Twisted Ways to Find Plane Structures in Simple Drawings of Complete Graphs

Oswin Aichholzer
Institute of Software Technology, Graz University of Technology, Austria

Alfredo García
Departamento de Métodos Estadísticos and IUMA, Universidad de Zaragoza, Spain

Javier Tejel
Departamento de Métodos Estadísticos and IUMA, Universidad de Zaragoza, Spain

Birgit Vogtenhuber
Institute of Software Technology, Graz University of Technology, Austria

Alexandra Weinberger
Institute of Software Technology, Graz University of Technology, Austria

Abstract

Simple drawings are drawings of graphs in which the edges are Jordan arcs and each pair of edges share at most one point (a proper crossing or a common endpoint). We introduce a special kind of simple drawings that we call generalized twisted drawings. A simple drawing is generalized twisted if there is a point O such that every ray emanating from O crosses every edge of the drawing at most once and there is a ray emanating from O which crosses every edge exactly once.

Via this new class of simple drawings, we show that every simple drawing of the complete graph with n vertices contains $\Omega(n^{\frac{1}{2}})$ pairwise disjoint edges and a plane path of length $\Omega(\frac{\log n}{\log \log n})$. Both results improve over previously known best lower bounds. On the way we show several structural results about and properties of generalized twisted drawings. We further present different characterizations of generalized twisted drawings, which might be of independent interest.

2012 ACM Subject Classification Mathematics of computing → Combinatorics; Mathematics of computing → Graph theory

Keywords and phrases Simple drawings, simple topological graphs, disjoint edges, plane matching, plane path

Funding *Oswin Aichholzer*: Partially supported by the Austrian Science Fund (FWF): W1230 and by H2020-MSCA-RISE project 734922 - CONNECT.
Alfredo García: Supported by H2020-MSCA-RISE project 734922 - CONNECT and Gobierno de Aragón project E41-17R.
Javier Tejel: Supported by H2020-MSCA-RISE project 734922 - CONNECT, Gobierno de Aragón project E41-17R and project PID2019-104129GB-I00 / AEI / 10.13039/501100011033 of the Spanish Ministry of Science and Innovation.
Birgit Vogtenhuber: Partially supported by Austrian Science Fund within the collaborative DACH project *Arrangements and Drawings* as FWF project I 3340-N35 and by H2020-MSCA-RISE project 734922 - CONNECT.
Alexandra Weinberger: Supported by the Austrian Science Fund (FWF): W1230 and by H2020-MSCA-RISE project 734922 - CONNECT.

This work (without appendix) is available at the 38th International Symposium on Computational Geometry (SoCG 2022). Some results of this work have also been presented at the Computational Geometry: Young Researchers Forum in 2021 [3] and at the Encuentros de Geometría Computacional 2021 [4].

1 Introduction

Simple drawings are drawings of graphs in the plane such that vertices are distinct points in the plane, edges are Jordan arcs connecting their endpoints, and edges intersect at most once either in a proper crossing or in a shared endpoint. The edges and vertices of a drawing partition the plane (or, more exactly, the plane minus the drawing) into regions, which are called the *cells* of the drawing. If a simple drawing is plane (that is, crossing-free), then its cells are classically called *faces*.

In the past decades, there has been significant interest in simple drawings. Questions about plane subdrawings of simple drawings of the complete graph on n vertices, K_n, have attracted particularly close attention.

Rafla [20] conjectured that every simple drawing of K_n contains a plane Hamiltonian cycle. The conjecture has been shown to hold for $n \leq 9$ [1], as well as for several special classes of simple drawings, like straight-line, monotone, and cylindrical drawings, but remains open in general. If Rafla's conjecture is true, then this would immediately imply that every simple drawing of the complete graph contains a plane perfect matching. However, to-date even the existence of such a matchging is still unknown.

Ruiz-Vargas [22] showed in 2017 that every simple drawing of K_n contains $\Omega(n^{\frac{1}{2}-\varepsilon})$ pairwise disjoint edges for any $\varepsilon > 0$, which improved over a series of previous results: $\Omega((\log n)^{\frac{1}{6}})$ in 2003 [17], $\Omega(\frac{\log n}{\log \log n})$ in 2005 [18], $\Omega((\log n)^{1+\varepsilon})$ in 2009 [10], and $\Omega(n^{\frac{1}{3}})$ in 2013 and 2014 [11, 13, 23].

Pach, Solymosi, and Tóth [17] showed that every simple drawing of K_n contains a subdrawing of $K_{c \log^{\frac{1}{8}} n}$, for some constant c, that is either *convex* or *twisted*[1]. They further showed that every simple drawing of K_n contains a plane subdrawing isomorphic to any fixed tree with up to $c \log^{\frac{1}{6}} n$ vertices, for some constant c. This implies that every simple drawing of K_n contains a plane path of length $\Omega((\log n)^{\frac{1}{6}})$, which has been the best lower bound known prior to this paper.

Concerning general plane substructures, it follows from a result of Ruiz-Vargas [22] that every simple drawing of K_n contains a plane subdrawing with at least $2n-3$ edges. Further, García, Pilz, and Tejel [14] showed that every maximal plane subdrawing of a simple drawing of K_n is biconnected. Note that, in contrast to straight-line drawings, simple drawings of K_n in general do not contain triangulations, that is, plane subdrawings where all faces (except at most one) are 3-cycles.

In this paper, we introduce a new family of simple drawings, which we call *generalized twisted* drawings. The name stems from the fact that one can show that any twisted drawing is weakly isomorphic to a generalized twisted drawing (but not every generalized twisted drawing is weakly isomorphic to a twisted drawing). It follows, that for any n there exists a generalized twisted drawing. Two drawings D and D' are *weakly isomorphic* if there is a bijection between the vertices and edges of D and D' such that a pair of edges in D crosses exactly when the corresponding pair of edges in D' crosses.

▶ **Definition 1.** *A simple drawing D is **c-monotone** (short for circularly monotone) if there is a point O such that any ray emanating from O intersects any edge of D at most once.*

[1] In their definition for simple drawings, *convex* means that there is a labeling of the vertices to $v_1, v_2, ..., v_n$ such that (v_i, v_j) $(i < j)$ crosses (v_k, v_l) $(k < l)$ if and only if $i < k < j < l$ or $k < i < l < j$, and *twisted* means that there is a labeling of the vertices to $v_1, v_2, ..., v_n$ such that (v_i, v_j) $(i < j)$ crosses (v_k, v_l) $(k < l)$ if and only if $i < k < l < j$ or $k < i < j < l$.

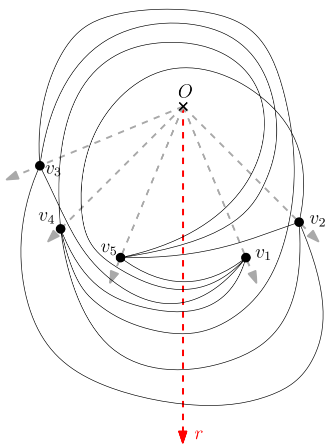


Figure 1 A generalized twisted drawing of K_5. All edges cross the (red) ray r.

A simple drawing D of K_n is ***generalized twisted*** *if there is a point O such that D is c-monotone with respect to O and there exists a ray r emanating from O that intersects every edge of D.*

We label the vertices of c-monotone drawings $v_1, \ldots, v_n$ in counterclockwise order around O. In generalized twisted drawings, they are labeled such that the ray r emerges from O between the ray to v_1 and the one to v_n. Figure 1 shows an example of a generalized twisted drawing of K_5.

Generalized twisted drawings turn out to have quite surprising structural properties. We show some crossing properties of generalized twisted drawings in Section 2 and with that also prove that they always contain plane Hamiltonian paths (Theorem 3). This result is an essential ingredient for showing that any simple drawing of K_n contains $\Omega(\sqrt{n})$ pairwise disjoint edges (Theorem 9 in Section 3), as well as a plane path of length $\Omega(\frac{\log n}{\log \log n})$ (Theorem 10 in Section 4). In Section 5, we present different characterizations of generalized twisted drawings that are of independent interest. We conclude with an outlook on further work and open problems in Section 6.

2 Twisted Preliminaries

In this section, we show some properties of generalized twisted drawings, which will be used in the following sections.

▶ **Lemma 2.** *Let D be a generalized twisted drawing of K_4, with vertices $\{v_1, v_2, v_3, v_4\}$ labeled counterclockwise around O. Then the edges v_1v_3 and v_2v_4 do not cross.*

The full proof of Lemma 2 can be found in Appendix A.

Proof Sketch. Assume, for a contradiction, that the edge v_1v_3 crosses the edge v_2v_4. There are (up to strong isomorphism) two possibilities to draw the crossing edges v_1v_3 and v_2v_4, depending on whether v_1v_3 crosses the (straight-line) segment from O to v_4 or not; cf. Figure 2. In both cases, there is only one way to draw v_1v_2 such that the drawing stays generalized twisted, yielding two regions bounded by all drawn edges (v_1v_3, v_2v_4, v_1v_2). The vertices v_3 and v_4 lie in the same region. It is well-known that every simple drawing of K_4 has at most one crossing. Thus, the edge v_3v_4 cannot leave this region. However, it is impossible

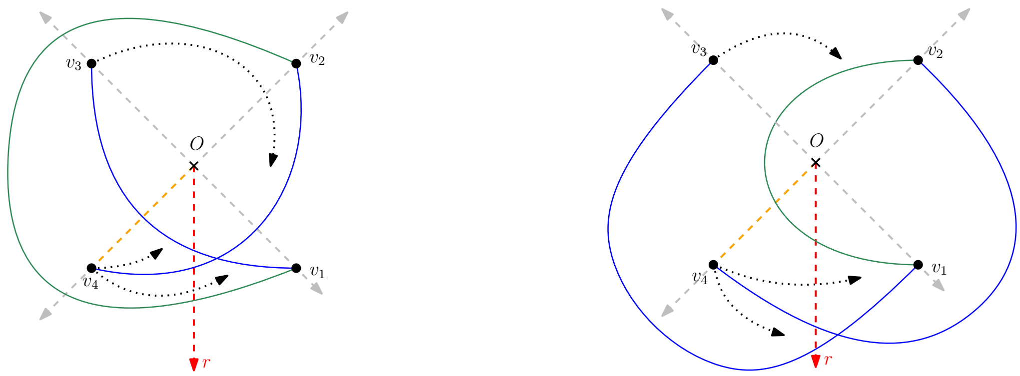


Figure 2 The two possibilities to draw v_1v_3 and v_2v_4 crossing and generalized twisted.

to draw v_3v_4 without leaving the region such that it is c-monotone and crosses the ray r (see the dotted arrows in Figure 2 for necessary emanating directions of v_3v_4). ◀

Using the crossing property of Lemma 2, it follows directly that generalized twisted drawings always contain plane Hamiltonian paths.

▶ **Theorem 3.** *Every generalized twisted drawing of K_n contains a plane Hamiltonian path.*

Proof of Theorem 3. Let D be a generalized twisted drawing of K_n. Consider the Hamiltonian path $v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_2, v_{\lceil \frac{n}{2} \rceil+2}, v_3, \ldots, v_{\lceil \frac{n}{2} \rceil-1}, v_n, v_{\lceil \frac{n}{2} \rceil}$ if n is odd or the Hamiltonian path $v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_2, v_{\lceil \frac{n}{2} \rceil+2}, v_3, \ldots, v_{n-1}, v_{\lceil \frac{n}{2} \rceil}, v_n$ if n is even. See for example the Hamiltonian path v_1, v_4, v_2, v_5, v_3 in Figure 1. Take any pair of edges (v_i, v_j) and (v_k, v_l) of the path, where we can assume without loss of generality that $i < j$ and $k < l$. If the two edges share an endpoint, they are adjacent and do not cross. Otherwise, if they do not share an endpoint, either $i < k < j < l$ or $k < i < l < j$ by definition of the path. In any of the two cases, (v_i, v_j) and (v_k, v_l) cannot cross by Lemma 2. Therefore, no pair of edges cross, and the Hamiltonian path is plane. ◀

Analogous to the proof of Theorem 3, one can argue that in every generalized twisted drawing of K_n with n odd, the Hamiltonian cycle $v_1, v_{\lceil \frac{n}{2} \rceil+1}, v_2, v_{\lceil \frac{n}{2} \rceil+2}, \ldots, v_{\lceil \frac{n}{2} \rceil-1}, v_n, v_{\lceil \frac{n}{2} \rceil}, v_1$ is plane. We strongly conjecture that every generalized twisted drawing of K_n contains a plane Hamiltonian cycle, but its structure for even n is still an open problem.

Theorem 3 will be used heavily in the next two sections. Further, the following statement, which has been implicitly shown in [11] and [13], will be used in all remaining sections. For completeness, we include a proof in Appendix B.

▶ **Lemma 4.** *Let D be a simple drawing of a complete graph containing a subdrawing D', which is a plane drawing of $K_{2,n}$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2\}$ be the sides of the bipartition of D'. Let D_A be the subdrawing of D induced by the vertices of A. Then D_A is weakly isomorphic to a c-monotone drawing. Moreover, if all edges in D_A cross the edge b_1b_2, then D_A is weakly isomorphic to a generalized twisted drawing.*

3 Disjoint Edges in Simple Drawings

In this section, we show that every simple drawing of K_n contains at least $\left\lfloor\sqrt{\frac{n}{48}}\right\rfloor$ pairwise disjoint edges, improving the previously known best bound of $\Omega(n^{\frac{1}{2}-\varepsilon})$, for any $\varepsilon > 0$, by Ruiz-Vargas [22]. In addition to the properties of generalized twisted drawings from Section 2, we use the following theorems and observations to prove this new lower bound.

▶ **Theorem 5** ([14]). *For $n \geq 3$, every maximal plane subdrawing of any simple drawing of K_n is biconnected.*

The following theorem is a direct consequence of Corollary 5 in [21].

▶ **Theorem 6.** *Let D be a simple drawing of K_n with $n \geq 3$. Let H be a connected plane subdrawing of D containing at least two vertices, and let v be a vertex in $D \setminus H$. Then D contains two edges incident to v that connect v with H and do not cross any edges of H.*

▶ **Observation 7.** *For any $n \geq 3$, the number of edges in a planar graph with n vertices is at most $3n - 6$.*

A drawing is *outerplane* if it is plane, and all vertices lie on the unbounded face of the drawing. A graph is *outerplanar* if it can be drawn outerplane. Outerplanar graphs have a smaller upper bound on their number of edges than planar graphs.

▶ **Observation 8.** *For any $n \geq 3$, the number of edges in an outerplanar graph with n vertices is at most $2n - 3$.*

▶ **Theorem 9.** *Every simple drawing of K_n contains at least $\left\lfloor\sqrt{\frac{n}{48}}\right\rfloor$ pairwise disjoint edges.*

Proof. Let D be a simple drawing of K_n, and let M be a maximal plane matching of D. If $m := |M| \geq \sqrt{\frac{n}{48}}$, then Theorem 9 holds. So assume that $|M| < \sqrt{\frac{n}{48}}$. We will show how to find another plane matching, whose size is at least $\lfloor\sqrt{\frac{n}{48}}\rfloor$.

The overall idea is the following: Let H be a maximal plane subdrawing of D whose vertex set is exactly the vertices matched in M and that contains M. We will find a face f in H that contains much more unmatched vertices inside than matched vertices on its boundary. Then we will show that there exists a subset of the vertices inside that face, which induces a subdrawing of D that is weakly isomorphic to a generalized twisted drawing and contains enough vertices to guarantee the desired size of the plane matching.

We start towards finding the face f. By Theorem 5, H is biconnected. Thus, H partitions the plane into faces, where the boundary of each face is a simple cycle. Note that the vertices of H are exactly the vertices that are matched in M, and the vertices inside faces are the vertices that are unmatched in M. Let U be the set of vertices of D that are not matched by any edge of M. We denote the set of vertices of U inside a face f_i by $U(f_i)$, the number of vertices in $U(f_i)$ by $u(f_i)$, and the number of vertices on the boundary of the face f_i by $|f_i|$.

We next show that there exists a face f of H such that $u(f) \geq \frac{\sqrt{48n}}{12}|f|$. Assume for a contradiction that for every face f_i it holds that

$$u(f_i) < \frac{\sqrt{48n}}{12}|f_i|.$$

There are exactly $n - 2m$ unmatched vertices. As every unmatched vertex is in the interior of a face of H (that might be the unbounded face), we can count the unmatched vertices by summing over the number of vertices in each face (including the unbounded face). Thus,

$$n - 2m \leq \sum_{f_i} u(f_i) < \frac{\sqrt{48n}}{12} \sum_{f_i} |f_i|. \tag{1}$$

The number of edges in H is $\frac{1}{2}\sum_{f_i}|f_i|$. Since H is plane, we can use Observation 7 to bound the number of edges of H by $3n'-6$, where n' is the number of vertices in H. As the vertices of H are exactly the matched vertices, their number is $n'=2m$. Hence,

$$\sum_{f_i}|f_i| \leq 6\cdot 2m - 12.$$

From $m < \sqrt{\frac{n}{48}}$ it follows that

$$\sum_{f_i}|f_i| < 12\sqrt{\frac{n}{48}} - 12 \tag{2}$$

and

$$n - 2\sqrt{\frac{n}{48}} < n - 2m. \tag{3}$$

Putting equations (1) to (3) together we obtain that

$$n - 2\sqrt{\frac{n}{48}} < \frac{\sqrt{48n}}{12}(12\sqrt{\frac{n}{48}} - 12) = n - \sqrt{48n}.$$

However, this inequality cannot be fulfilled by any $n \geq 0$. Thus, there exists at least one face f_i with $u(f_i) \geq \frac{\sqrt{48n}}{12}|f_i|$. We call that face f. (If there are several such faces, we take an arbitrary one of them and call it f.)

As a next step, we will find two vertices on the boundary of f to which many vertices inside f are connected via edges that do not cross each other or H. From f and the set $U(f)$, we construct a plane subdrawing H' as follows; cf. Figure 3 (left). We add the vertices and edges on the boundary of f. Then we iteratively add all the vertices in $U(f)$, where for each added vertex v we also add two edges of D incident to v such that the resulting drawing stays plane. Two such edges exist by Theorem 6. Since the matching M is maximal, any edges between two unmatched vertices must cross at least one edge of M and thus must cross the boundary of f. Hence, no edge in H' can connect two vertices of $U(f)$ (as they are unmatched). Consequently, every vertex in $U(f)$ is connected in H' to exactly two vertices that both lie on the boundary of f.

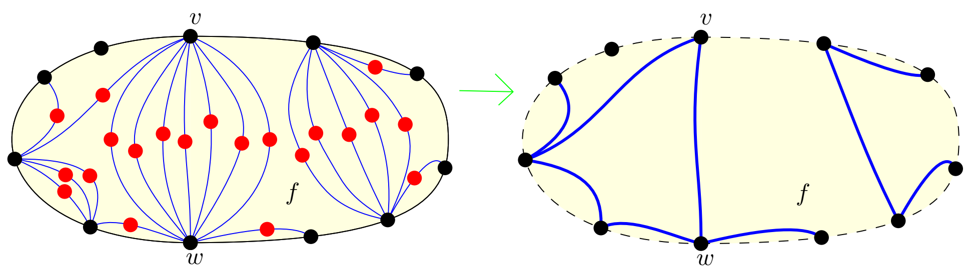


Figure 3 Left: The face f in H containing the plane drawing H' (blue lines) inside. Right: We can obtain an outerplane drawing from H' by interpreting bundles of edge pairs incident to the same black vertices as plane edges.

We consider the edges in H' that connect a vertex in $U(f)$ as a pair of edges. Every edge in such a pair is contained in exactly one pair, since it is incident to exactly one unmatched vertex. Thus, we can see every such pair of edges as one *long edge* incident to two vertices on the boundary of f. If several of those long edges have the same endpoints, we call them a bundle of edges; see Figure 3 (right).

From the long edges, we can define a graph G' as follows. The vertices of G' are the vertices of D that lie on the boundary of f. Two vertices u and v are connected in G' if there is at least one long edge in H' that connects them. By the definition of long edges, G' is outerplanar (as can be observed in Figure 3 (right)). Note that every unmatched vertex in $U(f)$ defines a long edge, so the number of long edges is $u(f) \geq \frac{\sqrt{48n}}{12}|f|$. From Observation 8, it follows that G' has at most $2|f|-3$ edges. As a consequence, there is a pair of vertices on the boundary of f such that the number of long edges in its bundle is at least

$$\frac{1}{(2|f|-3)} \frac{\sqrt{48n}}{12}|f| > \frac{\sqrt{48n}}{24}.$$

This implies that there are two vertices, say v and w, to which more than $\frac{\sqrt{48n}}{24}$ vertices inside f have two plane incident edges. We call the set of vertices in $U(f)$ that have plane edges to both vertices v and w the set U_{vw}. This set is marked in Figure 4 (left). We denote the subdrawing of D induced by U_{vw} by D_{vw}; see Figure 4 (right).

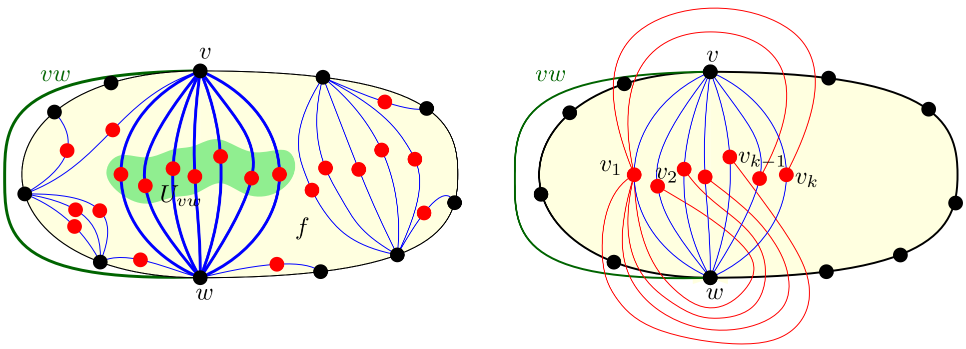


Figure 4 The subdrawing D' induced by U_{vw} and the edges in D_{vw}. Left: The set U_{vw}. Right: The edges adjacent to the leftmost vertex, v_1, are drawn (in red).

We show that all edges between vertices in U_{vw} cross the edge vw. Let x and y be two vertices of D_{vw}. Let R_1 be the region bounded by the edges xv, vy, yw, and wx that lies inside the face f; see Figure 5. We show that xy and vw lie completely outside R_1. The edge xy has to lie either completely inside or completely outside R_1, because it is adjacent to all edges on the boundary of R_1. As M is maximal and the edge xy connects two unmatched vertices, it has to cross at least one matching edge. Thus, xy has to lie completely outside R_1. (There can be no matching edges in R_1, as R_1 is contained inside the face f.) As H is a maximal plane subdrawing, vw cannot lie inside the face f and thus has to be outside R_1. Since both edges vw and xy lie completely outside R_1 and the vertices along the boundary of R_1 are sorted $vxwy$, the two edges have to cross. Thus, all edges of D_{vw} cross the edge vw.

Since the edges from vertices in U_{vw} to v and w are plane, it follows from Lemma 4 that D_{vw} is weakly isomorphic to a generalized twisted drawing. Thus, D_{vw} contains at least $\lfloor \frac{1}{2} \frac{\sqrt{48n}}{24} \rfloor$ pairwise disjoint edges by Theorem 3. Hence, D contains at least $\lfloor \sqrt{\frac{n}{48}} \rfloor$ pairwise disjoint edges. ◀

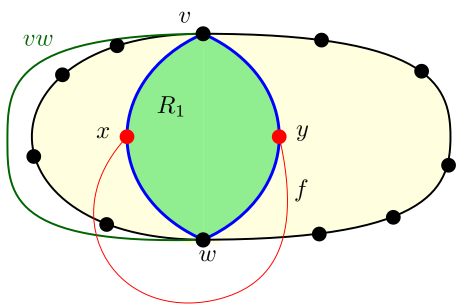


Figure 5 The edge xy has to cross the edge vw.

4 Plane Paths in Simple Drawings

In the previous section, we used generalized twisted drawings to improve the lower bound on the number of disjoint edges in simple drawings of K_n. In this section, we show that generalized twisted drawings are also helpful to improve the lower bound on the length of the longest path in such drawings, where the length of a path is the number of its edges, to $\Omega(\frac{\log n}{\log \log n})$. This improves the previously known best bound of $\Omega((\log n)^{\frac{1}{6}})$, which follows from a result of Pach, Solymosi, and Tóth [17].

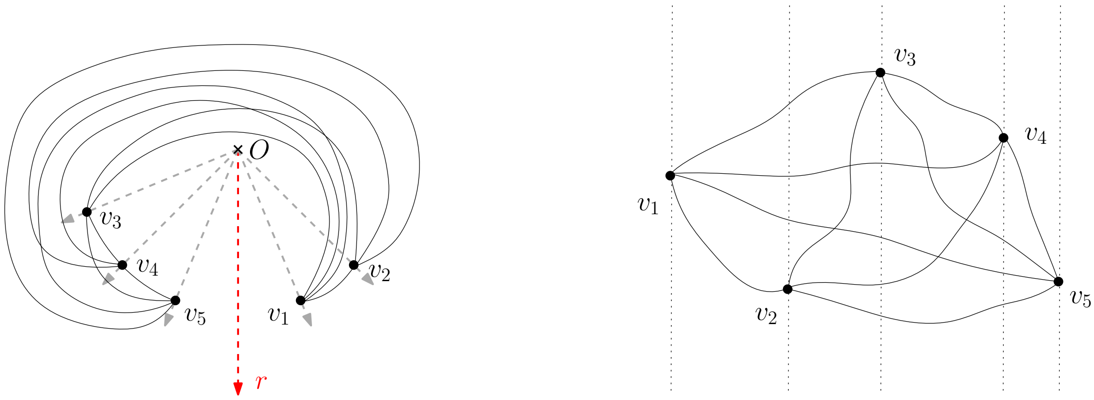
▶ **Theorem 10.** *Every simple drawing D of K_n contains a plane path of length $\Omega(\frac{\log n}{\log \log n})$.*

To prove the new lower bound, we first show that all c-monotone drawings on n vertices contain either a generalized twisted drawing on $\sqrt{n}$ vertices or a drawing weakly isomorphic to an x-monotone drawing on $\sqrt{n}$ vertices. We know that drawings weakly isomorphic to generalized twisted drawings or x-monotone drawings contain plane Hamiltonian paths (by Theorem 3 and Observation 11 below). We conclude that c-monotone drawings contain plane paths of the desired size. We then show that every simple drawing of the complete graph contains either a c-monotone drawing or a plane d-ary tree. With easy observations about the length of the longest path in d-ary trees and by putting all results together, we obtain that every simple drawing D of K_n contains a plane path of length $\Omega(\frac{\log n}{\log \log n})$.

4.1 Plane Paths in C-Monotone Drawings

A simple drawing is x-monotone if any vertical line intersects any edge of the drawing at most once (see Figure 6b). This family of drawings has been studied extensively in the literature (see for example [2, 5, 7, 12, 19]). By definition, c-monotone drawings in which there exists a ray emanating from O, which crosses all edges of the drawing, are generalized twisted. In contrast, consider a c-monotone drawing D such that there exists a ray r emanating from O that crosses no edge of D. Then it is easy to see that D is strongly isomorphic to an x-monotone drawing. (A c-monotone drawing on the sphere can be cut along the ray r and the result drawn on the plane such that all rays are vertical lines and the ray r is to the very left of the drawing.) Figure 6a shows a c-monotone drawing D of K_5 where no edge crosses the ray r, and Figure 6b shows an x-monotone drawing of K_5 strongly isomorphic to D. We will call simple drawings that are strongly isomorphic to x-monotone drawings *monotone* drawings. In particular, any c-monotone drawing for which there exists a ray emanating from O that crosses no edge of the drawing is monotone.

It is well-known that any x-monotone drawing of K_n contains a plane Hamiltonian path. For instance, assuming that the vertices are ordered by increasing x-coordinates, the set of edges $v_1v_2, v_2v_3 \ldots, v_{n-1}v_n$ form a plane Hamiltonian path.



(a) A c-monotone drawing D of K_5 such that the ray r crosses no edge of D.

(b) An x-monotone drawing of K_5 strongly isomorphic to D of Figure 6a.

Figure 6 Two strongly isomorphic monotone drawings of K_5.

▶ **Observation 11.** *Every monotone drawing of K_n contains a plane Hamiltonian path.*

We will show that c-monotone drawings contain plane paths of size $\sqrt{n}$, by showing that any c-monotone drawing of K_n contains a subdrawing of $K_{\sqrt{n}}$ that is either generalized twisted or monotone. To do so, we will use Dilworth's Theorem on chains and anti-chains in partially ordered sets. A *chain* is a subset of a partially ordered set such that any two distinct elements are comparable. An *anti-chain* is a subset of a partially ordered set such that any two distinct elements are incomparable.

▶ **Theorem 12** (Dilworth's Theorem, [9]). *Let P be a partially ordered set of at least $(s-1)(t-1)+1$ elements. Then P contains a chain of size s or an antichain of size t.*

▶ **Theorem 13.** *Let s, t be two integers, $1 \leq s, t \leq n$, such that $(s-1)(t-1)+1 \leq n$. Let D be a c-monotone drawing of K_n. Then D contains either a generalized twisted drawing of K_s or a monotone drawing of K_t as subdrawing. In particular, if $s = t = \lceil\sqrt{n}\rceil$, D contains a complete subgraph K_s whose induced drawing is either generalized twisted or monotone.*

The full proof of Theorem 13 can be found in Appendix C

Proof Sketch. Without loss of generality we may assume that the vertices of D appear counterclockwise around O in the order $v_1, v_2, \ldots, v_n$. Let r be a ray emanating from O, keeping v_1 and v_n on different sides. We define an order, $\preceq$, in this set of vertices as follows: $v_i \preceq v_j$ if and only if either $i = j$ or $i < j$ and the edge (v_i, v_j) crosses r.

We show that $\preceq$ is a partial order. The relation is clearly reflexive and antisymmetric. Besides, if $v_i \preceq v_j$ and $v_j \preceq v_k$, then $v_i \preceq v_k$, because $i < j$ and $j < k$ imply $i < k$, and if v_iv_j and v_jv_k cross r, then v_iv_k also crosses r (see Figure 7). Hence, the relation is transitive.

In this partial order $\preceq$, a chain consists of a subset $v_{i_1}, \ldots, v_{i_{s-1}}$ of pairwise comparable vertices, that is, a subset of vertices such that their induced subdrawing is generalized twisted (all edges cross r). An antichain, $v_{j_1}, \ldots, v_{j_{t-1}}$, consists of a subset of pairwise incomparable vertices, that is, a subset of vertices such that their induced subdrawing is monotone (no edge crosses r). Therefore, the first part of the theorem follows from applying Theorem 12 to the set of vertices of D and the partial order $\preceq$.

Finally, observe that if $s = t \leq \lceil\sqrt{n}\rceil$, then $(s-1)(t-1)+1 \leq n$. Thus, D contains a complete subgraph $K_{\lceil\sqrt{n}\rceil}$ whose induced subdrawing is either generalized twisted or monotone. ◀

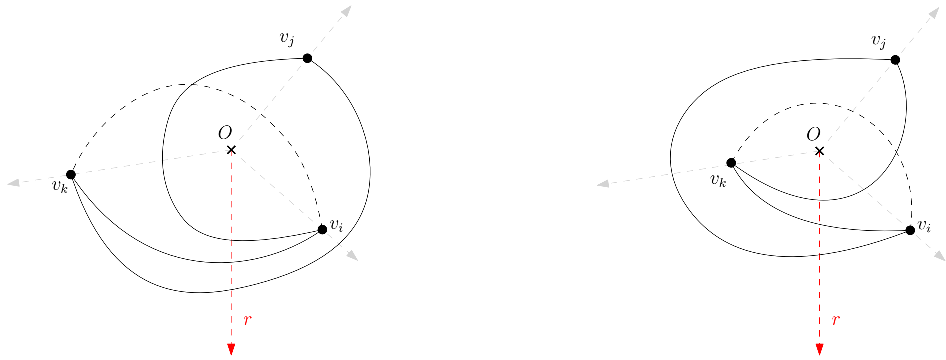


Figure 7 If edges v_iv_j and v_jv_k cross r in a c-monotone drawing, then v_iv_k must also cross r.

Combining Theorems 3 and 13 with Observation 11, we obtain the following theorem.

▶ **Theorem 14.** *Every c-monotone drawing of K_n contains a plane path of length $\Omega(\sqrt{n})$.*

4.2 Plane Paths in Simple Drawings

To show that any simple drawing of K_n contains a plane path of length $\Omega(\frac{\log n}{\log \log n})$, we will use d-ary trees. A d-ary tree is a rooted tree in which no vertex has more than d children. It is well-known that the height of a d-ary tree on n vertices is $\Omega(\frac{\log n}{\log d})$.

Proof of Theorem 10. Let v be a vertex of D and let $S(v)$ be the star centered at v, that is, the set of edges of D incident to v. $S(v)$ can be extended to a maximal plane subdrawing H that must be biconnected by Theorem 5. See Figure 8 for a depiction of $S(v)$ and H.

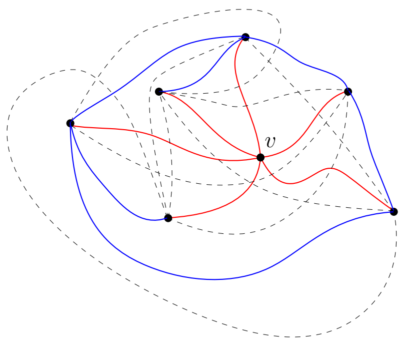


Figure 8 A simple drawing of K_7. The red edges show the star $S(v)$, the red and blue edges together form a maximal plane subdrawing H. Dashed edges are edges of K_7 that are not in H.

Assume first that there is a vertex w in $H \setminus v$ that has degree at least $(\log n)^2$ in H. Let U_{vw} be the set of vertices neighboured in H to both, v and w. Note that $|U_{vw}| \geq (\log n)^2$. The subdrawing H' of H consisting of the vertices in U_{vw}, the vertices v, and w, and the edges from v to vertices in U_{vw}, and from w to vertices in U_{vw} is a plane drawing of $K_{2,|U_{vw}|}$. From Lemma 4, it follows that the subdrawing of D induced by U_{vw} is weakly isomorphic to a c-monotone drawing. Therefore, by Theorem 14, the subdrawing induced by U_{vw} contains a plane path of length $\Omega(\sqrt{|U_{vw}|}) = \Omega(\log n)$.

Assume now that the maximum degree in $H \setminus v$ is less than $(\log n)^2$. Since H is biconnected, $H \setminus v$ contains a plane tree T of order $n-1$ whose maximum degree is at most $(\log n)^2$. Thus, considering that T is rooted, the diameter of T is at least $\Omega(\frac{\log n}{\log\log n})$. Therefore, since T is plane, it contains a plane path of length at least $\Omega(\frac{\log n}{\log\log n})$ and the theorem follows. ◀

5 Characterizing Generalized Twisted Drawings

In previous sections, we have seen how generalized twisted drawings were used to make progress on open problems of simple drawings. In addition to this, generalized twisted drawings are also interesting in their own right and have some quite surprising structural properties. Despite the fact that research on generalized twisted drawings is rather recent and still ongoing, there are already several interesting characteristics and structural results. Some of them will be presented in this section.

One characterization involves curves crossing every edge once. From the definition of generalized twisted drawing (see Figure 1), there always exists a simple curve that crosses all edges of the drawing exactly once (for instance, a curve that starts at O and follows r until it reaches a point Z on r in the unbounded cell). In Theorem 15, we show that the converse is also true. That is, every simple drawing D of K_n in which we can add a simple curve that crosses every edge of D exactly once is weakly isomorphic to a generalized twisted drawing.

Another characterization is based on what we call *antipodal vi-cells*. For any three vertices in a simple drawing D of K_n, the three edges connecting them form a simple cycle which we call a *triangle*. Every such triangle partitions the plane (or sphere) into two disjoint regions which are the *sides* of the triangle (in the plane a bounded and an unbounded one). Two cells of D are called *antipodal* if for each triangle of D, they lie on different sides. Further, we call a cell with a vertex on its boundary a vertex-incident-cell or, for short, a *vi-cell*.

By definition, every generalized twisted drawing D contains two antipodal cells, namely, the cell containing the starting point of the ray r and the unbounded cell. This follows from the fact that the ray r crosses every edge exactly once. Hence, r crosses the boundary of any triangle exactly three times, so the cells containing the "endpoints" of r must be on different sides of the triangle.

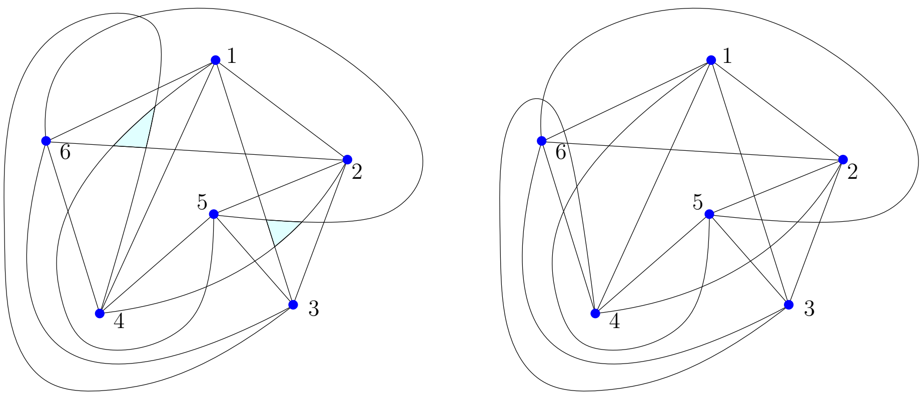


Figure 9 Two weakly isomorphic drawings of K_6 that are not weakly isomorphic to any generalized twisted drawing. Antipodal cells are marked in blue.

It turns out that the converse (existence of two antipodal cells implies weakly isomorphic to generalized twisted) is not true. Figure 9 (left) shows a drawing of K_6 that contains two antipodal cells, but no antipodal vi-cells. From Theorem 15 bellow it will follow that such drawings cannot be weakly isomorphic to a generalized twisted drawing. However, we observed that for all generalized twisted drawings of K_n with $n \leq 6$, both, the cell containing the startpoint of the ray r and the unbounded cell, are vi-cells. Figure 10 shows all (up to strong isomorphism) simple drawings of K_6 that are weakly isomorphic to generalized twisted drawings. We show that this is true in general. More than that, we show in Theorem 16 that every drawing of K_n that is weakly isomorphic to a generalized twisted drawing contains a pair of antipodal vi-cells. In the other direction, we show in Theorem 15 that every simple drawing containing a pair of antipodal vi-cells is weakly isomorphic to a generalized twisted drawing.

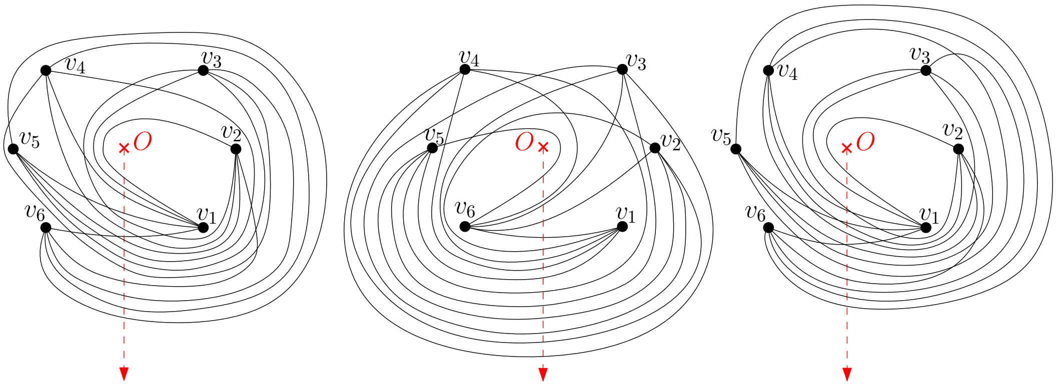


Figure 10 All different generalized twisted drawings of K_6 (up to weak isomorphism). The rightmost drawing is twisted.

The final characterization is based on the extension of a given drawing of the complete graph to a drawing containing a spanning, plane bipartite graph that has all vertices of the original drawing on one side of the bipartition. From the definition of generalized twisted drawings, it follows that any genereralized twisted drawing D of K_n can be extended to a simple drawing D' of K_{n+2} including new vertices O and Z such that D' contains a plane drawing of a spanning bipartite graph. One side of the bipartition consists of all vertices in D and the other side of the bipartition consists of the new vertices O and Z. Moreover, the edge OZ crosses all edges of D. One way to add the new vertices and edges incident to them is to draw (1) the vertex O at point O, (2) the vertex Z in the unbounded cell on the ray r, (3) the edge OZ straight-line (along the ray r), (4) edges from O to the vertices of D straight-line (along the inner segment of the rays crossing through the vertices), and (5) edges from Z to the vertices of D first far away in a curve and the final part straight-line (along the outer segment of the rays crossing through the vertices). The converse, that every drawing that can be extended like this is weakly isomorphic to a generalized twisted drawing, has already been shown in Lemma 4.

We show the following characterizations.

▶ **Theorem 15** (Characterizations of generalized twisted drawings). *Let D be a simple drawing of K_n. Then, the following properties are equivalent.*

Property 1 *D is weakly isomorphic to a generalized twisted drawing.*

Property 2 *D contains two antipodal vi-cells.*

Property 3 *D can be extended by a simple curve c such that c crosses every edge of D exactly once.*

Property 4 *D can be extended by two vertices, O and Z, and edges incident to the new vertices such that D together with the new vertices and edges is a simple drawing of K_{n+2}, the edge OZ crosses every edge of D, and no edge incident to O crosses any edge incident to Z.*

To prove Theorem 15, we will first show that *Property 1* implies *Property 2* (Theorem 16). We next show that *Property 2* implies *Property 3* (Theorem 17). Then, we show that *Property 3* implies *Property 4* (Theorem 18). By Lemma 4, *Property 4* implies *Property 1*. Thus, all properties are equivalent. In a full version of this work, we will extend the theorem to show that also strong isomorphism to a generalized twisted drawing is equivalent to the properties of Theorem 15. We show this by proving that any simple drawing of K_n fulfilling *Property 4* is strongly isomorphic to a generalized twisted drawing. However, the reasoning for strong isomorphism is quite lengthy and would exceed the space constraints of this submission.

In the remaining parts of this section, we will show sketches of the proofs of the above mentioned theorems. The full proofs can be found in the Appendix (Theorem 16 in Appendix D, Theorem 17 in Appendix E, and Theorem 18 in Appendix F).

▶ **Theorem 16.** *Every simple drawing of K_n which is weakly isomorphic to a generalized twisted drawing of K_n, with $n \geq 3$, contains a pair of antipodal vi-cells. In generalized twisted drawings the cell containing O and the unbounded cell form such a pair.*

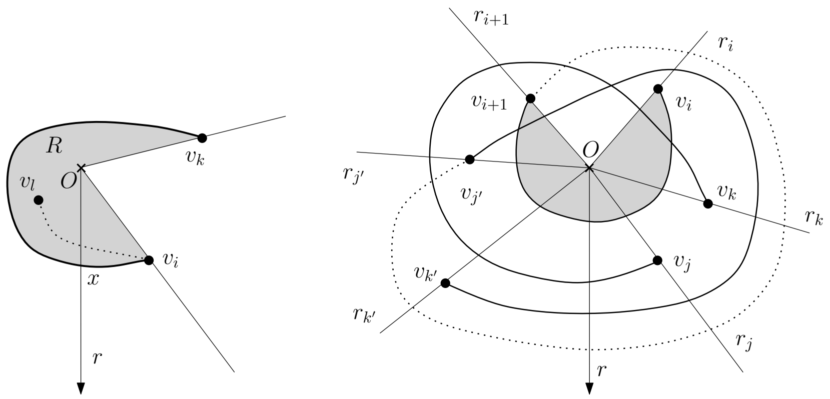


Figure 11 Left: If there is a vertex v_l in R, it cannot be connected to v_i without crossing r before x. Right: If the edge $v_j v_k$ crosses the segment $\overline{Ov_i}$ and the edge $v_{j'} v_{k'}$ crosses the segment $\overline{Ov_{i+1}}$, then there is no way of connecting v_{i+1} and $v_{j'}$.

Proof sketch. We first show that every generalized twisted drawing D of K_n, with $n \geq 3$, contains a pair of antipodal vi-cells, where O lies in a cell of that pair. Let c be the segment OZ, where Z is a point on r in the unbounded cell. By definition of generalized twisted, c crosses every edge of D once, so O and Z are in two antipodal cells C_1 and C_2, respectively.

To prove that C_1 is a vi-cell, we use the following properties. First, if we take the first edge $v_i v_k$ that crosses c (as seen from O) at point x, then we can prove that $k = i + 1$ and the bounded region R defined by the edge $v_i v_{i+1}$ and the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is empty (see Figure 11, left). Second, using this empty region we can prove that D cannot contain simultaneously an edge $v_j v_k$ crossing $\overline{Ov_i}$ and another edge $v_{j'} v_{k'}$ crossing $\overline{Ov_{i+1}}$ (see Figure 11, right). Therefore, at least one of the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is uncrossed, and O necessarily lies in a vi-cell (with either v_i or v_{i+1} on the boundary). Finally, arguing on the last edge crossing c and the unbounded cell, we can show that Z also lies in a vi-cell.

To show that also every drawing which is weakly isomorphic to a generalized twisted drawing contains a pair of antipodal vi-cells, we use Gioan's Theorem [6, 15]. By Gioan's Theorem, any two weakly isomorphic drawings of K_n can be transformed into each other with a sequence of triangle-flips and at most one reflection of the drawing. A *triangle-flip* is

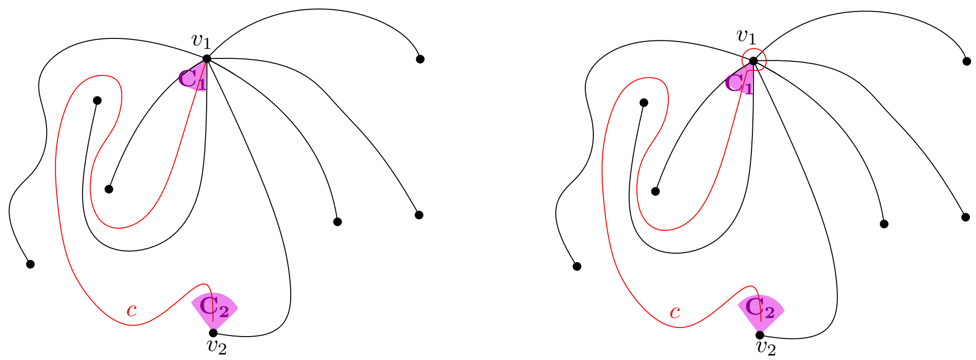


Figure 12 Building a curve such that it crosses every edge of D once and its endpoints do not lie on any edges or vertices of D.

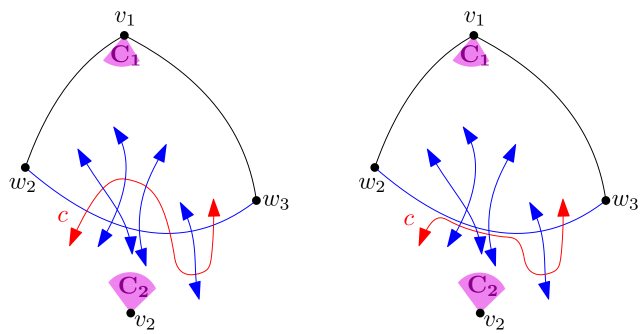


Figure 13 Decreasing the number of crossings between c and the edge w_2w_3.

an operation which transforms a triangular cell $\triangle$ that has no vertex on its boundary by moving one of its edges across the intersection of the two other edges of $\triangle$. We show that if a drawing D_1 contains two antipodal vi-cells, then after performing a triangle flip on D_1, the resulting drawing D_2 still has two antipodal vi-cells. The main argument is that triangle-flips are only applied to cells without vertices on their boundary, and thus the antipodality of the vi-cells cannot change. ◀

▶ **Theorem 17.** *In any simple drawing D of K_n that contains a pair of antipodal vi-cells, it is possible to draw a curve c that crosses every edge of D exactly once.*

Proof sketch. Let (C_1, C_2) be a pair of antipodal vi-cells of D. Let v_1 be a vertex on the boundary of C_1 and v_2 a vertex on the boundary of C_2. We construct the curve as follows: First, we draw a simple curve c from C_1 to C_2 such that (1) it emanates from v_1 in C_1 and ends in C_2 very close to v_2, (2) does not cross any edge incident to v_1, (3) only intersects edges of D in proper crossings, and (4) has the minimum number of crossings with edges of D among all curves that fulfill (1), (2) and (3). This curve c always exists since $S(v_1)$ is a plane drawing that has only a face in which both v_1 and v_2 lie (see Figure 12, left).

Then, we prove that c crosses every edge w_2w_3 in D that is not incident to v_1 exactly once. On the one hand, since c connects two antipodal cells, the endpoints of c have to be on two different sides of the triangle T formed by v_1, w_2 and w_3. Thus, c has to cross w_2w_3 an odd number of times because it does not cross $S(v_1)$ and must cross the boundary of T an odd number of times. On the other hand, if c crosses w_2w_3 at least three times, then we can prove that c can be redrawn as shown in Figure 13, decreasing the number of crossings,

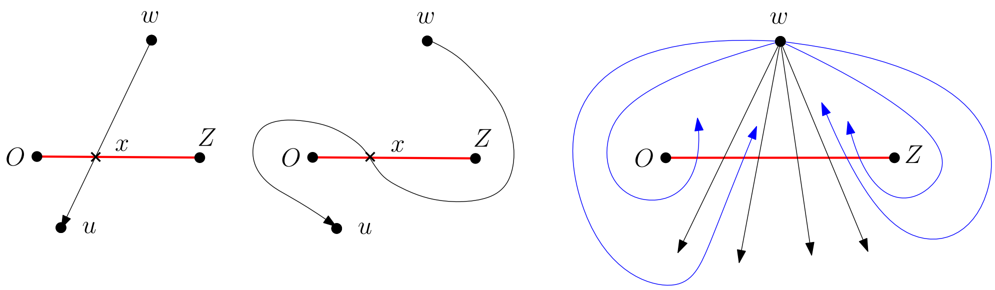


Figure 14 Top and bottom edges. For simplicity, the curve OZ is drawn as a horizontal line. Left: A top edge wu. Centre: A bottom edge wu. Right: The (black) top and (blue) bottom edges of $S(w)$.

which contradicts (4). Therefore, c crosses every edge w_2w_3 at most twice and, consequently, only once.

Finally, we change the end of c from v_1 to a point in C_1 in the following way (see Figure 12, right). From some point of c sufficiently close to v_1 and inside C_1, we reroute c by going around v_1 such that only the edges incident to v_1 are crossed, and end at a point in C_1. ◀

▶ **Theorem 18.** *Let D be a simple drawing of K_n in which it is possible to draw a simple curve c that crosses every edge of D exactly once. Then, D can be extended by two vertices O and Z (at the position of the endpoints of the curve), and edges incident to those vertices such that the obtained drawing is a simple drawing of K_{n+2}, no edge incident to O crosses any edge incident to Z, and all edges in D cross the edge OZ.*

Proof sketch. Let $c = OZ$ be the curve crossing every edge of D once, oriented from O to Z. Let wu be an edge of D, oriented from w to u, crossing OZ at a point x. We say that wu is a *top* (respectively *bottom*) edge if the clockwise order of w, Z, u and O around x is w, Z, u, O (respectively w, O, u, Z). See Figure 14. With these definitions, we can prove that there is a vertex w_1 in D such that all the oriented edges emanating from w_1 are top in relation to c. Thus, by removing w_1 and all its incident edges from D, there is a vertex w_2 in the new drawing such that all its incident edges are top, and so on. As a consequence, there is a *natural* order $w_1, w_2, \ldots, w_n$ of the vertices of D such that for any vertex w_i, the edges w_iw_j with $j > i$ are top, and the edges w_iw_j with $j < i$ are bottom.

Given the natural order $w_1, w_2, \ldots, w_n$, our construction of the extended drawing is as follows. Let D'_0 be the simple drawing formed by the vertices and edges of D, O and Z as new vertices, and c as the edge connecting O and Z. From D'_0, we build new drawings $D'_1, D'_2, \ldots, D'_n$, by adding in step i the edges w_iO and w_iZ. These two edges are added very close to some edges in D'_{i-1}. Figure 15 illustrates how these two edges are added in each step.

In the first step, the edge Ow_1 follows the curve OZ until the crossing point between OZ and the first top edge w_1u emanating from w_1, and then it follows this top edge until reaching w_1. The edge Zw_1 is built in an analogous way, taking the last top edge emanating from w_1. See Figure 15 top-left. For $i = 2, \ldots, n-1$, in step i we do different constructions depending on whether the first and last top edges of $S(w_i)$ cross the edges $w_{i-1}O$ and $w_{i-1}Z$. If the first top edge w_iu_1 crosses $w_{i-1}O$ at a point x and the last top edge w_iu_k crosses $w_{i-1}Z$ at a point y (see Figure 15 top-right), then Ow_i follows Ow_{i-1} until x, and then it follows u_1w_i until w_i. The edge Zw_i is built following Zw_{i-1} until y and then following u_kw_i. On the contrary, if the first and the last top edges of $S(w_i)$ only cross one of $w_{i-1}O$

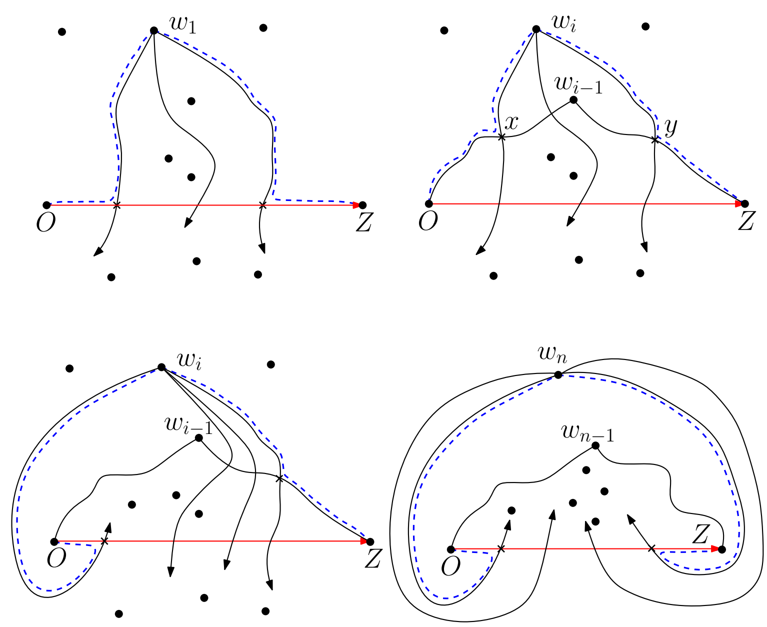


Figure 15 Building the (dashed) edges w_iO and w_iZ.

and $w_{i-1}Z$, say $w_{i-1}Z$ (see Figure 15 bottom-left), then Ow_i follows OZ until the crossing point between OZ and the last bottom edge of $S(w_i)$, and then it follows this bottom edge until w_i. The edge Zw_i is built as in the first step, using the last top edge of $S(w_i)$. In the last step, we build Ow_n and Zw_n as in the first step, but using the first and the last bottom edges of $S(w_n)$ instead of the first and last top edges. See Figure 15 bottom-right.

By a detailed analysis of cases, we can prove for $i = 1, \dots, n$ that D'_i is a simple drawing such that no edge incident to O crosses any edge incident to Z. Therefore, D'_n is the drawing of K_{n+2} satisfying the required properties. ◀

6 Conclusion and Outlook

Generalized twisted drawings have a suprisingly rich structure and many useful properties. We showed several of those properties in Section 2 and different characterizations of generalized twisted drawings in Section 5. We have proven in Section 2 that every generalized twisted drawing on an odd number of vertices contains a plane Hamiltonian cycle, and therefore one especially interesting open question is the following.

▶ **Conjecture 19.** *Every generalized twisted drawing of K_n contains a plane Hamiltonian cycle.*

Using properties of generalized twisted drawings has turned out to be helpful for investigating simple drawings in general. We first improved the lower bound on the number of disjoint edges in simple drawings of K_n to $\Omega(\sqrt{n})$ (Section 3). Then generalized twisted drawings played the central role to improve the lower bound on the length of plane paths contained in every simple drawing of K_n to $\Omega(\frac{\log n}{\log \log n})$ (Section 4).

On the other hand, from Theorem 17 it immediately follows that no drawing that is weakly isomorphic to a generalized twisted drawing can contain three interior-disjoint triangles (since the endpoints of the curve crossing every edge once must be on opposite sides of every triangle, the maximum number of interior-disjoint triangles is two). Up to strong isomorphism, there are only two simple drawings of K_4. The plane drawing contains three interior-disjoint triangles. Thus, (up to strong isomorphism) the only drawing of K_4 that is weakly isomorphic to a generalized twisted drawing, is the drawing with a crossing. Hence, in every generalized twisted drawing all subdrawings induced by 4 vertices contain a crossing and thus every generalized twisted drawing is crossing maximal. Up to strong isomorphism, there are two crossing maximal drawings of K_5: the convex drawing of K_5 and the twisted drawing of K_5. Since the convex drawing contains three interior-disjoint triangles, the only (up to strong isomorphism) drawing of K_5 that is weakly isomorphic to a generalized twisted drawing is the twisted drawing of K_5 (that is drawn generalized twisted in Figure 1).

It is part of our ongoing work to show that for $n \geq 7$, a drawing is weakly isomorphic to a generalized twisted drawing if and only if all subdrawings induced by five vertices are weakly isomorphic to the twisted K_5. Interestingly, the $n \geq 7$ is necessary as there is a drawing with 6 vertices that contains only twisted drawings of K_5 but is not weakly isomorphic to a generalized twisted drawing (see the drawings in Figure 9). There are (up to strong isomorphism) three more simple drawings of K_6 that consist of only twisted drawings of K_5 and they are all weakly isomorphic to generalized twisted drawings (see Figure 10).

References

1. Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Thomas Hackl, Jürgen Pammer, Alexander Pilz, Pedro Ramos, Gelasio Salazar, and Birgit Vogtenhuber. All good drawings of small complete graphs. In *Proc. 31^{st} European Workshop on Computational Geometry EuroCG '15*, pages 57–60, Ljubljana, Slovenia, 2015. URL: `http://www.ist.tu-graz.ac.at/files/publications/geometry/aafhpprsv-agdsc-15.pdf`.
2. Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. Shellable drawings and the cylindrical crossing number of K_n. *Discrete & Computational Geometry*, 52(4):743–753, 2014. `doi:10.1007/s00454-014-9635-0`.
3. Oswin Aichholzer, Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Plane matchings in simple drawings of complete graphs. In *Abstracts of the Computational Geometry: Young Researchers Forum*, pages 6–10, 2021. URL: `https://cse.buffalo.edu/socg21/files/YRF-Booklet.pdf#page=6`.
4. Oswin Aichholzer, Alfredo García, Javier Tejel, Birgit Vogtenhuber, and Alexandra Weinberger. Plane paths in simple drawings of complete graphs. In *Abstracts of XIX Encuentros de Geometría Computacional*, page 4, 2021. URL: `https://quantum-explore.com/wp-content/uploads/2021/06/Actas_egc21.pdf#page=11`.
5. Oswin Aichholzer, Thomas Hackl, Alexander Pilz, Gelasio Salazar, and Birgit Vogtenhuber. Deciding monotonicity of good drawings of the complete graph. In *Abstracts XVI Spanish Meeting on Computational Geometry (XVI EGC*, pages 33–36, 2015.
6. Alan Arroyo, Dan McQuillan, R. Bruce Ritcher, and Gelasio Salazar. Drawings of K_n with the same rotation scheme are the same up to Reidemeister moves (Gioan's theorem). *Australasian Journal of Combinatorics*, 67:131–144, 2017.
7. Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete Comput. Geom.*, 53(1):107–143, 2015. `doi:10.1007/s00454-014-9644-z`.
8. Morton Brown. A proof of the generalized Schoenflies theorem. *Bulletin of the American Mathematical Society*, 66(2):74 – 76, 1960. `doi:10.1090/S0002-9904-1960-10400-4`.
9. Robert P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161–166, 1950. `doi:10.2307/1969503`.
10. Jacob Fox and Benny Sudakov. Density theorems for bipartite graphs and related ramsey-type results. *Combinatorica*, 29(2):153–196, 2009. `doi:10.1007/s00493-009-2475-5`.

11 Radoslav Fulek. Estimating the number of disjoint edges in simple topological graphs via cylindrical drawings. *SIAM Journal on Discrete Mathematics*, 28(1):116–121, 2014. doi:10.1137/130925554.

12 Radoslav Fulek, Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Hanani-Tutte, monotone drawings, and level-planarity. In *Thirty essays on geometric graph theory*, pages 263–287. Springer, New York, NY, 2013. doi:10.1007/978-1-4614-0110-0_14.

13 Radoslav Fulek and Andres J. Ruiz-Vargas. Topological graphs: empty triangles and disjoint matchings. In *Proceedings of the 29th Annual Symposium on Computational Geometry (SoCG'13)*, pages 259–266, 2013. doi:10.1145/2462356.2462394.

14 Alfredo García, Alexander Pilz, and Javier Tejel. On plane subgraphs of complete topological drawings. *ARS MATHEMATICA CONTEMPORANEA*, 20:69–87, 2021. doi:10.26493/1855-3974.2226.e93.

15 Emeric Gioan. Complete graph drawings up to triangle mutations. In *Graph-Theoretic Concepts in Computer Science. WG 2005. Lecture Notes in Computer Science, vol 3787*, pages 139–150. Springer, 2005. doi:10.1007/11604686_13.

16 Barry Mazur. On embeddings of spheres. *Bulletin of the American Mathematical Society*, 65:59–65, 1959. doi:10.1090/S0002-9904-1959-10274-3.

17 János Pach, József Solymosi, and Gézak Tóth. Unavoidable configurations in complete topological graphs. *Discrete Comput Geometry*, 30:311–320, 2003. doi:10.1007/s00454-003-0012-9.

18 János Pach and Géza Tóth. Disjoint edges in topological graphs. In *Proceedings of the 2003 Indonesia-Japan Joint Conference on Combinatorial Geometry and Graph Theory (IJCCGGT'03)*, volume 3330, pages 133–140, 2005. doi:10.1007/978-3-540-30540-8_15.

19 János Pach and Géza Tóth. Monotone crossing number. In *Graph Drawing*, pages 278–289. Springer Berlin Heidelberg, 2012. doi:10.1007/978-3-642-25878-7_27.

20 Nabil H. Rafla. *The good drawings D_n of the complete graph K_n*. PhD thesis, McGill University, Montreal, 1988. URL: https://escholarship.mcgill.ca/concern/file_sets/cv43nx65m?locale=en.

21 Andres J. Ruiz-Vargas. Empty triangles in complete topological graphs. In *Discrete Computational Geometry*, volume 53, pages 703–712, 2015. doi:10.1007/s00454-015-9671-4.

22 Andres J. Ruiz-Vargas. Many disjoint edges in topological graphs. *Computational Geometry*, 62:1–13, 2017. doi:10.1016/j.comgeo.2016.11.003.

23 Andrew Suk. Disjoint edges in complete topological graphs. *Discrete & Computational Geometry*, 49(2):280–286, 2013. doi:10.1007/s00454-012-9481-x.

A Proof of Lemma 2

▶ **Lemma 2.** *Let D be a generalized twisted drawing of K_4, with vertices $\{v_1, v_2, v_3, v_4\}$ labeled counterclockwise around O. Then the edges v_1v_3 and v_2v_4 do not cross.*

Proof. Assume, for a contradiction, that the edge v_2v_4 crosses the edge v_1v_3. Since any simple drawing of K_4 has at most one crossing, no other edges of D can cross. Recall that in any generalized twisted drawing, all edges are drawn c-monotone and intersect the ray r. For every edge, this determines in which direction it emanates from its vertices. Hence there are (up to strong isomorphism) two possibilities how the crossing edges v_1v_3 and v_2v_4 can be drawn in D, depending on whether v_1v_3 crosses the ray from O through v_4 at a point x_3 before or after v_4; cf. Figure 16. In both cases, v_1v_2 has to cross the ray from O through v_4 at a point x_2. This point x_2 has to lie after v_4 in the first case and before v_4 in the second case. In both cases, as the edge v_3v_4 has to cross r, it must emanate from v_4 in the interior of the triangular region bounded by the segment x_2x_3, the portion v_1x_3 of v_1v_3, and the portion v_1x_2 of v_1v_2. However, the vertex v_3 is in the exterior of that triangular region, and therefore v_3v_4 would have to cross the segment x_2x_3, contradicting that D is c-monotone, or one of v_1v_2 and v_1v_3, contradicting the simplicity of D. ◀

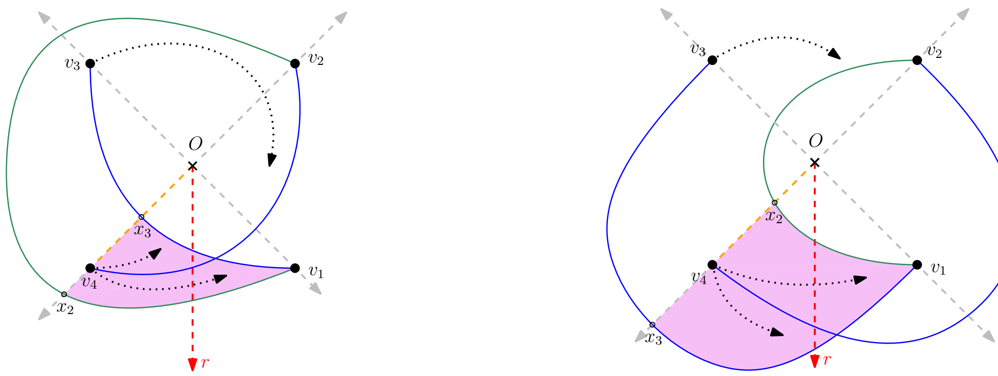


Figure 16 The two possibilities to draw v_1v_3 and v_2v_4 crossing and generalized twisted.

B Proof of Lemma 4

Lemma 4 has been implicitly shown in [11] and [13]. For completeness, we include a detailed proof of the lemma in this appendix. We remark that the proof presented here is in parts similar to the one in [13].

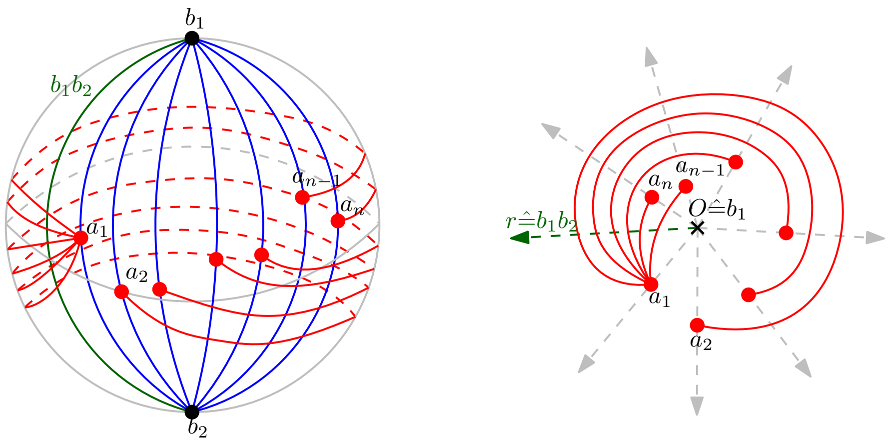


Figure 17 The homeomorphisms of D'. Left: D_A, the edges in R_A and b_1b_2 are drawn on the sphere, such that R_A and b_1b_2 are meridians. Right: The steographic projecton from b_2.

▶ **Lemma 4.** *Let D be a simple drawing of a complete graph containing a subdrawing D', which is a plane drawing of $K_{2,n}$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2\}$ be the sides of the bipartition of D'. Let D_A be the subdrawing of D induced by the vertices of A. Then D_A is weakly isomorphic to a c-monotone drawing. Moreover, if all edges in D_A cross the edge b_1b_2, then D_A is weakly isomorphic to a generalized twisted drawing.*

Proof. We call the pair of edges in D' incident to a_i, $1 \le i \le n$, the long edge r_i. Let R_A be the set of long edges. We first show that any edge between vertices in A crosses any long edge at most once. Then we show how to draw D' such that b_1 can be taken as origin O in a c-monotone drawing weakly isomorphic to D_A, where the long edges of R_A, as well as the edge b_1b_2, emanate as rays to infinity.

We now show that every edge between two vertices of A crosses every edge of R_A at most once. Let a_1, a_2, and a_3 be vertices in A. Let R_1 be the region bounded by the edges

b_1a_1, a_1b_2, b_2a_2 and a_2b_1 that does not contain a_3. Let R_2 be the region bounded by the edges b_1a_2, a_2b_2, b_2a_3 and a_3b_1 that does not contain a_1. Since D' is plane, these regions are disjoint.

As the edge $e = a_1a_2$ is incident to all edges on the boundary of R_1, it cannot cross it. Thus, e has to lie either completely inside or completely outside R_1 (and meet the boundary only in its endvertices). If e lies inside R_1, it can cross neither a_3b_1 nor a_3b_2. If it lies outside R_1, it has to cross the boundary of R_2 an odd number of times. (Since e must begin at a_1 outside R_2 and finish at a_2 inside R_2, and passing through R_1 is not possible.) As e cannot cross edges incident to a_2, this means it has to cross exactly one of the edges a_3b_1 or a_3b_2. Thus, e crosses the long edge r_3 at most once, for any vertex a_3.

We can draw D' such that b_1 is functioning as the origin and R_A as rays emerging from it by doing the following transformations; see Figure 17. We draw the subdrawing induced by the vertices of D' on the sphere such that b_1 and b_2 are antipodes, and the long edges of R_A, as well as the edge b_1b_2, are meridians. By the general Jordan-Schoenflies theorem [16, 8], the drawing on the sphere is homeomorphic to the original drawing on the plane. We then apply a stereographic projection from b_2 onto the plane. This way, the long edges in R_A and the edge b_1b_2 correspond to rays emerging from vertex b_1, where the long edges in R_A are exactly the rays through the vertices of D'.

Finally, we can obtain a c-monotone drawing that is weakly isomorphic to D_A. We consider the stereographic projection. As all edges of D_A cross the long edges in R_A only once, they cross in between two long edges (or rays in the projection) r_1 and r_2 if and only if their order along the rays changes (that is, the edge closer to b_1 at r_1 is further away from b_1 at r_2). Consequently, we can draw the edge-segments between every two rays as straight-lines and obtain a c-monotone drawing that is weakly isomorphic to D_A. If all edges of D_A cross the edge b_1b_2, they cross a ray to infinity in the weakly isomorphic c-monotone drawing, and thus the c-monotone drawing is also generalized twisted. ◀

C Proof of Theorem 13

▶ **Theorem 13.** *Let s, t be two integers, $1 \le s, t \le n$, such that $(s-1)(t-1)+1 \le n$. Let D be a c-monotone drawing of K_n. Then D contains either a generalized twisted drawing of K_s or a monotone drawing of K_t as subdrawing. In particular, if $s = t = \lceil \sqrt{n} \rceil$, D contains a complete subgraph K_s whose induced drawing is either generalized twisted or monotone.*

Proof. Without loss of generality we may assume that the vertices of D appear counterclockwise around O in the order $v_1, v_2, \ldots, v_n$. Let r be a ray emanating from O, keeping v_1 and v_n on different sides. We define an order, $\preceq$, in this set of vertices as follows: $v_i \preceq v_j$ if and only if either $i = j$ or $i < j$ and the edge (v_i, v_j) crosses r.

We show that $\preceq$ is a partial order. The relation is clearly reflexive and antisymmetric. Besides, if $v_i \preceq v_j$ and $v_j \preceq v_k$, then $i < j$ and $j < k$ imply $i < k$, so for the transitive property, we only have to prove that if v_iv_j and v_jv_k cross r, then v_iv_k also crosses r. We denote by r_i, r_j, r_k the rays emanating from O and passing trough v_i, v_j, v_k, respectively. We have two cases depending on where v_jv_i crosses the ray r_k at a point x_k; in the first case, x_k is located before v_k on r_k, while in the second one it is located after v_k. Then v_jv_k has to cross the ray r_i at a point x_i, which is after v_i in the first case and before v_i in the second case (see Figure 18). Let Q be the region bounded by the segments Ox_i, Ox_k and the portions v_jx_i, v_jx_k of the edges v_jv_k, v_jv_i, respectively. In both cases, the edge v_kv_i cannot be contained in the counterclockwise wedge from r_i to r_k, because v_iv_k should connect a vertex placed outside Q with points placed inside that region, contradicting either

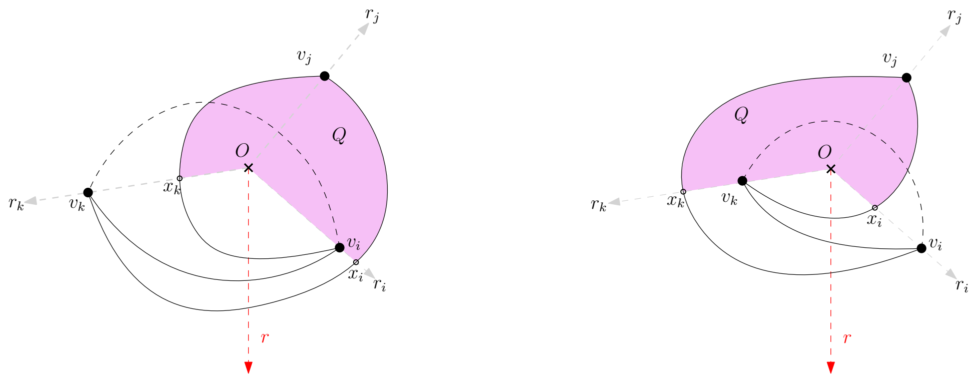


Figure 18 If edges v_iv_j and v_jv_k cross r in a c-monotone drawing, then v_iv_k must also cross r.

the simplicity or the c-monotonicity of D. Therefore, v_iv_k must be in the clockwise wedge from r_i to r_k and thus crosses the ray r.

In this partial order $\preceq$, a chain consists of a subset $v_{i_1}, \ldots, v_{i_{s-1}}$ of pairwise comparable vertices, that is, a subset of vertices such that their induced subdrawing is generalized twisted (all edges cross r). An antichain, $v_{j_1}, \ldots, v_{j_{t-1}}$, consists of a subset of pairwise incomparable vertices, that is, a subset of vertices such that their induced subdrawing is monotone (no edge crosses r). Therefore, the first part of the theorem follows from applying Theorem 12 to the set of vertices of D and the partial order $\preceq$.

Finally, observe that if $s = t \leq \lceil\sqrt{n}\rceil$, then $(s-1)(t-1)+1 \leq n$. Thus, D contains a complete subgraph $K_{\lceil\sqrt{n}\rceil}$ whose induced subdrawing is either generalized twisted or monotone. ◀

D Generalized twisted drawings contain a pair of antipodal vi-cells

In this section, we will show that every drawing weakly isomorphic to a generalized drawing of K_n contains a pair of antipodal vi-cells (Theorem 16). Before proving the theorem, we will see some useful properties of generalized twisted drawings. Recall that in a generalized twisted drawing, vertices are labeled $v_1, v_2, \ldots, v_n$ counterclockwise around the origin O, the ray emanating from O and passing through a vertex v_i is denoted by r_i, and the ray r that emanates from O and crosses every edge once is between r_n and r_1, counterclockwise from r_n.

▶ **Lemma 20.** *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Suppose the two edges v_iv_j and v_kv_l of D cross, and $i < j < k < l$. Then the crossing point between these two edges is in the wedge W defined by r_j and r_k, counterclockwise from r_j to r_k.*

Proof. Assume for contradiction that the crossing point is not in W, so it is in the wedge defined by r_l and r_i, counterclockwise from r_l. There are four cases, depending on whether v_k and v_l are to the left or the right of the directed edge v_jv_i; see Figure 19. In any of the four cases, there is no way of connecting v_k and v_j without crossing either v_iv_j or v_kv_l, which is a contradiction. ◀

▶ **Lemma 21.** *For every generalized twisted drawing D with $n \geq 3$ vertices, the following statements hold.*

i) There exists a vertex v_i, with $1 \leq i \leq n-1$, such that the bounded region RB defined by the edge v_iv_{i+1} and the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is empty.

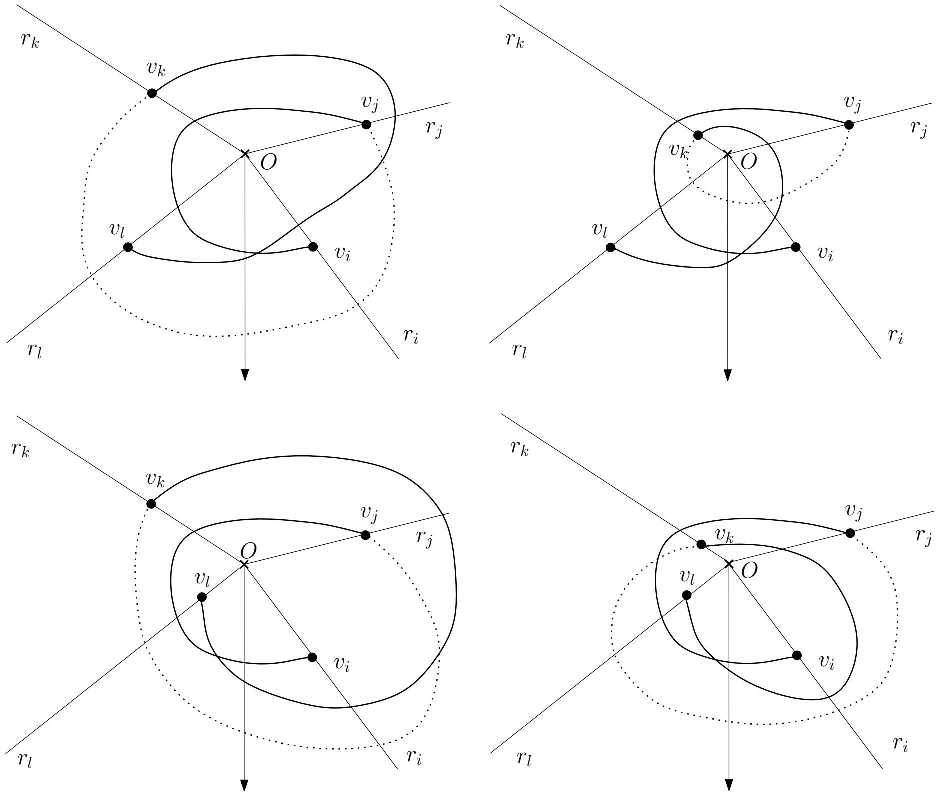


Figure 19 Illustrating the proof of Lemma 20.

ii) There exists a vertex $v_j \neq v_i$, with $1 \leq j \leq n-1$, such that the unbounded region RU defined by the edge $v_j v_{j+1}$ and the segments $\overline{Ov_j}$ and $\overline{Ov_{j+1}}$ is empty.

Proof. We show statement *i)*, we take the first edge $v_i v_k$ (with $i < k$) that crosses r and we show that v_i satisfies *i)*. Let x be the crossing point between $v_i v_k$ and r, and let R be the bounded region defined by the edge $v_i v_k$ and the segments $\overline{Ov_i}$ and $\overline{Ov_k}$.

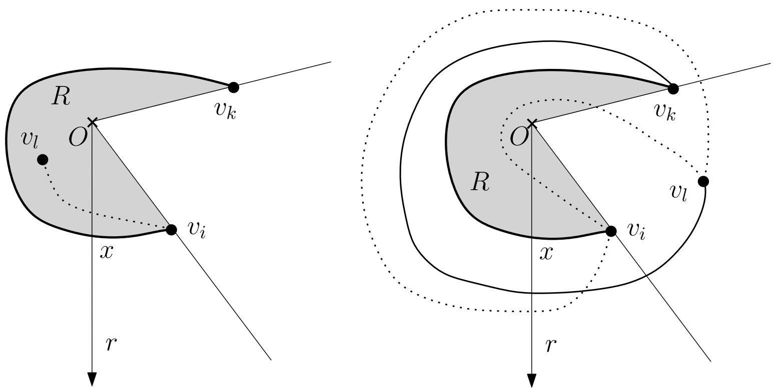


Figure 20 Illustrating the proof of Lemma 21.

Suppose that there is a vertex v_l inside R. See Figure 20, left. Then there is no way of connecting v_l and v_i without crossing r before x, which contradicts that $v_i v_k$ is the first edge crossing r. Thus, R must be empty.

Suppose now that $k \neq i+1$, so there is a vertex v_l with $i < l < k$. See Figure 20, right. The edge $v_l v_k$ must cross r at a point after x. But then there is no way of adding the edge $v_i v_l$ without crossing r before x or without crossing $v_l v_k$. Therefore, $k = i+1$ and *i)* follows. The proof of *ii)* is analogous by taking the last edge crossing r. In addition, v_i and v_j must be different since a same edge $v_i v_{i+1}$ cannot be at the same time the first and the last edge crossing r. ◀

▶ **Lemma 22.** *Let D be a generalized twisted drawing of K_n with $n \geq 3$ vertices. Then the cell containing O and the unbounded cell have at least one vertex on their boundaries.*

Proof. We show that the cell containing O is a vi-cell. The proof for the unbounded cell follows analogously.

By Lemma 21 *i)*, there exists a vertex v_i, with $1 \leq i \leq n-1$, such that the bounded region RB defined by the edge $v_i v_{i+1}$ and the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is empty. We now show that either the segment $\overline{Ov_i}$ or the segment $\overline{Ov_{i+1}}$ is uncrossed. Thus, it follows immediately that O lies in a vi-cell (with either v_i or v_{i+1} on the boundary).

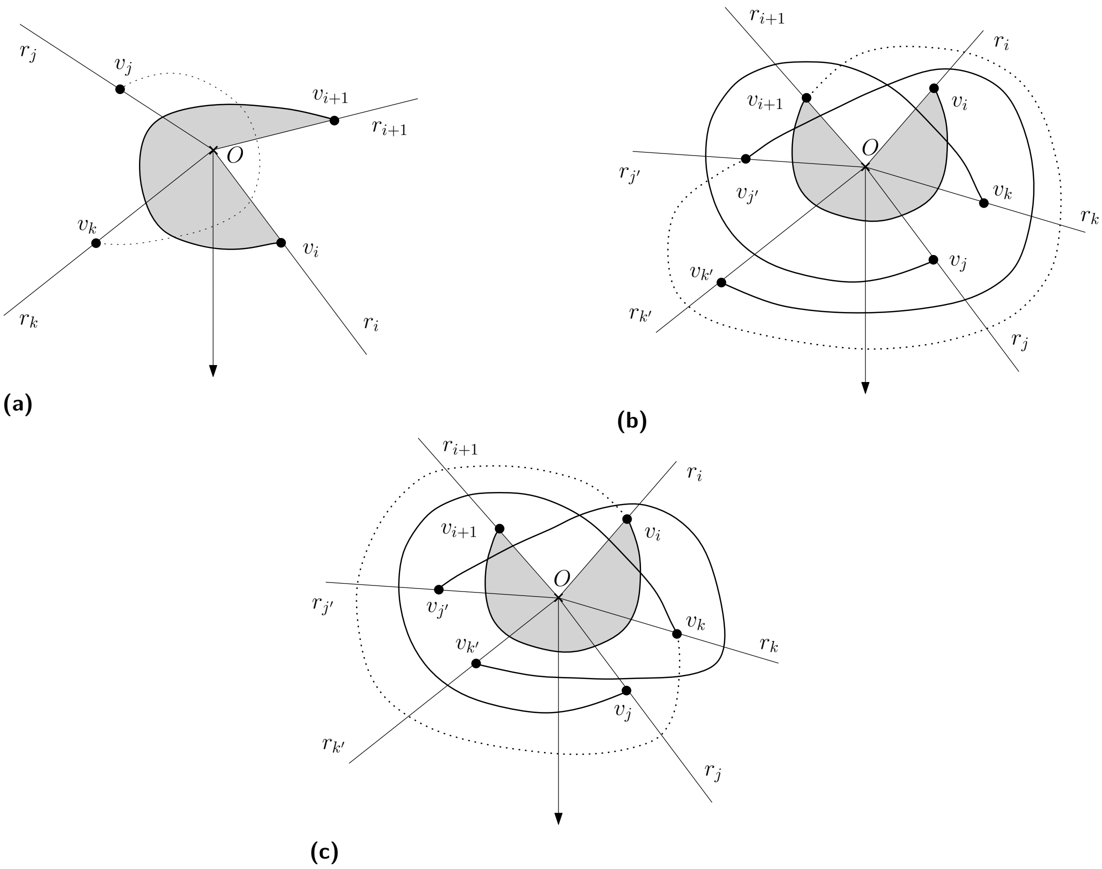


Figure 21 One of the segments $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is uncrossed.

Suppose to the contrary that neither segments are uncrossed, so there is an edge $v_j v_k$, with $j < k$, crossing the segment $\overline{Ov_i}$, and another edge $v_{j'} v_{k'}$, with $j' < k'$, crossing the segment $\overline{Ov_{i+1}}$.

Observe now the following. First, since RB is empty, an edge $v_{i+1} v_l$ cannot cross $\overline{Ov_i}$ for any l, so neither v_j nor v_k can be v_{i+1}. Second, suppose that $j, k > i+1$. Since both vertices are outside RB, if v_jv_k crosses $\overline{Ov_i}$, then it must also cross v_iv_{i+1} (see Figure 21a). Hence, by Lemma 20, the crossing point between v_jv_k and v_iv_{i+1} is in the wedge defined by r_{i+1} and r_j. But then, after crossing v_iv_{i+1} and $\overline{Ov_i}$, the edge v_jv_k must cross v_iv_{i+1} a second time to reach v_k, which is a contradiction. Therefore, $j, k < i$.

Using an analogous reasoning, we also obtain that if an edge $v_{j'}v_{k'}$ crosses $\overline{Ov_{i+1}}$, then $j', k' > i + 1$. As a consequence, the relative position of these three edges of D is as shown in Figures 21b and 21c. After emanating in v_k, the edge v_kv_j crosses v_iv_{i+1} at a point in the wedge defined by r_k and r_i; then it crosses $\overline{Ov_i}$, and reaches v_j surrounding v_iv_{i+1} by the exterior. The edge $v_{j'}v_{k'}$ must do the same but in opposite direction, crossing first v_iv_{i+1} at a point in the wedge defined by r_{i+1} and $r_{j'}$, then crossing $\overline{Ov_{i+1}}$ and reaching $v_{k'}$ surrounding v_iv_{i+1} by the exterior. Note that v_jv_k and $v_{j'}v_{k'}$ necessarily cross, and that the crossing point must be in the wedge defined by r_i and r_{i+1}. Hence, $v_{j'}$ must be to the left of the oriented edge v_kv_j and v_k must be to the right of the oriented edge $v_{j'}v_{k'}$. Otherwise, if $v_{j'}$ is to the right of the oriented edge v_kv_j or v_k is to the left of the oriented edge $v_{j'}v_{k'}$, then edges v_jv_k and $v_{j'}v_{k'}$ would cross twice.

For the vertices v_j and $v_{k'}$, there are two possibilities, depending on whether $v_{k'}$ is to the right (see Figure 21b) or to the left (see Figure 21c) of the oriented edge v_kv_j. But in the first case, the edge $v_{j'}v_{i+1}$ would cross v_jv_k twice, and in the second case, the edge v_iv_k would cross $v_{j'}v_{k'}$ twice. Therefore, at least one of $\overline{Ov_i}$ and $\overline{Ov_{i+1}}$ is uncrossed. ◀

▶ **Lemma 23.** *Let D be a generalized twisted drawing of K_n with $n \geq 3$ vertices. Then the cell of D containing O an the unbounded cell are a pair of antipodal vi-cells.*

Proof. Let c be the segment $\overline{OZ}$, where O is the origin and Z is a point on r in the unbounded cell. By Lemma 22, both O and Z lie in vi-cells. We will show that those cells are antipodal. Since r crosses every edge of D exactly once and Z lies in the unbounded cell, also the segment c crosses every edge exactly once. Consequently, c crosses the boundary of every triangle of D exactly three times. Since every triangle of D is plane, this means c starts and ends at different sides of every triangle. Thus, O and Z have to lie in antipodal cells. ◀

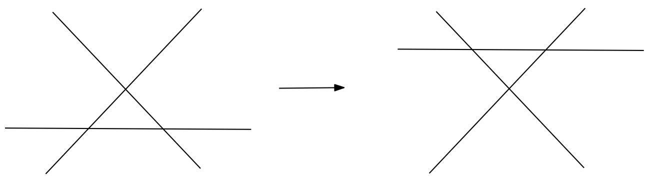


Figure 22 A triangle-flip.

We extend Lemma 23 to drawings weakly isomorphic to generalized twisted drawings using Gioan's Theorem [6, 15] that any two weakly isomorphic drawings of K_n can be transformed into each other with a sequence of triangle-flips and at most one reflection of the drawing. A *triangle-flip* is the operation that transforms a triangular cell $\triangle$ that has no vertex on its boundary, by moving one of its edges across the intersection of the two other edges of $\triangle$ (see Figure 22).

▶ **Theorem 16.** *Every simple drawing of K_n which is weakly isomorphic to a generalized twisted drawing of K_n, with $n \geq 3$, contains a pair of antipodal vi-cells. In generalized twisted drawings the cell containing O and the unbounded cell form such a pair.*

Proof. Let D be a simple drawing of K_n that is weakly isomorphic to a generalized twisted drawing D'. By Lemma 23, the cell in which O lies in D and the unbounded cell of D

are antipodal vi-cell pairs. Without loss of generality, we can assume that O is very close to a vertex v on the boundary of C_1 and Z is very close to a vertex w on the boundary of C_2. Using Gioan's Theorem, it is enough to show that afer every triangle flip that can be transformed on a drawing $\tilde{D}$ containing antipodal vi-cells, the resulting drawing $\tilde{D}_2$ still contains antipodal vi-cells.

As triangle-flips are only applied to cells without vertices on its boundary, and points O and Z are close enough to vertices on the boundary of their cell in $\tilde{D}$, they stay in vi-cells (with the vertices they are close to) after every triangle-flip. What remains to be shown is that the vi-cells stay antipodal.

Let T be a triangle of $\tilde{D}$. (Note that a triangle of the drawing is the simple cycle formed by the three edges connecting three vertices of the graph, and not the triangular cells on which we perform triangle flips.) Whenever a flip is performed on a cell $\triangle$, the cell $\triangle$ disappears and a new cell appears. The new cell might (but not has to) be on the other side of T, but no other cells are affected. In particular, since triangle flips are never applied on vi-cells, any pair of vi-cells that was antipodal before stays antipodal after the flip. ◀

E Characterizing via antipodal vi-cells

In this section, we prove that in any simple drawing D of K_n containing antipodal vi-cells it is possible to add a simple curve crossing every edge of D exactly once (Theorem 17). To this end, we will use the following lemmata that show some properties of antipodal vi-cells.

▶ **Lemma 24.** *Let D be a drawing of K_n with $n \geq 4$, and let (C_1, C_2) be a pair of antipodal vi-cells. Then there is no vertex that lies on the boundary of both cells.*

Proof. Assume, for a contradiction, that a vertex v_1 lies on the boundary of both cells C_1 and C_2. See Figure 23. Consider another cell C' different from C_1 and C_2, with v_1 on its boundary. Since v_1 has degree $n - 1 \geq 3$, this cell exists. Let u_1 and u_2 be the two vertices whose edges to v_1 are on the boundary of that cell C'. Then the triangle formed by vertices u_1, u_2, and v_1 always have the cells C_1 and C_2 on the same side, contradicting that C_1 and C_2 are antipodal. ◀

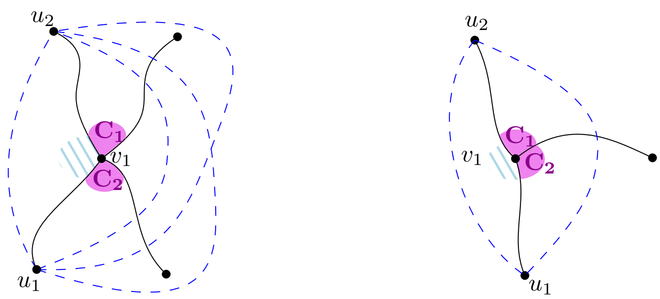


Figure 23 Edges of the star $S(v_1)$ incident to v_1 are drawn black; the antipodal cells are filled purple; the additional cell is indicated in cyan; the different ways to draw the edge u_1u_2 are drawn dashed in blue.

▶ **Lemma 25.** *Let D be a simple drawing of K_n that contains two antipodal vi-cells C_1 and C_2. Let v_1 be a vertex on the boundary of C_1. Let T be a triangle formed by v_1 and two other vertices u_2 and u_3. If there is a vertex u that lies on the same side of T as C_1, then the edge v_1u lies completely on that side.*

Proof. Assume, for a contradiction, that the edge v_1u does not lie completely on the same side of T as C_1, and thus crosses the boundary of T. See Figure 24. The only edge it can cross is u_2u_3, as the other edges are incident to v_1. Thus, the drawing induced by v_1, u_2, u_3 and u contains a crossing between v_1u and u_2u_3. Since any simple drawing of K_4 contains at most one crossing, the edges uu_2 and uu_3 cannot cross the boundary of T. Thus, the triangle T' formed by u, u_2 and u_3 has to lie on the same side of T as C_1, but keeping C_1 and C_2 on one of its sides, which is a contradiction to the definition of antipodal. ◀

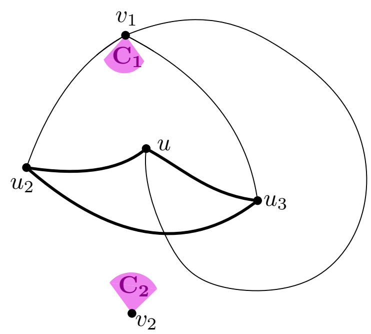


Figure 24 An illustration of Lemma 25. The vertex v_2 is placed as an example in one of the possible faces and could be at another place, but C_1 and C_2 lie on different sides of the triangle $v_1u_2u_3$ by definition. Thus, by construction of the triangle uu_2u_3, the cells C_2 and C_1 lie on the same side of uu_2u_3.

Now, we can prove Theorem 17.

▶ **Theorem 17.** *In any simple drawing D of K_n that contains a pair of antipodal vi-cells, it is possible to draw a curve c that crosses every edge of D exactly once.*

Proof. Let (C_1, C_2) be a pair of antipodal vi-cells of D, v_1 a vertex on the boundary of C_1, v_2 a vertex on the boundary of C_2, and $S(v_1)$ the star of v_1. Note that by Lemma 24, v_1 and v_2 are different. We draw a simple curve c from v_1 to v_2 such that it emerges from v_1 in the cell C_1 and ends in the cell C_2 very close to v_2, and the following holds:

1. The curve c does not cross any edges of $S(v_1)$.
2. All intersections of c with edges of D are proper crossings.
3. Over all curves for which 1 and 2 hold, the curve c has the minimum number of crossings with edges of D.

Since $S(v_1)$ is a plane drawing that has only one face in which both v_1 and v_2 lie, drawing c is always possible. See for example Figure 25. We will prove that c crosses every edge of $D \setminus S(v_1)$ exactly once. To show that c crosses an arbitrary edge w_2w_3 exactly once, we will first show that c crosses w_2w_3 an odd number of times and then show that c crosses w_2w_3 at most twice.

To observe that c has to cross w_2w_3 an odd number of times, consider the triangle T formed by v_1, w_2 and w_3. Since c connects two antipodal cells, the endpoints of c have to be on two different sides of T. Thus, c has to cross the boundary of T an odd number of times. Since c does not cross $S(v_1)$, it has to cross w_2w_3 an odd number of times.

We show now that c crosses w_2w_3 at most twice. Assume to the contrary that c crosses w_2w_3 at least three times. Without less of generality, we may assume that C_1 is inside T and C_2 is outside. Given two crossing points x and y between c and w_2w_3 that are consecutive

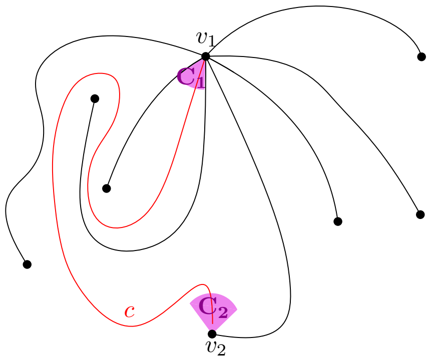


Figure 25 Edges of the star $S(v_1)$ incident to v_1 are drawn black; the antipodal cells are filled purple; the curve c is drawn in red.

on c when going from v_1 to v_2, a *lens* is the region to the left of the cycle formed by the arc xy on c and the arc yx on w_2w_3. See Figure 26 for an illustration. The pairs of two consecutive crossing points between c and w_2w_3 on c define a set of lenses on both sides of T, possibly nested (see Figure 26). Note that since c crosses w_2w_3 at least 3 times, there is at least one lens on each side of T. Among all the lens on the same side of T as C_1, we take one that does not contain any other lens in its interior. This lens L always exists by taking the "innermost" one in a set of nested lenses.

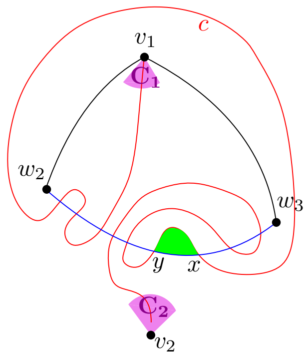


Figure 26 The curve c crossing several times the edge w_2w_3. If there are nested lenses in T, we take a minimal one that does not contain any nested lenses. This minimal lens L is shaded green.

Let x and y be the two crossing points defining the lens, so the boundary of L consists of arc xy on c and arc yx on w_2w_3. We claim that there is no vertex in L. Assume that there is a vertex u inside L (see Figure 27). By Lemma 25, the edge uv_1 cannot cross the edge w_2w_3. Thus, it has to cross c in order to get from u inside the lens to v_1 outside the lens. This is a contradiction to c being drawn such that it does not cross $S(v_1)$. Thus, there is no vertex in L, as claimed.

Since L does not contain any vertex, then every edge that crosses the arc yx on w_2w_3 has to also cross at least once the arc xy on the curve c (as there is no vertex in the lens where it could stop and it cannot cross the edge w_2w_3 more than once); see Figure 28 (left). Thus, c can be drawn such that it stops before x, follows an arc very close to the arc xy on w_2w_3 until a point very close to y, and then continues as c did before; see Figure 28 (right). This way the new drawing of the curve is still simple and does not have any crossings that the original one did not, but two less crossings with w_2w_3, which is a contradiction to the

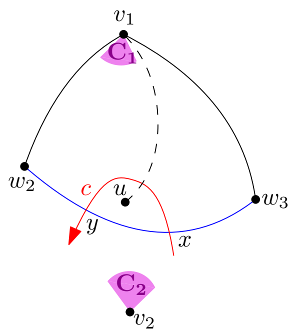


Figure 27 There is no vertex in L. If there is a vertex u inside L, then the edge from u to v_1 (drawn dashed) would have to cross c, which it cannot by construction.

minimality of c. In conclusion, the curve c cannot cross any edge w_2w_3 more than twice.

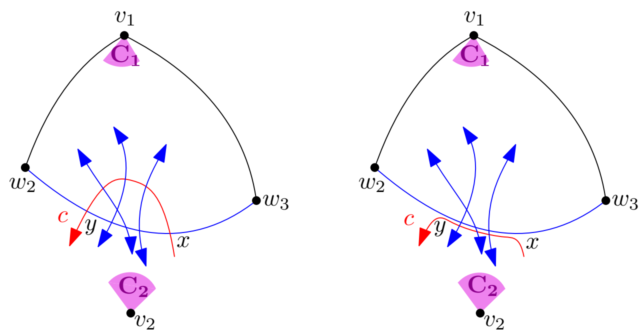


Figure 28 Left: The lens L does not contain any vertices, thus all edges crossing w_2w_3 within the lens have to leave the lens crossing c. Right: The curve c is redrawn such that it has fewer crossings.

Therefore, since c crosses all edges in $D \setminus S(v_1)$ an odd number of times and at most twice, it follows that it crosses all edges in $D \setminus S(v_1)$ exactly once, while (per construction) it does not cross any edges of $S(v_1)$.

We can transform c to a curve that crosses all edges exactly once in the following way: Instead of the starting point being v_1 we remove an ε of the curve on this end such that it starts very close to v_1 in the cell C_1 (and consequently still crosses exactly the edges it crossed before). Then, on that start in C_1, we extend the curve by going around the vertex v_1 so close to v_1 that the extension crosses exactly the edges of $S(v_1)$, and then ending again in C_1; see Figure 29. This way the extension crosses all edges of $S(v_1)$ exactly once and consequently, we obtained a curve crossing all edges exactly one, with its endpoints not lying on any edges or vertices of D. ◀

F Characterizing via a Curve Crossing Everything

In this section we prove Theorem 18.

▶ **Theorem 18.** *Let D be a simple drawing of K_n in which it is possible to draw a simple curve c that crosses every edge of D exactly once. Then, D can be extended by two vertices O and Z (at the position of the endpoints of the curve), and edges incident to those vertices such that the obtained drawing is a simple drawing of K_{n+2}, no edge incident to O crosses any edge incident to Z, and all edges in D cross the edge OZ.*

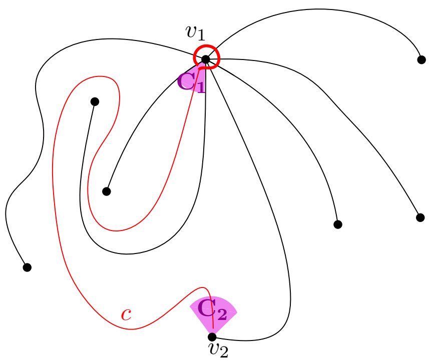


Figure 29 The resulting curve after the extension. The last part crossing $S(v_1)$ is drawn bold.

We first show several properties for drawings such that there is a simple curve $c = OZ$ crossing every edge once. Then, we show that we can extend the drawing $D = D_n$ to a drawing D_{n+2} of K_{n+2} by adding O and Z as vertices, the curve c as an edge, and edges from O and Z to each vertex w of D_n in such a way that D_{n+2} fulfills the following properties.

(P1) D_{n+2} is a simple drawing. (This implies in particular that none of the curves incident to O crosses another curve incident to O and no curve incident to Z crosses another curve incident to Z.)

(P2) No edge incident to O crosses any edge incident to Z.

Notation and Basic Properties

Assume that D_n is a simple drawing of K_n and $c = OZ$ a simple curve crossing every edge of D_n once.

We will consider two orientations for each edge of D_n. By uv we denote the edge oriented from u to v, and by vu the same edge oriented from v to u. If x, y are points placed in that order on the edge uv of D_{n+2}, then the portion of the curve uv placed between x and y is called the *arc* xy. We also consider the arcs oriented from the first point to the second point. Consequently, yx has the same points as xy but with the opposite orientation. We orient c from O to Z. When considering the star $S(w)$ of a vertex w, we will always consider the edges oriented from w to the other endpoint.

When the edge wu crosses OZ in a crossing point x we know that this crossing can be of two different ways, depending on the radial order of the arcs xO, xw, xZ, xu around point x. We will say that wu is a *top edge* if around x the arcs xu, xO, xw, xZ appear clockwise in this order, and wu is a *bottom edge* when that clockwise order is xu, xZ, xw, xO. In the figures we draw the curve $c = OZ$ as a horizontal line, thus the directed edges reaching that line by its top side are precisely the top edges. Note that if wu is a top edge, then uw is a bottom edge.

Three arcs, xy on the edge e_1, yz on the edge e_2 and zx on the edge e_3, form a cycle and divide the plane (or the sphere) into two regions A, B. By triangular region xyz we mean the region (A or B) found on the left side when we walk the cycle in the order x, then y, then z, and returning to x, using the corresponding arcs in e_1, e_2, e_3. In the same way, if the arcs $x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1$ form a simple cycle, the region found on the left side when we walk the cycle in the order $x_1, x_2, \dots, x_k, x_1$ will be denoted by $x_1x_2 \dots x_k$. We suppose that the drawings are on the sphere S^2, so we consider drawings homeomorphic in S^2 as topologically identical, like the left and right drawings of Figure 30. However, in the figures,

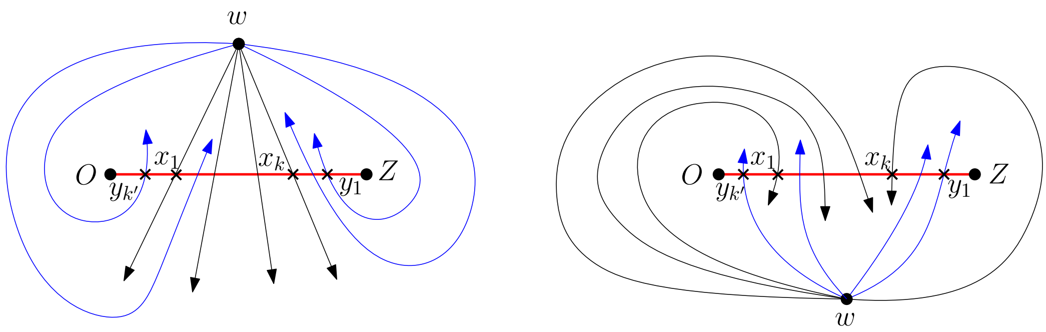


Figure 30 Top edges are drawn in black, bottom edges are drawn in blue.

as the drawings are shown on the plane, a given region can be bounded or unbounded. For example, region wx_1y_1 is bounded in the left drawing of Figure 30 and unbounded in the right drawing. But in both drawings, the arcs, vertices and crossings inside region wx_1y_1 are the same.

In our constructions, we are going to draw new arcs that are close to (or glued to) arcs of D_n. A new arc a' is *close* or *glued* to the arc ux if

(g1) An edge crosses a' if and only if it crosses ux.
(g2) All the crossing points on a' and on ux have the same order.
(g3) The arcs a' and ux do not cross each other. (They can share the endpoints, points u, x, but do not have to.)

▶ **Lemma 26.** *Let w be a vertex of D_n. The edges of $S(w)$ satisfy the following properties:*

($\mathbf{a_1}$) When exploring counterclockwise around w the edges of $S(w)$, the top edges are consecutive, $wu_1, wu_2 \dots, wu_k$, and cross the curve $c = OZ$ at points $x_1, \dots, x_k$ in that order. Then the bottom edges are consecutive, $wv_1, \dots, wv_{k'}$ (where $k' = n - k - 1$), and cross the curve $c' = ZO$ at points $y_1, \dots, y_{k'}$ in that order. See Figure 30.
($\mathbf{a_2}$) Let z_1 and z_{n-1} be the first and the last crossing points of $S(w)$ on the curve c. Then the endpoints of the bottom edges of $S(w)$ are inside the triangular region $z_1z_{n-1}w$, and the endpoints of the top edges are outside that region. See Figure 31.

Proof. $(\mathbf{a_1})$ Draw a new arc a glued to OZ by its top part, and another arc a' glued to ZO on the bottom part. Both have endpoints O and Z, and thus a, a' define a cycle C. The edges of the star $S(w)$ in counterclockwise order have to reach (that is, have their first crossing point with) the cycle C at points placed in clockwise order on C. The top edges of $S(w)$ are the ones reaching C on the arc a, and the corresponding crossing points $x_1, \dots, x_k$ on OZ are in increasing order (from O to Z). Then come the bottom edges reaching C on the arc a', their corresponding crossing points $y_1, \dots, y_{k'}$ in this order on ZO.

$(\mathbf{a_2})$ Notice that z_1 can be either x_1 or $y_{k'}$. Similarly, z_{n-1} can be x_k or y_1. In any case, the only way to connect w to a vertex v placed inside the triangular region $z_1z_{n-1}w$ and crossing OZ, is crossing the arc z_1z_{n-1} from its bottom part. For the same reason, all the vertices placed outside that region are precisely the endpoints of the top edges of $S(w)$. See Figure 31. ◀

▶ **Lemma 27.** *Assume $wu_1, \dots, wu_k$ are the top edges of $S(w)$, $k \geq 2$, and let $x_1, \dots, x_k$ be the corresponding crossing points on OZ. Consider two of those edges wu_i, wu_j, with x_i placed before x_j. Then*

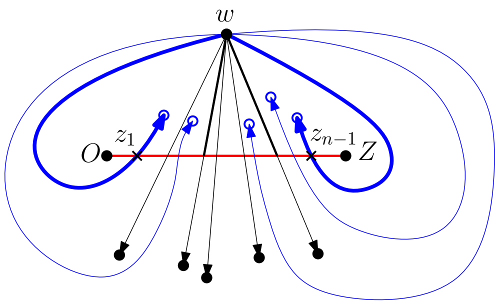


Figure 31 Property a_2: The vertices placed in the regions with a blue circle can be reached from w only with bottom edges.

(b1) *If $e = u_i u_j$ is bottom, it must cross c at a point x placed on Ox_i. If it is top, it must cross c at a point y placed on $x_j Z$. See Figure 32.*

(b2) *Suppose that there is an edge e of D_n crossing both arc wx_1 and arc wx_k. Then the two endpoints of e, vertices v, v', have to be endpoints of bottom edges of $S(w)$, and not both vertices can be inside the triangle $x_1 x_k w$.*

Proof. **(b1)** Suppose $e = u_i u_j$ is a bottom edge; the other case follows analogously. Since the two endpoints u_i, u_j are outside the triangular region $x_i x_j w$, the edge $u_i u_j$ cannot enter in that region without breaking the simplicity, and therefore $u_i u_j$ cannot cross the arc $x_i x_j$. If the edge e crosses OZ at a point x on $x_j Z$, just after crossing OZ, e is outside the region $u_i x x_j w$, but vertex u_j is inside that region, so it is impossible to reach u_j without either breaking the simplicity of D_n or crossing the curve OZ twice. Hence, e must cross OZ on the arc Ox_i. See Figure 32 left.

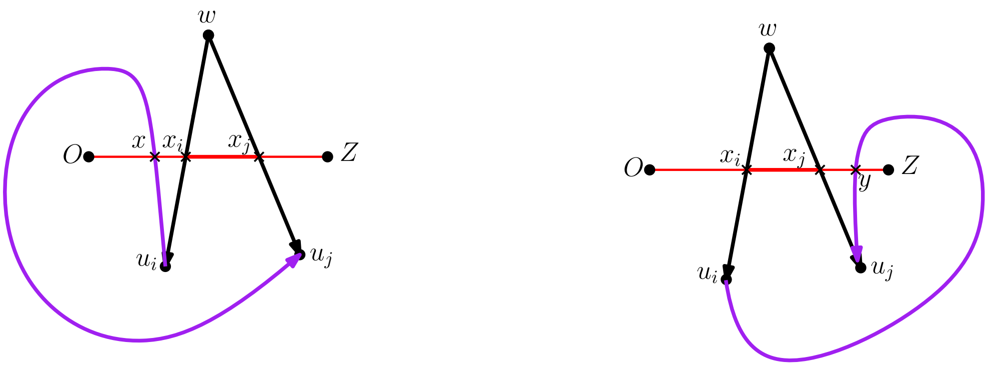


Figure 32 Property b_1: The edge $u_i u_j$ must cross OZ as in the left figure or as in the right figure.

(b2) Suppose that the edge $e = vv'$ crosses both wx_1 and wx_k.

We first analyze the case when both endpoints of e, vertices v, v', are inside the triangular region $x_1 x_k w$, and therefore, by Property **(a_2)**, both are endpoints of bottom edges of $S(w)$. Let y be the crossing point of wv with OZ. Then, $e = vv'$ cannot cross the arc $x_1 x_k$ because otherwise the boundary of $x_1 x_k w$ is crossed three times, contradicting that both v, v' are in that region. So, vv' has to first cross either the arc wx_1 or wx_k. In the first case, after crossing wx_1, the edge is in the region wyx_1 and the vertex v' is outside that region, hence the edge cannot leave that region keeping the simplicity. See Figure 33 left. Similarly, if vv' first crosses wx_k, it enters in the region $wx_k y$, but the vertex v' is outside that region, therefore the edge cannot leave that region without breaking the simplicity.

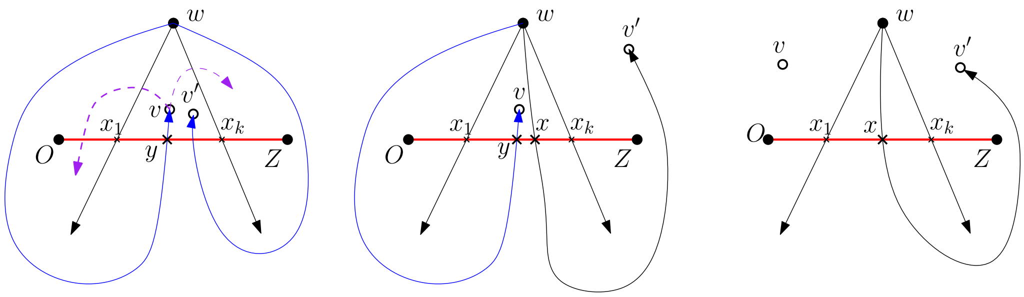


Figure 33 Property b_2: Edge e cannot cross both wx_1, wx_k if its endpoints are inside x_1x_kw or when one of them is an endpoint of a top edge.

Suppose now that v is inside the region x_1x_kw, and v' is an endpoint of a top edge wv' crossing OZ at a point x. As above, let y be the crossing point of wv with OZ, and suppose that y is placed before x on OZ. See Figure 33 centre. Then, v is in the region wyx and v' is outside that region, therefore the edge $e = vv'$ has to cross the arc yx. On the other hand, since neither v' nor v are in the region xx_kw and e crosses wx_k, it has to cross also the arc xx_k, hence e should cross OZ twice, a contradiction. A similar analysis can be done in the symmetric case, when y is placed after x.

Finally, suppose that v is outside the region x_1x_kw, and v' is an endpoint of a top edge wv' crossing OZ at point x. As both vertices v, v' are outside the region x_1x_kw, the edge vv' cannot cross three times the boundary of that region, so e cannot cross the arc x_1x_k. However, when e enters in that region, by crossing wx_1 or wx_k, as it cannot cross the arc wx, it should cross x_1x_k, again a contradiction. See Figure 33 right. ◀

If we consider a mirror drawing of D_n on the horizontal line OZ, all the top edges become bottom and vice versa, then, $(\mathbf{b_1})$ has a symmetric Property $(\mathbf{b_1'})$:

If wu_i, wu_j are bottom edges and the crossing point x_i is placed before x_j, then if $e = u_iu_j$ is top, it must cross c at a point x placed on Ox_i, and if it is bottom, it must cross c at a point y placed on x_jZ.

▶ **Lemma 28.** *There is one vertex w_1 such that all the edges emanating from w_1 are top.*

Lemma 28 is depicted in Figure 34.

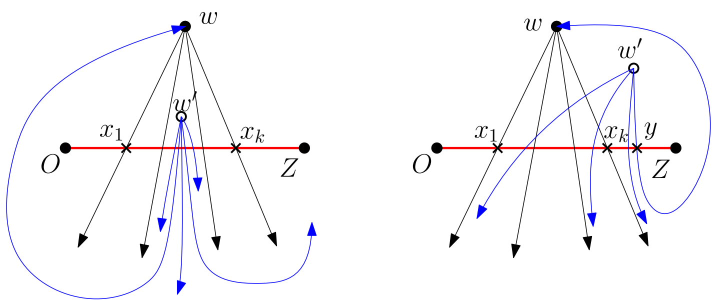


Figure 34 Lemma 28: There is a vertex w_1 such that all the edges of $S(w_1)$ are top.

Proof. We prove the existence of such a vertex by induction on the number of vertices n. For $n = 2$ the lemma obviously is true. Now, consider a simple drawing D_n of K_n and assume that the lemma holds for any simple drawing of K_{n-1} (with all their edges crossed by a curve). By removing a vertex w' of D_n, we obtain a simple drawing of K_{n-1}, by induction, containing a vertex w such that all the edges of $S(w)$ are top. If ww' is also top in D_n, all the edges of $S(w)$ are top in D_n, and w is the sought vertex. Suppose now that ww' is bottom and let x_1, x_k be the crossing points of the first and last edges of $S(w)$. If ww' crosses the arc x_1x_k, w' has to be inside the region x_1x_kw. Then, as all the other vertices are outside that region, by Property **(b₂)**, all the edges of $S(w')$ must cross first x_1x_k and therefore they are top. Finally, if ww' is bottom and does not cross x_1x_k it has to cross OZ through a point y on the arc x_kZ or on the arc Ox_1. Suppose y is on x_kZ (the other case is symmetric), then w' is in the triangular region x_1yw, and all the other vertices out of that region. Thus, again by Property **($\mathbf{b_2}$)**, all edges incident to w' have to cross first the top part of OZ, and therefore w' is the sought vertex. ◀

By removing w_1, we obtain a subdrawing with one vertex w_2 of which all the edges are top, by removing w_2 another vertex w_3, and so on. Thus, we obtain an order of the vertices $w_1, \ldots, w_n$ such that for each vertex w_i **the edges** w_iw_j **with** $j > i$ **are top edges** ($n - i$ edges) **and the edges** w_iw_l**, with** $l < i$ **are bottom edges** ($i - 1$ edges).

Construction of the drawing D_{n+2}

Beginning with the simple drawing D'_0 formed by D_n and the curve $c = OZ$, we build new drawings $D'_1, \ldots, D'_i, \ldots, D'_n$, where drawing D'_i is obtained from drawing D'_{i-1} by adding two simple curves Ow_i, w_iZ, following the order $w_1, w_2, \ldots w_n$ explained above. Hence, we start adding Ow_1, w_1Z, where w_1 is the unique vertex such that all the edges of $S(w_1)$ are top, and we finish by adding Ow_n, w_nZ, where w_n is the unique vertex such that all the edges of $S(w_n)$ are bottom. We build the new drawings in such a way that in D'_i, the following invariants are satisfied:

1. The curves OZ, Zw_i, w_iO form a well defined triangular region (region $R_i = OZw_i$) containing the triangular region $R_{i-1} = OZw_{i-1}$, and containing no vertex w_j with $j > i$. Notice that this invariant implies that R_i contains precisely the vertices w_l with $l < i$, and that neither Ow_i nor Zw_i can properly cross any edge Ow_l or Zw_l with $l < i$.
2. The drawing D'_i is a simple drawing.

The last drawing obtained, D'_n, taking O and Z as vertices, provides the sought drawing D_{n+2}, since Invariants 1 and 2 imply that it satisfies properties **(P1)** and **(P2)**.

To prove that D'_i satisfies Invariants 1 and 2, we suppose that D'_{i-1} satisfies these two invariants. Then, the crossing points of the edges of D'_{i-1} with the boundary of OZw_{i-1} must satisfy the properties given in the following lemma.

▶ **Lemma 29.** *Suppose that D'_{i-1} satisfies Invariants 1, 2, and let x_s, x_t, x be the crossing points of OZ with the edges $w_{i-1}w_s, w_{i-1}w_t, w_sw_t$, respectively. Then*

- **(c_1)** *All the top edges of $S(w_{i-1})$ are counterclockwise between $w_{i-1}O$ and $w_{i-1}Z$, and the bottom edges are counterclockwise between $w_{i-1}Z$ and $w_{i-1}O$.*
- **(c_2)** *Any edge w_sw_t with $s < t < i - 1$ crosses first OZ then one of the curves Ow_{i-1} or Zw_{i-1}.*
  Besides, if w_sw_t crosses Zw_{i-1}, the order of the above crossing points on OZ is x_t, x_s, x, and if w_sw_t crosses Ow_{i-1}, this order is x, x_s, x_t.
- **(c_3)** *Any edge w_sw_t with $s < i - 1 < t$ crosses first OZ and it does not cross any other arc of the boundary of OZw_i.*

Besides, if w_s is in the region Ox_tw_{i-1}, the order of the above crossing points on OZ is x_s, x, x_t, and if w_s is in the region x_tZw_{i-1}, the order is x_t, x, x_s.

(c₄) *Any edge w_sw_t with $i-1 < s < t$ first crosses one of Ow_{i-1} or Zw_{i-1}, then OZ. Besides, if w_sw_t crosses Zw_{i-1}, the order of the crossing points is x_s, x_t, x, and if it crosses Ow_{i-1}, the order is x, x_t, x_s.*

Proof. $(\mathbf{c_1})$ A top edge $w_{i-1}w_t$ with $t > i-1$ of $S(w_{i-1})$ has to reach OZ by inside of OZw_{i-1}, and since D'_{i-1} is simple, it must start entering in that region, hence it must be counterclockwise between $w_{i-1}O$ and $w_{i-1}Z$. The same reasoning works for bottom edges $w_{i-1}w_s$ with $s < i-1$.

$(\mathbf{c_2})$ Since $s < t < i-1$, by Invariant 1, the vertices w_s, w_t are both in OZw_{i-1}. Hence, the edge w_sw_t must cross the boundary of OZw_{i-1} an even number of times. And since it has to cross OZ, it has to cross also only one of the boundary curves $Ow_{i-1}, w_{i-1}Z$. Finally, since w_sw_t is a top edge, it has to cross first OZ then the other boundary curve.

On the other hand, suppose that the edge w_sw_t crosses $w_{i-1}Z$ at a point z. Then, the vertex w_{i-1} is in the region xZz, like the vertices w_s, w_t. Therefore, if w_tw_{i-1} (or w_sw_{i-1}) leaves that region crossing the arc xZ, it cannot reach w_{i-1} without crossing again the boundary of that region, breaking the simplicity of D'_{i-1}. Therefore, x_s, x_t have to be placed before x on OZ. Finally, w_tw_{i-1} cannot cross the arc x_sw_s (on the edge $w_{i-1}w_s$) or the arc w_sx (on the edge w_sw_t), therefore it cannot cross OZ between x_s and x, and the order of the crossing points must be x_t, x_s, x. The same arguments can be used when the edge w_sw_t crosses Ow_{i-1}

$(\mathbf{c_3})$ By Invariant 1, w_s is inside the region OZw_{i-1} and w_t is outside, so the edge w_sw_t has to cross the boundary of OZw_{i-1} an odd number of times. Suppose that it crosses the three curves of the boundary, curves $OZ, Ow_{i-1}, w_{i-1}Z$. It cannot cross first Ow_{i-1} (or $w_{i-1}Z$) then OZ because w_sw_t is a top edge. It cannot cross first OZ then Ow_{i-1} and finally $w_{i-1}Z$ because then, by $(\mathbf{c_1})$, it has to cross the top edge $w_{i-1}w_t$. By the same reason it cannot cross the boundary in the order first OZ, next $w_{i-1}Z$ and then Ow_{i-1}. Finally, if it crosses first Ow_{i-1}, next $w_{i-1}Z$ and then OZ, then, again by $(\mathbf{c_1})$, it has to cross the bottom edge $w_{i-1}w_s$.

Besides, since $w_{i-1}w_t$ is a top edge, the arc $w_{i-1}x_t$ divides the region OZw_{i-1} into two disjoint regions Ox_tw_{i-1} and x_tZw_{i-1}. Then, if w_s is in the region Ox_tw_{i-1}, the edge w_swi-1 has to cross by the arc Ox_t. And then, necessarily the crossing point x must be between x_s and x_t. The same argument can be used when w_s is in the region x_tZw_{i-1}. $(\mathbf{c_4})$ Since $i-1 < s < t$, both vertices w_s, w_t are outside region OZw_{i-1}, and we use the same reasonings as in $(\mathbf{c_2})$: The boundary of OZw_{i-1} has to be crossed twice, and since w_sw_t is a top edge, first one of Ow_{i-1} or Zw_{i-1} is crossed, then OZ.

Now, if w_sw_t crosses $w_{i-1}Z$ at a point z, then none of the edges $w_{i-1}w_s, w_{i-1}w_t$ can cross the arc xz (placed on w_sw_t), so they cannot cross the arc xZ either. Therefore the crossing points x_s, x_t have to be placed before x on OZ. Finally, w_{i-1} and w_t are in different sides of the region x_sxw_s and the edge $w_{i-1}w_t$ cannot cross the arc w_sx_s (on edge $w_{i-1}w_s$) or the arc w_sx (on the edge w_sw_t), therefore it has to cross OZ between x_s and x, and the order of the crossing points must be x_s, x_t, x. The same arguments can be used when the edge w_sw_t crosses Ow_{i-1}. ◀

Observe that, as we suppose that D'_{i-1} satisfies Invariants 1 and 2, properties $(\mathbf{c_2})$, $(\mathbf{c_3})$, $(\mathbf{c_4})$ imply that no edge of D_n can cross simultaneously the curve Ow_{i-1} and the curve $w_{i-1}Z$.

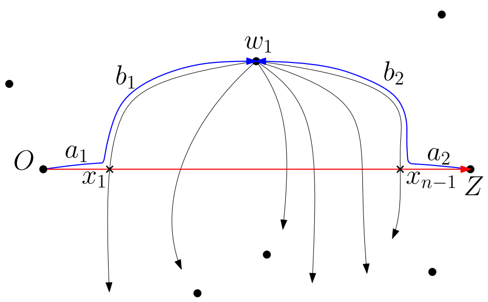


Figure 35 Step 1: How to draw Ow_1, Zw_1.

For the construction of the drawings D'_i, we will make a first step (for D'_1), a generic step (for D'_i with $1 < i < n$), and a final step (for D'_n).

Step 1. Let $x_1, \ldots, x_{n-1}$ denote the crossing points of $S(w_1)$ with OZ. We draw Ow_1 following (slightly counterclockwise) the curve OZ until the point x_1, then we turn counterclockwise following the arc x_1w_1; see Figure 35. We draw Zw_1 analogously, following ZO until x_{n-1}, then following the arc $x_{n-1}w_1$. If we denote the arcs $Ox_1, x_1w_1, Zx_{n-1}, x_{n-1}w_1$ by a_1, b_1, a_2, b_2 respectively, then Ow_1 has a first part glued to a_1, then a second part glued to b_1, and in the same way, Zw_1 consists of two parts glued to a_2 and b_2 respectively. Clearly, OZw_1 is a well defined region and by Property **($\mathbf{a_2}$)**, there are no vertices in the triangular region $w_1x_1x_{n-1}$, hence region OZw_1 does not contain any vertices. That establishes Invariant 1.

Suppose now that the edge $w_sw_t, 1 < s < t \leq n-1$ crosses the arc a_1, and let x_s, x_t be the crossings points on OZ of w_1w_s, w_1w_t, respectively. Notice that x_s can be placed before or after x_t, but x is placed before these two points. Then, by Property **($\mathbf{b_1}$)**, since the bottom edge is w_tw_s, x_t has to be placed before x_s, and the order of the crossing points on OZ must be x, x_1, x_t, x_s. Besides, the arc b_1 must start at w_1 by outside the triangular region $w_1w_tw_s$ (because it is the first top arc), and it finishes at x_1, a point also placed outside $w_1w_tw_s$. Therefore, the arc b_1 cannot cross w_sw_t, the unique edge of the triangle $w_1w_tw_s$ not incident to w_1. A symmetric argument can be used to prove that an edge $w_sw_t, 1 \leq s < t < n-1$ crossing a_2 cannot cross b_2. This establishes Invariant 2 for D'_1.

Step i ($2 \leq i \leq n-1$). We will draw Ow_i in two different ways, Way 1 and Way 2, depending on whether the first top edge of $S(w_i)$, edge e_1, first crosses Ow_{i-1} or $w_{i-1}Z$. When e_1 first crosses Ow_{i-1} at a point z'_1, we draw Ow_i in **Way 1**, which is the following: Ow_i follows the curve Ow_{i-1} (very close to that curve, slightly counterclockwise) until it reaches the crossing point z'_1, then Ow_i continues close to z'_1w_i until it reaches its endpoint w_i; see Figure 36 left.

When e_1 crosses first $w_{i-1}Z$, by Property **($\mathbf{c_4}$)** applied to e_1 (e_1 is some w_iw_t with $i-1 < i < t$), the edge $w_{i-1}w_i$ has to cross OZ before x'_1 (the crossing point of e_1 with OZ), and therefore, the last bottom edge of $S(w_i)$, edge $e'_{k'}$, crosses OZ at a point $y'_{k'}$ placed before x'_1 (or $e'_{k'}$ coincides with w_iw_{i-1}). In this case, we draw Ow_i in **Way 2**, which is the following: Ow_i follows the curve OZ (very close to that curve, slightly clockwise) until it reaches the crossing point $y'_{k'}$, then Ow_i continues close to the arc $y'_{k'}w_i$ on $e'_{k'}$ until it reaches the endpoint w_i; see Figure 36 right.

In both ways, we consider the curve Ow_i as consisting of two arcs a_1 and b_1. In Way 1, the arcs of Ow_i are first a_1 glued to Oz'_1 and second b_1 glued to z'_1w_i. In Way 2, the arcs are first a_1 glued to $Oy'_{k'}$ and then b_1 glued to $y'_{k'}w_i$.

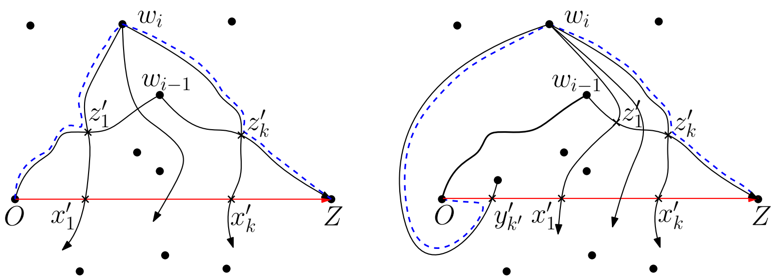


Figure 36 Case 1 and Case 2.

Symmetric constructions are used for drawing Zw_i. When e_k, the last top edge of $S(w_i)$, first crosses $w_{i-1}Z$ at a point z'_k, Zw_i is drawn in Way 1: Zw_i consists of the arcs a_2 and b_2 glued to Zz'_k and z'_kw_i respectively. When e_k first crosses Ow_{i-1}, then OZ at a point x'_k, the first bottom edge of $S(w_i)$, edge e'_1, has to cross OZ at a point y'_1 placed after x'_k. Then, we build Zw_i in Way 2: it consists of an arc a_2 (counterclockwise) close to Zy'_1, then an arc b_2 glued to y'_1w_i on edge e'_1.

As construction Way 2 can only be used in one of the two curves Ow_i, Zw_i, we only need to see that Invariants 1, 2 hold in two cases: **Case 1**, when construction Way 1 is used for both Ow_i and w_iZ, and **Case 2**, when Way 2 is used for Ow_i and Way 1 for w_iZ. The case when Way 2 is used for Zw_i and Way 1 for Ow_i is symmetric to Case 2; see Figure 36.

To prove that Invariants 2 holds for D'_i, we will see that in both Case 1 and Case 2, each edge of D_n crosses at most one of the arcs a_1, b_1 and at most one of the arcs a_2, b_2 of D'_i.

Case 1.-

By construction, the curves Ow_i and w_iZ do not cross each other, the triangular region OZw_i contains the region OZw_{i-1}, and Ow_i, w_iZ cannot properly cross any edge Ow_l, w_lZ, with $l < i$, because these last edges are inside OZw_i. Notice that the order of the edges around O and Z are counterclockwise $OZ, Ow_1, \ldots, Ow_i$ and $ZO, Zw_i, \ldots, Zw_1$, respectively. Moreover, by Property **(a$_2$)**, all the vertices in the triangular region $x'_1x'_kw_i$ must be reached from w_i via a bottom edge (where $x'_1, \ldots, x'_k$ are the crossing points of the top edges of $S(w_i)$ with curve OZ). On the other hand, all the vertices $w_s, s \leq i-1$ are in OZw_{i-1} (w_{i-1} on the boundary), and these are precisely the endpoints of the bottom edges of $S(w_i)$. Therefore, the subregion $z'_1w_{i-1}z'_kw_i$ must be empty, and OZw_i contains all w_s with $s < i$ and does not contain vertices $w_t, t > i$.

To prove the simplicity of D'_i, it is enough to prove that for any edge w_sw_t of D_n, the drawing formed by Ow_i, w_iZ, w_sw_t is simple. The following subcases $1, 2, 3, 4, 5$ prove that simplicity for edges w_sw_t when $s = i$, $s = i-1$, $s < t < i-1$, $s < i-1 < i < t$ and $i < s < t$, respectively.

-1 No edge of $S(w_i)$ can cross any of the arcs a_1, b_1, a_2, b_2.
For the arcs b_1, b_2, this is obvious since these arcs are glued to edges of $S(w_i)$. Besides, no top edge of $S(w_i)$ can cross arc a_1 because this arc follows the curve Ow_{i-1} until precisely the first crossing point with a top edge of $S(w_i)$. By the same reason, top edges of $S(w_i)$ cannot cross a_2 either. Finally, by the simplicity of D'_{i-1} the bottom edge w_iw_{i-1} cannot cross a_1 or a_2, and by Property **(c$_3$)** applied to an edge w_iw_s with $s < i-1 < i$, this bottom edge of $S(w_i)$ cannot cross Ow_{i-1} or Zw_{i-1}, and therefore it cannot cross a_1 or a_2.

-2 Any edge of $S(w_{i-1})$ crosses at most one of the arcs a_1, b_1 and at most one of the arcs a_2, b_2.
Since the edges of $S(w_{i-1})$ cannot cross the arc a_1 nor the arc a_2, the result follows.

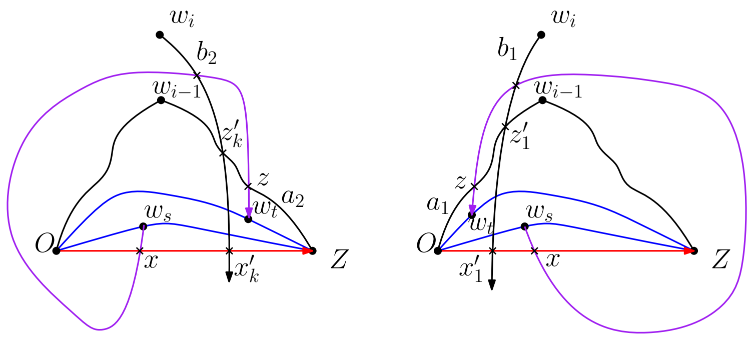


Figure 37 Left: w_sw_t crosses in the order OZ, b_2, a_2. Right: It crosses in the order OZ, b_1, a_1.

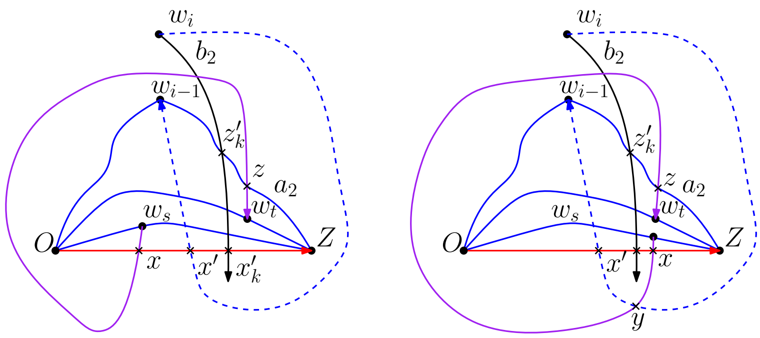


Figure 38 w_sw_t crosses b_2 and a_2. The edge w_tw_{i-1} cannot be drawn.

-3 Any edge $e = w_sw_t$, $s < t < i-1$ crosses at most one of the arcs a_1, b_1 and at most one of the arcs a_2, b_2.

Suppose that the edge w_sw_t crosses both a_2 and b_2. By Property $(\mathbf{c_2})$, this edge first crosses OZ at a point x, then reaches a_2 at a point z before finishing at w_t. Since $b_2 = w_iz'_k$ is outside OZw_{i-1} the crossing of b_2 with w_sw_t must be at a point in the arc xz, as shown in Figure 37 left. Observe that after crossing b_2, which is an arc on the last top edge of $S(w_i)$, the edge w_sw_t enters in OZw_{i-1} crossing a_2, therefore it cannot cross again the last top edge of $S(w_i)$, and thus w_t has to be placed inside the region $x'_kZz'_k$. A totally symmetric case occurs when the edge w_sw_t crosses both a_1 and b_1, then w_t must be inside the region $Ox'_1z'_1$, see Figure 37 right.

Let us analyze only the first case, when $e = w_sw_t$ crosses OZ, b_2, a_2 in that order, because the other case is totally symmetric. We are going to prove that the edge w_tw_{i-1} cannot be drawn without breaking the simplicity. By Property $(\mathbf{c_2})$ applied to w_sw_t, the crossing point of w_tw_{i-1} with OZ, must be placed before x on OZ. On the other hand, if x' is the crossing point of $w_{i-1}w_i$ with OZ, by Property $(\mathbf{c_4})$ applied to e_k (the last top edge of $S(w_i)$), the point x' has to be placed before x'_k. Then, by Property $(\mathbf{c_3})$ applied to w_tw_i $(t < i-1 < i)$, as w_t is in region $x'Zw_{i-1}$, the edge w_tw_{i-1} has to cross OZ after x'. Therefore, the edge w_tw_{i-1} should cross OZ after x' and before x, and this is not possible when x is placed before x'; see Figure 38 (left). Finally, if x is placed after x', then x' is inside the region xZz, w_i outside that region, and therefore the arc $x'w_i$ has to cross the arc xz at some point y, see Figure 38 (right). But then, the edge w_tw_{i-1} has to cross OZ on the arc $x'x$, entering into the region $x'xy$, bounded by arcs on the edges w_sw_t and w_iw_{i-1}, that cannot be crossed by w_tw_{i-1} without breaking the simplicity of D_n.

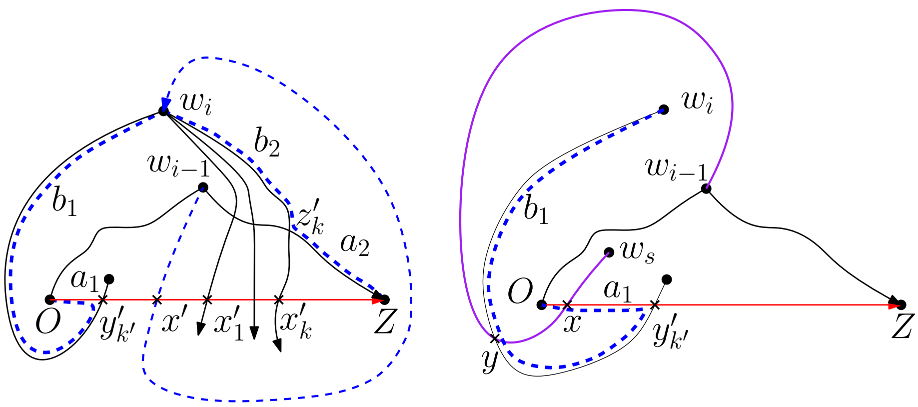


Figure 39 Left: Case 2 for drawing Ow_i, Zw_i. Right: Edges of $S(w_{i-1})$ at most cross one of a_1, b_1.

-4 Any edge w_sw_t, $s < i-1 < i < t$ crosses at most one of the arcs a_1, b_1 or a_2, b_2.
Observe that w_s is inside the triangular region OZw_{i-1} and w_t is outside that region. Then, by Property **(c₃)**, the edge w_sw_t must first cross OZ and it cannot enter the region OZw_{i-1} again. Therefore, it cannot cross a_1 or a_2.

-5 Any edge w_sw_t, $i < s < t$ crosses at most one of the arcs a_1, b_1 or a_2, b_2.
First observe that by Property **($\mathbf{b_2}$)** applied to $S(w_i)$, the edge w_sw_t cannot cross both b_1 and b_2 (because w_iw_s and w_iw_t are top edges of that star). Besides, w_s and w_t are outside the triangular region OZw_i, therefore if edge w_sw_t crosses b_1 or b_2 it has to cross also z'_1w_{i-1} or $w_{i-1}z'_k$. Finally, according to Property **($\mathbf{c_4}$)**, the edge w_sw_t must enter to OZw_{i-1} crossing only one of a_1,z'_1w_{i-1},a_2,$w_{i-1}z'_k$, then leaving through OZ, therefore it cannot cross both a_1, b_1 or both a_2, b_2.

Case 2.-
Again, by the method that the construction is done, the curves Ow_i and w_iZ do not cross each other and the triangular region OZw_i contains the region OZw_{i-1}, hence Ow_i, w_iZ cannot properly cross any edge Ow_l, w_lZ, with $l < i$. See Figure 36 right. Moreover, by Property **($\mathbf{a_2}$)** all the vertices in region $Ow_{i-1}z'_kw_iy'_{k'}$ are endpoints of bottom edges of $S(w_i)$, but all these endpoints $w_1, \ldots, w_{i-1}$ of the bottom edges of $S(w_i)$ must be inside the region OZw_{i-1} (w_{i-1} on the boundary). Therefore, region $Ow_{i-1}z'_kw_iy'_{k'}$ is empty and OZw_i only contains inside the $i-1$ endpoints $w_1, \ldots, w_{i-1}$. So, Invariant 1, holds.

Like in Case 1, to prove the simplicity of D'_i we analyze the same five subcases.

-1 No edge of $S(w_i)$ can cross any of the arcs a_1, b_1, a_2, b_2.
As in Case 1, no edge of $S(w_i)$ can cross the arcs b_1, b_2, no a top edge of that star can cross a_2. Besides, by Property **($\mathbf{c_3}$)**, a bottom edge w_iw_s, $s < i-1 < i$, cannot cross Zw_{i-1}, therefore it cannot cross a_2 either. Finally, a top edge cannot cross either arc a_1, (because the crossing points $x'_1, \ldots, x'_k$ are after $y'_{k'}$), and a bottom edge cannot cross $a_1 = Oy'_{k'}$ because $y'_{k'}$ is the first crossing point of those edges.

-2 Any edge of $S(w_{i-1})$ crosses at most one of the arcs a_1, b_1 or one of a_2, b_2.
See Figure 39 left. Since they cannot cross a_2, we only have to prove that one of this edges cannot cross both a_1 and b_1. In the definition of Way 2, we have seen that the crossing point x' of w_iw_{i-1} with OZ must be placed before x'_1 and after $y'_{k'}$. Hence, if x_t is the crossing point on OZ of a top edge $w_iw_t, t > i$, then x' is placed between $y'_{k'}$ and x_t. But then, by Property **($\mathbf{c_4}$)** applied to $w_iw_t, i-1 < i < t$, the crossing point of

$w_{i-1}w_t$ on OZ must be placed between x' and x_t, so after $y'_{k'}$. Therefore, the top edges $w_{i-1}w_t, t > i-1$ of $S(w_{i-1})$ cannot cross a_1.
Finally, suppose that an edge $w_s w_{i-1}, s < i-1$ crosses both a_1 and b_1. Necessarily, $w_s w_{i-1}$ first crosses OZ at a point x, then b_1 at a point y, finishing at w_{i-1}. See Figure 39 right. Then, the vertices w_i, w_s must be both inside the region xZw_{i-1}. However, the bottom edge $w_i w_s$ cannot cross a_1, (remember that $y'_{k'}$ is the first crossing point of those bottom edges). Therefore, by Property **(c₃)**, $w_i w_s$ cannot cross Ow_{i-1} or $w_{i-1}Z$, and it has to enter in OZw_{i-1} crossing through the bottom of the arc $y'_{k'}Z$, but then it should first cross the arc xw_{i-1} (on $w_s w_{i-1}$), contradicting the simplicity of D_n.

-3 Any edge $e = w_s w_t$, $s < t < i$ crosses at most one of the arcs a_1, b_1 or one of a_2, b_2.
A scheme of this situation is shown in Figure 40, where the two blue curves mark the boundary of regions OZw_s and OZw_t.

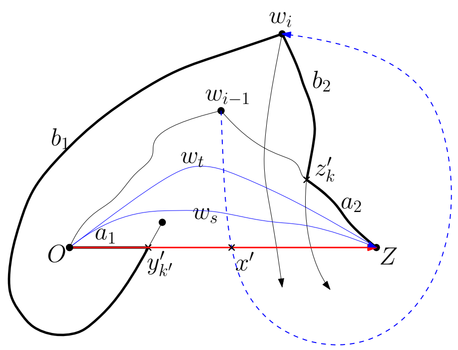


Figure 40 Case 2-**3**.

On the contrary, suppose that $w_s w_t$ crosses both a_1 and b_1. Necessarily, it first crosses a_1 at a point x, next b_1 at a point y, then entering in region OZw_{i-1} by crossing either Ow_{i-1} or $w_{i-1}Z$ at a point z, until reaching w_t. See Figure 41 left. But then, by the same reasonings as in the previous Case 2-**2**, the edge $w_i w_s$ cannot be drawn. As above, w_s and w_i are in the region xZz, if z is on $w_{i-1}Z$, or in the region $xZw_{i-1}z$ when z is on Ow_{i-1}. However, $w_i w_s$ cannot cross a_1 or Ow_{i-1} or $w_{i-1}Z$, so it has to enter in OZw_{i-1} crossing through the arc $y'_{k'}Z$, and therefore so it should first cross the arc xz (on $w_s w_t$), a contradiction.

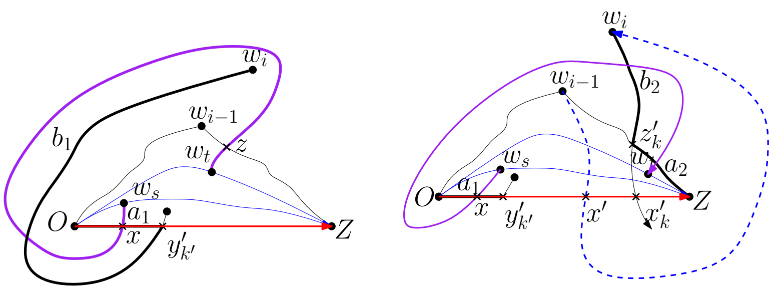


Figure 41 Case 2-**3**: Left, $w_s w_t$ crosses a_1, then b_1. Right, $w_s w_t$ crosses b_2, then a_2.

Finally, suppose that $w_s w_t$ crosses a_2 and b_2. We are exactly in the same situation as in Case 1-**3** (see Figure 38): necessarily the edge first crosses OZ at a point x, next crosses b_2, then reaches a_2 at a point z before finishing at w_t. And we have seen that a contradiction is reached in this situation. It does not matter if the edge $w_s w_t$ crosses a_1 (like in Figure 41 right) or not.

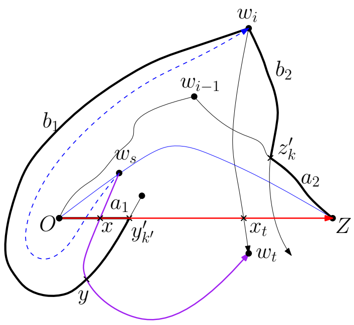


Figure 42 Case 2-**4**.

-4 Any edge w_sw_t, $s < i-1 < i < t$ crosses at most one of the arcs a_1, b_1 or a_2, b_2.
By Property **(c₃)**, an edge w_sw_t with $s < i < t$ cannot cross $w_{i-1}Z$, hence, it cannot cross a_2. So we have to prove that it cannot cross both a_1 and b_1. Let us see that in this situation the edge w_sw_i cannot be drawn without breaking the simplicity of the drawing. If $w_sw_t, s < i-1 < t$ crosses both a_1 and b_1, necessarily it starts crossing OZ through a_1, then crossing b_1 at a point y, finishing at vertex w_t, see Figure 42. Then, w_s has to be inside the region w_iyw_t. On the other hand, the bottom edge w_iw_s must start from w_i counterclockwise after the arc b_2 (on the last top edge of $S(w_i)$), and before the arc b_1 (on the last bottom edge of $S(w_i)$). Therefore, w_iw_s should start outside the region w_iyw_t, and should finish at w_s, placed inside that region. However, w_iw_s cannot cross any of the arcs of the boundary of w_iyw_t, because they are on edges incident to either w_i or w_s.

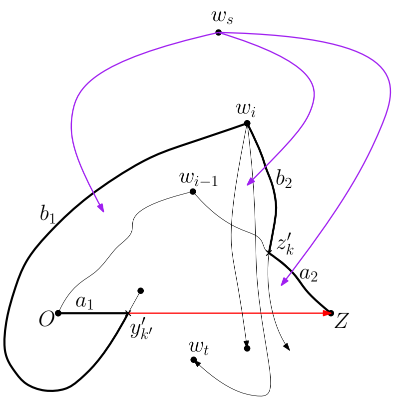


Figure 43 Case 2-**5**:w_sw_t enters into region R_i crossing either b_1 or b_2 or a_2.

-5 Any edge w_sw_t,$i < s < t$ crosses at most one of the arcs a_1, b_1 or a_2, b_2.
Since w_s and w_t are outside the triangular region OZw_{i-1}, by Property **(c₄)**, w_sw_t must cross first either Ow_{i-1} or $w_{i-1}Z$, and then OZ. On the other hand, w_s and w_t are outside the region $R = y'_{k'}Zz'_kw_i$, bounded by the four arcs $y'_{k'}Z$,a_2,b_2,b_1, so edge w_sw_t has to enter into that region crossing first either b_1 or b_2 or a_2 and it has to cross either two or four of those arcs. Suppose that it enters by crossing b_1, then it cannot exit by crossing b_2 or a_2, because then all the top edges of $S(w_i)$ would be crossed, contradicting Property **(b₂)**. So, it has to exit crossing first Ow_{i-1} or $w_{i-1}Z$, then crossing $y'_{k'}Z$. After crossing $y'_{k'}Z$ it cannot cross again the boundary $Ow_{i-1}, w_{i-1}Z$, so it cannot cross a_2, and therefore in this case, only b_1 and $y'_{k'}Z$ can be crossed. See Figure 43.
Similarly, if w_sw_t enters in R crossing through b_2, then it cannot exit by crossing b_1,

that contradicts the Property $(\mathbf{b_2})$, not by crossing a_2, because then the boundary $Ow_{i-1}, w_{i-1}Z$ would be crossed twice, so, it has to exit crossing Ow_{i-1} or $w_{i-1}Z$, then crossing $y'_{k'}Z$, and again after that crossings, the edge cannot enter again in R. Therefore in this case, only b_2 and $y'_{k'}Z$ are crossed.
Finally, if $w_s w_t$ enters in R by crossing a_2, again, it cannot exit by crossing b_1, that contradicts the Property $(\mathbf{b_2})$, not by crossing b_2, the boundary $Ow_{i-1}, w_{i-1}Z$ would be crossed twice, so, it has to exit crossing $y'_{k'}Z$. After crossing $y'_{k'}Z$, the edge cannot cross b_1 or b_2, because it should cross both, contradicting again Property $(\mathbf{b_2})$. Therefore in this last case, only a_2 and $y'_{k'}Z$ are crossed.

These 5 subcases prove that Invariant 2 holds also for D'_i in Case 2.

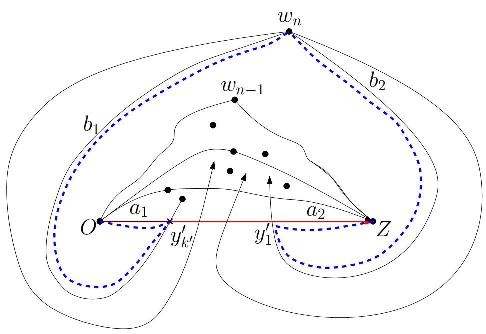


Figure 44 Final Step.

Final Step.- After D'_{n-1} has been built, we have to add the curves Ow_n, w_nZ to D'_{n-1}, where w_n is the last vertex, the one with only bottom edges in $S(w_n)$. That is done as in **Step 1**, changing bottom for top, and counterclockwise by clockwise; see Figure 44. Again, by construction, the curves Ow_n and w_nZ do not cross each other, the triangular region OZw_n contains the region OZw_{n-1}, and Ow_n, w_nZ cannot properly cross any edge Ow_l, w_lZ, with $l < n$. Since the arcs a_1, b_1 forming the curve Ow_n are build in **Way 2**, the reasonings used in Cases 2-**1**,2-**2**,2-**3**, to prove the simplicity for the arcs a_1, b_1, also work in this final step, with $i = n$. By symmetry, the same arguments prove the simplicity for the arcs a_2, b_2.

This finishes the proof: The last drawing obtained, D'_n, satisfies Invariants 1, 2. Then, taking O and Z as vertices, it is the sought drawing D_{n+2}, the one satisfying the Properties **(P1)** and **(P2)**.