), \tag{13.4}
\end{equation*}
$$

and also that

$$
\begin{equation*}
S\left(X_{\pi}\right)=-X_{\pi}-\sum_{\substack{\sigma \in N C(n), \pi<\sigma<1_{n}}} S\left(M_{\pi, \sigma}\right) X_{\sigma}, \tag{13.5}
\end{equation*}
$$

where the monomials $M_{\pi, \sigma}$ are as introduced in Notation 12.2.
Proof. The relation $\operatorname{Id} * S=\widehat{\epsilon}$ implies in particular that $\operatorname{Id} * S\left(X_{\pi}\right)=\epsilon\left(X_{\pi}\right) 1_{\mathcal{T}}=0$. But on the other hand, the explicit description (13.2) used for $\operatorname{Id} * S$ says that:

$$
\operatorname{Id} * S\left(X_{\pi}\right)=\sum_{\sigma \geq \pi} M_{\pi, \sigma} S\left(X_{\sigma}\right)=M_{\pi, \pi} S\left(X_{\pi}\right)+M_{\pi, 1_{n}} S\left(X_{1_{n}}\right)+\sum_{\pi<\sigma<1_{n}} M_{\pi, \sigma} S\left(X_{\sigma}\right) .
$$

Upon recalling (cf. Remark (12.2) that $M_{\pi, \pi}=1_{\mathcal{T}}$ and $M_{\pi, 1_{n}}=X_{\pi}$, we thus find that

$$
\begin{equation*}
0=S\left(X_{\pi}\right)+X_{\pi}+\sum_{\pi<\sigma<1_{n}} M_{\pi, \sigma} S\left(X_{\sigma}\right), \tag{13.6}
\end{equation*}
$$

where separating the term $S\left(X_{\pi}\right)$ on the right-hand side leads to the formula (13.4).
The derivation of (13.5) is analogous, where we now start from the fact that $S * \operatorname{Id}=\widehat{\epsilon}$.
Remark 13.4. $1^{o}$ In the statement of Proposition 13.3 we excluded the case when $\pi=1_{n}$. In that case we have $X_{\pi}=1_{\mathcal{T}}$ and taking the antipode just gives $S\left(X_{1_{n}}\right)=S\left(1_{\mathcal{T}}\right)=1_{\mathcal{T}}$.

Note also that, in the case when $|\pi|=2$, the sum over $\left\{\sigma \in N C(n) \mid \pi<\sigma<1_{n}\right\}$ is an empty sum. In that case, either (13.4) or (13.5) gives that $S\left(X_{\pi}\right)=-X_{\pi}$; this is in agreement with the fact, observed in Section 12.3, that $X_{\pi}$ is a primitive element of $\mathcal{T}$.
$2^{o}$ Both (13.4) and (13.5) can be used for a recursive computation of values $S\left(X_{\pi}\right)$, but the setting of the recursion is different in the two situations. Formula (13.4) works when we fix an $n \in \mathbb{N}$, taken in isolation, and compute $S\left(X_{\pi}\right)$ for $\pi \in N C(n)$, by induction on $|\pi|$. Formula (13.5) works when we already know how $S$ works on some partitions from $N C(m)$ 's with $m<n$ - for instance, this works neatly when we fix an $\ell \geq 1$ and we are interested in $S\left(X_{\pi}\right)$ for all $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ such that every block $V$ of $\pi$ has $|V| \leq \ell$.

The next example (continued in the subsequent Examples 13.8 and 13.14) illustrates how these two recursive methods work towards computing $S\left(X_{0_{n}}\right)$ for some small values of $n$.

Example 13.5. Recall that $0_{n} \in N C(n)$ is the partition with $n$ singleton blocks. From Remark 13.4. 1 we infer that $S\left(X_{0_{1}}\right)=1_{\mathcal{T}}$ (because $X_{0_{1}}=1_{\mathcal{T}}$ ) and that $S\left(X_{0_{2}}\right)=-X_{0_{2}}$ (because $X_{0_{2}}$ is primitive).

For the computation of $S\left(X_{0_{3}}\right)$, let us record that the set of intermediate partitions $\left\{\sigma \in N C(3) \mid 0_{3}<\sigma<1_{3}\right\}$ consists of $\sigma_{1}, \sigma_{2}, \sigma_{3}$, where:

$$
\sigma_{1}=\{\{1\},\{2,3\}\}, \sigma_{2}=\{\{1,3\},\{2\}\}, \sigma_{3}=\{\{1,2\},\{3\}\} .
$$

For every $1 \leq i \leq 3$ we have that $S\left(X_{\sigma_{i}}\right)=-X_{\sigma_{i}}$, because $\left|\sigma_{i}\right|=2$, and (directly from the definition of the monomials $M_{\pi, \sigma}$ ) we see that $M_{0_{3}, \sigma_{i}}=X_{0_{2}}$. We leave to the reader the immediate verification that, based on this information, either of the two Bogoliubov formulas shown in Proposition 13.3 leads to:

$$
\begin{equation*}
S\left(X_{0_{3}}\right)=-X_{0_{3}}+X_{0_{2}}\left(X_{\sigma_{1}}+X_{\sigma_{2}}+X_{\sigma_{3}}\right) \tag{13.7}
\end{equation*}
$$

The sum of 4 terms that appeared on the right-hand side of Equation (13.7) can be viewed as a sum indexed by all possible chains that go from $0_{3}$ to $1_{3}$ in the poset $N C(3)$. This is clarified in Proposition 13.7 below, which is a special case of a result of Schmitt [26] holding in the general framework of an incidence Hopf algebra. For the proof of Proposition 13.7 (which is, essentially, an induction on $|\pi|$ based on the recursion formula (13.4)) we refer to [26, Theorem 6.1] or [27, Theorem 4.1].

Definition 13.6. Let $n$ be a positive integer and let $\pi, \sigma \in N C(n)$ be such that $\pi<\sigma$. A chain from $\pi$ to $\sigma$ is a tuple

$$
\begin{equation*}
c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right), \text { where } \pi=\pi_{0}<\pi_{1}<\cdots<\pi_{k}=\sigma . \tag{13.8}
\end{equation*}
$$

The number $k$ appearing in (13.8) is called the length of $c$.
For a chain $c$ as in (13.8) it will be convenient to use the shorthand notation

$$
\begin{equation*}
M_{c}:=M_{\pi_{0}, \pi_{1}} M_{\pi_{1}, \pi_{2}} \cdots M_{\pi_{k-1}, \pi_{k}} \in \mathcal{T} . \tag{13.9}
\end{equation*}
$$

Proposition 13.7. For $n \geq 1$ and $\pi \in N C(n) \backslash\left\{1_{n}\right\}$ one has:

$$
\begin{equation*}
S\left(X_{\pi}\right)=\sum_{\substack{c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right), \\ \text { chain from } \pi \text { to } 1_{n}}}(-1)^{k} M_{c} . \tag{13.10}
\end{equation*}
$$

Example 13.8. In continuation of Example 13.5, let us now compute what is $S\left(X_{0_{4}}\right)$. Proposition 13.7 gives an explicit formula for this antipode, as a sum indexed by chains in $N C(4)$ which go from $0_{4}$ to $1_{4}$. There are 29 such chains:

- 1 chain of length 1 , the chain $c=\left(0_{4}, 1_{4}\right)$;
- 12 chains of length 2 , of the form $c=\left(0_{4}, \sigma, 1_{4}\right)$ with $\sigma \in N C(4) \backslash\left\{0_{4}, 1_{4}\right\}$;
- 16 chains of length 3 , of the form $c=\left(0_{4}, \sigma, \sigma^{\prime}, 1_{4}\right)$ where $\sigma, \sigma^{\prime} \in N C(4)$ are such that $|\sigma|=3,\left|\sigma^{\prime}\right|=2$ and $\sigma<\sigma^{\prime}$.
Hence Proposition 13.7 gives $S\left(X_{0_{4}}\right)$ written as a sum of 29 terms.
Now, recall that Proposition 13.7 is based on the Bogoliubov formula (13.4), which "has $S$-factors on the right". The computation of $S\left(X_{0_{4}}\right)$ can also be done by using the formula (13.5), which has $S$-factors on the left:

$$
\begin{equation*}
S\left(X_{0_{4}}\right)=-X_{0_{4}}-\sum_{\substack{\sigma \in N C(4), 0_{4}<\sigma<1_{4}}} S\left(M_{0_{4}, \sigma}\right) X_{\sigma} . \tag{13.11}
\end{equation*}
$$

It is immediate that, for every $\sigma \in N C(4)$ with $0_{4}<\sigma<1_{4}$, the monomial $M_{0_{4}, \sigma}$ is a product of factors $X_{0_{2}}$ and $X_{0_{3}}$; so, consequently, $S\left(M_{0_{4}, \sigma}\right)$ can be computed explicitly
by using the formulas for $S\left(X_{0_{2}}\right), S\left(X_{0_{3}}\right)$ found in Example 13.5. In this way, the righthand side of (13.11) is turned into an explicit formula for $S\left(X_{0_{4}}\right)$. The reader who has the patience to really write the latter formula will discover the interesting detail that it only has 25 terms (instead of 29 , as we got from applying Proposition 13.7). This happens because the formula (13.10) isn't generally cancellation-free. In the case at hand, of $\pi=0_{4}$, we can pin down precisely where it is that the cancellations in (13.10) take place. There are two terms disappearing because the chains of length 3

$$
\left\{\begin{array}{l}
c^{\prime}=\left(0_{4},\{\{1,2\},\{3\},\{4\}\},\{\{1,2\},\{3,4\}\}, 1_{4}\right) \text { and }  \tag{13.12}\\
c^{\prime \prime}=\left(0_{4},\{\{1\},\{2\},\{3,4\}\},\{\{1,2\},\{3,4\}\}, 1_{4}\right)
\end{array}\right.
$$

have the same contribution (but with opposite sign) as the shorter chain $\left(0_{4},\{\{1,2\},\{3,4\}\}\right.$, $\left.1_{4}\right)$. Then there are two other terms that disappear, in a similar way, in connection to the chain $\left(0_{4},\{\{1,4\},\{2,3\}\}, 1_{4}\right)$.

The method based on (13.5) can be shown to give a cancellation-free formula for $S\left(X_{0_{n}}\right)$, for every $n \geq 1$. The number $t_{n}$ of terms which appears in the cancellation-free formula satisfies a recursion presented in Example 13.14 below. According to the calculations we showed so far, the sequence $\left(t_{n}\right)_{n=1}^{\infty}$ starts with $1,1,4,25$; but this promising start turns out to not continue towards some known integer sequence.

Remark 13.9. We will next show how one can re-structure the summation over chains from (13.10) in order to obtain a cancellation-free summation formula. This will be done by pruning the index set used in (13.10) to a smaller collection of chains in $N C(n)$, which we call "efficient chains" - cf. Definition 13.10, Theorem 13.13.

We mention that our identifying of the notion of efficient chain retrieves a special case of a notion identified in the thesis [12], in the general framework of incidence Hopf algebras, where the terms of the cancellation-free summations arrive to be described by objects called "forests of lattices" (cf. [12, Chapter 5]). While it would be possible to review the fairly substantial background and terminology developed in [12] and then invoke the result from there, we find it easier to write down a direct inductive argument which covers the special case needed in Theorem 13.13.

We would also like to signal that another path towards obtaining a cancellation-free summation formula for the antipode of $\mathcal{T}$ is offered by the work in [23]. This would exploit the fact that $\mathcal{T}$ is an example of so-called left-handed polynomial Hopf algebra, a term which refers to the fact that the formula (12.4) defining comultiplication merely has an " $X_{\sigma}$ " (rather than a product of $X_{\sigma}$ 's) on the right side of the tensor product.

Definition 13.10. Let $n$ be a positive integer and let $\pi, \sigma \in N C(n)$ be such that $\pi<\sigma$.
$1^{o}$ To every chain $c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right)$ from $\pi$ to $\sigma$ we associate two collections of subsets of $\{1, \ldots, n\}$, as follows:

$$
\operatorname{Blocks}(c):=\left\{V \subseteq\{1, \ldots, n\} \mid \exists 0 \leq j \leq k \text { such that } V \text { is a block of } \pi_{j}\right\}, \text { and }
$$

$$
\operatorname{Blocks}^{+}(c):=\left\{V \in \operatorname{Blocks}(c) \mid V \text { is not a block of } \pi_{0}\right\} .
$$

$2^{o}$ A chain $c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right)$ from $\pi$ to $\sigma$ will be said to be efficient when it satisfies:
$\left\{\begin{array}{l}\text { For every set } V \in \operatorname{Blocks}^{+}(c) \text { there exists } \\ \text { a unique } j \in\{1, \ldots, k\} \text { such that } V \text { is a block of } \pi_{j} .\end{array}\right.$
$3^{o}$ We denote by $\mathcal{E C}(\pi, \sigma)$ the set of all efficient chains from $\pi$ to $\sigma$.

Remark 13.11. $1^{o}$ In order to explain the term "efficient" used in the preceding definition, let $\pi<\sigma$ be as above and let $c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right)$ be a chain from $\pi$ to $\sigma$. Pick an $m \in\{1, \ldots, n\}$ and for every $0 \leq j \leq k$ let us denote by $V^{(j)}$ the block of $\pi_{j}$ which contains the number $m$. Then we have

$$
\begin{equation*}
V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(k)} \quad(\text { subsets of }\{1, \ldots, n\}), \tag{13.13}
\end{equation*}
$$

where some of the inclusions in (13.13) may actually be equalities. The property of $c$ described in Definition 13.10 , 2 amounts to the fact that once we run into an inclusion $V^{(i-1)} \subseteq V^{(i)}$ which is strict, all the subsequent inclusions $V^{(j-1)} \subseteq V^{(j)}$ with $j \geq i$ have to be strict as well - in a certain sense, one "moves efficiently" towards the last set $V^{(k)}$ indicated in that list.
$2^{o}$ Given $\pi<\sigma$ in $N C(n)$ and a chain $c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right) \in \mathcal{E C}(\pi, \sigma)$ we will be interested in the quantity

$$
\begin{equation*}
(-1)^{\mid \text {Blocks }^{+}(c) \mid} M_{c}=(-1)^{\mid \text {Blocks }^{+}(c) \mid} M_{\pi_{0}, \pi_{1}} M_{\pi_{1}, \pi_{2}} \cdots M_{\pi_{k-1}, \pi_{k}}, \tag{13.14}
\end{equation*}
$$

which will be featured in Theorem 13.13 below. For illustration, let us look at how this quantity plays out in connection to the cancellations we spotted in Example 13.8. The chains $c^{\prime}, c^{\prime \prime}$ of length 3 shown in (13.12) are not efficient: for instance for the first of them we find that the set $V=\{1,2\} \in$ Blocks $^{+}\left(c^{\prime}\right)$ belongs to both partitions $\pi_{1}$ and $\pi_{2}$ of $c^{\prime}$, where $\pi_{1}=\{\{1,2\},\{3\},\{4\}\}$ and $\pi_{2}=\{\{1,2\},\{3,4\}\}$. On the other hand:

$$
c:=\left(0_{4},\{\{1,2\},\{3,4\}\}, 1_{4}\right) \text { is efficient, with } \operatorname{Blocks}^{+}(c)=\{\{1,2\},\{3,4\},\{1,2,3,4\}\} .
$$

The issue observed in Example 13.8 was this: when plugged into the summation on the right-hand side of (13.10), both $c^{\prime}$ and $c^{\prime \prime}$ have contributions of $-X_{0_{2}} X_{\sigma}$, for $\sigma=$ $\{\{1,2\},\{3,4\}\}$, while $c$ has a contribution of $+X_{0_{2}} X_{\sigma}$. (Cancellation!) In the formula featured in Theorem 13.13, the chains $c^{\prime}$ and $c^{\prime \prime}$ will no longer appear, while $c$ will appear with a contribution of $-X_{0_{2}} X_{\sigma}$; we note the different sign in the contribution of $c$ (coming from the fact that $\left|\operatorname{Blocks}^{+}(c)\right|$ is of different parity than the length of $c$ ), and accounting for the cancellations " $(-1)+(-1)+1=-1$ " that we had encountered before.
$3^{o}$ Let $\pi<\sigma$ be in $N C(n)$ and consider a chain $c=\left(\pi_{0}, \pi_{1}, \ldots, \pi_{k}\right) \in \mathcal{E C}(\pi, \sigma)$. Upon tallying what indeterminates " $X_{\rho}$ " are taken into the monomials $M_{\pi_{0}, \pi_{1}}, M_{\pi_{1}, \pi_{2}}, \ldots, M_{\pi_{k-1}, \pi_{k}}$ multiplied in (13.14), one finds the following interpretation for the cardinality of the set Blocks $^{+}(c)$ : it counts the total number of factors $X_{\rho}$ when the product $M_{\pi_{0}, \pi_{1}} M_{\pi_{1}, \pi_{2}} \ldots$ $\cdots M_{\pi_{k-1}, \pi_{k}}$ is simply treated as a product of $X_{\rho}$ 's, and we eliminate the units " $X_{1_{m}}$ " which may have appeared in the description of the monomials $M_{\pi_{j-1}, \pi_{j}}$.

The observation made in the preceding paragraph ensures that the summation formula stated in Theorem 13.13 is cancellation-free! Indeed, if two chains appearing on the righthand side of that summation formula turn out to have the same " $M_{c}$ " contribution, then they will also have the same sign in the " $(-1)^{\mid \text {Blocks }^{+}(c) \mid "}$ part of the formula; hence the terms indexed by the two chains in question will not cancel, but will rather add up.

For a concrete example, suppose we make $n=7$ and we consider the chains

$$
\begin{array}{r}
c^{\prime}:=\left(0_{7},\{\{1,2\},\{3\},\{4,5,6\},\{7\}\},\{\{1,2,3\},\{4,5,6,7\}\}, 1_{7}\right) \text { and } \\
c^{\prime \prime}:=\left(0_{7},\{\{1\},\{2\},\{3\},\{4,5\},\{6\},\{7\}\},\{\{1\},\{2\},\{3\},\{4,5,6\},\{7\}\},\right. \\
\left.\{\{1,2,3\},\{4,5,6,7\}\}, 1_{7}\right),
\end{array}
$$

which are efficient chains going from $0_{7}$ to $1_{7}$. In the summation formula (13.18) of Theorem 13.13, $c^{\prime}$ and $c^{\prime \prime}$ have identical contributions, of $(-1)^{5} X_{\rho_{1}} \cdots X_{\rho_{5}}$ where

$$
\rho_{1}=\{\{1,2,3\},\{4,5,6,7\}\}, \rho_{2}=\{\{1,2,3\},\{4\}\}, \rho_{3}=\{\{1,2\},\{3\}\}, \rho_{4}=0_{3}, \rho_{5}=0_{2}
$$

The point to note is that the contributions of $c^{\prime}$ and $c^{\prime \prime}$ to the right-hand side of (13.18) do not cancel each other, but rather get to be added together, as mentioned above.

The proof of Theorem 13.13 is based on the following lemma.
Lemma 13.12. Let $\pi, \sigma$ be in $N C(n)$ for some $n \geq 1$, such that $\pi<\sigma<1_{n}$, and where we write explicitly $\sigma=\left\{V_{1}, \ldots, V_{r}\right\}$. Consider the set of ${ }^{3}$ efficient chains

$$
\begin{equation*}
\widetilde{\mathcal{E C}}:=\left\{c \in \mathcal{E C}\left(\pi, 1_{n}\right) \mid c=\left(\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_{n}\right) \text { with } k \geq 2 \text { and } \pi^{(1)}=\sigma\right\} \tag{13.15}
\end{equation*}
$$

One has a bijection

$$
\begin{equation*}
\widetilde{\mathcal{E C}} \ni c \mapsto\left(c_{1}, \ldots, c_{r}\right) \in \mathcal{E C}\left(\pi_{V_{1}}, 1_{\left|V_{1}\right|}\right) \times \cdots \times \mathcal{E C}\left(\pi_{V_{r}}, 1_{\left|V_{r}\right|}\right) \tag{13.16}
\end{equation*}
$$

where, for $c=\left(\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_{n}\right) \in \widetilde{\mathcal{E C}}$ and $1 \leq s \leq r$, we put $c_{s}:=\left(\pi_{V_{s}}^{(k)}, \ldots, \pi_{V_{s}}^{(1)}\right)$. (If it happens that we have $\pi_{V_{2}}=\pi_{V_{s}}^{(k)}=\pi_{V_{s}}^{(k-1)}=\cdots=\pi_{V_{s}}^{(j)}$ for some $1 \leq j \leq k-1$, then $\left(\pi_{V}^{(k)}, \ldots, \pi_{V}^{(2)}, \pi_{V}^{(1)}\right)$ is not properly a chain, so we rather take $c_{s}=\left(\pi_{V}, \pi_{V}^{(j-1)}, \ldots, \pi_{V}^{(1)}\right)$.) Furthermore, for $c \mapsto\left(c_{1}, \ldots, c_{r}\right)$ as in (13.16), one has

$$
\begin{equation*}
(-1)^{\mid \text {Blocks }^{+}(c) \mid} M_{c}=-X_{\pi^{(1)}} \prod_{s=1}^{r}\left((-1)^{\mid \text {Blocks }^{+}\left(c_{s}\right) \mid} M_{c_{s}}\right) . \tag{13.17}
\end{equation*}
$$

Proof. We first prove that $c_{1}, \ldots, c_{r}$ from (13.16) are efficient chains. Pick an $s \in\{1, \ldots, r\}$ and, for the sake of contradiction, assume that $c_{s}$ is not efficient. This implies that there exist a block $W \in \operatorname{Blocks}^{+}\left(c_{s}\right)$ and indices $1 \leq i<j \leq k$ such that $W \in \pi_{V}^{(i)}$ and $W \in \pi_{V}^{(j)}$. Since $\pi^{(j)}<\pi^{(i)} \leq \pi^{(1)}$, this implies that $W \in \operatorname{Blocks}^{+}(c)$ with $W \in \pi^{(i)}$ and $W \in \pi^{(j)}$, contradicting the fact that $c$ is efficient. Therefore, $c^{\prime} \in \mathcal{E C}\left(\pi_{V}, 1_{|V|}\right)$ for all $c \in \mathcal{E C}\left(\pi, 1_{n}\right)$ and $V \in \pi^{(1)}$.

In order to prove that the map indicated in (13.16) is bijective, we will describe how its inverse works. For this, suppose we have an $r$-tuple of chains, $c_{s}=\left(\pi_{s}^{\left(j_{s}\right)}, \ldots, \pi_{s}^{(1)}\right) \in$ $\mathcal{E C}\left(\pi_{V_{s}}, 1_{\left|V_{s}\right|}\right)$. To reconstruct the chain $c \in \widetilde{\mathcal{E C}}$ which corresponds to $\left(c_{1}, \ldots, c_{r}\right)$, we first consider the size of the largest chain $j:=\max _{1 \leq s \leq r} j_{s}$. Then, we enlarge the other chains so that all have the largest size, by denoting $\pi_{s}^{(i)}:=\pi_{V_{s}}$ for every $s=1, \ldots, r$ and $j_{s}<$ $i \leq j$. Then, for every $i=1, \ldots, j$, we construct the partition $\pi^{(i)} \in N C(n)$ uniquely determined by the fact that $\pi^{(i)} \leq \sigma$ and $\pi_{V_{s}}^{(i)}=\pi_{s}^{(i)}$ for $s=1, \ldots, r$. Finally, we define $c:=\left(\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_{n}\right)$. It is not hard to show (left as exercise to the reader) that this $c$ is in $\widetilde{\mathcal{E C}}$, is mapped by (13.16) into the $\left(c_{1}, \ldots, c_{r}\right)$ we started from, and is uniquely determined by this property.

Finally, Equation (13.17) follows easily from the bijection (13.16). Indeed, for the equality of signs we break $c$ by taking apart $\left(\pi^{(1)}, 1_{n}\right)$, and then regroup the remaining chain in terms of the blocks of $\pi^{(1)}$. Since $\operatorname{Blocks}^{+}\left(\pi^{(1)}, 1_{n}\right)=1$ we get that

$$
\operatorname{Blocks}^{+}(c)=\operatorname{Blocks}^{+}\left(\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}\right)+1=1+\sum_{s=1}^{r} \operatorname{Blocks}^{+}\left(c_{r}\right) .
$$

[^3]For the term $M_{c}$, the idea is the same, although the computation is a bit more involved:

$$
\begin{aligned}
M_{c} & =M_{\pi^{(1)}, 1_{n}} \prod_{i=1}^{k-1} M_{\pi^{(i+1)}, \pi^{(i)}}=X_{\pi^{(1)}} \prod_{i=1}^{k-1} \prod_{s=1}^{r} M_{\pi_{V_{s}}^{(i+1)}, \pi_{V_{s}}^{(i)}}=X_{\pi^{(1)}} \prod_{s=1}^{r} \prod_{i=1}^{k-1} M_{\pi_{V_{s}}^{(i+1)}, \pi_{V_{s}}^{(i)}} \\
& =X_{\pi^{(1)}} \prod_{s=1}^{r} \prod_{i=1}^{j_{s}} M_{\pi_{s}^{(i+1)}, \pi_{s}^{(i)}}=X_{\pi^{(1)}} \prod_{s=1}^{r} M_{c_{s}}
\end{aligned}
$$

Putting together the sign and the computation for $M_{c}$ yields Equation (13.17).

Theorem 13.13. For every $\pi \in \sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right)$ one has:

$$
\begin{equation*}
S\left(X_{\pi}\right)=\sum_{c \in \mathcal{E C}\left(\pi, 1_{n}\right)}(-1)^{\mid \text {Blocks }^{+}(c) \mid} M_{c} . \tag{13.18}
\end{equation*}
$$

Proof. The proof is by induction on $|\pi|$. For the base case: consider a $\pi$ in some $N C(n)$, such that $|\pi|=2$. In this case we know that $S\left(X_{\pi}\right)=-X_{\pi}$. On the other hand, the set $\mathcal{E C}\left(\pi, 1_{n}\right)$ consists of only one chain, namely $c=\left(\pi, 1_{n}\right)$, which has $\left|\operatorname{Blocks}^{+}(c)\right|=1$ and $M_{c}=M_{\pi, 1_{n}}=X_{\pi}$; hence the right-hand side of Equation (13.18) also comes out as $-X_{\pi}$, as required.

For the inductive step we fix a $j \geq 3$, we assume that the formula (13.18) holds for every $\sigma \in \sqcup_{n=1}^{\infty}\left(N C(n) \backslash\left\{1_{n}\right\}\right)$ with $|\sigma|<j$, and we prove that the same formula also holds for a $\pi$ with $|\pi|=j$.

By the Bogoliubov recursion indicated in Equation (13.5), we have

$$
\begin{align*}
S\left(X_{\pi}\right) & =-X_{\pi}-\sum_{\substack{\sigma \geq \pi \\
\pi \neq \sigma \neq 1_{n}}}\left(\prod_{V \in \sigma} S\left(X_{\pi \mid V}\right)\right) X_{\sigma}  \tag{13.19}\\
& =-X_{\pi}-\sum_{\substack{\sigma=\left\{V_{1}, \ldots, V_{r}\right\} \\
1_{n}>\sigma>\pi}} X_{\sigma} \prod_{s=1}^{r}\left(\sum_{c_{s} \in \mathcal{E C}\left(\pi_{V_{s}},\left.\right|_{\left|V_{s}\right|} \mid\right.}(-1)^{\mid \text {Blocks }^{+}\left(c_{s}\right) \mid} M_{c_{s}}\right), \tag{13.20}
\end{align*}
$$

where for the latter equality we used the induction hypothesis on $S\left(X_{\pi_{V_{s}}}\right)$, for each $V_{s} \in \sigma$. Finally, from the bijection in Lemma 13.12, equation (13.20) can be concisely written as

$$
\begin{equation*}
-X_{\pi}+\sum_{\substack{\sigma=\left\{V_{1}, \ldots, V_{r}\right\} \\ 1 n>\sigma>\pi}} \sum_{c \in \widetilde{\mathcal{E C}_{\sigma}}} X_{\pi^{(1)}}(-1)^{\mid \text {Blocks }^{+}(c) \mid} M_{c}, \tag{13.21}
\end{equation*}
$$

where the notation $\widetilde{\mathcal{E C}}_{\sigma}$ is just to acknowledge that the set $\widetilde{\mathcal{E C}}$ from Lemma 13.12 depends on the partition $\sigma$.

The conclusion follows from observing that the sum in (13.21) is a sum over all chains in $\mathcal{E C}\left(\pi, 1_{n}\right)$ and thus coincides with the right hand side of (13.18). Indeed, given a chain $c \in \mathcal{E C}\left(\pi, 1_{n}\right)$, we either have $c=\left(\pi, 1_{n}\right)$, in which case we get the term $-X_{\pi}$, or else we have $c=\left(\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_{n}\right)$ with $k \geq 2$ and $\pi^{(1)}=\sigma$ for some $1_{n}>\sigma>\pi$, implying that $c \in \widetilde{\mathcal{E C}}_{\sigma}$.

Example 13.14. In continuation of the last paragraph of Example 13.8, let $t_{n}$ denote the number of terms in the cancellation-free summation giving $S\left(X_{0_{n}}\right), n \geq 1$. In view of Theorem 13.13, $t_{n}$ can also be viewed as $\left|\mathcal{E C}\left(0_{n}, 1_{n}\right)\right|$, the number of efficient chains from $0_{n}$ to $1_{n}$ in $N C(n)$.

We will derive a recursion satisfied by the numbers $t_{n}$. It is possible (left as exercise to the reader) to do so by examining the method of proof used for Theorem 13.13, and by extracting out of it a recursion among the cardinalities of the sets $\mathcal{E C}\left(0_{n}, 1_{n}\right)$. Here we will take the alternative path of getting the desired recurrence for the $t_{n}$ 's via a direct analysis of the Bogoliubov formula (13.5), which says that

$$
S\left(X_{0_{n}}\right)=-X_{0_{n}}-\sum_{\substack{\sigma \in N C(n), 0_{n}<\sigma<1_{n}}} S\left(M_{0_{n}, \sigma}\right) X_{\sigma} .
$$

Every monomial $M_{0_{n}, \sigma}$ is equal, by definition, to $\prod_{W \in \sigma} X_{0_{|W|}}$. Since $S$ is multiplicative, we thus find that

$$
\begin{equation*}
S\left(X_{0_{n}}\right)=-X_{0_{n}}-\sum_{\substack{\sigma \in N C(n), 0_{n}<\sigma<1_{n}}}\left(\prod_{W \in \sigma} S\left(X_{0_{|W|}}\right)\right) X_{\sigma} . \tag{13.22}
\end{equation*}
$$

Suppose that on the right-hand side of (13.22) we write every $S\left(X_{0_{|W|}}\right)$ as a cancellationfree sum of $t_{|W|}$ terms, then cross-multiply these sums. For every $\sigma \in N C(n) \backslash\left\{0_{n}, 1_{n}\right\}$ we thus get a sum of $\prod_{W \in \sigma} t_{|W|}$ terms, which (very importantly) get to be also multiplied by an additional factor of $X_{\sigma}$. Now, the latter factor of $X_{\sigma}$ appears only once in the whole expression on the right-hand side of (13.22). Multiplying with it will therefore prevent any cancellations with terms that appear from the analogous discussion related to some other $\sigma^{\prime} \in N C(n) \backslash\left\{0_{n}, 1_{n}\right\}$.

Altogether, the discussion in the preceding paragraph shows how on the right-hand side of (13.22) we arrive to a cancellation-free summation, where we can count the terms, in order to arrive to the conclusion that

$$
\begin{equation*}
t_{n}=1+\sum_{\substack{\sigma \in N C(n), 0_{n}<\pi<1_{n}}} \prod_{W \in \sigma} t_{|W|}, \quad n \geq 1 \tag{13.23}
\end{equation*}
$$

(the empty sums appearing for $n=1$ and $n=2$ correspond to the fact that $t_{1}=t_{2}=1$ ). Equation (13.23) is the recursion we wanted for the numbers $t_{n}$. If we read the separate term of 1 on the right-hand side as $t_{1}^{n}$, and we add on both side a term of $t_{n}$, we arrive to the nicer form

$$
\begin{equation*}
2 t_{n}=\sum_{\sigma \in N C(n)} \prod_{W \in \sigma} t_{|W|}, \quad n \geq 2 \tag{13.24}
\end{equation*}
$$

Finally, Equation (13.24) strongly suggests using the functional equation of the $R$ transform from free probability (very closely related to free cumulants - cf. 24, Lecture $16]$ ), in order to find an equation satisfied by the generating function

$$
\begin{equation*}
T(z):=\sum_{n=1}^{\infty} t_{n} z^{n}=z+z^{2}+4 z^{3}+25 z^{4}+\cdots \tag{13.25}
\end{equation*}
$$

More precisely: let $\mu: \mathbb{C}[X] \rightarrow \mathbb{C}$ be the linear functional which has $\mu(1)=1$ and has its sequence of free cumulants equal to $\left(t_{n}\right)_{n=1}^{\infty}$, hence has $R$-transform $R_{\mu}(z)$ equal to the above $T(z)$. From (13.24) it follows that the moment series $M_{\mu}(z)=\sum_{n=1}^{\infty} \mu\left(X^{n}\right) z^{n}$ is then equal to $2 T(z)-z$. The functional equation of the $R$-transform says that

$$
R_{\mu}\left(z\left(1+M_{\mu}(z)\right)=M_{\mu}(z) \quad(\text { cf. [24, Remark 16.18] })\right.
$$

which becomes

$$
\begin{equation*}
T(z(2 T(z)-z+1))=2 T(z)-z \tag{13.26}
\end{equation*}
$$

It is nicer to record this equation in terms of the series

$$
\begin{equation*}
U(z):=2 T(z)-z+1=1+z+2 z^{2}+8 z^{3}+50 z^{4}+\cdots, \tag{13.27}
\end{equation*}
$$

which satisfies:

$$
\begin{equation*}
U(z U(z))=(2-z) U(z)-1 . \tag{13.28}
\end{equation*}
$$

The first few $t_{n}$ 's come out as $1,1,4,25,206,2060,23920,314065,4582300, \ldots$ This, unfortunately, doesn't seem to match the beginning of some sequence previously recorded in the research literature.

## 14. The Hopf algebra view on the inclusion $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$

There exists a result parallel to the above Theorem 12.6. concerning the smaller group $\mathcal{G}$ of multiplicative functions which was reviewed in Section 5 . This result was established in [22] and recognizes $\mathcal{G}$ as the group of characters $\mathbb{X}(\mathrm{Sym})$ of the Hopf algebra Sym of symmetric functions. In the present section we put into evidence a natural Hopf algebra homomorphism $\Psi: \mathcal{T} \rightarrow$ Sym, with $\mathcal{T}$ as in Section 12, such that the dual group homomorphism $\Psi^{*}: \mathbb{X}(\mathrm{Sym}) \rightarrow \mathbb{X}(\mathcal{T})$ corresponds in a canonical way to the inclusion of $\mathcal{G}$ into $\widetilde{\mathcal{G}}$.

### 14.1. Review of the group isomorphism $\mathcal{G} \approx \mathbb{X}(\mathbf{S y m})$.

Notation and Remark 14.1. We use the incarnation of the Hopf algebra Sym as

$$
\begin{equation*}
\text { Sym }=\mathbb{C}\left[Y_{2}, Y_{3}, \ldots, Y_{n}, \ldots\right] \quad \text { (commutative algebra of polynomials) } \tag{14.1}
\end{equation*}
$$

where $Y_{2}, Y_{3}, \ldots, Y_{n}, \ldots$ are the so-called parking-function symmetric functions. In the same spirit as for the considerations on the Hopf algebra $\mathcal{T}$, we will also denote

$$
\begin{equation*}
Y_{1}:=1_{\text {Sym }} \quad(\text { the unit of Sym }) . \tag{14.2}
\end{equation*}
$$

A description of how the $Y_{n}$ 's relate to other (more commonly used) families of generators of Sym can e.g. be found in [31, Proposition 2.2]. But here the only thing we need to know about the $Y_{n}$ 's is how the comultiplication $\Delta: \operatorname{Sym} \rightarrow \mathrm{Sym} \otimes \operatorname{Sym}$ operates on them. The original motivation for featuring the $Y_{n}$ 's in [22] was that the formula giving $\Delta\left(Y_{n}\right)$ follows the same pattern as we saw in Section 11 in connection to the multiplication of free elements: one has

$$
\begin{equation*}
\Delta\left(Y_{n}\right)=\sum_{\pi \in N C(n)}\left(\prod_{V \in \pi} Y_{|V|}\right) \otimes\left(\prod_{W \in \operatorname{Kr}(\pi)} Y_{|W|}\right), \quad \forall n \geq 1 . \tag{14.3}
\end{equation*}
$$

[For instance $\Delta\left(Y_{3}\right)=Y_{3} \otimes Y_{1}^{3}+3 Y_{1} Y_{2} \otimes Y_{1} Y_{2}+Y_{1}^{3} \otimes Y_{3}=Y_{3} \otimes 1_{\mathrm{Sym}}+3 Y_{2} \otimes Y_{2}+1_{\mathrm{Sym}} \otimes Y_{3}$, a sum of 5 terms, corresponding to the 5 partitions in $N C(3)$.]

We also mention that, when described in terms of the $Y_{n}$ 's:

- The counit of Sym is the character $\varepsilon: \operatorname{Sym} \rightarrow \mathbb{C}$ uniquely determined by the requirement that $\varepsilon\left(Y_{n}\right)=0$ for all $n \geq 2$.
- The grading of Sym is determined by the fact that $Y_{n}$ has degree $n-1$, for every $n \geq 1$, with the usual follow-up defining the degree of a monomial $Y_{n_{1}} \cdots Y_{n_{k}}$ to be $n_{1}+\cdots+n_{k}-k$.

Notation and Remark 14.2. In the framework of Notation 14.1, it is convenient that for every $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ we denote

$$
\begin{equation*}
Y_{\pi}:=\prod_{V \in \pi} Y_{|V|}(\text { a monomial in the algebra Sym }) \tag{14.4}
\end{equation*}
$$

[For example, $\pi=\{\{1,2,6\},\{3,4\},\{5\},\{7,8\}\} \in N C(8)$ has $\left.Y_{\pi}=Y_{3} Y_{2} Y_{1} Y_{2}=Y_{3} Y_{2}^{2}.\right]$
The formula (14.3) describing comultiplication can then be written concisely as

$$
\begin{equation*}
\Delta\left(Y_{n}\right)=\sum_{\pi \in N C(n)} Y_{\pi} \otimes Y_{\operatorname{Kr}(\pi)}, \quad n \geq 1 \tag{14.5}
\end{equation*}
$$

By using the fact that $\Delta$ is an algebra homomorphism, it is easy (see [22, Lemma 3.3]) to extend (14.5) to

$$
\begin{equation*}
\Delta\left(Y_{\sigma}\right)=\sum_{\substack{\pi \in N C(n) \\ \pi \leq \sigma}} Y_{\pi} \otimes Y_{\mathrm{Kr}_{\sigma}(\pi)}, \quad \forall n \geq 1 \text { and } \sigma \in N C(n), \tag{14.6}
\end{equation*}
$$

where $\operatorname{Kr}_{\sigma}(\pi)$ stands, as usual, for the relative Kreweras complement of $\pi$ in $\sigma$.

Remark 14.3. Now let us look at the group $\mathcal{G}$ of multiplicative functions and at the group $\mathbb{X}(\mathrm{Sym})$ of characters of Sym. For every $f \in \mathcal{G}$ one can consider a character $\widehat{\chi}_{f} \in \mathbb{X}(\mathrm{Sym})$, defined by requiring that

$$
\begin{equation*}
\widehat{\chi}_{f}\left(Y_{n}\right)=f\left(0_{n}, 1_{n}\right), \quad \forall n \geq 1 \tag{14.7}
\end{equation*}
$$

It is clear that the map $\mathcal{G} \ni f \mapsto \widehat{\chi}_{f} \in \mathbb{X}(\mathrm{Sym})$ is bijective, and it is easy to check that

$$
\widehat{\chi}_{f_{1} * f_{2}}=\widehat{\chi}_{f_{1}} * \widehat{\chi}_{f_{2}}, \quad \forall f_{1}, f_{2} \in \mathcal{G}
$$

where on the left-hand side we invoke the convolution operation on $\mathcal{G}$, while on the righthand side we use the convolution of characters of Sym. Thus $f \mapsto \widehat{\chi}_{f}$ gives a group isomorphism $\mathcal{G} \approx \mathbb{X}(\mathrm{Sym})$, analogous to the isomorphism $\widetilde{\mathcal{G}} \approx \mathbb{X}(\mathcal{T})$ from Theorem 12.9,

### 14.2. The surjective homomorphism $\Psi: \mathcal{T} \rightarrow$ Sym.

Consider now the Hopf algebra $\mathcal{T}$ from Section 12 and recall that $\mathcal{T}$ enjoys a universality property, stated in (12.3), which makes it very easy to define unital algebra homomorphisms having $\mathcal{T}$ as domain. We use that to make the following definition.

Definition 14.4. We let $\Psi: \mathcal{T} \rightarrow$ Sym be the unital algebra homomorphism defined by using the universality property (12.3) and the requirement that

$$
\begin{equation*}
\Psi\left(X_{\pi}\right)=Y_{\operatorname{Kr}(\pi)}=\prod_{W \in \operatorname{Kr}(\pi)} Y_{|W|}, \quad \forall \pi \in \sqcup_{n=1}^{\infty} N C(n) \tag{14.8}
\end{equation*}
$$

Note: in order for the universality property of $\mathcal{T}$ to apply, the right-hand side of (14.8) must be equal to $1_{\text {Sym }}$ whenever $\pi=1_{n}$ for some $n \geq 1$. This is indeed the case, since $\operatorname{Kr}\left(1_{n}\right)=0_{n}$ and $Y_{0_{n}}=Y_{1}^{n}=1_{\text {Sym }}$.

Remark 14.5. For every $n \geq 1$, the definition of $\Psi$ gives $\Psi\left(X_{0_{n}}\right)=Y_{1_{n}}=Y_{n}$. This immediately implies that the homomorphism $\Psi$ is surjective.

The point about $\Psi$ is that it also respects the coalgebra structure, as we show next.

Theorem 14.6. The map $\Psi$ introduced in Definition 14.4 is a homomorphism of graded connected bialgebras.

Proof. We have to check that $\Psi$ respects: (i) comultiplications; (ii) counits; (iii) gradings on the Hopf algebras $\mathcal{T}$ and Sym.
For (i): we have to verify the equality

$$
\begin{equation*}
\Delta_{\mathrm{Sym}} \circ \Psi=(\Psi \otimes \Psi) \circ \Delta_{\mathcal{T}}, \tag{14.9}
\end{equation*}
$$

where $\Delta_{\text {Sym }}$ and $\Delta_{\mathcal{T}}$ are the comultiplications of Sym and of $\mathcal{T}$, respectively. Since both sides of (14.9) are unital algebra homomorphisms from $\mathcal{T}$ to $\operatorname{Sym} \otimes \mathrm{Sym}$ ), it suffices to check their agreement on a generator $X_{\pi}$ of $\mathcal{T}$, with $\pi \in N C(n) \backslash\left\{1_{n}\right\}$ for some $n \geq 1$.

Let us then pick an $X_{\pi}$ as mentioned above, plug it into the left-hand side of (14.9), and compute:

$$
\begin{aligned}
\left(\Delta_{\mathrm{Sym}} \circ \Psi\right)\left(X_{\pi}\right) & =\Delta_{\mathrm{Sym}}\left(Y_{\mathrm{Kr}(\pi)}\right) \\
& =\sum_{\rho \leq \operatorname{Kr}(\pi)} Y_{\rho} \otimes Y_{\operatorname{Kr}}^{\mathrm{Kr}(\pi)}(\rho)
\end{aligned} \quad \text { (by Equation (14.6)). } .
$$

We next do the same on the right-hand side of (14.9):

$$
\begin{aligned}
\left((\Psi \otimes \Psi) \circ \Delta_{\mathcal{T}}\right)\left(X_{\pi}\right) & =(\Psi \otimes \Psi)\left(\Delta_{\mathcal{T}}\left(X_{\pi}\right)\right) \\
& =(\Psi \otimes \Psi)\left(\sum_{\sigma \geq \pi}\left(\prod_{W \in \sigma} X_{\pi_{W}}\right) \otimes X_{\sigma}\right) \\
& =\sum_{\sigma \geq \pi}\left(\prod_{W \in \sigma} \Psi\left(X_{\pi_{W}}\right)\right) \otimes \Psi\left(X_{\sigma}\right) \\
& =\sum_{\sigma \geq \pi}\left(\prod_{W \in \sigma} Y_{\mathrm{Kr}\left(\pi_{W}\right)}\right) \otimes Y_{\mathrm{Kr}(\sigma)},
\end{aligned}
$$

with the relabeled-restrictions $\pi_{W}$ as in Notation 2.2. In the latter summation over $\sigma$ : when we put together the Kreweras complements of all the partitions $\pi_{W}$ with $W$ running in $\sigma$, what comes out is the relative Kreweras complement of $\pi$ in $\sigma$. Thus the conclusion for the right-hand side of (14.9) reads:

$$
\begin{equation*}
\left((\Psi \otimes \Psi) \circ \Delta_{\mathcal{T}}\right)\left(X_{\pi}\right)=\sum_{\sigma \geq \pi} Y_{\mathrm{Kr}_{\sigma}(\pi)} \otimes Y_{\mathrm{Kr}(\sigma)} . \tag{14.10}
\end{equation*}
$$

In order to reconcile the results of our calculations on the two sides of (14.9), we perform the change of variable $\rho:=\operatorname{Kr}_{\sigma}(\pi)$ on the right-hand side of (14.10). It fits perfectly to invoke here the considerations on relative Kreweras complements from [24, Lecture 18], and specifically Lemma 18.9 from that lecture, which tells us that:

$$
\left\{\begin{array}{l}
\text { if } \sigma \text { runs in the interval }\left[\pi, 1_{n}\right] \text { of } N C(n), \\
\text { then } \rho=\operatorname{Kr}_{\sigma}(\pi) \text { runs (bijectively) in the interval }\left[0_{n}, \operatorname{Kr}(\pi)\right] \text { of } N C(n), \\
\text { and one has the relation } \operatorname{Kr}(\sigma)=\operatorname{Kr}_{\operatorname{Kr}(\pi)}(\rho) .
\end{array}\right.
$$

The change of variable from $\sigma$ to $\rho$ thus transforms (14.10) into

$$
\left((\Psi \otimes \Psi) \circ \Delta_{\mathcal{T}}\right)\left(X_{\pi}\right)=\sum_{\rho \leq \operatorname{Kr}(\pi)} Y_{\rho} \otimes Y_{\mathrm{Kr}_{\mathrm{Kr}(\pi)}(\rho)} ;
$$

this brings us to precisely the same expression as we found when we processed the left-hand side of (14.9).
For (ii): we must check that $\varepsilon_{\text {Sym }} \circ \Psi=\varepsilon_{\mathcal{T}}$, where $\varepsilon_{\text {Sym }}$ and $\varepsilon_{\mathcal{T}}$ are the counits of Sym and of $\mathcal{T}$, respectively. Both $\varepsilon_{\text {Sym }} \circ \Psi$ and $\varepsilon_{\mathcal{T}}$ are unital algebra homomorphisms from $\mathcal{T}$ to $\mathbb{C}$, hence it suffices to check that they agree on every $X_{\pi}$ with $\pi \in \sqcup_{n=1}^{\infty} N C(n) \backslash\left\{1_{n}\right\}$. But for any such $\pi$ we have that

$$
\begin{equation*}
\left(\varepsilon_{\text {Sym }} \circ \Psi\right)\left(X_{\pi}\right)=0=\varepsilon_{\mathcal{T}}\left(X_{\pi}\right) . \tag{14.11}
\end{equation*}
$$

Indeed, the second equality (14.11) holds by the definition of $\varepsilon_{\mathcal{T}}$; while for the first equality (14.11) we write, for $\pi \in N C(n) \backslash\left\{1_{n}\right\}$ :

$$
\begin{aligned}
\pi \neq 1_{n} & \Rightarrow \operatorname{Kr}(\pi) \neq 0_{n} \Rightarrow \exists W_{o} \in \operatorname{Kr}(\pi) \text { with }\left|W_{o}\right| \geq 2 \text { and hence with } \varepsilon_{\mathrm{Sym}}\left(Y_{\left|W_{o}\right|}\right)=0 \\
& \Rightarrow\left(\varepsilon_{\mathrm{Sym}} \circ \Psi\right)\left(X_{\pi}\right)=\varepsilon_{\mathrm{Sym}}\left(Y_{K r(\pi)}\right)=\prod_{W \in \operatorname{Kr}(\pi)} \varepsilon_{\mathrm{Sym}}\left(Y_{|W|}\right)=0 .
\end{aligned}
$$

For (iii): since $\Psi$ is a unital algebra homomorphism, it will suffice to check that

$$
\begin{equation*}
\operatorname{deg}_{\mathrm{Sym}}\left(\Psi\left(X_{\pi}\right)\right)=\operatorname{deg}_{\mathcal{T}}\left(X_{\pi}\right), \quad \forall n \geq 1 \text { and } \pi \neq 1_{n} \text { in } N C(n), \tag{14.12}
\end{equation*}
$$

where $\operatorname{deg}_{\text {Sym }}$ and $\operatorname{deg}_{\mathcal{T}}$ denote the degree functions for Sym and $\mathcal{T}$, respectively. And indeed, direct computation yields that both sides of (14.12) are equal to $|\pi|-1$, where on the left-hand side we first write that $\left.\operatorname{deg}_{\operatorname{Sym}}\left(Y_{\mathrm{Kr}(\pi)}\right)\right)=n-|K r(\pi)|$, and then we invoke the known fact that $|\operatorname{Kr}(\pi)|=n+1-|\pi|$.

Corollary 14.7. Let $\Psi: \mathcal{T} \rightarrow$ Sym be as above, and consider the groups of characters $\mathbb{X}($ Sym $)$ and $\mathbb{X}(\mathcal{T})$ of the Hopf algebras Sym and $\mathcal{T}$.
$1^{\circ}$ One has an injective group homomorphism $\Psi^{*}: \mathbb{X}(S y m) \rightarrow \mathbb{X}(\mathcal{T})$ defined by

$$
\begin{equation*}
\Psi^{*}(\chi):=\chi \circ \Psi, \quad \chi \in \mathbb{X}(\text { Sym }) . \tag{14.13}
\end{equation*}
$$

$2^{o}$ Consider the identifications $\mathbb{X}(\mathrm{Sym})=\left\{\widehat{\chi}_{f} \mid f \in \mathcal{G}\right\}$ from Remark 14.3 and $\mathbb{X}(\mathcal{T})=$ $\left\{\chi_{g} \mid g \in \widetilde{\mathcal{G}}\right\}$ from Theorem 12.9. In terms of these identifications, the group homomorphism $\Psi^{*}$ is just the inclusion of $\mathcal{G}$ into $\widetilde{\mathcal{G}}$; that is, one has

$$
\begin{equation*}
\Psi^{*}\left(\widehat{\chi}_{f}\right)=\chi_{f}, \quad \forall f \in \mathcal{G} \tag{14.14}
\end{equation*}
$$

Proof. The property of $\Psi^{*}$ of being a group homomorphism is a general Hopf algebra fact and the injectivity of $\Psi^{*}$ is implied, in particular, by (14.14). We are thus left to fix an $f \in \mathcal{G}$ and to verify that the two characters $\chi_{f}, \widehat{\chi}_{f} \circ \Psi \in \mathbb{X}(\mathcal{T})$ are equal to each other. To that end, it suffices to also fix an $n \geq 1$ and a $\pi \in N C(n) \backslash\left\{1_{n}\right\}$, and to check that the two characters in question agree on the generator $X_{\pi}$ of $\mathcal{T}$. We know that

$$
\chi_{f}\left(X_{\pi}\right)=f\left(\pi, 1_{n}\right)=\prod_{W \in \operatorname{Kr}(\pi)} f\left(0_{|W|}, 1_{|W|}\right),
$$

where the second equality sign uses the fact that $f$ is multiplicative. On the other hand, we have

$$
\left(\widehat{\chi}_{f} \circ \Psi\right)\left(X_{\pi}\right)=\widehat{\chi}_{f}\left(Y_{\operatorname{Kr}(\pi)}\right)=\prod_{W \in \operatorname{Kr}(\pi)} \widehat{\chi}_{f}\left(Y_{|W|}\right)=\prod_{W \in \operatorname{Kr}(\pi)} f\left(0_{|W|}, 1_{|W|}\right),
$$

and this completes the required verification.

Remark 14.8. There was another subgroup of $\widetilde{\mathcal{G}}$ which played a significant role throughout this paper, namely the group $\widetilde{\mathcal{G}}_{\text {c-c }}$ of semi-multiplicative functions of cumulant-to-cumulant type. The group $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ can also be identified, in a natural way, as character group of a Hopf algebra $\mathcal{Z}$, where the latter Hopf algebra is some kind of "truncation of $\mathcal{T}$ to irreducible non-crossing partitions". Without going into details, we give here some highlights on what is $\mathcal{Z}$ and of how it comes that $\mathbb{X}(\mathcal{Z}) \approx \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$.

As an algebra, $\mathcal{Z}$ is just a commutative algebra of polynomials:

$$
\begin{equation*}
\mathcal{Z}:=\mathbb{C}\left[Z_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty}\left(N C_{\text {irr }}(n) \backslash\left\{1_{n}\right\}\right)\right], \tag{14.15}
\end{equation*}
$$

where for every $n \in \mathbb{N}$ we use the shorthand notation

$$
N C_{\mathrm{irr}}(n):=\{\pi \in N C(n) \mid \pi \text { is irreducible }\} .
$$

Analogously to how we went when we defined $\mathcal{T}$ in Section 12.1, we also put

$$
\begin{equation*}
Z_{1_{n}}:=1_{\mathcal{Z}}, \quad \forall n \geq 1, \tag{14.16}
\end{equation*}
$$

and we record the universality property enjoyed by $\mathcal{Z}$, which says:

$$
\left\{\begin{array}{l}
\text { If } \mathcal{A} \text { is a unital commutative algebra over } \mathbb{C} \text { and we are given }  \tag{14.17}\\
\text { elements }\left\{a_{\pi} \mid \pi \in \sqcup_{n=1}^{\infty} N C_{\text {irr }}(n)\right\} \text { in } \mathcal{A} \text {, with } a_{1_{n}}=1_{\mathcal{A}} \text { for all } n \geq 1, \\
\text { then there exists a unital algebra homomomorphism } \Phi: \mathcal{Z} \rightarrow \mathcal{A} \text {, uniquely } \\
\text { determined, such that } \Phi\left(Z_{\pi}\right)=a_{\pi} \text { for all } \pi \in \sqcup_{n=1}^{\infty} N C_{\text {irr }}(n) \text {. }
\end{array}\right.
$$

The universality property (14.17) yields in particular a recipe for how to construct characters of $\mathcal{Z}$ (i.e. unital algebra homomomorphisms from $\mathcal{Z}$ to $\mathbb{C}$ ). For every function $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ let $\check{\chi}_{g}: \mathcal{Z} \rightarrow \mathbb{C}$ be the character defined via universality and the requirement that

$$
\check{\chi}_{g}\left(Z_{\pi}\right)=g\left(\pi, 1_{n}\right) \text { for every } n \geq 1 \text { and } \pi \in N C_{\mathrm{irr}}(n) .
$$

It is easy to verify that in this way we get a bijective map

$$
\begin{equation*}
\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}} \ni g \mapsto \check{\chi}_{g} \in \mathbb{X}(\mathcal{Z}), \tag{14.18}
\end{equation*}
$$

where $\mathbb{X}(\mathcal{Z})$ is the set of all characters of $\mathcal{Z}$.
Now, in a nutshell, one has that:

$$
\left\{\begin{array}{c}
\mathcal{Z} \text { also carries a coalgebra structure, }  \tag{14.19}\\
\text { which makes } \mathbb{X}(\mathcal{Z}) \text { become a group under convolution, } \\
\text { and makes the bijection (14.18) become a group isomorphism. }
\end{array}\right.
$$

The statements made in (14.19) require a bunch of verifications which are pretty much a repeat of the arguments shown in connection to $\mathcal{T}$ in Sections 12.1 and 12.2 of the paper. We will leave these (not difficult) verifications as an exercise to the reader, and only provide here the definitions for the comultiplication, counit and grading on $\mathcal{Z}$.

Comultiplication: this is the unital algebra homomorphism $\Delta: \mathcal{Z} \rightarrow \mathcal{Z} \otimes \mathcal{Z}$ defined via the universality property (14.17) and the requirement that for every $n \geq 1$ and $\pi \in N C_{\mathrm{irr}}(n)$ we have

$$
\begin{equation*}
\Delta\left(Z_{\pi}\right)=\sum_{\substack{\sigma \in N C(n), \sigma \gg \pi}}\left(\prod_{W \in \sigma} Z_{\pi_{W}}\right) \otimes Z_{\sigma} \tag{14.20}
\end{equation*}
$$

where " $\gg$ " is in reference to the partial order from Notation 2.5.1. We take a moment here to emphasize the importance of having $\sigma \gg \pi$ (rather than a plain " $\sigma \geq \pi$ ") on the right-hand side of Equation (14.20): the condition $\sigma \gg \pi$ amounts precisely to the fact that $\pi_{W} \in N C_{\mathrm{irr}}(|W|)$ for every block $W \in \sigma$, which is crucial in order to be able to talk about the element $Z_{\pi_{W}} \in \mathcal{Z}$.

Counit: this is the character $\check{\epsilon}:=\check{\chi}_{e} \in \mathbb{X}(\mathcal{Z})$, where $e$ is the unit of $\widetilde{\mathcal{G}}_{c-c}$. Spelled explicitly, $\check{\epsilon}$ is the character defined via the requirement that it has $\check{\epsilon}\left(Z_{\pi}\right)=e\left(\pi, 1_{n}\right)=0$ for every $n \geq 1$ and $\pi \in N C_{\text {irr }}(n) \backslash\left\{1_{n}\right\}$.

Grading: this is obtained by postulating that every $Z_{\pi}$ has degree $|\pi|-1$, and hence that every monomial $Z_{\pi_{1}} \cdots Z_{\pi_{k}}$ has degree $\left|\pi_{1}\right|+\cdots+\left|\pi_{k}\right|-k$.

We conclude the discussion around $\mathcal{Z}$ by pointing out that one has a result analogous to the above Corollary 14.7, concerning the inclusion of groups $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}} \subseteq \widetilde{\mathcal{G}}$. More precisely, let $\Phi: \mathcal{T} \rightarrow \mathcal{Z}$ be the unital algebra homomorphism obtained by invoking the universality
property (12.3) of $\mathcal{T}$ in connection to the requirement that for every $\pi \in \sqcup_{n=1}^{\infty} N C(n)$ we have:

$$
\Phi\left(X_{\pi}\right)= \begin{cases}Z_{\pi}, & \text { if } \pi \text { is irreducible } \\ 0, & \text { if } \pi \text { is reducible }\end{cases}
$$

It turns out $\Phi$ also respects the coalgebras structures on $\mathcal{T}$ and $\mathcal{Z}$; this statement is analogous to the statement of the above Theorem 14.6, but has a much simpler (nearly immediate, in fact) proof. As a consequence of $\Phi$ being a Hopf algebra homomorphism, we get a group homomorphism $\Phi^{*}: \mathbb{X}(\mathcal{Z}) \rightarrow \mathbb{X}(\mathcal{T})$, defined by putting $\Phi^{*}(\check{\chi}):=\check{\chi} \circ \Phi$, for $\check{\chi} \in \mathbb{X}(\mathcal{Z})$. Directly from definitions it follows that for every $g \in \widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ one has

$$
\check{\chi}_{g} \circ \Phi=\chi_{g}, \quad \text { or in other words that } \Phi^{*}\left(\check{\chi}_{g}\right)=\chi_{g},
$$

where $\check{\chi}_{g} \in \mathbb{X}(\mathcal{Z})$ is as above, while $\chi_{g} \in \mathbb{X}(\mathcal{T})$ is picked from Theorem 12.9, Thus, when the canonical identifications $\widetilde{\mathcal{G}} \approx \mathbb{X}(\mathcal{T})$ and $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}} \approx \mathbb{X}(\mathcal{Z})$ are considered, $\Phi^{*}$ is just the inclusion of $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ into $\widetilde{\mathcal{G}}$.

## Appendix.

Consider the framework and notation used in (11.18) of Remark 11.10, where we specialize $x_{1}=\cdots=x_{n}=: x$, with $x \in \mathcal{M}$ picked such that $\varphi(x)=1$. This appendix shows the output of some computer calculations which check the difference of the quantities on the two sides of (11.18),

$$
\rho_{n}(x y, \ldots, x y)-\sum_{\pi \in N C(n)} \rho_{\pi}(x, \ldots, x) \cdot \rho_{\mathrm{Kr}(\pi)}(y, \ldots, y)=?
$$

for $5 \leq n \leq 8$. As mentioned in Remark 11.10, the above difference is equal to 0 for $n \leq 4$.
For every $n \geq 1$, we use the shorthand notation $\rho_{n}(x):=\rho_{n}(x, \ldots, x)$ and $\rho_{n}(y):=$ $\rho_{n}(y, \ldots, y)$.

Since $\rho_{1}(x)=\varphi(x)=1$ and $\rho_{1}(y)=\varphi(y)=1$, in the formulas listed below we have omitted the occurrence of the powers of $\rho_{1}(x)$ and $\rho_{1}(y)$. Putting in these powers would make the various products appearing there become homogeneous (for instance for $n=5$, the product $\rho_{2}(x) \rho_{2}(y)$ would become $\rho_{1}(x)^{3} \rho_{2}(x) \rho_{1}(y)^{3} \rho_{2}(y)$, homogeneous of degree 5 with respect to each of $x$ and $y$ ).

$$
\begin{aligned}
& \boldsymbol{n}=\text { 5. } \rho_{5}(x y, \ldots, x y)-\sum_{\pi \in N C(5)} \rho_{\pi}(x, \ldots, x) \rho_{\operatorname{Kr}(\pi)}(y, \ldots, y)=-\frac{1}{12} \rho_{2}(x) \rho_{2}(y) \text {. } \\
& \boldsymbol{n}=\text { 6. } \rho_{6}(x y, \ldots, x y)-\sum_{\pi \in N C(6)} \rho_{\pi}(x, \ldots, x) \rho_{\operatorname{Kr}(\pi)}(y, \ldots, y) \\
& =-\frac{1}{4} \rho_{2}(x)^{2} \rho_{2}(y)-\frac{1}{4} \rho_{2}(x) \rho_{2}(y)^{2}-\frac{1}{3} \rho_{2}(x) \rho_{3}(y)-\frac{1}{3} \rho_{3}(x) \rho_{2}(y) . \\
& \boldsymbol{n}=\text { 7. } \rho_{7}(x y, \ldots, x y)-\sum_{\pi \in N C(7)} \rho_{\pi}(x, \ldots, x) \rho_{\operatorname{Kr}(\pi)}(y, \ldots, y) \\
& =-\rho_{3}(x) \rho_{3}(y)-\frac{4}{3} \rho_{2}(x)^{2} \rho_{2}(y)^{2}+\frac{7}{180} \rho_{2}(x) \rho_{2}(y) \\
& -\frac{19}{12} \rho_{2}(x)^{2} \rho_{3}(y)-\frac{19}{12} \rho_{3}(x) \rho_{2}(y)^{2}-\frac{3}{4} \rho_{2}(x) \rho_{4}(y)-\frac{3}{4} \rho_{4}(x) \rho_{2}(y) \\
& -\frac{17}{12} \rho_{2}(x) \rho_{3}(x) \rho_{2}(y)-\frac{17}{12} \rho_{2}(x) \rho_{2}(y) \rho_{3}(y)-\frac{1}{6} \rho_{2}(x)^{3} \rho_{2}(y)-\frac{1}{6} \rho_{2}(x) \rho_{2}(y)^{3} .
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{n}=\text { 8. } & \rho_{8}(x y, \ldots, x y)-\sum_{\pi \in N C(8)} \rho_{\pi}(x, \ldots, x) \rho_{\operatorname{Kr}(\pi)}(y, \ldots, y) \\
= & +\frac{7}{30} \rho_{2}(x) \rho_{3}(y)+\frac{7}{30} \rho_{3}(x) \rho_{2}(y)+\frac{43}{180} \rho_{2}(x) \rho_{2}(y)^{2}+\frac{43}{180} \rho_{2}(x)^{2} \rho_{2}(y) \\
& -\frac{4}{3} \rho_{2}(x) \rho_{5}(y)-\frac{4}{3} \rho_{5}(x) \rho_{2}(y)-\frac{4}{3} \rho_{2}(x) \rho_{2}(y)^{2} \rho_{3}(y)-\frac{4}{3} \rho_{2}(x)^{2} \rho_{3}(x) \rho_{2}(y) \\
& -\frac{4}{3} \rho_{2}(x) \rho_{3}(y)^{2}-\frac{4}{3} \rho_{3}(x)^{2} \rho_{2}(y)-\frac{32}{3} \rho_{2}(x)^{2} \rho_{2}(y) \rho_{3}(y)-\frac{32}{3} \rho_{2}(x) \rho_{3}(x) \rho_{2}(y)^{2} \\
& -\frac{8}{3} \rho_{3}(x) \rho_{2}(y)^{3}-\frac{8}{3} \rho_{2}(x)^{3} \rho_{3}(y)-\frac{9}{2} \rho_{2}(x)^{2} \rho_{4}(y)-\frac{9}{2} \rho_{4}(x) \rho_{2}(y)^{2} \\
& -\frac{20}{3} \rho_{3}(x) \rho_{2}(y) \rho_{3}(y)-\frac{20}{3} \rho_{2}(x) \rho_{3}(x) \rho_{3}(y)-2 \rho_{2}(x)^{2} \rho_{2}(y)^{3}-2 \rho_{2}(x)^{3} \rho_{2}(y)^{2} \\
& -3 \rho_{2}(x) \rho_{2}(y) \rho_{4}(y)-3 \rho_{2}(x) \rho_{4}(x) \rho_{2}(y)-2 \rho_{3}(x) \rho_{4}(y)-2 \rho_{4}(x) \rho_{3}(y)
\end{aligned}
$$

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[^1]:    ${ }^{1}$ It would be possible to introduce and use here the notion of "non-crossing partition of $W$ ". After weighing the pros and cons of doing so, we decided to rather let $\pi_{W}$ be a partition in $N C(m)$, for $m=|W|$.

[^2]:    ${ }^{2} \mathrm{In} g_{\mathrm{fc}-\mathrm{m}}$, the subscript "fc-m" is a reminder that we are doing a transition from free cumulants to moments. Similar conventions will be used for other such special functions, e.g. " $g_{\mathrm{bc}-\mathrm{m}}$ " for the function in $\widetilde{\mathcal{G}_{c-m}}$ which encodes the transition from Boolean cumulants to moments, or " $g_{\mathrm{fc}-\mathrm{bc}}$ " for the function in $\widetilde{\mathcal{G}}_{\mathrm{c}-\mathrm{c}}$ which encodes the transition from free cumulants to Boolean cumulants.

[^3]:    ${ }^{3}$ Note that in the chain $c$ indicated in (13.15) the partition $\pi$ appears as $\pi^{(k)}$. We chose this way of denoting $c$ because it simplifies the write-up of the proof of the lemma.

