Bichromatic Perfect Matchings with Crossings *

Oswin Aichholzer[1], Stefan Felsner[2], Rosna Paul[1], Manfred Scheucher[2],
and Birgit Vogtenhuber[1]

[1] Institute of Software Technology, Graz University of Technology, Graz, Austria
{oaich,ropaul,bvogt,weinberger}@ist.tugraz.at
[2] Institute for Mathematics, Technical University of Berlin, Berlin, Germany
{felsner,scheucher}@math.tu-berlin.de

Abstract. We consider bichromatic point sets with n red and n blue points and study straight-line bichromatic perfect matchings on them. We show that every such point set in convex position admits a matching with at least $\frac{3n^2}{8} - \frac{n}{2} + c$ crossings, for some $-\frac{1}{2} \le c \le \frac{1}{8}$. This bound is tight since for any $k > \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$ there exist bichromatic point sets that do not admit any perfect matching with k crossings.

Keywords: Perfect matchings · Bichromatic point sets · Crossings

1 Introduction

Let $P = R \cup B$, $|R| = |B| = n$ be a point set in *general position*, that is, no three points of P are collinear. We refer to R and B as the set of red and blue points, respectively. A straight-line matching M on P where every point in R is uniquely matched to a point in B is called a *straight-line bichromatic perfect matching* (all matchings considered in this work are straight-line, so we will mostly omit this term). In this work, we study the existence of bichromatic perfect matchings on P with a fixed number k of crossings, where $0 \le k \le \binom{n}{2}$. It is folklore that any P of even size admits a crossing-free perfect matching. Perfect matchings with k crossings in the uncolored setting have been considered in [2]. There it is shown that for every $k \le \frac{n^2}{16} - O(n\sqrt{n})$, every point set of size $2n$ admits a perfect matching with exactly k crossings and that there exist point sets where every perfect matching has at most $\frac{5n^2}{18}$ crossings. As a direct consequence, there exist bichromatic point sets which do not admit bichromatic perfect matchings with k crossings for $k > \frac{5n^2}{18}$. For $2n$ (uncolored) points in convex position it was shown in [2] that they admit perfect matchings with k crossings for every k in the range from 0 to $\binom{n}{2}$.

For bichromatic point sets, this situation changes quite significantly. Consider a set P of $2n$ points in convex position (convex point set, for short) with an alternating coloring, that is, every second point along the convex hull is red (and the other points are blue). Moreover, let the number n of red (and blue) points be even. Then the number of crossings in a bichromatic perfect matching M on P is at most $\frac{n(n-2)}{2} = \binom{n}{2} - \frac{n}{2}$. The idea is as follows: Label the points of P as $p_0, p_1, \ldots, p_{2n-1}$ along the boundary of the convex hull. Then p_i cannot be matched to p_{i+n} since both points have the same color. Hence, for any edge e in M, the number of crossings of e is at most $n - 2$. As every crossing involves two edges, the number of crossings of M is at most $\frac{n(n-2)}{2} = \binom{n}{2} - \frac{n}{2}$. This bound is tight, since it is possible to construct a bichromatic perfect matching on P with exactly $\binom{n}{2} - \frac{n}{2}$ crossings as follows. For $0 \leq i \leq n - 1$, match the point p_i to the point p_{i+n+1}, when i is even. Otherwise, match p_i to p_{i+n-1}. Based on the above observations we state the following question.

Question 1. *For which values of k does every bichromatic convex point set $P = R \cup B$, $|R| = |B| = n$, admit a straight-line bichromatic perfect matching with exactly k crossings?*

The above example implies that if $k > \binom{n}{2} - \frac{n}{2}$, there exist bichromatic point sets with n red and n blue points that do not have any bichromatic perfect matching with k crossings. Thus, the answer to Question 1 can be true only for $k \leq \binom{n}{2} - \frac{n}{2}$. As a main result of this paper, we prove the following theorem.

Theorem 1. *For every n and for every $k > \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$, there exists a bichromatic convex point set with n red and n blue points that does not have a straight-line bichromatic perfect matching with k crossings.*

To show this, we study bichromatic convex point sets and matchings on them with the maximum number of crossings. In Section 2, we first determine the maximum number of crossings for certain bichromatic convex point sets, depending on their cardinality modulo 4. Then we prove that this number gives a tight lower bound on the maximum number of crossings in any bichromatic convex point set. We further show some positive results for Question 1 in Section 3.

Related work. A survey by Kano and Urrutia [6] gives an overview of various problems on bichromatic point sets, including matching problems. Crossing-free bichromatic perfect matchings have been studied from various perspectives such as their structure [5,8], linear transformation distance [1], and matchings compatible to each other [3,4]. Sharir and Welzl [9] proved that the number of crossing-free bichromatic perfect matchings on $2n$ points is at most $O(7.61^n)$. However, on bichromatic perfect matchings with crossings much less is known. Pach et al. [7] showed that every straight-line drawing of $K_{n,n}$ contains a crossing family of size at least $n^{1-o(1)}$, where a *crossing family* is a set of pairwise crossing edges. This implies that for any $P = R \cup B$, $|R| = |B| = n$, there exists a bichromatic perfect matching with at least $n^{2-o(1)}$ crossings.

2   Bichromatic Convex Point Sets

Let $\mathcal{C}_{n,n}$ be the collection of all bichromatic convex point sets $P = R \cup B$ with $|R| = |B| = n$. For a point set $P \in \mathcal{C}_{n,n}$, we label the points in P in clockwise direction along the convex hull as $p_0, p_1, \ldots, p_{2n-1}$ and refer to this as the *clockwise ordering*. We will consider all indices modulo $2n$. The number of crossings in a bichromatic perfect matching M_P on P is denoted by $\overline{\mathrm{cr}}(M_P)$. If M_P has the maximum number of crossings among all such matchings on P, then M_P is called a *max-crossing* matching on P. Among all max-crossing matchings for all $P \in \mathcal{C}_{n,n}$, we are interested in matchings with the minimum number of crossings. We call such a matching a *min-max-crossing matching* of $\mathcal{C}_{n,n}$. From now on, we mostly refer to bichromatic perfect matchings just as *matchings*.

A *block* of $P \in \mathcal{C}_{n,n}$ is a maximal set of consecutive points of P of the same color. (If $R_1 = \{p_a, p_{a+1}, \ldots, p_{a+s}\}$ is a red block then p_{a-1} and p_{a+s+1} are blue.) Collecting the blocks of P in clockwise order yields a cyclically ordered partition $(R_1, B_1, R_2, B_2, \ldots, R_s, B_s)$ of P. The coloring of P is called $2s$–*block* if it induces s red and s blue blocks. In particular a $2n$–block coloring is alternating. A $2s$–block coloring where all blocks have the same cardinality is *balanced*; see e.g. Fig. 4(a). A balanced $2s$–block coloring only exists if s divides n. To overcome this restriction, we also consider $2s$–block colorings in which block sizes differ by at most 1 as *balanced $2s$–block* colorings. Note that for $s = 2$ and any given value of n, there is a unique balanced $2s$–block coloring (up to symmetry).

Theorem 2. *Let $P \in \mathcal{C}_{n,n}$ and $\mathrm{M}_P^\vee$ be a max-crossing matching on P. Then*

$$\overline{\mathrm{cr}}(\mathrm{M}_P^\vee) \geq \begin{cases} \frac{3n^2}{8} - \frac{n}{2} & \text{if } n \equiv 0 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} & \text{if } n \equiv 1 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} - \frac{1}{2} & \text{if } n \equiv 2 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Moreover, equality holds if P has a balanced 4–block coloring. In this case, $\mathrm{M}_P^\vee$ is a min-max-crossing matching of $\mathcal{C}_{n,n}$.

Theorem 1 is implied by Theorem 2: consider any point set with balanced 4–block coloring. The bound for the crossings in a max-crossing matching of this point set implies Theorem 1. Theorem 2 will follow directly from Lemma 2 and Lemma 3, which are stated and shown in the next sections.

2.1   Max-Crossing Matching of a Balanced 4–Block Coloring

In this section, we determine the number of crossings in any max-crossing matching of a point set with balanced 4–block coloring. A *crossing family* of a point set P is a set of edges spanned by points from P that pairwise cross.

Lemma 1. *Let $P \in \mathcal{C}_{n,n}$ have a 4–block coloring with blocks R_1, B_1, R_2, B_2 and let $\mathrm{M}_P^\vee$ be a max-crossing matching on P. Then for each block $X \in \{R_1, R_2, B_1, B_2\}$, the edges emanating from X form a crossing family.*

Proof. Consider a block X of P and assume w.l.o.g. that $X = \{p_1, p_2, \ldots, p_x\}$. If in a matching M on P, there are two non-crossing edges with endpoints in X, then there also exist two such non-crossing edges with adjacent endpoints in X, i.e., M contains edges (p_i, p_k) and (p_{i+1}, p_j) for some $1 \leq i < x \leq j < k \leq 2n$. Let M' be obtained from M by replacing the two edges by the two crossing edges (p_i, p_j) and (p_{i+1}, p_k) and note that $\overline{cr}(M') = \overline{cr}(M) + 1$. □

Consider a max-crossing matching $M_P^\vee$ on P. Lemma 1 implies that there is some $a \geq 0$ such that in $M_P^\vee$, the first a points of R_1 are matched to points in B_1 while the last $|R_1| - a$ points of R_1 are matched to points in B_2. Analogously, the first $|B_1| - a$ points of B_1 are matched to points in R_2. Since $|R_2| = n - |R_1|$ and the $|B_1| - a$ points of R_2 are matched to B_1, the first $n - |R_1| - |B_1| + a$ points of R_2 are matched to points in B_2; see Fig. 1. Hence, to get a max-crossing matching on P, it is sufficient to determine the optimal value of a.
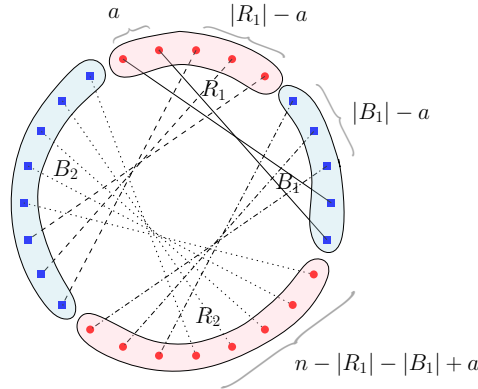

Fig. 1: Structure of a max-crossing matching on a set P with 4–block coloring.

By determining the optimum value of a, we next construct a max-crossing matching on any set P with balanced 4–block coloring and compute its exact crossing number.

Lemma 2. *Let $P \in \mathcal{C}_{n,n}$ have a balanced 4–block coloring and let $M_P^\vee$ be a max-crossing matching on P. Then*

$$\overline{cr}(M_P^\vee) = \begin{cases} \frac{3n^2}{8} - \frac{n}{2} & \text{if } n \equiv 0 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} & \text{if } n \equiv 1 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} - \frac{1}{2} & \text{if } n \equiv 2 \mod 4 \\ \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8} & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Proof. Let $P \in \mathcal{C}_{n,n}$ have a balanced 4–block coloring with blocks R_1, B_1, R_2, B_2, labeled such that $|R_1|, |B_1| \leq \frac{n}{2}$. Let $r_1 = |R_1| = \lfloor \frac{n}{2} \rfloor$ and $b_1 = |B_1| = \lfloor \frac{n}{2} \rfloor$.

Let M_P be a matching on P such that the first x points of the set R_1 are matched to the last x points of B_1, as a crossing family, for some $x \in \mathbb{N}_0$. The number of pairs of non-crossing edges in M_P is obtained by $(r_1 - x)(b_1 - x) + x(n - r_1 - b_1 + x) = r_1 b_1 - 2xb_1 - 2xr_1 + nx + 2x^2$. To determine the value $x \in \mathbb{N}_0$ that gives the maximum number of crossings, we first calculate the value $x^* \in \mathbb{R}$ for which $f(x) = (n - 2r_1 - 2b_1)x + 2x^2 + r_1 b_1$ attains its minimum. This is achieved by $x^* = \frac{1}{2}(r_1 + b_1 - \frac{n}{2})$. Note that x^* might not be in $\mathbb{N}_0$. Since f is a quadratic function, its minimum over all $x \in \mathbb{N}_0$ is reached for $x = \lfloor x^* \rceil$, where $\lfloor x^* \rceil$ denotes the closest integer of $x^* \in \mathbb{R}$. Then the max-crossing matching $M_P^\vee$ on P has $\overline{\mathrm{cr}}(M_P^\vee) = \binom{n}{2} - (n - 2r_1 - 2b_1)x - 2x^2 - r_1 b_1$ many crossings, where $x = \lfloor \frac{r_1 + b_1}{2} - \frac{n}{4} \rceil = \lfloor \lfloor \frac{n}{2} \rfloor - \frac{n}{4} \rceil$.

Note that the blocks in P may differ in size by 1 and that also the rounding for obtaining x depends on the value of $n \bmod 4$. To account for this, we evaluate each case separately.

Case 1: Let $n \equiv 0 \bmod 4$. Then there exists an integer m such that $n = 4m$. Then $r_1 = b_1 = 2m$ and $x = \lfloor \frac{4m}{2} - \frac{4m}{4} \rceil = m$. Hence the number of crossings of $M_P^\vee$ is $\overline{\mathrm{cr}}(M_P^\vee) = \binom{4m}{2} - (4m - 4(2m))m - 2m^2 - (2m)^2 = 6m^2 - 2m$. Replacing m by $\frac{n}{4}$ gives $\overline{\mathrm{cr}}(M_P^\vee) = \frac{3n^2}{8} - \frac{n}{2}$.

Case 2: Let $n \equiv 1 \bmod 4$. Then there exists an integer m such that $n = 4m+1$. Then $r_1 = b_1 = 2m$ and $x = \lfloor \frac{4m}{2} - \frac{4m+1}{4} \rceil = m$. Hence the number of crossings of $M_P^\vee$ is $\overline{\mathrm{cr}}(M_P^\vee) = \binom{4m+1}{2} - (4m + 1 - 4(2m))m - 2m^2 - (2m)^2 = 6m^2 + m$. Replacing m by $\frac{n-1}{4}$ gives $\overline{\mathrm{cr}}(M_P^\vee) = \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$.

Case 3: Let $n \equiv 2 \bmod 4$. Then there exists an integer m such that $n = 4m+2$. Then $r_1 = b_1 = 2m + 1$ and $x = \lfloor \frac{4m+2}{2} - \frac{4m+2}{4} \rceil = m$. Hence the number of crossings of $M_P^\vee$ is $\overline{\mathrm{cr}}(M_P^\vee) = \binom{4m+2}{2} - (4m + 2 - 4(2m + 1))m - 2m^2 - (2m + 1)^2 = 6m^2 + 4m$. Replacing m by $\frac{n-2}{4}$ gives $\overline{\mathrm{cr}}(M_P^\vee) = \frac{3n^2}{8} - \frac{n}{2} - \frac{1}{2}$.

Case 4: Let $n \equiv 3 \bmod 4$. Then there exists an integer m such that $n = 4m+3$. Then $r_1 = b_1 = 2m + 1$ and $x = \lfloor \frac{4m+2}{2} - \frac{4m+3}{4} \rceil = m$. Hence the number of crossings of $M_P^\vee$ is $\overline{\mathrm{cr}}(M_P^\vee) = \binom{4m+3}{2} - (4m + 3 - 4(2m + 1))m - 2m^2 - (2m + 1)^2 = 6m^2 + 7m + 2$. Replacing m by $\frac{n-2}{4}$ gives $\overline{\mathrm{cr}}(M_P^\vee) = \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$.

Altogether this completes the proof of the lemma. $\qquad\square$

2.2   Min-Max-Crossing Matching for All Colorings

In the following, we show that the maximum number of crossings of a bichromatic matching on $P \in \mathcal{C}_{n,n}$ is minimized by sets with balanced 4–block coloring. Let $P \in \mathcal{C}_{n,n}$. For any point $v \in P$, the point $w \in P$ is called the antipodal pair of v, if the line through v and w partitions P into two equal sized halves (antipodals exist because the number of points is even). If antipodal pairs v and w are of the same color, then they are *monochromatic antipodal* pairs (in short *m-antipodal* pairs) and if they have different colors then they are *bichromatic antipodal* pairs (in short *b-antipodal* pairs).

Lemma 3. *Let $P, Q \in \mathcal{C}_{n,n}$, where Q has a balanced 4–block coloring and let $\mathrm{M}_P^\vee, \mathrm{M}_Q^\vee$ be max-crossing matchings on P and Q, respectively. Then $\overline{\mathrm{cr}}(\mathrm{M}_P^\vee) \geq \overline{\mathrm{cr}}(\mathrm{M}_Q^\vee)$. That is, $\mathrm{M}_Q^\vee$ is a min-max-crossing matching of the set $\mathcal{C}_{n,n}$.*

To prove Lemma 3, we make use of a variant of the classic ham sandwich theorem [10]. A full proof can be found in Appendix A.

Proof (sketch). We define a matching M_P on P in three steps and then compare its crossings with those of $\mathrm{M}_Q^\vee$, where we will distinguish two cases.

Step 1: Let S be the point set obtained by removing all the b-antipodal pairs from P. If S is empty then all the points in P are b-antipodal pairs. Thus P admits a crossing family of size n (which is a perfect matching) that has more crossings than $\mathrm{M}_Q^\vee$. Hence we may assume that S is non-empty.

Step 2: Partition the set S into four groups as follows. First, we arbitrarily partition S into two consecutive sets S_L and S_R of equal size and note that each part contains half of the blue and half of the red points. Using the ham sandwich theorem, partition S_L into $S_{L,1}$ and $S_{L,2}$ such that each of them has an equal number of red and blue points. Due to the symmetry of S_L and S_R, this partition can be duplicated on $S_{R,1}$. Depending on the ham sandwich cut, $S_{L,1}$, $S_{L,2}$, $S_{R,1}$, and $S_{R,2}$ form four or six bundles of consecutive points along the convex hull. If we have only four bundles, we are done with the partition. So assume that we have six bundles. Then one partition in the part S_R, say $S_{R,2}$, is split into $S_{R,2a}$ and $S_{R,2b}$ by $S_{R,1}$ along the convex hull; see Fig. 2 (left). A similar splitting occurs for S_L. As the points in $S_{R,2b}$ and $S_{L,2a}$, and $S_{R,2a}$ and $S_{L,2b}$ are m-antipodal pairs, the composition of points in $S_{R,2}$ is same as in $S_{L,2a} \cup S_{R,2a}$ and the composition of points in $S_{L,2}$ is same as $S_{L,2b} \cup S_{R,2b}$. That is, if we have four or six bundles, they can always be (re)assembled into four cyclically connected groups $S_{RT}, S_{RB}, S_{LB}, S_{LT}$ of S such that each of these groups has the same number of red and blue points; see again Fig. 2. Moreover, each group is antipodal to another group: S_{RT} is antipodal to S_{LB} and S_{LT} is antipodal to S_{RB}. We call these pairs of groups *matching-pair* groups.

Step 3: Add all the removed b-antipodal pairs back to S to get P. The partition of S induces a partition $P_{RT}, P_{RB}, P_{LB}, P_{LT}$ in P. Note that the number of red (blue) points of P_{RT} and the number of blue (red) points of P_{LB} are equal. The same holds for P_{RB} and P_{LT}. Define the matching M_P on P as follows. For any matching-pair group $(X, Y) \in \{(P_{RT}, P_{LB}), (P_{RB}, P_{LT})\}$, the points in X are matched to points in Y such that any two of the matching edges emanating from the points of the same color on X cross each other.

Assuming that S is non-empty, there are two possible structures for S, depending on whether $|S_L|/2$ is even or odd. We consider them as two separate cases; see Fig. 3. Case 1: All groups in the partition of S have the same number m of red and blue points. Case 2: Each group in one matching-pair group has m red (and blue) points and each group in the other matching-pair has $m + 1$ red (and blue) points.
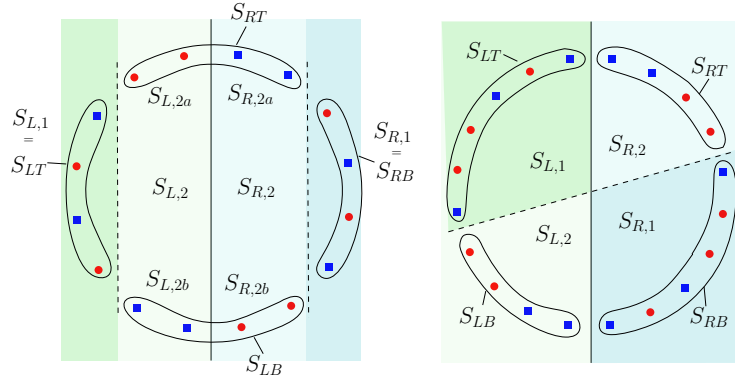
Fig. 2: A bichromatic point set S with 16 points (left) and 20 points (right). In both cases, dotted lines represent the partition of S_L obtained by the ham sandwich theorem, and its mirror on S_R.
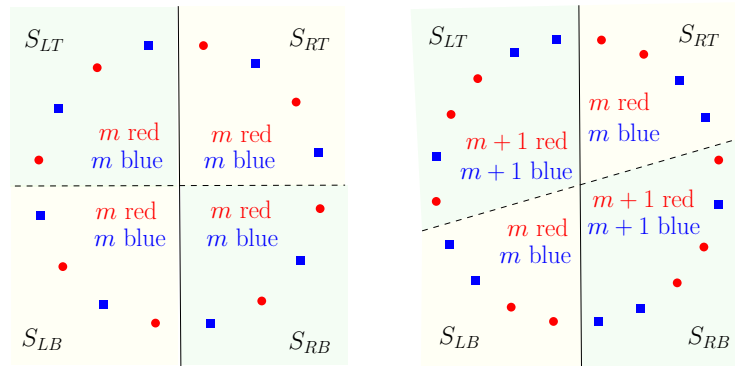

Fig. 3: Possible distribution of red and blue points between the groups of S.

For Case 1, the number of crossings in M_P is given by $\overline{\mathrm{cr}}(M_P) \geq \binom{m+x_1}{2} + \binom{m+x_2}{2} + \binom{m+y_1}{2} + \binom{m+y_2}{2} + (2m+x_1+x_2)(2m+y_1+y_2) + x_1x_2 + y_1y_2$, where x_1, x_2 are the number of b-antipodal points in P_{RT} of the colors red and blue, respectively and y_1, y_2 are the number of b-antipodal points in P_{LT} of the colors red and blue. These b-antipodal pairs always cross in M_P and contribute $x_1x_2 + y_1y_2$ crossings in M_P. Then for a balanced 4–block coloring $Q \in \mathcal{C}_{n,n}$ we have, $\overline{\mathrm{cr}}(\mathrm{M}_Q^\vee) = \frac{3n^2}{8} - \frac{n}{2} + c$, where $n = 4m + x_1 + x_2 + y_1 + y_2$ and $c \in \{\frac{1}{8}, -\frac{1}{2}, 0\}$ by Lemma 2. Comparing the number of crossings in M_P and $\mathrm{M}_Q^\vee$ gives $\overline{\mathrm{cr}}(M_P) - \overline{\mathrm{cr}}(\mathrm{M}_Q^\vee) \geq \frac{(x_1y_1+x_2y_1+x_1y_2+x_2y_2)}{4} + \frac{(x_1^2+x_2^2)}{8} + \frac{(y_1^2+y_2^2)}{8} + \frac{(x_1x_2+y_1y_2)}{4} - c$. As $c \leq \frac{1}{8}$ and since S is a proper subset of P, $x_1 + x_2 + y_1 + y_2 \geq 1$. Thus $\overline{\mathrm{cr}}(M_P) \geq \overline{\mathrm{cr}}(\mathrm{M}_Q^\vee)$.

The reasoning for Case 2 is similar to the one for Case 1 sketched above. □

As mentioned, Theorem 2 follows directly from Lemma 2 and Lemma 3. We remark that balanced 4–block colorings are not the only colorings that admit min-max-crossing matchings. For example, for $n \equiv 0 \mod 4$, a 4-block coloring with block sizes $\frac{n}{2} + 1, \frac{n}{2}, \frac{n}{2} - 1, \frac{n}{2}$ always induces the same number of crossings as the according balanced 4–block coloring; see Fig. 4.
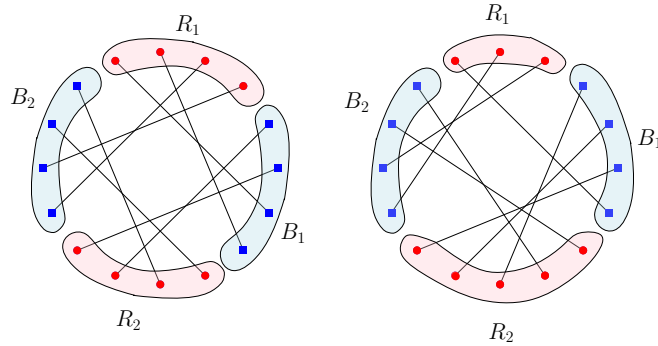

Fig. 4: A balanced 4–block coloring (left) and a slightly unbalanced 4–block coloring (right) on 16 points and max-crossing matchings on them, each with 20 crossings.

3 Further Results

We have shown that for any n and any $k > \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$ there exist point sets $P \in \mathcal{C}_{n,n}$ that do not admit any bichromatic perfect matching with k crossings. It is natural to ask what happens in the range $[1, \frac{3n^2}{8}]$. By straight-forward calculations, any $P \in \mathcal{C}_{n,n}$ with $2n$–block coloring cannot have a bichromatic perfect matching with k crossings for $k \in \{1, 2\}$. Using computers (with SAT framework) we obtained that every $P \in \mathcal{C}_{7,7}$ admits bichromatic perfect matchings with k crossings for any $k \in \{0, 1, \ldots, 15\} \setminus \{1, 2\}$. Based on this, we can show the following proposition. Its proof is deferred to Appendix B.

Proposition 1. *For $n \geq 7$, every $P \in \mathcal{C}_{n,n}$ admits bichromatic perfect matchings with k crossings for any $k \in \{0, 1, \ldots, \frac{15(n-6)}{7}\} \setminus \{1, 2\}$.*

For bichromatic point sets in general (non-convex) position with n red and n blue points, computer assisted results shows that for $n = 6$, every such set admits bichromatic perfect matchings with k crossings for $k \in \{0, 3, 4\}$. With a similar proof as for Proposition 1, it follows that every bichromatic point set with n red and n blue points admits bichromatic perfect matchings with k crossings for $k \in \{0, 1, \ldots, \frac{n}{3}\} \setminus \{1, 2\}$. In ongoing work, we study the range $k \in [\frac{15(n-6)}{7}, \frac{3n^2}{8}]$ for bichromatic convex point sets and the range $k \in [\frac{n}{3}, \frac{5n^2}{18}]$ for bichromatic point sets in general position.

4 Conclusion

We considered max-crossing matchings of bichromatic convex point sets in the plane with n red and n blue points. We gave the exact number of crossings in max-crossing matchings of point sets with balanced 4–block coloring and showed that these matchings are min-max-crossing matchings of bichromatic convex point sets. This result implies a negative answer to Question 1 on the existence of matchings with k crossings in the convex case for $k > \frac{3n^2}{8} - \frac{n}{2} + \frac{1}{8}$. We further answered the question for $k \leq \frac{15(n-6)}{7}$. From a computational point of view, an interesting open question is the following:

Question 2. *Given a bichromatic (convex) point set P and an integer k, what is the computational complexity of deciding whether there is a matching with exactly (or at least) k crossings?*

References

1. Aichholzer, O., Barba, L., Hackl, T., Pilz, A., Vogtenhuber, B.: Linear transformation distance for bichromatic matchings. Computational Geometry **68**, 77–88 (2018). https://doi.org/10.1016/j.comgeo.2017.05.003
2. Aichholzer, O., Fabila-Monroy, R., Kindermann, P., Parada, I., Paul, R., Perz, D., Schnider, P., Vogtenhuber, B.: Perfect matchings with crossings. In: Combinatorial Algorithms (IWOCA 2022). LNCS, vol. 13270, pp. 46–59. Springer (2022). https://doi.org/10.1007/978-3-031-06678-8_4
3. Aichholzer, O., Hurtado, F., Vogtenhuber, B.: Compatible matchings for bichromatic plane straight-line graphs. Proceedings of EuroCG'12 pp. 257–260 (2012), https://www.eurocg.org/2012/booklet.pdf
4. Aloupis, G., Barba, L., Langerman, S., Souvaine, D.L.: Bichromatic compatible matchings. Computational Geometry **48**(8), 622–633 (2015). https://doi.org/10.1016/j.comgeo.2014.08.009
5. Asinowski, A., Miltzow, T., Rote, G.: Quasi-parallel segments and characterization of unique bichromatic matchings. J. Comput. Geom. **6**(1), 185–219 (2015). https://doi.org/10.20382/jocg.v6i1a8, https://doi.org/10.20382/jocg.v6i1a8
6. Kano, M., Urrutia, J.: Discrete geometry on colored point sets in the plane — a survey. Graphs and Combinatorics **37**(1), 1–53 (Jan 2021). https://doi.org/10.1007/s00373-020-02210-8, https://doi.org/10.1007/s00373-020-02210-8
7. Pach, J., Rubin, N., Tardos, G.: Planar point sets determine many pairwise crossing segments. Advances in Mathematics **386**, 107779 (2021). https://doi.org/10.1016/j.aim.2021.107779
8. Savić, M., Stojaković, M.: Structural properties of bichromatic non-crossing matchings. Applied Mathematics and Computation **415**, 126695 (2022)
9. Sharir, M., Welzl, E.: On the number of crossing-free matchings, cycles, and partitions. SIAM Journal on Computing **36**(3), 695–720 (2006). https://doi.org/10.1137/050636036
10. Tóth, C.D., O'Rourke, J., Goodman, J.E.: Handbook of Discrete and Computational Geometry. CRC press, third edn. (2017). https://doi.org/10.1201/9781315119601

A   Proof of Lemma 3

Before proving Lemma 3, we first show that the max-crossing matching of a balanced 4–block coloring is a min-max-crossing matching for the set of all $P \in \mathcal{C}_{n,n}$ with a 4–block coloring.

Lemma 4. *Let $Q \in \mathcal{C}_{n,n}$ have a balanced 4–block coloring and let $\mathrm{M}_Q^\vee$ be a max-crossing matching on Q. Then $\mathrm{M}_Q^\vee$ is a min-max-crossing matching for all 4–block colored point sets of size $2n$.*

Proof. Let $P \in \mathcal{C}_{n,n}$ have a 4–block coloring with blocks R_1, B_1, R_2, B_2, labeled such that $|R_1|, |B_1| \leq \frac{n}{2}$. Let $r_1 = |R_1|$ and $b_1 = |B_1|$. As mentioned in the proof of Lemma 2, the function $h(r_1, b_1) = (n - 2r_1 - 2b_1)x + 2x^2 + r_1 b_1$ with $x = \max\{\left\lfloor \frac{1}{2}(r_1 + b_1 - \frac{n}{2})\right\rceil, 0\}$[3] gives the number of non-crossing edge pairs in a max-crossing matching of P. To determine a point set with 4–block coloring such that its max-crossing matchings minimizes the number of crossings among all max-crossing matchings of 4–block colored point sets, we need to maximize $h(r_1, b_1)$ over its domain, that is, over $r_1, b_1 \leq \frac{n}{2}$.

For that, we separately consider the cases when $x = \frac{1}{2}(r_1 + b_1 - \frac{n}{2}) + q > 0$, where q is the smallest real number required to make x an integer, and $x = 0$. The value q is the smallest real number required to make x an integer. Note that q depends on the parity of $r_1 + b_1$ and on the value of $n \mod 4$ (see Table 1).

$r_1 + b_1$	$n \equiv 0 \mod 4$	$n \equiv 1 \mod 4$	$n \equiv 2 \mod 4$	$n \equiv 3 \mod 4$
even	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
odd	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$

Table 1: Values of q for Lemma 4.

By substituting the value $x = \frac{r_1 + b_1}{2} - \frac{n}{4} + q$ in the function $h(r_1, b_1)$, we obtain $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + 2q^2$. The critical points of this function are obtained by setting its partial derivatives with respect to r_1 and b_1 to zero. This gives exactly one critical point at $(\frac{n}{2}, \frac{n}{2})$. The second partial derivatives show that this point is a local maximum of $h(r_1, b_1)$. But $\frac{n}{2}$ might be a real number or not have the correct parity for the requirement. Hence it might not be give valid maximum w.r.t. r_1, b_1, n. Thus, we have to separately consider the different parities of $r_1 + b_1$ in combination with the different values of $n \mod 4$. For that, we consider the following cases. Note that as $h(r_1, b_1)$ is symmetric in r_1 and b_1, w.l.o.g. we can assume that $r_1 \leq b_1$.

---

[3] Here we consider the maximum among $\left\lfloor \frac{1}{2}(r_1 + b_1 - \frac{n}{2})\right\rceil$ and zero to make sure that $x \geq 0$.

Case 1: If $n \equiv 0 \mod 4$:

Then $n = 4m$ for some integer m. In this case, the critical point $r_1 = b_1 = \frac{n}{2} = 2m$, is an integer. In order to find a valid maximum, we also need to consider the parity of $r_1 + b_1$. This leads to the following cases.

If $r_1 + b_1$ is even: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8}$. Then critical point $r_1 = b_1 = 2m$ gives the maximum of $h(r_1, b_1) = \frac{n^2}{8}$.

If $r_1 + b_1$ is odd: Here, the above critical point is not valid as the sum is not odd. In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + \frac{1}{2}$. The maximum is at $r_1 = 2m$ and $b_1 = 2m + 1$ and its value is $h(r_1, b_1) = \frac{n^2}{8}$.

Case 2: If $n \equiv 1 \mod 4$:

Then $n = 4m + 1$ for some integer m. In this case the critical point, $r_1 = b_1 = \frac{n}{2}$, does not give an integer value. In order to find the maximum, we also need to consider the parity of $r_1 + b_1$. This leads to the following cases.

If $r_1 + b_1$ is even: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + \frac{1}{8}$. Then the maximum is at $r_1 = b_1 = 2m$ and has the value $h(r_1, b_1) = \frac{n^2}{8} - \frac{1}{8}$.

If $r_1 + b_1$ is odd: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + \frac{1}{8}$. The maximum is obtained at $r_1 = 2m$ and $b_1 = 2m + 1$ and has the value $h(r_1, b_1) = \frac{n^2}{8} - \frac{1}{8}$.

Case 3: If $n \equiv 2 \mod 4$:

Then $n = 4m + 2$ for some integer m. In this case the critical point, $r_1 = b_1 = \frac{n}{2}$, is an integer value. However, for the maximum, we also need to consider the parity of $r_1 + b_1$. This leads to the following cases.

If $r_1 + b_1$ is even: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + \frac{1}{2}$. Then $r_1 = b_1 = 2m + 1$ gives a valid point and the resulting maximum value is $h(r_1, b_1) = \frac{n^2}{8} + \frac{1}{2}$.

If $r_1 + b_1$ is odd: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8}$. The maximum is obtained at $r_1 = 2m$ and $b_1 = 2m + 1$ and has the value $h(r_1, b_1) = \frac{n^2}{8} - \frac{1}{2}$.

Case 4: If $n \equiv 3 \mod 4$:

Then $n = 4m + 3$ for some integer m. In this case the critical point, $r_1 = b_1 = \frac{n}{2}$, does not have integer values. In order to find the points reaching the maximum, we again also consider the parity of $r_1 + b_1$. This leads to the following cases.

If $r_1 + b_1$ is even: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + \frac{1}{8}$. Then maximum is at $r_1 = b_1 = 2m + 2$ and has the value $h(r_1, b_1) = \frac{n^2}{8} - \frac{1}{8}$.

If $r_1 + b_1$ is odd: In this case, $h(r_1, b_1) = \frac{nb_1 + nr_1 - b_1^2 - r_1^2}{2} - \frac{n^2}{8} + \frac{1}{8}$. Then maximum is obtained at $r_1 = 2m + 1$ and $b_1 = 2m + 2$ and has the value $h(r_1, b_1) = \frac{n^2}{8} - \frac{1}{8}$.

In all the above cases, any point set P with balanced 4–block coloring gives a min-max-crossing matching.

The only remaining case is to check what happens if when $x = 0$. Here, $h(r_1, b_1) = r_1 b_1$. We have to maximize this function under the restriction that $r_1 + b_1 < \frac{n}{2}$ and $r_1, b_1 \geq 0$. Thus we get $r_1 = b_1 = \frac{n}{4}$, which again might not be an integer or not have the right parity of $r_1 + b_1$. However, doing an analogous case distinction as above, the maximum number of non-crossing edges always is less than and equal to $m^2 + am$, where a is a fixed integer depending on the case. This is relevantly smaller than the number of non-crossing edges in the maxima obtained above (by the balanced 4–block coloring), which is given by $2m^2 + cm + d$, for some integers c and d. Hence no 4–block colored point set P with $r_1 + b_1 \leq \frac{n}{2}$ minimizes the number of non-crossing edges in $M_P^{\vee}$ over all point sets with 4–block coloring. Altogether, this implies that the number of crossings in a max-crossing matching is minimized on a balanced 4–block coloring. □

Lemma 3. *Let $P, Q \in \mathcal{C}_{n,n}$, where Q has a balanced 4–block coloring and let $M_P^{\vee}, M_Q^{\vee}$ be max-crossing matchings on P and Q, respectively. Then $\overline{\mathrm{cr}}(M_P^{\vee}) \geq \overline{\mathrm{cr}}(M_Q^{\vee})$. That is, $M_Q^{\vee}$ is a min-max-crossing matching of the set $\mathcal{C}_{n,n}$.*

Proof. We define a matching M_P on P in three steps and then compare its crossings with those of $M_Q^{\vee}$, where we distinguish two cases.

Step 1: Let S be the subset of P obtained by removing all the b-antipodal pairs from P. Then S is also a bichromatic convex point set with an equal number of red and blue points. Since each point in S has an antipodal pair of the same color, the number of red (blue) points must be even.

If S is empty then all the points in P are b-antipodal pairs. Thus P admits a crossing family of size n. Then clearly this crossing family (which is a perfect matching) has more crossings than $M_Q^{\vee}$.

Step 2: Partition the set S into four groups $S_{RT}, S_{RB}, S_{LB}, S_{LT}$ as follows. First, we partition the points of S into two sets S_L and S_R of equal size. Note that each of these sets has an equal number of red and blue points. Using the ham sandwich theorem, we partition S_L into $S_{L,1}$ and $S_{L,2}$ such that $S_{L,1}$ has the same number of red and blue points, and also $S_{L,2}$ has the same number of red and blue points. Note that if the number of red (blue) points in S_L is odd, then the number on red (blue) points in $S_{L,1}$ and $S_{L,2}$ will differ by one. Due to the symmetry of S_L and S_R we have a corresponding partition $S_{R,1}$ and $S_{R,2}$ of S_R. Depending on the ham sandwich cut, $S_{L,1}$, $S_{L,2}$, $S_{R,1}$, and $S_{R,2}$ form four or six bundles of consecutive points along the convex hull. If we have only four bundles, we are done with the partition. So assume that we have six bundles. Then one partition in the part S_R, say $S_{R,2}$, is split into $S_{R,2a}$ and $S_{R,2b}$ by $S_{R,1}$ along the convex hull; see Fig. 2 (left). A similar splitting occurs for S_L. As the points in $S_{R,2b}$ and $S_{L,2a}$, and $S_{R,2a}$ and $S_{L,2b}$ are m-antipodal pairs, the composition of points in $S_{R,2}$ is same as in $S_{L,2a} \cup S_{R,2a}$ and the composition of points in $S_{L,2}$ is same as $S_{L,2b} \cup S_{R,2b}$. That is, if we have four or six bundles, they can always be (re)assembled into four cyclically connected groups $S_{RT}, S_{RB}, S_{LB}, S_{LT}$ of S such that each of these groups has the same number of red and blue points; see again Fig. 2. Moreover, each group is antipodal to another group:

S_{RT} is antipodal to S_{LB} and S_{LT} is antipodal to S_{RB}. We call these pairs of groups *matching-pair* groups.

If $S = P$, sort the points in S_{RT} and S_{LB} such that all the red points appear before the blue points w.r.t. clockwise order. Then sort the points in S_{RB} and S_{LT} such that all the blue points appear before the red points. This gives a bichromatic point set W with the partition $W_{RT}, W_{RB}, W_{LB}, W_{LT}$. Define the matchings M_S and M_W on S and W, respectively, as follows. For any $(X, Y) \in \{(S_{RT}, S_{LB}), (S_{RB}, S_{LT}), (W_{RT}, W_{LB}), (W_{RB}, W_{LT})\}$, the points in X are matched to points in Y such that any two of the matching edges emanating from the same colored points on X cross each other. Hence the matching edges of X give two crossing families, where the size of each family is determined by the number of points in X of each color. By our construction, the size of the crossing families is the same in both S and W. But in S these crossing families might cross each other, while in W they do not cross each other. Hence, $\overline{cr}(M_S) \geq \overline{cr}(M_W)$. By construction W has a balanced 4–block coloring (i.e., $W = Q$) and $M_W = M_Q^\vee$. Thus, $\overline{cr}(M_S) \geq \overline{cr}(M_Q^\vee)$. However, the constructed matching M_S might not be a max-crossing matching on S. Hence, if $M_S^\vee$ is a max-crossing matching for P, then $\overline{cr}(M_S^\vee) \geq \overline{cr}(M_S)$. This implies $\overline{cr}(M_S^\vee) \geq \overline{cr}(M_Q^\vee)$, which completes the proof for the case $S = P$.

For the remaining part, S is a non-empty proper subset of P.

Step 3: Add back the b-antipodal pairs to S to obtain P and assign each b-antipodal pairs to one of the matching-pair groups. We denote the new groups by $P_{RT}, P_{RB}, P_{LB}, P_{LT}$, respectively. This gives a partition of P into four groups, where each group may have a different number of red and blue points, but the number of red (blue) points in one group is equal to the number of blue (red) points in its matching-pair group. For calculation purposes we assume that there are x_1 new red points and x_2 new blue points added to the group P_{RT} and y_1 new red points and y_2 new blue points added to the group P_{LT}.

As we assume that S is non-empty, there are two possible structures for S as described in Step 2. We consider them as two separate cases. Case 1: All groups in the matching-pair groups have the same number m of red and blue points. Case 2: Each group in one matching-pair group has m red (blue) points whereas each group in the other matching-pair has $m + 1$ red (blue) points, see Fig. 3.

Case 1: All groups have m red and m blue points

Define the matching M_P on P as follows. For any matching-pair group $(X, Y) \in \{(P_{RT}, P_{LB}), (P_{RB}, P_{LT})\}$, the points in X are matched to points in Y such that any two of the matching edges emanating from the same colored points in X cross each other. Since P_{RT} has $m + x_1$ red points, the edges emanating from these points form a crossing family of size $m + x_1$ with $\binom{m+x_1}{2}$ crossings in M_P. Similarly, the edges emanating from the $m + x_2$ blue points in P_{RT} form a crossing family of size $m + x_2$ with $\binom{m+x_2}{2}$ crossings in M_P. Note that these crossing families can cross each other in M_P. Likewise, P_{LT}
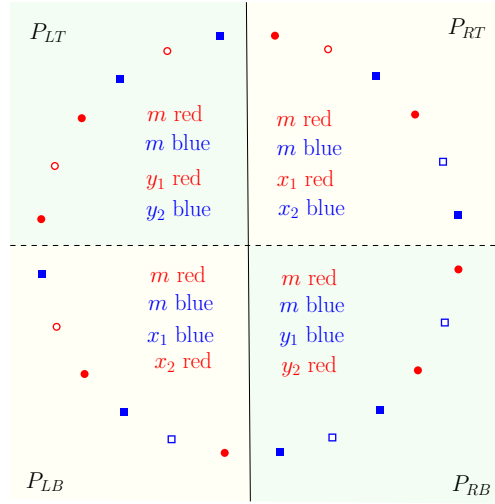
Fig. 5: Distribution of points in Case 1 of the proof of Lemma 3. Filled vertices represent m-antipodal pairs and unfilled vertices represent b-antipodal pairs.

admits a crossing family of red points size $m + y_1$ with $\binom{m+y_1}{2}$ crossings and a crossing family of blue points of size $m + y_2$ with $\binom{m+y_2}{2}$ crossings. Furthermore, all the edges emanating from P_{RT} cross all the edges emanating from P_{LT}, see Fig. 5.

The number of crossings in M_P is given by

$$\overline{\mathrm{cr}}(M_P) \geq \binom{m + x_1}{2} + \binom{m + x_2}{2} + \binom{m + y_1}{2} + \binom{m + y_2}{2}$$
$$+ (2m + x_1 + x_2)(2m + y_1 + y_2) + a + b.$$

Here, a is the number of crossings obtained between the bundle of edges emanating from the x_1 red points in P_{RT} with the edges emanating from the x_2 blue points in P_{RT}. Similarly, b is the number of crossings between the edges emanating from the b-antipodal points in P_{LT} of different color. Note that in this counting we did not count the crossings between the edges emanating from the red and blue points in S_{RT} and S_{LT}. Also, the crossings between the edges emanating from the x_1 (or x_2) newly added red (blue) points in P_{RT} and m blue (red) points in the corresponding S_{RT} were not considered in the above counting. Similarly for P_{LT}.

Claim. $a \geq x_1 \cdot x_2$ and $b \geq y_1 \cdot y_2$.

<u>Proof of Claim:</u> In the following we prove $a = x_1 \cdot x_2$. The proof for $b = y_1 \cdot y_2$ follows similarly. Assume that $\{(p_1, q_1), (p_2, q_2)\}$ with $p_1, p_2 \in P_{RT}$ and $q_1, q_2 \in P_{LB}$ are the b-antipodal pairs such that p_1, q_2 are red and p_2, q_1
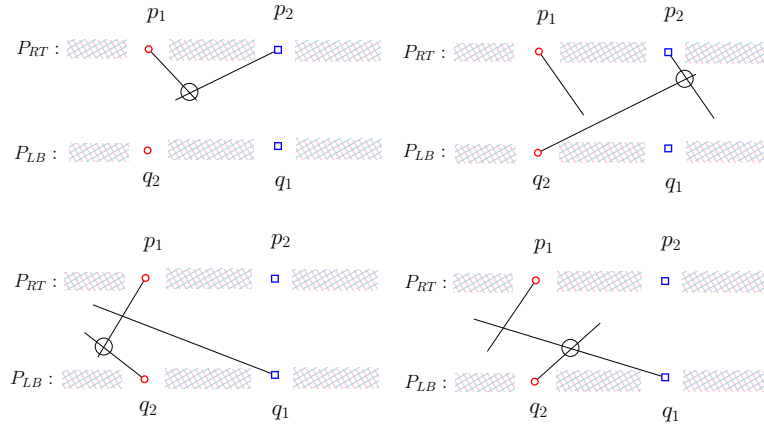
Fig. 6: Possible ways in which b-antipodal points cross in M_P. The crossings counted in the claim are marked by a circle. The crossed red-blue lines represents the remaining red-blue points in P_{RT} and P_{LB}.

are blue (see Fig. 6). Let e_v be the matching edge of M_P incident to point v. We will show that either e_{p_1} or e_{q_1} crosses with e_{p_2} or e_{q_2}.

Note that the edges e_{p_1} and e_{q_1} cross each other as they belong to the same crossing family and similarly, e_{p_2} and e_{q_2} cross each other. If the edge e_{p_1} separates the points p_2 and q_2, then either e_{p_2} or e_{q_2} has to cross e_{p_1} as e_{p_2} and e_{q_2} cross each other. If the edge e_{p_1} does not separate the points p_2 and q_2, then e_{q_1} separates them as e_{p_1} and e_{q_1} cross each other. Thus, either e_{p_1} or e_{q_1} has to cross e_{q_2}.   ∎

By substituting the values of a and b we get

$$\overline{cr}(M_P) \geq 6m^2 + 3m(x_1 + x_2 + y_1 + y_2) - 2m + x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$
$$+ \frac{x_2^2 + x_1^2 + y_1^2 + y_2^2 - x_1 - x_2 - y_1 - y_2}{2} + x_1x_2 + y_1y_2.$$

Consider a balanced 4–block coloring $Q \in \mathcal{C}_{n,n}$ with $n = 4m+x_1+x_2+y_1+y_2$. Then by Lemma 2, we get

$$\overline{cr}(M_Q^\vee) = \frac{3n^2}{8} - \frac{n}{2} + c$$
$$= \frac{3(4m + x_1 + x_2 + y_1 + y_2)^2}{8} - \frac{4m + x_1 + x_2 + y_1 + y_2}{2} + c$$
$$= 6m^2 + 3m(x_1 + x_2 + y_1 + y_2) + \frac{3(x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2)}{4}$$
$$+ \frac{3(x_1x_2 + y_1y_2)}{4} + \frac{3(x_1^2 + x_2^2)}{8} + \frac{3(y_1^2 + y_2^2)}{8} - 2m - \frac{x_1}{2} - \frac{x_2}{2}$$
$$- \frac{y_1}{2} - \frac{y_2}{2} + c,$$

where c takes one of the values $\frac{1}{8}, -\frac{1}{2},$ or 0, depending on the divisibility of n by four. Comparing the number of crossings in M_P and $M_Q^\vee$ gives

$$\overline{\mathrm{cr}}(M_P) - \overline{\mathrm{cr}}(M_Q^\vee) \geq \frac{(x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2)}{4} + \frac{(x_1^2 + x_2^2)}{8} + \frac{(y_1^2 + y_2^2)}{8}$$
$$+ \frac{(x_1 x_2 + y_1 y_2)}{4} - c.$$

As $c \leq \frac{1}{8}$ and since S is a proper subset of P at least one among x_1, x_2, y_1, y_2 is at least 1. Thus $\overline{\mathrm{cr}}(M_P) \geq \overline{\mathrm{cr}}(M_Q^\vee)$.

Case 2: Two groups in a matching-pair group have m red (blue) points and the two remaining groups have $m+1$ red (blue) points.
As in Case 1, define the matching M_P on P as follows. For any matching-pair group $(X, Y) \in \{(P_{RT}, P_{LB}), (P_{RB}, P_{LT})\}$, the points in X are matched to points in Y such that any two of the matching edges emanating from the same colored points on X cross each other. Since P_{RT} has $m + x_1$ red points, the edges emanating from these points form a crossing family of size $m + x_1$ with $\binom{m+x_1}{2}$ crossings in M_P. Similarly, the edges emanating from the $m + x_2$ blue points in P_{RT} form a crossing family of size $m + x_2$ with $\binom{m+x_2}{2}$ crossings in M_P. Note that these crossing families can cross each other in M_P. Likewise, P_{LT} admits a crossing family emanating from red points of size $m + y_1 + 1$ with $\binom{m+y_1+1}{2}$ crossings and a crossing family emanating from blue points of size $m + y_2 + 1$ with $\binom{m+y_2+1}{2}$ crossings. Furthermore, all the edges emanating from P_{RT} cross all the edges emanating from P_{LT}, see Fig. 7.
The number of crossings in M_P is given by

$$\overline{\mathrm{cr}}(M_P) \geq \binom{m + x_1}{2} + \binom{m + x_2}{2} + \binom{m + y_1 + 1}{2} + \binom{m + y_2 + 1}{2}$$
$$+ (2m + x_1 + x_2)(2m + y_1 + y_2 + 2) + a + b.$$

Here a is the number of crossings obtained between the bundle of edges emanating from the x_1 red points in P_{RT} with the edges emanating from the x_2 blue points in P_{RT}. Similarly, b is the number of crossings between the edges emanating from the b-antipodal points in P_{LT}. Note that in this counting we did not count the crossings between the edges emanating from the red and blue points in S_{RT} and S_{LT}. Also, the crossings between the edges emanating from the x_1 (or x_2) newly added red (blue) points in P_{RT} and m blue (red) points in the corresponding S_{RT} were not considered in the above counting. Similarly for P_{LT}. Analogous Case 1, we have $a \geq x_1 x_2$ and $b \geq y_1 y_2$. By substituting the values of a and b we get

$$\overline{\mathrm{cr}}(M_P) \geq 6m^2 + 3m(x_1 + x_2 + y_1 + y_2) + 4m + x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2$$
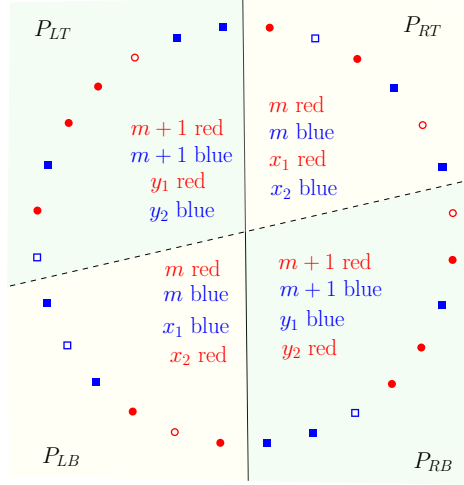$$+ \frac{x_2^2 + x_1^2 + y_1^2 + y_2^2}{2} + \frac{3(x_1 + x_2)}{2} + \frac{(y_1 + y_2)}{2} + x_1 x_2 + y_1 y_2.$$

Fig. 7: Distribution of points in Case 2 of the proof of Lemma 3. Filled vertices represents m-antipodal pairs and unfilled vertices represents b-antipodal pairs.

Consider a balanced 4–block coloring $Q \in \mathcal{C}_{n,n}$ with $n = 4m + 2 + x_1 + x_2 + y_1 + y_2$. Then by Lemma 2, we get

$$
\begin{aligned}
\overline{\mathrm{cr}}(\mathrm{M}_Q^\vee) &= \frac{3n^2}{8} - \frac{n}{2} + c \\
&= \frac{3(4m + x_1 + x_2 + y_1 + y_2 + 2)^2}{8} - \frac{4m + x_1 + x_2 + y_1 + y_2 + 2}{2} + c \\
&= 6m^2 + 3m(x_1 + x_2 + y_1 + y_2) + \frac{3(x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2)}{4} \\
&\quad + \frac{3(x_1 x_2 + y_1 y_2)}{4} + \frac{3(x_1^2 + x_2^2)}{8} + \frac{3(y_1^2 + y_2^2)}{8} + 4m + x_1 + x_2 \\
&\quad + y_1 + y_2 + \frac{1}{2} + c,
\end{aligned}
$$

where c takes one of the values $\frac{1}{8}, -\frac{1}{2}$, or 0, depending on the divisibility of n by four. Comparing the number of crossings in M_P and $\mathrm{M}_Q^\vee$ gives

$$
\begin{aligned}
\overline{\mathrm{cr}}(M_P) - \overline{\mathrm{cr}}(\mathrm{M}_Q^\vee) &\geq \frac{(x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2)}{4} + \frac{(x_1^2 + x_2^2)}{8} + \frac{(y_1^2 + y_2^2)}{8} \\
&\quad + \frac{(x_1 x_2 + y_1 y_2)}{4} + \frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2} - \frac{1}{2} - c.
\end{aligned}
$$

For $x_1 + x_2 \geq 2$, we have $\overline{\mathrm{cr}}(M_P) - \overline{\mathrm{cr}}(\mathrm{M}_Q^\vee) \geq \frac{(y_1 + y_2)^2}{8} + \frac{1}{2} - c$. As $c \leq \frac{1}{8}$, $\overline{\mathrm{cr}}(M_P) \geq \overline{\mathrm{cr}}(\mathrm{M}_Q^\vee)$. Thus we only have to consider the case when $x_1 + x_2 < 2$, i.e., either $x_1 = x_2 = 0$ or $x_1 = 1$ and $x_2 = 0$. Note that we do not have to

consider the case where $x_1 = 0$ and $x_2 = 1$ separately, as the inequality is symmetric in x_1 and x_2 (as well as in y_1 and y_2).

Case 2.1: Assume that $x_1 = 1$ and $x_2 = 0$. It implies that $\overline{cr}(M_P) - \overline{cr}(M_Q^\vee) \geq \frac{(y_1+y_2)(y_1+y_2-2)}{8} + \frac{1}{8} - c \geq \frac{(y_1+y_2)(y_1+y_2-2)}{8}$, where the last inequality holds as $c \leq \frac{1}{8}$. Thus, $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$ whenever $y_1 + y_2 \neq 1$. W.l.o.g. assume that $y_1 = 1$ and $y_2 = 0$. Then the total number of red points in such a point set P is $n = 4m + 4$. As $n \equiv 0 \mod 4$, $c = 0$ by Lemma 2. This implies that $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$.

Case 2.2: Assume that $x_1 = x_2 = 0$. Then $\overline{cr}(M_P) - \overline{cr}(M_Q^\vee) \geq \frac{(y_1+y_2)(y_1+y_2-4)}{8} - \frac{1}{2} - c \geq \frac{(y_1+y_2)(y_1+y_2-4)}{8} - \frac{1}{2} - \frac{1}{8}$, where the last inequality holds as $c \leq \frac{1}{8}$. Thus, $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$ whenever $y_1 + y_2 \geq 5$. Assume that $y_1 + y_2 = 4$, then the total number of red points in such a set P is $n = 4m + 6$, i.e., $c = -\frac{1}{2}$. Thus, $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$. Next, assume that $y_1 + y_2 = 1$. Then the total number of red points in such a set P is $n = 4m + 3$, i.e., $c = \frac{1}{8}$. Thus, $\overline{cr}(M_P) - \overline{cr}(M_Q^\vee) \geq -1$. Similarly, assume that $y_1 + y_2 = 2$. Then the total number of red points in such a set P is $n = 4m + 4$, i.e., $c = 0$. Thus, $\overline{cr}(M_P) - \overline{cr}(M_Q^\vee) \geq -1$. And finally if $y_1 + y_2 = 3$. Then the total number of red points in such a set P is $n = 4m + 5$, i.e., $c = \frac{1}{8}$. Thus, $\overline{cr}(M_P) - \overline{cr}(M_Q^\vee) \geq -1$. Note that $y_1 + y_2 = 0$ would imply $x_1 + x_2 + y_1 + y_2 = 0$ which imply $S = P$ which we considered already.

In short, if $y_1 + y_2 \leq 3$, $\overline{cr}(M_P) - \overline{cr}(M_Q^\vee) \geq -1$. In this situation, in order to show that the max-crossing matching on P has at least as many as crossings than the max-crossing matching of the corresponding Q, we either have to find one more crossing in M_P which is not counted before, or we should define a different matching on P and compare its crossings with the crossings in $M_Q^\vee$. If the edges emanating from red and blue points in some group of $S \subset P$ crosses each other, then we can add at least one more crossing to our counting of $\overline{cr}(M_P)$ (such crossings are not counted in the formula before). Then $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$. Thus, the only remaining possibility is that the crossing families formed by the red and blue points in S_{RT} (and S_{LT}) do not cross each other, i.e, S has a 4–block coloring. Here, we consider the following two cases.

Case 2.2.1: Assume that $y_1 = 0$ and $S \subset P$ has a 4–block coloring. Then we have $x_1 = x_2 = y_1 = 0$ and $y_2 \neq 0$. If the edges emanating from y_2 cross any other edges that emanates from a different colored point in the same group, then $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$. If not, then P has a 4–block coloring. Then by Lemma 4, $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$.

Case 2.2.2: Assume that $y_1, y_2 \neq 0$ and $S \subset P$ has a 4–block coloring. If the edges emanating from any of the b-antipodal points in P_{LT} crosses an edge emanating from a m-antipodal point in S_{LT} of different color, then $\overline{cr}(M_P) \geq \overline{cr}(M_Q^\vee)$. Otherwise, we define a different matching on P.

As $x_1 = x_2 = 0$, P has a 6–block coloring. W.l.o.g. assume that $|R_1| = 2m + 1 + y_1$, $|B_1| = 2m+1$, $|R_2| = y_2$, $|B_2| = y_1$, $|R_3| = 2m+1$, $|B_3| = 2m+1+y_2$. We define a matching M_P' on P as follows: (Note that when we say "match two collection of points", we mean to match it as a crossing family). Match

the first m points in R_1 to the last m points of B_1. Match next y_1 points of R_1 to B_2. Then match the remaining $m+1$ points of R_1 to the first $m+1$ points of B_3. Now match the next y_2 points of B_3 to R_2 and the remaining m points of B_3 is matched to the first m points of R_3. Next match the remaining unmatched points, that is, the first $m+1$ points in B_1 to the last $m+1$ points in R_3 (see Fig 8).
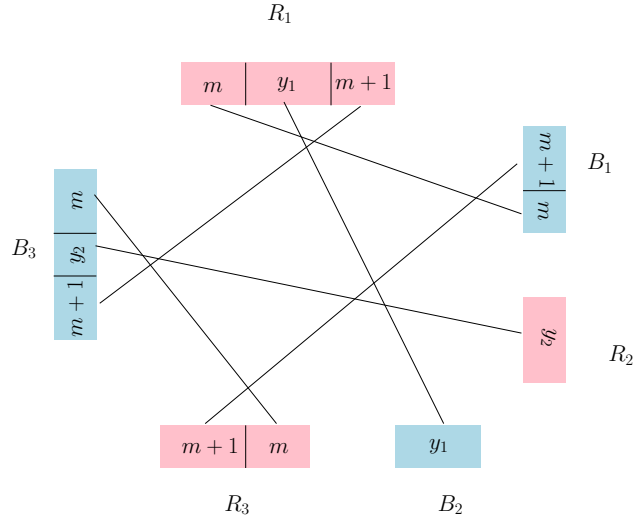

Fig. 8: 6–blocks and their corresponding matching.

The number of crossings in $M'_P = 2\binom{m}{2} + 2\binom{m+1}{2} + \binom{y_1}{2} + \binom{y_2}{2} + 4m^2 + m + 2y_1(m+1) + 2y_2(m+1) + my_1 + my_2 + y_1y_2$. For $y_1 = y_2 = 1$ we get $\overline{\text{cr}}(M'_P) = 6m^2 + 10m + 5$. The total number of red points in such a set P is $n = 4m+4$, i.e., $c = 0$. In this case, $\overline{\text{cr}}(M_Q^\vee) = 6m^2 + 10m + 4$. Thus we have $\overline{\text{cr}}(M_P) - \overline{\text{cr}}(M_Q^\vee) \geq 0$. For $y_1 = 1$ and $y_2 = 2$, we get $\overline{\text{cr}}(M'_P) = 6m^2 + 13m + 9$ and the total number of red points in such a set P is $n = 4m+5$, i.e., $c = \frac{1}{8}$. In this case, $\overline{\text{cr}}(M_Q^\vee) = 6m^2 + 13m + 7$. Thus we have $\overline{\text{cr}}(M_P) - \overline{\text{cr}}(M_Q^\vee) \geq 0$. This completes the last case and hence the whole proof. $\qquad\square$

B   Proof of Proposition 1

Proposition 1. *For $n \geq 7$, every $P \in \mathcal{C}_{n,n}$ admits bichromatic perfect matchings with k crossings for any $k \in \{0, 1, \ldots, \frac{15(n-6)}{7}\} \setminus \{1, 2\}$.*

Proof. Let $P \in \mathcal{C}_{n,n}$, where $n \geq 7$. We will first partition P into subsets $P_1, P_2, \ldots, P_\ell$ of 14 points each, such that each subset has 7 red and 7 blue points, plus one (possibly empty) point set $P_{\ell+1}$ of at most 6 red and blue points each. To do this, we will iteratively split of a set P_i from P, starting with $i = 1$, in the following way:

Starting with an empty set P_i, we traverse the points along the convex hull of P in clockwise order (starting from an arbitrary point) and add them to P_i one by one until P_i has 14 points. Let s be the difference between the number of red and blue points in P_i. While $s \neq 0$, we remove the first point that was added to P_i and add the next point along the convex hull of P. In each such step, the value of s either increases by 2 or decreases by 2 or remains unchanged. Note that since the number of red and blue points in P is the same, summing the values of s over all different subsets of 14 consecutive points along the convex hull of P gives zero. Hence we eventually obtain a set P_i with 7 red and 7 blue points and remove these points from P. Once the iteratively reduced set P has less than 14 points, we denote this remaining (possibly empty) set as $P_{\ell+1}$.

Recall that by computations, we know that for any $1 \leq i \leq \ell$ and any integer $k' \in \{0, 1, 2, \ldots, 15\} \setminus \{1, 2\}$, there exists a perfect matching of P_i with exactly k' crossings. Further, $P_{\ell+1}$ always admits a plane perfect matching.

Since the convex hulls of the subsets $P_1, P_2, \ldots, P_\ell, P_{\ell+1}$ are pairwise disjoint by construction, no crossing occurs between edges from different subsets. Therefore, we can construct a perfect matching of P with any number of crossings k such that $3 \leq k \leq 15 \cdot \ell = \frac{15}{7}(n - c)$, where $c \leq 6$ is the number of red (blue, respectively) points in $P_{\ell+1}$, by combining appropriate matchings of the subsets $P_1, P_2, \ldots, P_\ell$. $\qquad\square$

We remark that the partition of P into balanced bichromatic sets with pairwise disjoint convex hulls in the above proof strongly relies on the fact that P is in convex position. In fact, for a balanced bichromatic point set in general position, an according result is in general not possible. An alternative partitioning method that also works for general point sets is to recursively apply the ham sandwich theorem until all sets are sufficiently small (but not smaller than the desired value). Note, however, that the resulting balanced bichromatic sets will have cardinalities ranging from the desired minimum value to nearly twice that value. Using this method in the above proof would result in roughly half the desired bound for k.