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— Abstract -

For a drawing of a labeled graph, the rotation of a vertex or crossing is the cyclic order of its incident edges, represented by the labels of their other endpoints. The extended rotation system (ERS) of the drawing is the collection of the rotations of all vertices and crossings. A drawing is simple if each pair of edges has at most one common point. Gioan's Theorem states that for any two simple drawings of the complete graph K_n with the same crossing edge pairs, one drawing can be transformed into the other by a sequence of triangle flips (a.k.a. Reidemeister moves of Type 3). This operation refers to the act of moving one edge of a triangular cell formed by three pairwise crossing edges over the opposite crossing of the cell, via a local transformation.

We investigate to what extent Gioan-type theorems can be obtained for wider classes of graphs. A necessary (but in general not sufficient) condition for two drawings of a graph to be transformable into each other by a sequence of triangle flips is that they have the same ERS. As our main result, we show that for the large class of complete multipartite graphs, this necessary condition is in fact also sufficient. We present two different proofs of this result, one of which is shorter, while the other one yields a polynomial time algorithm for which the number of needed triangle flips for graphs on *n* vertices is bounded by $O(n^{16})$. The latter proof uses a Carathéodory-type theorem for simple drawings of complete multipartite graphs, which we believe to be of independent interest.

Moreover, we show that our Gioan-type theorem for complete multipartite graphs is essentially tight in the following sense: For the complete bipartite graph $K_{m,n}$ minus two edges and $K_{m,n}$ plus one edge for any $m, n \ge 4$, as well as K_n minus a 4-cycle for any $n \ge 5$, there exist two simple drawings with the same ERS that cannot be transformed into each other using triangle flips. So having the same ERS does not remain sufficient when removing or adding very few edges.

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1 Introduction

Gioan's Theorem states that any two simple drawings of the complete graph K_n in which the same pairs of edges cross can be transformed into each other (up to strong isomorphism) via a sequence of triangle flips. Informally, a triangle flip is the act of moving one edge of a triangular cell formed by three pairwise crossing edges over the opposite crossing of the cell; see Figure 1 for an illustration of this operation and Section 2 for the formal definition.



Figure 1 A sketch of a triangle flip.

Gioan's Theorem can be seen as a generalization of results on pseudolines by Ringel [29] from 1955 and Roudneff [30] from 1988 to simple drawings of K_n . Gioan's conference paper [15] from 2005 contained a proof sketch only. A full proof was first published in 2017 by Arroyo, McQuillan, Richter, and Salazar [4], who also coined the name "Gioan's Theorem". In 2021, Schaefer [31] generalized Gioan's Theorem to slightly sparser graphs, namely, simple drawings of K_n minus any non-perfect matching. A full version of Gioan's proof [16] finally appeared in 2022.

A priori it is not clear how to generalize Gioan's Theorem beyond Schaefer's result. For transforming drawings of general graphs via triangle flips, it is not sufficient to only have the same crossing edge pairs. We should also consider the rotation of a vertex or edge crossing, which is defined as the cyclic order of emanating edges. For example, Figure 2 shows two simple drawings of the complete bipartite graph $K_{3,3}$ with the same crossing edge pairs and the same rotations of vertices, but different rotations of the crossings involving b_1r_3 . Observe that triangle flips do not change the rotations of crossings or vertices. A take-away from this observation is that for a Gioan-type theorem to hold, the rotations of all crossings and vertices must be the same in both drawings. A concept capturing exactly this necessity is the extended rotation system. The extended rotation system (ERS) of a drawing of a graph is the collection of the rotations of all vertices and crossings. In this light, one of the contributions of Gioan's Theorem is that for drawings of the complete graph, having the same crossing edge pairs is equivalent to having the same ERS (up to global inversion) [15, 16]. This fact has been first stated by Gioan [15]; the first published proofs are by Kynčl [22, 23]. An analogous statement for K_n minus any non-perfect matching has been shown by Schaefer [31]. For complete multipartite graphs, this equivalence does not hold; see again Figure 2.

As our main result, we show that having the same ERS is sufficient to transform simple drawings of complete multipartite graphs into each other via triangle flips. We thus obtain a



Figure 2 Two simple drawings of $K_{3,3}$ with the same crossing edge pairs and same rotations at all vertices but different rotations at all crossings involving the edge b_1r_3 and hence different ERSs.

Gioan-type theorem for a large class of graphs that includes the before studied graphs, namely complete graphs [4, 15, 16, 31] and complete graphs minus a non-perfect matching [31].

▶ **Theorem 1.** Let D_1 and D_2 be two simple drawings of a complete multipartite graph on the sphere S^2 with the same ERS. Then there is a sequence of triangle flips that transforms D_1 into D_2 .

We also show that Theorem 1 is essentially tight in the sense that having the same ERS does not remain sufficient when removing or adding very few edges.

▶ **Theorem 2.** For any $m, n \ge 3$ and $K_{m,n}$ minus any two edges, there exist two simple drawings with the same ERS that cannot be transformed into each other using triangle flips. The same holds for any $n \ge 5$ and K_n minus any four-cycle C_4 , as well as for any $m \ge 4, n \ge 1$ and $K_{m,n}$ plus one edge between vertices in the bipartition class of size m.

The first part of Theorem 2 implies that an analogue to Schaefer's generalization of Gioan's Theorem for K_n minus a non-perfect matching cannot be achieved for complete bipartite graphs, not even for $K_{m,n}$ minus a matching of size two. Note that $K_{m,n}$ with $m \ge 4$ and $n \ge 1$ is a subgraph of K_{n+m} minus a 4-cycle. Hence, the second part of Theorem 2 implies that—perhaps counterintuitively—the set of graphs for which a Gioan-type theorem holds is not closed under adding edges. From the proof of Theorem 2 it follows that Theorem 1 cannot be extended to any graph that contains a K_5 minus a four-cycle C_4 or a $K_{3,2}$ minus two edges incident to the same vertex of the smaller partition class, as an induced subgraph.

We present two different proofs of Theorem 1. Our first proof uses a similar approach as the proof of Gioan's Theorem by Schaefer [31]. His proof heavily relies on a (plane) spanning star as a basis for transforming one drawing into the other. While plane spanning stars exist in any simple drawing of K_n , also minus a non-perfect matching, this is in general not the case for complete multipartite graphs. However, any simple drawing of a complete multipartite graph G contains a plane spanning tree [2]. We show that for drawings of Gwith the same ERS, such a plane spanning tree can be used for transforming one drawing into the other. The resulting proof is shorter and probably more elegant than the second proof. But it does not directly yield a polynomial time transformation algorithm, as it is still an open question [2] whether a plane spanning tree can be found in polynomial time.

Our second proof yields a polynomial time algorithm for the transformation. It uses a similar approach as the proof of Gioan's Theorem by Arroyo, McQuillan, Richter, and Salazar [4]. Several ingredients of their proof are known properties of drawings of complete graphs or follow directly from such properties, while it was unknown whether analogous statements hold for drawings of other graphs. Hence, for our proof we discover a number of useful, fundamental properties of simple drawings of complete multipartite graphs. For example, we establish a Carathéodory-type theorem for them.

The classic Carathéodory Theorem states that if a point $p \in \mathbb{R}^2$ lies in the convex hull of a set $A \subset \mathbb{R}^2$ of $n \ge 3$ points, then there exists a triangle spanned by points of A that

contains p. In the terminology of drawings, if a point p lies in a bounded cell of a straight-line drawing D of K_n in \mathbb{R}^2 , then there exists a 3-cycle C in D so that p lies in the bounded cell of C. This statement has been generalized to simple (not necessarily straight-line) drawings of K_n [6, 7]. However, it clearly does not generalize to arbitrary (non-complete) graphs; consider for example a simple drawing of a path with self-intersections that forms a bounded cell. A natural question is, for which classes of graphs this statement, or a variation of it, holds. We show that it holds for complete multipartite graphs if in addition to 3-cycles—which might not exist in those graphs—we also allow 4-cycles to contain p.

▶ **Theorem 3** (Carathéodory-type theorem for simple drawings of complete multipartite graphs). Let D be a simple drawing of a complete multipartite graph G in the plane. For every point p in a bounded cell of D, there exists a cycle C of length three or four in D such that p is contained in a bounded cell of C. This statement is tight in the sense that it may not hold for G minus one edge.

Number of triangle flips. Schaefer [31, Remark 3.3] showed that for K_n , polynomially many triangle flips are sufficient and gave an upper bound of $O(n^{20})$ for the number of required flips. Using a different approach in our second proof of Theorem 1, we show an upper bound of $O(n^{16})$ triangle flips for complete multipartite graphs on *n* vertices. We further present drawings which, regardless of the approach, require at least $\Omega(n^6)$ triangle flips.

Motivation and related work. Originally, rotation systems were invented to investigate embeddings of graphs on higher-genus surfaces [17]. Nowadays they are widely used to represent drawings of graphs in the plane and to derive their structural properties. Gioan's Theorem implies that for simple drawings of complete graphs, the set of crossing pairs of edges determines the drawing's ERS. Conversely, for drawings of complete graphs, the rotation system determines which pairs of edges cross [22, 27]. These relations are crucial in the study of simple drawings of complete graphs, their generation and enumeration [1, 22, 24].

For non-complete graphs, the literature on rotation systems for simple drawings is rather sparse. Besides the recent work of Schaefer [31], we are only aware of work by Cardinal and Felsner [8], who investigate the realization of complete bipartite graphs as outer drawings. The main reason why there are no further results on rotation systems beyond drawings of complete graphs is the lack of known properties in these cases. Our work contributes towards the generalization of rotation systems to drawings of wider graph classes, not only by the main statement but also due to the structural results obtained along the way.

We note that rotation systems of drawings also play a role in a wider context. For example, they are crucial in a recent breakthrough result devising an algorithm for the subpolynomial approximation of the crossing number for non-simple drawings of general graphs [10].

The study of triangle flips has a long history in several different contexts. In addition to the mentioned work on Gioan's Theorem [4, 15, 16, 31], this in particular includes work on arrangements of pseudolines [14, 29, 30, 32], knot theory [3, 20, 21, 25, 28, 35, 36], as well as on transforming curves on compact oriented surfaces [9].

Outline. In Section 2, we mainly state definitions, introduce notation, and give a characterization of complete multipartite graphs. In Sections 3 and 4 we sketch the proofs of the Carathéodory-type Theorem 3 and Theorem 2, respectively. Section 5 is devoted to proving Theorem 1, where the first proof is given nearly fully, and the second one is shortly sketched to explain the algorithm. In Section 6 we present bounds on the required number of triangle flips derived from the second proof. We conclude the paper with open questions in Section 7.

2 Definitions and preliminaries

A graph G = (V, E) is *multipartite* if its vertex set V can be partitioned into k nonempty subsets V_1, \ldots, V_k , for some $k \in \mathbb{N}$, such that each V_i , for $i \in \{1, \ldots, k\}$, induces an independent set in G, that is, no two vertices in V_i are adjacent. A *complete multipartite* graph G = (V, E) contains all edges outside of the independent sets, that is, we have $E = \{v_i v_j : v_i \in V_i \land v_j \in V_j \land 1 \leq i < j \leq k\}$. For a multiset $\{n_1, \ldots, n_k\}$ of natural numbers, there is a unique (up to isomorphism) complete multipartite graph K_{n_1,\ldots,n_k} with $|V_j| = n_j$, for all $j \in \{1, \ldots, k\}$. Note that both the empty graph on n vertices (with k = 1 and $n_1 = n$) and the complete graph K_n (with k = n and $n_1 = \cdots = n_k = 1$) are complete multipartite graphs. We also have the following useful characterization, whose proof is an easy graph-theoretic exercise. For completeness, we include it in Appendix A.1.

▶ Lemma 4. A graph G = (V, E) is complete multipartite if and only if for every edge $uv \in E$ and every vertex $w \in V \setminus \{u, v\}$ we have $uw \in E$ or $vw \in E$ (or both).

Drawings. A drawing γ of a graph G = (V, E) is a geometric representation of G by points and curves on an oriented surface S. More precisely, every vertex v of G is mapped to a point γ_v on S and every edge uv of G is mapped to a simple (that is, continuous and not self-intersecting) curve γ_{uv} on S with endpoints γ_u and γ_v , such that:

- 1. Any two vertices are mapped to distinct points $(\gamma_u = \gamma_v \Longrightarrow u = v, \text{ for all } u, v \in V).$
- 2. No vertex is mapped to the relative interior of an edge $(\gamma_{uv} \cap \gamma_w = \emptyset$, for all $uv \in E$ and $w \in V \setminus \{u, v\}$).
- 3. Every pair of curves γ_e, γ_f , for $e \neq f$, intersects in at most finitely many points, each of which is either a common endpoint or a proper, transversal crossing.

In this paper, we consider drawings on the sphere S^2 , except for a few places—specified explicitly—where we consider drawings in the plane \mathbb{R}^2 . All our graphs and drawings are labeled. Hence, we often identify vertices and edges with their geometric representation in a drawing. Any subgraph H of G induces a subdrawing $\gamma[H]$ that is obtained by restricting γ to the vertices and edges of H. For a graph F, an F-subdrawing of γ is a subdrawing $\gamma[H]$ that is induced by some subgraph H of G that is isomorphic to F. A drawing partitions S into vertices (endpoints) and crossings of the curves { $\gamma_e : e \in E$ }, edge fragments (the connected components of the curves { $\gamma_e : e \in E$ } after removing all vertices and crossings), and cells (the connected components of S after removing all vertices, crossings, and edge fragments). For a cell C we denote by ∂C the boundary of C. A cell that is bounded by exactly three edge fragments is called a tricell.

The class of drawings of a graph is vast and for many purposes too rich to be directly useful. To begin with, it is not clear in general how to represent a drawing using a finite amount of space. Two natural approaches to address this concern are to (1) further restrict the class of drawings or (2) study drawings on a much coarser level, up to some notion of isomorphism. In this work, we use a combination of both of these approaches.

Simple drawings. An example for the first approach are *straight-line drawings* in the Euclidean plane (also known as *geometric graphs*), where the geometry of an edge is uniquely determined by the location of its endpoints; see the *Handbook of Discrete and Computational Geometry* [34, Chapter 10] and references therein. In this work, we consider a more general class of drawings, which appear in the literature as *simple drawings* [11], *good drawings* [5, 12], *topological graphs* [26], *simple topological graphs* [22], and even just as *drawings* [18]. In a simple drawing, every pair of edges has at most one point in common, either a common endpoint or a proper crossing. Additionally, we may assume that no three edges meet



Figure 3 Two drawings of $K_{3,3}$ that have same ERS but are not strongly isomorphic (because ux crosses vy and wz in different order). The shaded tricell is an invertible triangle with different parities in the two drawings.

at a common point. Simple drawings are a combinatorial/topological generalization of straight-line drawings. If the graph G has n vertices, then every simple drawing of G has $O(n^4)$ crossings, edge fragments, and cells. Simple drawings are also important for crossing minimization because all crossing-minimal drawings are simple [33].

Strong isomorphism. An example for the second approach is the notion of strong isomorphism for drawings, defined as follows. Two drawings γ and η of a graph G = (V, E) are strongly isomorphic, denoted by $\gamma \cong \eta$, if there exists an orientation-preserving homeomorphism¹ of S that maps γ to η , that is, $\gamma_v \mapsto \eta_v$, for all $v \in V$, and $\gamma_e \mapsto \eta_e$, for all $e \in E$. A combinatorial formulation, which is equivalent for connected drawings, can be obtained as follows [22]:

- 1. The same pairs of edges cross. (This is called *weak isomorphism*.)
- 2. The order of crossings along each edge is the same.
- **3.** At each vertex and crossing the *rotation*, that is, the clockwise circular order of incident edges, is the same (see next paragraph for more details).

The notion of strong isomorphism encapsulates basically everything that can be said about a drawing from a topological or combinatorial point of view: the order of edges around vertices and cells, which pairs of edges cross, and in which order the crossings appear along an edge. For our purposes, we consider strongly isomorphic drawings to be equivalent.

Extended rotation systems. A coarser notion of equivalence can be obtained by requiring two drawings to have the same *rotation system*, which is the collection of the rotations of all vertices. Property 3 in the above-mentioned combinatorial description uses a slightly stronger notion of equivalence, where also the rotations at crossings are the same in both drawings. More formally, the *rotation of a crossing* χ is the clockwise cyclic order of the four vertices of the crossing edge pair which is induced by the cyclic order of edge fragments around χ . (In other words, the rotation of a crossing χ is the rotation of an additional degree-4 vertex v_{χ} obtained by splitting the crossing edge pair at χ and replacing χ by v_{χ} .) The *extended rotation system* (ERS) of a drawing is the collection of rotations of all vertices and crossings. Any two strongly isomorphic drawings have the same ERS [22]. But the converse is not true in general, as the example in Figure 3 demonstrates.

Crossing triangles. In fact, the only difference between the two drawings in Figure 3 with respect to strong isomorphism stems from the tricell formed by the triple ux, vy, wz of pairwise crossing edges, which is shaded gray in the figure: In the left drawing, this cell lies to

¹ Strong isomorphism can also be defined for unlabeled drawings; then a mapping for the vertex sets is needed. The homeomorphism is sometimes not required to be orientation-preserving; then, e.g., mirror-images of drawings are also considered to be strongly isomorphic.

the right of the oriented edge ux, whereas in the right drawing, it lies to the left of ux. Given a simple drawing, a tricell Δ in the subdrawing of three pairwise crossing edges e_1, e_2, e_3 is called a *crossing triangle*; the three edges e_1, e_2, e_3 are said to span Δ . Note that every edge triple in a simple drawing spans at most one crossing triangle. The following lemma shows that the crossing triangles are well-defined for complete multipartite graphs. We do not use it later on and it also follows from the proof of Theorem 1. However, for completeness, we include a direct (and much shorter) proof in Appendix A.2.

▶ Lemma 5. In every simple drawing of a complete multipartite graph, the set of edge triples that span crossing triangles is uniquely determined by the ERS.

Invertible triangles and triangle flips. To formally define the triangle flip operation, globally fix an orientation π of the edges of the abstract graph G. This orientation can be arbitrary, but once we fix the graph, we also fix its orientation. With this orientation π , we can assign every crossing triangle a parity as follows. The *parity* of a crossing triangle Δ in a drawing is the parity (odd or even) of the number of bounding edges of Δ such that Δ lies to the left of the edge (when going along the edge according to its orientation). See Figure 3 for two drawings with even (left) and odd (right) parity of the crossing triangle. A crossing triangle Δ in a drawing γ is *invertible* if there exists another simple drawing $\gamma' \neq \gamma$ of the same graph G with the same edge orientation π and with the same ERS in which Δ appears with the opposite parity. In Lemma 14, we show that any invertible triangle in a drawing of a complete multipartite graph is empty in the sense that it does not contain any vertices.

Locally redrawing the edges of an empty crossing triangle and thereby changing its parity is an elementary operation to transform a given drawing, say, the one in Figure 3(left), into a new drawing, such as the one in Figure 3(right). Up to strong isomorphism, there is a unique way for the redrawing. This operation is referred to as *triangle flip* [4], *triangle mutation* [15], *slide move* [31], *homotopy move* [9, 20], or *Reidemeister move of Type 3*, where the latter name has been extensively used² in knot theory [3, 21, 25, 28, 35, 36].

Triangle flip graphs. Based on the triangle flip as an elementary operation, we can define a meta graph whose vertices are drawings and whose edges correspond to triangle flips. We fix a graph G and consider all simple drawings of G on S up to strong isomorphism; these are the vertices of the triangle flip graph $\mathcal{T}(G)$. Any two such drawings γ, η are connected by an edge in $\mathcal{T}(G)$ if η can be obtained from γ by a single triangle flip. As triangle flips are reversible, edges are symmetric. So we consider $\mathcal{T}(G)$ as an undirected graph.

Observe that a triangle flip does not change the rotation of any vertex or crossing, only the order of crossings along the edges changes. Therefore only drawings that have the same ERS can be in the same component of $\mathcal{T}(G)$. In general, the flip graph $\mathcal{T}(G)$ may be disconnected. Consider, for instance, the two drawings of a path depicted in in Figure 4. As neither drawing contains any crossing triangle, both are isolated vertices in $\mathcal{T}(G)$.



Figure 4 Two drawings of a path with the same ERS, but the order of crossings along the edge *cd* differs, thus, the drawings are not strongly isomorphic. Neither drawing contains any tricell to flip.

 $^{^{2}}$ albeit in the context of knots also an above/below relationship among the curves is relevant

3 A Carathéodory-type theorem for complete multipartite graphs

This section is devoted to a proof outline of the Carathéodory-type Theorem 3. The corresponding statement for simple drawings of K_n , which is a direct generalization of the classic theorem for convex sets in \mathbb{R}^2 , was shown by Balko, Fulek, and Kynčl [6]. A simpler proof was given later by Bergold, Felsner, Scheucher, Schröder, and Steiner [7], whose proof idea we follow. The full proof of Theorem 3 can be found in Appendix B. We give a proof outline here.

Sketch of Proof. If G is empty or a star $K_{1,n}$, then the statement is vacuously true. So we assume that G is neither, and thus every pair of distinct vertices $u, v \in V$ with $uv \notin E$ has at least two distinct common neighbors. By studying a minimal counter-example we prove Theorem 3 by contradiction. To that aim, we consider a simple drawing D of G and a point p, such that the following holds:

- 1. The point p is in a bounded cell of D.
- 2. The point p is not contained in a bounded cell of any induced C_i -subdrawing of D, for $i \in \{3, 4\}$.
- 3. When removing any vertex from D, the point p lies in the unbounded cell.

Let a be a vertex of G, and let O be the smallest set of edges incident to a such that removal of all edges of O from D puts p into the unbounded cell of the resulting drawing D^- . Then in D^- one can draw a simple curve P from p to the interior of the unbounded cell of D so that P does not intersect any vertex or edge of D^- . Subject to this constraint, we select P to minimize the number of crossings with edges of D. We show that we can assume every edge in O crosses P exactly once. Finally we consider an edge $ab \in O$, which crosses P in a point p_{ab} , and analyze two cases depending on whether ab crosses another edge between a and p_{ab} or not. We show that in both cases, p is contained in a bounded cell of an induced C_i -subdrawing of D, for $i \in \{3, 4\}$.



Figure 5 Drawing of $K_{m,n}$ minus one edge $(r_2b_1, \text{ drawn dashed})$, based on Figure 6. The point p lies in a bounded cell, but in no C_i , for $i \in \{3, 4, 5\}$.

To see that the theorem may not hold if we remove one edge from G, consider the simple drawing of $K_{m,n}$, $m, n \ge 2$, depicted in Figure 5. When removing the edge b_1r_2 , the point p still lies in a bounded cell, but any cycle that encloses p has at least six vertices.

4 Theorem 1 is essentially tight

Theorem 2 implies that Theorem 1 is essentially tight: The removal or addition of very few edges may yield a graph for which the theorem does not hold. This implies that the class of graphs for which this Gioan-type theorem holds is not closed under the operation of taking (non-induced) subgraphs or supergraphs. We sketch the proof of Theorem 2 by depicting the drawings we use to show tightness. The full proof can be found in Appendix C.

Each of Figures 6–9 contains two simple drawings of a graph with the same ERS. In all of them, the crossing order along b_1r_1 differs between the two drawings. This order cannot be changed via triangle flips because the edges crossing b_1r_1 in different orders are pairwise non-crossing. Figures 6 and 7 cover the case of $K_{m,n}$ minus two adjacent or disjoint edges, Figure 8 is an extension of Figure 6 to K_m minus a 4-cycle, and Figure 9 shows subdrawings of Figure 8 that form a $K_{m-1,n+1}$ plus one edge.



Figure 6 Two drawings of $K_{m,n}$ minus two adjacent edges b_1r_2 and b_1r_3 (drawn as dashed lines) that have the same ERS but cannot be transformed into each other via triangle flips.



Figure 7 Two drawings of $K_{m,n}$ minus two independent edges b_2r_1 and b_1r_2 (drawn dashed) that have the same ERS but cannot be transformed into each other via triangle flips.

We remark that also two simple drawings with the same ERS that cannot be transformed into each other via triangle flips exist for any graph that contains (1) a K_5 minus a 4-cycle, or (2) a $K_{2,3}$ minus two edges sharing a vertex in the bipartition class of cardinality two (where the list of induced subgraphs is not exhaustive). This can be shown by choosing appropriate subdrawings in the construction from Figure 8.



Figure 8 Two drawings of K_m minus a 4-cycle (drawn dashed) that have the same ERS, but cannot be transformed into each other via triangle flips.



Figure 9 Two drawings of $K_{m-1,n+1}$ plus one edge (b_1r_1) that cannot be transformed into each other via triangle flips.

5 A Gioan-type theorem for complete multipartite graphs

In this section, we present our two proofs of Theorem 1 and include a short algorithmic discussion of the second one.

5.1 First proof of Theorem 1

For our first proof of Theorem 1, we use the same general approach as Schaefer [31]. To closely follow the lines of Schaefer, we also use homeomorphisms in this proof.

Proof. Let G be a complete multipartite graph, and let D_1 and D_2 be two simple drawings of G on S^2 with the same ERS. Let $R = \{r_1, r_2, \ldots, r_n\}$ be a maximal independent set in G and let $B = \{b_1, b_2, \ldots, b_m\}$ denote the set of the remaining vertices. Note that the graph on the vertex set $R \cup B$ together with all edges with an endpoint in R and one in B forms a complete bipartite graph $K_{n,m}$, and the set R is an independent set in G while B might not necessarily be an independent set.

By [2], the subdrawing of D_1 spanned by this $K_{n,m}$ contains a spanning tree T which is drawn crossing-free in this subdrawing and hence also in D_1 . As D_1 and D_2 have the same crossing edge pairs, T is drawn crossing-free in D_2 as well. Since the rotation systems of D_1 and D_2 are the same by assumption, the drawings of T in D_1 and D_2 are homeomorphic. Thus there exists a drawing $D \cong D_1$ with the following properties.

- 1. The drawing of T is the same for both drawings D and D_2 , implying that also the vertex locations are the same in both drawings.
- 2. Considering the set of the vertices and edges of D and D_2 together as the *combined* drawing of D and D_2 , we denote the cyclical order of edges in D and D_2 emanating from a vertex as *combined rotation* at that vertex. For each edge e of G (not in T) and each vertex v of e, the two drawings of e are consecutive in the combined rotation at v.
- 3. For each edge e of G, the two drawings of e are either identical or have only finitely many points in common (two are its endpoints and the others are proper crossings).

Our goal is to change D via triangle flips (and orientation-preserving homeomorphisms) until we obtain $D = D_2$. Since the vertex locations in both drawings are the same, we can speak about two drawings of an edge, one in D, and one in D_2 , being the same or not. As in Schaefer's proof, we iteratively reduce the number of edges that are drawn differently in Dand D_2 . Let E_{\pm} be the set of edges whose drawings in D and D_2 are the same. Initially, E_{\pm} contains at least all edges of T. If E_{\pm} contains all edges of G then we are done.

So suppose that this is not the case and consider an edge e that is drawn differently in D and D_2 . Let e_1 and e_2 denote the curves representing e in D and D_2 , respectively. Since D and D_2 have the same ERS, e_1 and e_2 cross the same edges of T and they do so with the same crossing rotations. Moreover, the following lemma implies that they also cross those edges in the same order. The lemma can be proven relying on Lemma 4 and using a case distinction for drawings with six vertices.

▶ Lemma 6. Let D be a simple drawing of a complete multipartite graph G on S^2 and let vw be an edge of G. Then for any pair of adjacent or disjoint edges crossed by vw, the ERS of D determines the order in which vw crosses them.

Hence e_1 and e_2 are equivalent with respect to the drawing of T (which is the same in D and D_2), that is, e_1 has the same sequence of directed crossings with T as e_2 . Let $\Gamma = e_1 \cup e_2$ be the (not necessarily simple) closed curve formed by e_1 and e_2 . A lens in Γ is a cell of Γ whose boundary is formed by exactly two edge fragments of Γ , where one is from e_1 and one is from e_2 . Next, consider the drawing D_T of T plus the drawings e_1 and e_2 of e. A lens of Γ is called *empty* if it contains no vertices of T (and hence also no vertices of G) in its interior. With the next lemma, we show that Γ forms an *empty lens*. This lemma is a special case of a result of Hass and Scott on intersecting curves on surfaces [19, Lemma 3.1], which is also known as the *bigon criterion* [13, Section 1.2.4]. Schaefer [31, Lemma 3.2] gives an elementary proof in the planar (or spherical) case when the plane spanning tree T is a star. However, he only uses that the star is a spanning subdrawing that is crossing-free and that e_1 and e_2 are equivalent with respect to the star. Thus, we can follow the proof line by line to obtain the result for any plane spanning tree T.

▶ Lemma 7 ([13, 19, 31]). Let D_1 and D_2 be two simple drawings of a graph on S^2 that contain the same crossing-free drawing D_T of a spanning tree T as a subdrawing. Let e be an edge for which the drawings e_1 and e_2 differ, but are equivalent with respect to D_T . Then $\Gamma = e_1 \cup e_2$ forms an empty lens.

Let L be an empty lens of Γ , which is formed by the edge fragments γ_1 of e_1 and γ_2 of e_2 , respectively. Each of the two points of $\gamma_1 \cap \gamma_2$ is either an endpoint or a crossing between e_1 and e_2 . Recall that, in the combined drawing of D and D_2 , e_1 and e_2 are consecutive in the combined rotation at each of their endpoints. Hence, independent of whether the points of $\gamma_1 \cap \gamma_2$ are crossings or endpoints, γ_2 is what Schaefer calls a "homotopic detour of γ_1 on e_1 ".

We next need his detour lemma, which we restate here using slightly different terminology (and for drawings on the sphere instead of in the plane).

▶ Lemma 8 (detour lemma [31, Lemma 2.1]). Let γ_2 be a homotopic detour of the arc γ_1 on the edge e_1 in a simple drawing of a graph. Let F be the set of edges which cross γ_2 at least twice. Then we can apply a sequence of triangle flips and homeomorphisms of the sphere S^2 so that in the resulting drawing, γ_1 is routed arbitrarily close to γ_2 , without intersecting it. The triangle flips and homeomorphisms only affect a small open neighborhood of the region bounded by $\gamma_1 \cup \gamma_2$, and only edges in F and the γ_1 part of e_1 are redrawn.

Note that the set F of edges that are affected by the transformation is disjoint from $E_{=}$, because any edge of $E_{=}$ is identical in D and D₂ and hence intersects γ_2 at most once.

If at least one of the points of $\gamma_1 \cap \gamma_2$ is a crossing, then after applying the detour lemma, we can redraw e_1 (via a homeomorphism) to have at least one fewer crossing with e_2 and repeat the process of applying Lemmas 7 and 8 with the redrawn edge.

If none of the points of $\gamma_1 \cap \gamma_2$ is a crossing, then $e_1 \cup e_2$ is a simple closed curve and $\gamma_1 = e_2$ is a homotopic detour of $\gamma_2 = e_1$. Hence, after one final application of Lemma 8, we can redraw e_1 to be identical to e_2 . With this step, e_2 is added to $E_{=}$ and we have reduced the number of edges differing between D and D_2 by one.

Repeating this process for the remaining differing edges we obtain two identical drawings. Omitting the homeomorphisms, the process yields a sequence of triangle flips for transforming D_1 into D_2 (up to strong isomorphism), which completes the proof of the theorem.

5.2 Second Proof of Theorem 1

Our second proof of Theorem 1, which we briefly outline here, uses the same general framework as the proof of Gioan's Theorem by Arroyo, McQuillan, Richter, and Salazar [4]. We present only a brief outline here. The full proof, starting with a more detailed outline, can be found in Appendix D.

Sketch of Proof. We consider two simple drawings D_1 and D_2 of a complete (multipartite) graph G = (V, E) with the same ERS, and one of them, say $D := D_1$, is iteratively transformed to become "more similar" to the other. Similarity is measured using a subgraph X of G for which we demand as an invariant that the induced subdrawings D[X] and $D_2[X]$ are strongly isomorphic. In each iteration, we will add one edge to X and then perform a sequence of triangle flips in D so as to reestablish the invariant.

Initially, we establish the invariant in the following way. As in the first proof, we consider an independent set $R \subseteq V$ of vertices such that G contains a complete bipartite subgraph between R and $B := V \setminus R$. If G is complete, then R contains a single vertex only; in general, it may contain several vertices. We then pick one vertex $r_0 \in R$ and start by taking X to be the maximal induced substar of G centered at r_0 (which includes all vertices of B). Then the invariant holds because both drawings have the same rotation system by assumption.

We then consider the (possibly) remaining vertices of R in an arbitrary order. Let $r \in R$ be the next vertex to be considered. First, we show that the position of r in the induced strongly isomorphic, by the invariant—subdrawings D[X] and $D_2[X]$ is consistent, that is, the vertex r lies in the same (according to isomorphism) face of these drawings (Lemma 11, whose proof uses the Carathéodory-type Theorem 3).

We add the edges incident to r one by one to X. When adding an edge rb to X to obtain $X' = X \cup \{rb\}$, the drawings D[X'] and $D_2[X']$ may not be strongly isomorphic because the edge rb may cross other edges in a different order in both drawings. We consider

a sort of overlay O of both drawings D[X'] and $D_2[X']$, in which the two versions of rb together form a closed curve Γ with $O(|V(X')|^4)$ self-crossings (Lemma 12), where |V(X')| is the number of vertices of X'. In Γ , we can identify a nice substructure, which we refer to as a free lens, and show that it always exists (Lemma 13). A lens in Γ is free if it does not contain any vertex of O; it may contain edge crossings, though. Each such edge crossing corresponds to an invertible triangle in D. Invertible triangles are empty of vertices (Lemma 14), not only of the vertices in X but also of the (possibly) not yet considered vertices of R. Hence, the edges of D that cross an invertible triangle Δ behave similarly to a collection of pseudolines inside Δ , except that not all pairs need to cross. Let m be the number of edges that cross Δ . Using a classic sweeping algorithm by Hershberger and Snoeyink [32, Lemma 3.1], all m edges can be "swept" out of Δ via triangle flips in D, where the total number of flips is bounded by $O(m^3)$. After these flips, Δ has become a crossing triangle and can be flipped in D. Processing all invertible triangles inside a selected free lens in this fashion effectively destroys this lens. And after iteratively destroying all free lenses, the resulting drawing D[X']

After all vertices in R and the complete bipartite subgraph of G between R and B have been added to X, we add the remaining edges (the ones with both endpoints in B) in exactly the same fashion as described above.

While the outline of the above proof mostly follows the one for K_n [4], its core challenges lie in the proofs of several statements, whose analogues are known for K_n but not for complete multipartite graphs. These include in particular the proofs of Lemmata 11, 13, and 14 (while the proof of Lemma 12 is quite straightforward). We discuss these challenges at the end of Appendix D.2.

Algorithmic complexity. The above proof yields an algorithm that can be implemented using standard computational geometry data structures. Its runtime is polynomial in the size of the input and the number of performed triangle flips.

6 On the number of triangle flips

The *flip distance* between two different drawings of a complete multipartite graph with the same ERS is the minimum number of triangle flips that are required to transform one drawing into the other. This section is devoted to obtain bounds on the flip distance.

For an upper bound, Schaefer [31, Remark 3.3] showed that any two simple drawings of K_n with the same rotation system can be transformed into each other with at most $O(n^{20})$ triangle flips. Using our second proof of Theorem 1, we can obtain an upper bound of $O(n^{16})$ on the flip distance between two simple drawings of any complete multipartite graph with nvertices and the same ERS (and thus also for such drawings of K_n).

▶ **Theorem 9.** Let D_1 and D_2 be two simple drawings of a complete multipartite graph G on S^2 with n vertices and with the same ERS. Then D_1 can be transformed into D_2 via a sequence of $O(n^{16})$ triangle flips, obtained via the algorithm in the second proof of Theorem 1.

Proof. We analyze the number of flips performed through the second proof of Theorem 1. Recall that in this proof, we iteratively consider the edges of G. We perform flips in a drawing D (initially set to D_1) so that the subdrawings of D and D_2 induced by the already considered edges become (strongly) isomorphic.

When considering a new edge e, we imagine to add both versions of it (the one from Dand the one from D_2) to the already isomorphic subdrawing X of D and D_2 . By Lemma 12,

this can be done in such a way that in the combined drawing, the two copies of e have $O(|V(X)|^4) = O(n^4)$ crossings, where |V(X)| is the number of vertices of X.

Let C be the closed curve formed by the two copies of e. In order to transform D to make the drawing of e in D isomorphic to the one in D_2 , we iteratively resolve a free lens of C. At every iteration, we reduce the number of crossings of C, except for the very last iteration (i.e, for the very last lens). Hence, the number of lenses we need to resolve when processing e is bounded by $O(n^4)$ as well. To resolve a free lens, we need to flip all inverted triangles in this lens that have e as an edge, of which there are at most $O(n^4)$ many. For one inverted triangle Δ intersected by $m = O(n^2)$ edges, this can be done with $O(m^3) = O(n^6)$ flips. Hence resolving one free lens can be achieved with $O(n^4) \cdot O(n^6) = O(n^{10})$ flips.

Repeating this for all lenses of C and for each of the $O(n^2)$ edges of G, we obtain an upper bound of $O(n^2) \cdot O(n^4) \cdot O(n^{10}) = O(n^{16})$ for the total number of triangle flips.



Figure 10 Two simple drawings of K_n with the same ERS whose flip distance is $\Omega(n^6)$.

▶ **Theorem 10.** Let G be a multipartite graph G with n vertices that contains two vertexdisjoint subgraphs each forming a $K_{m,m}$ for some $m = \Theta(n)$. Then G admits two drawings D_1 and D_2 with the same ERS that have flip distance $\Omega(n^6)$.

Proof idea. To transform the two drawings of K_n in Figure 10 into each other, each of the $\Theta(n^2)$ edges $b_i d_j$ needs to be moved over the $\Theta(n^4)$ crossings formed by edges $a_k c_\ell$, yielding the $\Omega(n^6)$ lower bound. An according example of two drawings of a $K_{m,m}$ can be obtained by disregarding all edges $a_i b_j$ and $c_i d_j$. A detailed proof can be found in Appendix E.

7 Conclusion & open questions

We have shown that Gioan's Theorem holds for complete multipartite graphs (Theorem 1), extending previous results [4, 15, 16, 31]. Further, we have shown that the class of graphs for which an analogue statement holds is not closed under addition or removal of edges (Theorem 2). We also provide several obstructions such that Gioan's Theorem does not hold for any graph that contains any of these obstructions as a substructure. However, the list of obstructions is probably incomplete. A full characterization of graphs for which a Gioan-type statement for drawings with the same ERS holds remains open.

▶ Question 1. Can we completely characterize all graphs for which a Gioan-type theorem holds for drawings with the same ERS?

Further, having the same ERS is not the only necessary condition for a Gioan-type statement to hold. Another example of such a condition is that incident or disjoint edges must have the same crossing orders over all drawings. The constructions in the proof of Theorem 2 rely on violating this condition.

▶ Question 2. Can we characterize all graphs for which a Gioan-type theorem holds for classes of drawings which fulfill (subsets of) obviously necessary conditions?

In Section 3, we have proven a Carathéodory-type theorem for simple drawings of complete multipartite graphs with the same ERS (Theorem 3). It would be interesting to know for which further classes of graphs a similar statement is true.

Naturally, we would also like to narrow or even close the gap between the lower bound of $\Omega(n^6)$ and the upper bound of $O(n^{16})$ for the flip distance, obtained in Section 6.

▶ Question 3. What is the worst case flip distance between two simple drawings of a complete multipartite graph on n vertices with a given ERS?

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A Missing proofs of Section 2

A.1 Characterizing complete multipartite graphs (Proof of Lemma 4)

▶ Lemma 4. A graph G = (V, E) is complete multipartite if and only if for every edge $uv \in E$ and every vertex $w \in V \setminus \{u, v\}$ we have $uw \in E$ or $vw \in E$ (or both).

Proof. " \Rightarrow ": If $uv \in E$, then u and v are in different sets of the partition. So w can be in the same partition set with at most one of u or v. Without loss of generality assume that u and w are in different partition sets. Then $uw \in E$.

" \Leftarrow ": Define the non-adjacency relation \approx on $V \times V$ by $u \approx v \iff uv \notin E$. We claim that \approx is an equivalence relation. Reflexivity and symmetry are obvious. For transitivity suppose that $u \approx v$ and $v \approx w$. Then $uv \notin E$ and $vw \notin E$. If $v \in \{u, w\}$, then $u \approx w$ is immediate. So suppose that $v \in V \setminus \{u, w\}$. We want to show that $u \approx w$, that is, $uw \notin E$. Suppose to the contrary that $uw \in E$. Then by the assumption in the statement of the lemma, applied to the edge uw and the vertex v, we have $vu \in E$ or $vw \in E$, in contradiction to $u \approx v$ and $v \approx w$. Therefore, we have $uw \notin E$ and $u \approx w$, which proves that \approx is transitive.

Let V_1, \ldots, V_k denote the partition of V into equivalence classes according to \approx . Clearly, by definition of \approx , each V_i , for $i \in \{1, \ldots, k\}$, is an independent set in G, and for each pair $u \in V_i$ and $v \in V_j$ with $1 \le i < j \le k$ we have $u \not\approx v$, that is, $uv \in E$. In other words, the graph G is complete multipartite with vertex partition V_1, \ldots, V_k .

A.2 ERS determines crossing triangles (Proof of Lemma 5)

▶ Lemma 5. In every simple drawing of a complete multipartite graph, the set of edge triples that span crossing triangles is uniquely determined by the ERS.

Proof. Consider a multipartite graph G = (V, E) on n vertices. If there is no simple drawing of G that contains a crossing triangle, then the statement of the lemma holds. In particular, this is the case for $n \leq 5$ because six vertices are required to span a crossing triangle. So we may suppose that $n \geq 6$, and that there exists a drawing Γ of G that contains a crossing triangle T. Let $ab, cd, ef \in E$ be the three edges that define T. Three pairwise crossing edges in a simple drawing can only form two possible arrangements, up to labeling and (strong) isomorphism: Type 1 (Figure 11a) corresponds to a crossing triangle (that is, all vertices are on the boundary of one cell) and Type 2 (Figure 11b) has two vertices on the boundary of one cell and four vertices on the boundary of the other cell.



Figure 11 Possible arrangements formed by three pairwise crossing edges.

So suppose for the sake of contradiction that there exists another drawing Γ' of G with the same ERS as Γ in which the edges ab, cd, and ef form an arrangement of Type 2. As the Type 1 arrangement is completely symmetric, we may fix its labeling without loss of generality as indicated in Figure 11a and select ef to be the edge in Γ' that has both vertices on the same cell of the Type 2 arrangement. Then the remaining vertices are uniquely determined by the ERS; the two possible cases are depicted in Figures 11b and 11c. We will consider both cases in order. In each case we add some of the remaining edges of G, which we know to exist by Lemma 4, to arrive at a contradiction in all cases.

Case 1: In Γ' , the three edges form the arrangement depicted in Figure 11b. By Lemma 4 we have $ae \in E$ or $af \in E$.

Case 1.1: $ae \in E$. See Figures 12a and 12b for an illustration. Then in Γ' the edge ae crosses cd. Consider the subdrawing $\Gamma'[\{a, b, e, f\}]$: Vertex c is inside the tricell that is bounded by the edge ae and vertex d is not. There are two different ways to draw the edge ae in Γ so that it crosses cd, see the red and blue curve in Figure 12a: In one case (red curve), both vertices c and d are in the tricell of $\Gamma[\{a, b, e, f\}]$ that is bounded by the edge ae; in the other case (blue curve), no vertex is inside this tricell. In either case this yields a contradiction to Lemma 16.



Figure 12 Case 1 in the proof of Lemma 5.

Case 1.2: $af \in E$. See Figures 12c and 12d for an illustration. Then in Γ' the edge af crosses cd, and there is only one way to draw the edge af in Γ so that it crosses cd with the same rotation, see the blue curve in Figure 12c. Observe that in $\Gamma[\{a, b, e, f\}]$, the vertex c is contained in the tricell that is bounded by the edge af, whereas in $\Gamma'[\{a, b, e, f\}]$ the tricell that is bounded by the edge af does not contain any vertex. This yields a contradiction to Lemma 16.

Case 2: In Γ' , the three edges form the arrangement depicted in Figure 11c. By Lemma 4 we have $de \in E$ or $df \in E$.

Case 2.1: $de \in E$. See Figures 13a and 13b for an illustration. Then in Γ' the edge de crosses ab. Consider the subdrawing $\Gamma'[\{c, d, e, f\}]$: Vertex b is inside the tricell that is bounded by the edge de and vertex a is not. There are two different ways to draw the edge de in Γ so that it crosses cd with the same rotation, see the red and blue curve in Figure 12a: In one case (red curve), both vertices a and b are in the tricell of $\Gamma[\{c, d, e, f\}]$ that is bounded by the edge de; in the other case (blue curve), no vertex is inside this tricell. In either case this yields a contradiction to Lemma 16.

Case 2.2: $df \in E$. See Figures 13c and 13d for an illustration. Then in Γ' the edge df crosses ab, and there is only one way to draw the edge df in Γ so that it crosses cd with the same rotation, see the blue curve in Figure 13c. Observe that in $\Gamma[\{c, d, e, f\}]$, the vertex b is contained in the tricell that is bounded by the edge df, whereas this tricell does not contain any vertex in $\Gamma'[\{c, d, e, f\}]$. This yields a contradiction to Lemma 16.

This completes the proof of Lemma 5.



Figure 13 Case 2 in the proof of Lemma 5.

B Missing proof of Section 3: A Carathéodory-type theorem

In this section, we present the proof of Theorem 3. As stated before, the corresponding result has been shown for simple drawings of K_n by Balko, Fulek, and Kynčl [6], and by Bergold, Felsner, Scheucher, Schröder, and Steiner [7]. We follow the proof idea of the latter, but the adaptation to the multipartite setting—specifically, the proof of the explicit *claim* in the proof below—is nontrivial.

▶ **Theorem 3** (Carathéodory-type theorem for simple drawings of complete multipartite graphs). Let D be a simple drawing of a complete multipartite graph G in the plane. For every point p in a bounded cell of D, there exists a cycle C of length three or four in D such that p is contained in a bounded cell of C. This statement is tight in the sense that it may not hold for G minus one edge.

Proof. If G is empty or a star $K_{1,n}$, then the statement is vacuously true, since D contains no bounded cell and hence the set of possible choices for p is empty. So we may assume that G is neither empty nor a star, and hence every pair of distinct vertices $u, v \in V$ with $uv \notin E$ has at least two distinct common neighbors.

Suppose for the sake of contradiction that there exists a simple drawing D of G such that there is a point p in a bounded cell of D but p is not contained in a bounded cell of any induced C_i -subdrawing of D, for $i \in \{3, 4\}$. We choose D to be minimal with respect to the number of vertices. That is, if we remove any vertex (and all its incident edges) from D, then p lies in the unbounded cell.

Let a be any vertex of the graph, and let O be a smallest set of edges incident to a so that removal of all edges of O from D puts p into the unbounded cell of the resulting drawing D^- . Then in D^- one can draw a simple curve P from p to *infinity* (any point in the interior of the unbounded cell of D) so that P does not intersect any vertex or edge of D^- . Subject to this constraint, we select P to minimize the number of crossings with edges of D. Observe that by the minimality of O, every edge $o \in O$ is crossed at least once by P, and adding o to D^- puts p into a bounded cell of the resulting drawing. Also note that the edges in O are pairwise non-crossing because D is a simple drawing and all edges in O are incident to the common vertex a.

 \triangleright Claim. We may assume that every edge in O crosses P exactly once.

Proof. Suppose that there is an edge $ab \in O$ that crosses P at least twice. Trace the edge ab from b to a and denote by χ_i , for i = 1, ..., k, the *i*-th crossing with P encountered along the way. By assumption we have $k \geq 2$. Now we have two curves between χ_1 and χ_2 , one along the edge ab and another one along P. Denote the former curve by γ_{ab} and the latter one by γ_P . Together they form a closed Jordan curve $\Gamma_{ab\cup P} = \gamma_{ab} \cup \gamma_P$. Denote the bounded region of $\Gamma_{ab\cup P}$ by R.

We claim that there is an edge uw in D such that $u \in R$ and $w \notin R$. To see this, observe that $P' = (P \setminus \gamma_p) \cup \gamma'_{ab}$ is a curve from p to *infinity*, where γ'_{ab} is a close copy of γ_{ab} such that $\gamma'_{ab} \cap ab = \{\chi_1, \chi_2\}$. If no edge of D crosses γ_{ab} , then P' would have fewer crossings than P (at least one), in contradiction to the minimality of P. Hence, there is an edge uwof D that crosses γ_{ab} . As D is simple, the crossing edges ab and uw do not share an endpoint and cross exactly once. As only edges incident to a may cross P, the edge uw does not cross P. Thus, the edge uw has exactly one crossing with $\Gamma_{ab\cup P}$, and so exactly one of its endpoints is in R. Without loss of generality we take $u \in R$ and $w \notin R$. We distinguish two cases, depending on whether or not $b \in R$.

Case 1: $b \in R$. Then bw is not an edge of G, as any simple curve from b to w crosses $\Gamma_{ab\cup P}$, but an edge bw can neither cross ab (as an incident edge) nor P (as not being incident to a). Hence both aw and bu are edges of G (by Lemma 4 applied to w with ab and to b with uw, respectively), implying that abuw forms a C_4 in G. As χ_1 and χ_2 are the first two crossings of ab with P when traversing ab from b to a, we have $p \in R$. Further, as neither bu nor uw crosses P (not being adjacent to a), we conclude that p lies in a bounded cell of the C_4 -subdrawing induced by abuw, which completes the proof in this case.

Case 2: $b \notin R$. Then, for analogous reasons as in the first case, bu is not an edge of G and both bw and au are edges of G, implying that abwu forms a C_4 in G. As the edge bw is not incident to a, it must not cross P. Thus, the closed curve that, starting from b, follows the edge ba up to its crossing with uw, then follows uw to w, and then returns to b via the edge bw, crosses P exactly once, at χ_1 , and hence contains p in its interior. Therefore, the point p lies in a bounded cell of the C_4 -subdrawing induced by abwu, which concludes the proof of this case.

Altogether, this completes the proof of the claim.

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For an edge $ab \in O$ denote by p_{ab} the unique (by the claim) point in $ab \cap P$. We conclude by again considering two cases (numbered 3 and 4 for convenience).

Case 3: There exist edges $ab \in O$ and cd in D such that cd crosses ab between a and p_{ab} . Then by Lemma 4 at least one of c or d is adjacent to b in G. Without loss of generality assume that bc is an edge. Consider the closed curve Γ_{Δ} that starts at the crossing $ab \cap cd$, follows cd to c, then follows cb to b, and then follows ab back to $ab \cap cd$. The path Pcrosses Γ_{Δ} exactly once, namely on the part along ab. (It crosses there by assumption, only once along ab by the claim, and not at all along the edges bc and cd because these are not incident to a.) Thus, p is bounded by Γ_{Δ} (or, equivalently, p and *infinity* are on different sides of Γ_{Δ}). If a and d are adjacent in G, it follows that p lies in a bounded cell of the C_4 -subdrawing induced by abcd. If a and d are not adjacent in G, then, by Lemma 4, the vertex a must be adjacent to c in G, and d must be incident to b. Since ab crosses cd, the sides of the 3-cycles abc and abd containing p intersect exactly in the region bounded by Γ_{Δ} , which contains p and consequently does not contain *infinity*. Hence, *infinity* can be on the same side of p in at most one of the 3-cycles abc and abd. Thus, p lies in a bounded cell in (at least) one of those 3-cycles, which concludes the proof in this case.

Case 4: In D, every edge $ab \in O$ is uncrossed between a and p_{ab} . Let ab be the first edge from O that is crossed by P. Then we can reroute P to not cross ab at p_{ab} but instead follow ab to a, without crossing any edge of D. Let ac be the next edge incident to a along the same cell. We claim that $ac \notin O$. To see this, suppose for the sake of contradiction that $ac \in O$. Then we continue to route P from a along ac to p_{ac} , without crossing any edge

of D, and from there along its original track to *infinity*. The resulting path from p to *infinity* crosses strictly fewer edges from D than P, in contradiction to the crossing-minimality of P. Thus, the claim holds and $ac \notin O$; in particular, $P \cap ac = \emptyset$. If $bc \in E$, then p lies in the bounded cell of the 3-cycle abc because P crosses abc exactly once, at p_{ab} . Otherwise, we have $bc \notin E$. As stated at the very beginning of the proof, every pair of non-adjacent distinct vertices has at least two distinct common neighbors, and so b and c have at least two distinct common neighbors is a; let $d \neq a$ denote another common neighbor of b and c. As P crosses exactly one edge, namely ab, of the C_4 -subdrawing induced by abcd, it follows that p lies in a bounded cell of this subdrawing.

In summary, in every case we have established a C_i -subdrawing of D, for $i \in \{3, 4\}$ so that p lies in a bounded cell of the subdrawing. This concludes the main part of the proof.



Figure 14 Drawing of $K_{m,n}$ minus one edge $(r_2b_1, \text{drawn dashed})$, based on Figure 15. The point p lies in a bounded cell, but in no C_i , for $i \in \{3, 4, 5\}$.

To see that the theorem may not hold if we remove one edge from G, consider the simple drawing of $K_{m,n}$, for $m, n \ge 2$, depicted in Figure 14. The only missing edge is r_2b_1 . The point p lies in a bounded cell. To form a cycle that encloses p, we need the edge r_1b_1 and the vertex r_2 , which, in a cycle, has two incident edges connecting to b_i and b_j , for $2 \le i < j$. Thus, any cycle that encloses p has at least six vertices.

C Missing proof of Section 4: Tightness of Theorem 1

This section is devoted to the proof of Theorem 2, which is sketched in Section 4. The figures of Section 4 are repeated here for better readability. For convenience, we also repeat the statement of the theorem before its proof.

▶ **Theorem 2.** For any $m, n \ge 3$ and $K_{m,n}$ minus any two edges, there exist two simple drawings with the same ERS that cannot be transformed into each other using triangle flips. The same holds for any $n \ge 5$ and K_n minus any four-cycle C_4 , as well as for any $m \ge 4, n \ge 1$ and $K_{m,n}$ plus one edge between vertices in the bipartition class of size m.

Proof. Let us first consider $K_{m,n}$ minus two edges. The case where the two missing edges are adjacent is depicted in Figure 15. It shows two drawings of $K_{m,n}$ minus two edges b_1r_2 and b_1r_3 that have the same ERS. First note that the two drawings are not strongly isomorphic because the order in which r_1b_1 crosses the other edges differs in both drawings. We claim that the two drawings cannot be transformed into each other via triangle flips. The edge r_1b_1 only crosses edges incident to r_2 or r_3 . The subdrawing of all edges incident to r_2 or r_3 is plane. Thus, there is no crossing triangle that involves the edge r_1b_1 . Consequently, no triangle flip can change the order of crossings along r_1b_1 . (The ERS determines the set of crossing edge pairs, and this set remains invariant under triangle flips. Therefore, the edge r_1b_1 can never be part of a crossing triangle.)



Figure 15 Two drawings of $K_{m,n}$ minus two adjacent edges b_1r_2 and b_1r_3 (drawn as dashed lines) that have the same ERS but cannot be transformed into each other via triangle flips. The drawing for the smallest case, $K_{2,3}$, is shown bold. No triangle flip can change the crossing order along the green edge r_1b_1 , which is different in the two drawings.

It remains to consider the case where the two missing edges are independent, which is depicted in Figure 16. The vertices labeled r_3, \ldots and b_4, \ldots stand for an arbitrarily large (possibly empty) cluster of red and blue vertices, respectively, that are connected to the remaining vertices in the same way (topologically). The edge b_1r_1 crosses b_2r_2 and b_3r_3 in a different order in both drawings. Among those edges that cross b_1r_1 no two cross. So there is no crossing triangle that involves b_1r_1 and therefore no sequence of triangle flips can change the order of crossings along b_1r_1 . Thus, there is no way to transform one drawing into the other using triangle flips.



Figure 16 Two drawings of $K_{m,n}$ minus two independent edges b_2r_1 and b_1r_2 (shown dashed) that have the same ERS but cannot be transformed into each other via triangle flips.

To obtain the statement for K_n , we extend the drawings in Figure 15 to drawings of a complete graph minus 4 edges that form a C_4 such that both drawings have the same ERS. The extension is shown in Figure 17, where the original edges are drawn black and the edges of the extension are drawn purple. This drawing contains the drawing of Figure 15 as a subdrawing. As before, the edge r_1b_1 crosses the original (black) edges incident to r_2 and r_3 in a different order, and there are no crossing triangles that involve the edge r_1b_1 and the edges incident to r_2 and r_3 . Thus, there is no way to transform one drawing into the other using triangle flips.

Finally, the statement for $K_{m,n}$ plus one edge in a partition class of size at least four can be obtained from the construction depicted in Figure 17 in the following way: color the four



Figure 17 Two drawings of K_m minus four edges building a C_4 that have the same ERS, but cannot be transformed into each other via triangle flips. The missing edges are b_1r_2 , r_2r_1 , r_1r_3 , and r_3r_1 and drawn as dashed lines. (The drawing for K_5 is marked in bold lines.) No triangle flip can change the crossing order in which the green edge r_1b_1 crosses the black edges incident to r_1 and r_2 .



Figure 18 Two drawings of $K_{m-1,n+1}$ plus one edge (b_1r_1) that cannot be transformed into each other via triangle flips.

vertices r_1, r_2, r_3 , and b_1 of the missing C_4 with one color, the vertex b_2 with the other color, and the remaining vertices arbitrarily, see for an example Figure 18. Then disregarding all edges between two vertices of the same color except for the edge r_1b_1 yields two different drawings of $K_{m,n}$ plus one edge. With the same reasoning as for K_n minus C_4 , those two drawings cannot be transformed into each other via triangle flips.

D Missing proofs of Section 5

D.1 Proof of Lemma 6

In this section, we proof Lemma 6, which is part of the first proof of Theorem 1. For convenience, we repeat the lemma before its proof.

▶ Lemma 6. Let D be a simple drawing of a complete multipartite graph G on S^2 and let vw be an edge of G. Then for any pair of adjacent or disjoint edges crossed by vw, the ERS of D determines the order in which vw crosses them.

Proof. We prove the statement by showing that if in two different drawings of a complete multipartite graph G, an edge vw crosses a pair of adjacent or disjoint edges in different orders, then the according drawings must have different ERSs. We distinguish two cases, depending on whether the two edges crossed by vw are adjacent or disjoint.

Case 1: The edge vw crosses two adjacent edges ab and ac in different orders. We distinguish two subcases, depending on the topology of the drawing of vw, ab, and ac; see Figure 19.



Figure 19 Case 1: the two topologically different cases how *vw* can intersect *ab* and *ac* (up to relabelling).

Case 1.1: ab and ac cross vw from the same side. By Lemma 4, at least one of va and wa is an edge of G. Assume w.l.o.g. that va is in G (the other choice is symmetric). As va must not cross any of vw, ab, and ac, the drawing of va is unique in each drawing. The resulting drawings have different rotations at a and hence different ERSs.

Case 1.2: ab and ac cross vw from different sides. By Lemma 4, at least one of vb and wb is an edge of G. Assume w.l.o.g. that vb is in G (the other choice is symmetric). As vb must not cross any of vw and ab, the drawing of vb is again unique in each drawing. However, in one of the drawings vb does not cross any edge, while in the other one it must cross ac, again implying different ERSs.

Case 2: The edge vw crosses two disjoint edges ab and cd in different orders. The only possibility for the topology of the drawing of vw, ab, and ac in this case is depicted in Figure 20. We will again use other existing edges for our reasoning.



Figure 20 Case 2: Depiction of how *vw* intersects *ab* and *cd* (up to relabelling).

Consider the two edges vw and ab. By Lemma 4, each of the four vertices must have an edge to at least one of the vertices of the other edge. Hence, out of the four edges va, wa, vb, and wb, either va and wb or vb and wa are in G. Assume w.l.o.g. that va and wb are in G (the other choice is symmetric). Note that each of va and wb might cross cd, but none of them can cross any of vw and ab. Further, by Case 1, we know that the crossing order of any of the edges with two adjacent edges must be the same in both drawings.

Case 2.1: va crosses cd between c and the crossing of vw and cd. See Figure 21a. Then in each of the drawings, the crossing rotation of va and cd is fixed, as the other rotation would

require a second crossing (which is not allowed by the simplicity of the drawing). As these fixed crossing rotations are different in the two drawings, the ERSs are different as well. \triangleleft

Case 2.2: va crosses cd between d and the crossing of vw and cd. See Figure 21b. Then it follows from the left drawing that wb must cross cd. However, in the right drawing wb must not cross cd.



Figure 21 Case 2: The two cases where *va* crosses *cd*.

Note that, if va does not cross cd but wb crosses cd, Figures 21a and 21b apply analogously. Hence the only missing case is that none of the edges crosses cd; see Figure 22.



Figure 22 Case 2.3: The situation where none of *va* and *wb* crosses *cd*.

Case 2.3: None of va and wb crosses cd. Consider the two edges vw and cd. By Lemma 4, each of the four vertices must have an edge to at least one of the vertices of the other edge. Hence, out of the four edges vc, wc, vd, and wd, either vc and wd or vd and wc are in G, which yields two subcases.

Case 2.3.1: vc and wd are edges of G. We can assume that none of the edges vc and wd crosses ab, as otherwise the two drawings have different ERSs by Case 2.1 and Case 2.2. (independent of how va and wb are drawn). Hence, for the two drawings to have the same crossings, vc must be added without crossing any edge (due to the right drawing), which gives a unique way to add it to each of the drawings. These unique ways force different rotations at v and hence again different ERSs of the two drawings; see Figure 23a.

Case 2.3.2: vd and wc are edges of G. As in the previous case, we can assume that none of the edges vd and wc crosses ab by Case 2.1 and Case 2.2. Hence in the left drawing, vd must cross wb, while in the right drawing, vd cannot cross any of the edges, again implying different ERSs of the two drawings; see Figure 23b.

As the statement holds in each of the cases this completes the proof of the lemma. \blacktriangleleft



(a) Case 2.3.1

(b) Case 2.3.2

Figure 23 The two possibilities in Case 2.3

D.2 Second Proof of Theorem 1

For our second proof of Theorem 1, we use the same general proof framework as [4] where we iteratively transform one of the drawings in order to increase the strongly isomorphic parts of both drawings. However, many steps of the proof in [4] rely on known properties of drawings of complete graphs which do not hold for complete multipartite graphs in general (or were not known to do so prior to the work at hand). In the following, we present the big picture of the proof, while deferring the proofs of some key lemmata to subsequent sections. We then highlight the novel contributions of the proof and discuss the key differences to [4].

▶ **Theorem 1.** Let D_1 and D_2 be two simple drawings of a complete multipartite graph on the sphere S^2 with the same ERS. Then there is a sequence of triangle flips that transforms D_1 into D_2 .

Proof. Let G be a complete multipartite graph, and let D_1 and D_2 be two simple drawings of G on S^2 with the same ERS. Let $R = \{r_1, r_2, \ldots, r_n\}$ be a maximal independent set in G, and let $B = \{b_1, b_2, \ldots, b_m\}$ denote the set of the remaining vertices. We call the vertices in R and B the red and blue vertices, respectively. Note that the graph on vertex set $R \cup B$ together with all edges with a red and a blue endpoint forms a complete bipartite graph $K_{n,m}$, and the set R is an independent set in G while B might not necessarily be an independent set. Let $D \cong D_1$. We change D by performing triangle flips, until we obtain $D \cong D_2$. We iteratively consider the vertices r_1, \ldots, r_n . For each vertex r_i , we iteratively consider the incident edges $r_i b_1, \ldots, r_i b_m$. After that the remaining edges, which connect two vertices in B, are considered in some arbitrary but fixed order. Formally, we consider the edges of G in an order (a_1, \ldots, a_t) , where for every $1 \le i \le t$, $a_i = r_j b_k$ or $a_i = b_k b_\ell$ for some $1 \le j \le n$ and $1 \le k, \ell \le m$, and where

- the edge $r_i b_j$ precedes $r_i b_k$ for all $1 \le i \le n$ and $1 \le j < k \le m$,
- the edge $r_i b_j$ precedes $r_k b_\ell$ for all $1 \le i < k \le n$ and $1 \le j, \ell \le m$, and
- the edge $r_i b_j$ precedes $b_k b_\ell$ for all $1 \le i \le n$ and $1 \le j, k, \ell \le m$.

For $1 \leq i \leq t$, we denote by X_i the subgraph of G induced by the edge set $\{a_1, \ldots, a_i\}$. When considering an edge a_i , the goal is to establish $D[X_i] \cong D_2[X_i]$, where $D[X_i]$ and $D_2[X_i]$ are the corresponding subdrawings of D and D_2 , respectively.

For the base case $i \leq m$, observe that $D[K_{1,i}] \cong D_2[K_{1,i}]$ because there is only one simple drawing of $K_{1,i}$ (our graphs are labeled but the ERS is given).

For the general case i > m, assume that $D[X_{i-1}] \cong D_2[X_{i-1}]$. Observe that, since i > m, all vertices of B are already present in X_{i-1} . In the case when a vertex r_i is introduced

for the first time in X_i , (i.e., r_j is present in X_i but not in X_{i-1}), we first argue that the position of vertex r_j is consistent between $D[X_{i-1}]$ and $D_2[X_{i-1}]$. To show this, we use the following lemma, whose proof relies on Theorem 3 (Carathéodory's Theorem) and is deferred to Appendix D.3.

▶ Lemma 11. Let F be a simple drawing of a complete multipartite graph on S^2 . For any vertex v in F, the ERS of F uniquely determines which cell of $F' := F \setminus \{v\}$ contains v.

Since $D[X_{i-1}] \cong D_2[X_{i-1}]$, the two drawings topologically have the same cells. By the order of the edges, if r_j is introduced for the first time in X_i , then X_{i-1} is a complete bipartite graph between the vertices in B and the vertices in $\{r_1, \ldots, r_{j-1}\}$. Further, X_{i-1+m} is then a complete bipartite graph between the vertices in B and the vertices in $\{r_1, \ldots, r_j\}$. Hence, as D and D_2 have the same ERS, by Lemma 11 applied to $F = D[X_{i-1+m}]$ and to $D_2[X_{i-1+m}]$, both times with $v = r_i$, we conclude that r_i lies in the same cell in $D[X_{i-1}]$ and $D_2[X_{i-1}]$.

Now consider the edge $a_i = xb_j$, where x can be in either R or B. The aim is to use a sequence of triangle flips to transform D such that $D[X_i] \cong D_2[X_i]$. Let e_1 denote the curve that represents xb_j in D. We imagine adding another copy \tilde{e}_2 of xb_j to D, which corresponds to the curve e_2 that represents the edge xb_j in D_2 and serves as a "target" curve which we aim to transform e_1 into. The following lemma, which is proven in Appendix D.4, guarantees the existence of a suitable target curve.

▶ Lemma 12. There exists a simple curve \tilde{e}_2 such that $D[X_{i-1}] \cup \tilde{e}_2 \cong D_2[X_i]$ and e_1 and \tilde{e}_2 have $O(|V(X_{i-1})|^4)$ intersections in $D[X_i] \cup \tilde{e}_2$, where $|V(X_{i-1})|$ is the number of vertices of X_{i-1} .

Now fix such a curve $\tilde{e_2}$. Then $\Gamma = e_1 \cup \tilde{e_2}$ forms a (not necessarily simple) closed curve. A *lens* in Γ is a cell whose boundary is formed by exactly two edge fragments of Γ , one from e_1 and one from $\tilde{e_2}$. With the next lemma, we show that there is a lens in Γ which we can use as a starting point for transforming e_1 into $\tilde{e_2}$ via triangle flips in D. Its proof is presented in Appendix D.5.

▶ Lemma 13. In Γ there is a free lens, i.e., a lens that does not contain any vertex of $D[X_i]$.

Now consider a free lens L that exists by Lemma 13. While L does not contain any vertex of $D[X_{i-1}]$, it may contain crossings of $D[X_{i-1}]$. As a next step, we aim to transform Dusing triangle flips such that L does not contain any crossings of $D[X_{i-1}]$. Let $\chi \in L$ be a crossing of two edges a', a'' in $D[X_{i-1}]$. As x and b_j are the only vertices of $e_1 \cup \tilde{e_2}$, it follows that each of a', a'' crosses ∂L twice; as both D and D_2 are simple drawings, one of these crossings is with e_1 and the other is with $\tilde{e_2}$. Thus, a', a'', and e_1 form a crossing triangle Δ_{e_1} in D. Moreover, the corresponding crossing triangle in D_2 has the opposite parity, and hence Δ_{e_1} is invertible. By the following lemma, whose proof is presented in Appendix D.6, the triangle Δ_{e_1} is empty of all vertices of D if D is a complete bipartite graph (we already knew this for the vertices already in X_i , but not for vertices in R that are not present in $D[X_i]$).

▶ Lemma 14 (Invertible triangles are empty). Let D be a simple drawing of a complete bipartite graph G, and let Δ be an invertible triangle in D. Then all vertices of D lie outside Δ .

Thus, while $D[X_i]$ contains only edges between R and B, the crossing triangle Δ_{e_1} has to be empty of all vertices of D. If $D[X_i]$ also contains edges between two vertices of B, then all vertices of D are in $D[X_i]$ and thus Δ_{e_1} is empty of all vertices of D by Lemma 13.

We claim that also all edges that cross Δ_{e_1} can be "swept" out of Δ_{e_1} . To see this, consider the set Ξ of all edges of D that cross Δ_{e_1} . Note that every edge from Ξ crosses

exactly two sides (that is, edges) of Δ_{e_1} . For two sides a, b of Δ_{e_1} let $\Xi_{a,b} \subseteq \Xi$ denote the set of edges that cross both a and b. Pick some pair a, b for which $\Xi_{a,b} \neq \emptyset$ (if none exists, we are done), and let η be the edge from $\Xi_{a,b}$ that crosses a closest to the common endpoint of aand b. Let R_η denote the closed triangular region that is bounded by $a, b, \text{ and } \eta$ and whose interior lies inside Δ_{e_1} , and let Ξ_η denote the set of edges from Ξ that cross the interior of R_η . Note that by the choice of η , every edge in Ξ_η crosses each of η and $\partial R_\eta \setminus \eta$ exactly once. Hence, locally in R_η , the edges in $\Xi_\eta \cup \{a\}$ behave similarly to a collection of pseudolines, except that not every pair may cross (also called an *arrangement of bi-infinite curves having the one-intersection property* by Hershberger and Snoeyink [32]). We will use the following lemma for sweeping R_η with η .

▶ Lemma 15 ([32, Lemma 3.1]). Any arrangement of bi-infinite curves having the oneintersection property can be swept starting with any curve from the arrangement using three operations: passing a triangle, passing the first ray, and taking the first ray.

Note that the operation passing a triangle is a triangle flip. The other two operations are only applied if the arrangement contains either lines that do not intersect the sweeping curve or lines that intersect nothing but the sweeping curve. Thus, the other two operations are not applied, since all edges in $\Xi_{\eta} \cup \{a\}$ cross each of b and η . Since our sweeping curve η intersects all edges in $\Xi_{\eta} \cup \{a\}$, we can sweep η through R_{η} via a sequence of triangle flips, in this way sweeping η once over each crossing among the edges from $\Xi_{\eta} \cup \{a, b\}$. After the last flip in this process, which is for the crossing triangle ηab , the edge η does not cross Δ_{e_1} anymore. Hence, after repeating this process $|\Xi|$ times—each time for an appropriate choice of η —the crossing triangle Δ_{e_1} is empty and, therefore, can be flipped as well. We remark that the number of flips to empty Δ_{e_1} with this process is bounded from above by the number of crossings between edges in Ξ and Δ_{e_1} times the number of edges in Ξ . Hence at most $O(|\Xi|^3)$ flips are required.

Processing all remaining crossings inside L in the described fashion, we establish that in the resulting drawing, the lens L does not contain any vertex or crossing of $D[X_{i-1}]$. In other words, locally around L, the edge e_1 is topologically identical to $\tilde{e_2}$ with respect to $D[X_{i-1}]$. Thus, we can adapt $\tilde{e_2}$ by replacing its edge part on ∂L with a close copy of the edge part of e_1 on ∂L , effectively removing the lens L from Γ . As a result, the edges e_1 and $\tilde{e_2}$ have fewer crossings than before in D, and the parameters D and $\Gamma = e_1 \cup \tilde{e_2}$ again meet the conditions of Lemma 13. Repeatedly applying this procedure, we eventually obtain a drawing $D[X_i] \cup \tilde{e_2}$ where e_1 and $\tilde{e_2}$ do not cross and hence Γ forms a simple closed curve. By Lemma 13, one of the two cells bounded by Γ contains no vertices of $D[X_i]$. So after one last round of transformations as described above, we obtain a drawing $D[X_i] \cup \tilde{e_2}$ in which all vertices and crossings lie on one side of Γ . Hence we have obtained $D[X_i] \cong D_2[X_i]$. Processing all vertices r_i , for $i = 2, \ldots, n$, and in turn handling all edges incident to r_i eventually yields a drawing $D \cong D_2$.

Key differences to the proof [4] for complete graphs. The key differences to the proof in [4] are concentrated in the statements of Lemma 11, 13, and 14.

Firstly, when extending the 'isomorphic subgraph' by a vertex v, we require that v lies in the same cell in both drawings induced by the subgraph, as stated in Lemma 11. For the complete graph this statement is an easy corollary of the known fact that each cell is the intersection of 3-cycles, and that the position of a fourth vertex with respect to a 3-cycle is determined by the rotation system. However, multipartite graphs may not have any 3-cycles. We circumvent this issue by also allowing 4-cycles—which, in contrast to triangles, may

self-cross—and developing a corresponding Carathéodory-type theorem for simple drawings of complete multipartite graphs (Theorem 3).

Secondly, for extending the 'isomorphic subgraph' X_{i-1} by an edge a_i , which is drawn with different crossing orders as e_1 in $D[X_i]$ and as e_2 in $D_2[X_i]$, we think of e_2 drawn virtually into $D[X_i]$ as $\tilde{e_2}$ and aim to untangle e_1 and $\tilde{e_2}$ via triangle flips. For this we require that the curve formed by both edge incarnations encloses at least one 'free lens'; see Lemma 13. The corresponding statement for the complete graph follows from the facts that the drawings are simple and that each pair of vertices is adjacent. However, in a graph where not all pairs of vertices are adjacent, there are fewer restrictions on how the edges e_1 and e_2 can be drawn, which allows for a possibly complicated interaction between them. As a result, more deliberate considerations and new structural insights are needed in order to prove this property. We consider Lemma 13 as the core lemma for our proof.

Finally, another essential element of the proof of Gioan's Theorem for the complete graph [4] is that for any invertible triangle T all vertices lie on the same side of T. For the complete graph, this statement again can be shown using the fact that all vertices are pairwise adjacent. This does not hold for general multipartite graphs, and therefore in order to prove the according statement of Lemma 14, we need a different approach.

D.3 ERS determines positions of vertices (Proof of Lemma 11)

To show Lemma 11, we first consider the crucial base case of subgraphs on up to four vertices.

▶ Lemma 16. Let D be a simple drawing of a complete multipartite graph on S^2 , let D' be a subdrawing of D on at most four vertices, and let v be a vertex in $D \setminus D'$. Then the ERS of D uniquely determines which cell of D' contains v.

Proof. Let G = (V, E) denote the (complete multipartite) graph represented by D'. If G has no edges or if G is a tree, then the statement is vacuously true because then D' has only one cell. Hence, we may suppose that G is either $K_{1,1,1} = K_3 = C_3$, $K_{2,2} = C_4$, $K_{1,1,2}$ or K_4 .

First consider the case that G has exactly three vertices. Then $G = K_3$. Let $e \in E$. Then by Lemma 4 we have vu in D, for some endvertex u of e. We trace the edge uv in $D' \cup D[uv]$ starting from u. The given ERS tells us which of the two cells incident to u the edge uventers when leaving u. As uv is adjacent to two of the three edges of G, it may only cross the third edge and only once. The ERS tells us whether or not this crossing exists and thereby the cell of D' that contains v, as claimed.

So we may assume that G has exactly four vertices. Then G is one of $K_{2,2}, K_{1,1,2}$, or K_4 . Up to (strong) isomorphism, each of these three graphs admits exactly two different drawings on S^2 , one without crossings and one with exactly one crossing, as depicted in Figure 24. We pick an edge $e \in E$ as follows: If $G = K_{1,1,2}$, then let e be the edge between the two degree three vertices; otherwise, select $e \in E$ arbitrarily. By Lemma 4 we have vu in D, for some endvertex u of e. We trace the edge uv in $D' \cup D[uv]$ starting from u. The given ERS tells us which cell incident to u the edge uv enters when leaving u. It also tells us which other edges to cross and the rotations of all those crossings. We claim that this information suffices to uniquely determine the cell of D' that contains v.

Note that uv may cross each edge of D' at most once and each edge incident to u not at all. Thus, tracing uv corresponds to a trail³ in the dual, more precisely, in the graph D^*

³ A *trail* in a graph is a walk that uses every edge at most once. In contrast to a path, vertices may be visited several times.

that is obtained from the dual of D' by removing the edges that are dual to an edge incident to u. Again, see Figure 24 for an illustration, where the graphs D^* are shown in pink.



Figure 24 The three graphs and their two drawings each to consider in Lemma 16. Their respective duals are drawn with square vertices in pink. The vertex u is marked with an orange circle, and its incident edges, which must not be crossed by the edge uv, are also marked in orange. The dual edges shown bold are in 1-to-1 correspondence to edges of G.

Consider first the case $G \in \{K_{1,1,2}, K_4\}$, and observe that then the graph D^* is always a tree. Moreover, most of the dual edges in these four drawings are in 1-to-1 correspondence to edges of G; we call these edges *bold*, and they are also shown bold in Figure 24. The only non-bold edges are those that are incident to the leaves of D^* in the drawings D' with one crossing, as both of these edges cross the same edge of G. As D^* is a tree, whenever the dual trail that we trace encounters a bold edge that corresponds to an edge of G that must be crossed, it must take this edge. As all non-bold edges are incident to leaves of D^* (where there is only one option to continue, anyway) and no two non-bold edges are adjacent in D^* (so that there is never a choice between the two), it follows that the trail traced by uv in D^* is uniquely determined by the list of crossed edges in these cases.

It remains to consider the case $G = K_{2,2}$. The drawing without crossings has only two cells, and every crossing switches between them. So the target cell is uniquely determined by the number of crossings along uv (which is zero, one or two). The drawing with one crossing is more interesting because here we need the crossing rotation. Note that our trace starts in a cell incident to u, for which there are two options. The tricell incident to u is a leaf of D^* and using the incident edge rules out the other non-bold edge of D^* . Therefore, the trace is uniquely determined if it starts into this tricell. Otherwise, the edge uv enters the other cell incident to u, which is bounded by six edge-fragments. If the bold edge must be used, things are clear because it cannot be combined with the non-bold edge that is incident to a leaf of D^* . Otherwise, the bold edge must not be used and then we can select the correct non-bold edge to take—if any—by considering the crossing rotation.

With Lemma 16 and Theorem 3, we can now prove Lemma 11. The idea to the next proof is similar to the independently developed proof of Lemma 3.4 in Gioan's full proof [16].

▶ Lemma 11. Let F be a simple drawing of a complete multipartite graph on S^2 . For any vertex v in F, the ERS of F uniquely determines which cell of $F' := F \setminus \{v\}$ contains v.

Proof. If F' has only one cell, then the statement is vacuously true. Hence, we may suppose that F' is neither empty nor a star. Let c_1 and c_2 be any two distinct (hence disjoint) cells of F'. To prove the lemma it suffices to show that at least one of c_1 or c_2 cannot contain v.

By marking one cell as the unbounded cell, we may consider F' as a drawing Γ' in the plane \mathbb{R}^2 . We select c_1 to take the role of the unbounded cell in Γ' , which makes c_2 a bounded cell in Γ' . Then Theorem 3 implies that for any point $p \in c_2$ there is a C_i -subdrawing C in Γ' , for some $i \in \{3, 4\}$, such that p lies in a bounded cell of C. As c_2 is a cell of Γ' , it follows that C does not depend on the choice of p, that is, the whole cell c_2 is contained in a bounded cell of C. In contrast, the cell c_1 is contained in the unbounded cell of C by construction. As

by Lemma 16 the cell of C (in F' and thus also in Γ') that contains v is uniquely determined, at most one of c_1 or c_2 may contain v.

D.4 A new curve can be drawn nicely (Proof of Lemma 12)

We use the notation introduced in Section 5. Recall that we have two drawings D and D_2 of the given graph G, where we assume $D[X_{i-1}] \cong D_2[X_{i-1}]$ for some i > m. We consider the curves e_1 and e_2 of the edge xb_i in D and D_2 , respectively.

▶ Lemma 12. There exists a simple curve $\tilde{e_2}$ such that $D[X_{i-1}] \cup \tilde{e_2} \cong D_2[X_i]$ and e_1 and $\tilde{e_2}$ have $O(|V(X_{i-1})|^4)$ intersections in $D[X_i] \cup \tilde{e_2}$, where $|V(X_{i-1})|$ is the number of vertices of X_{i-1} .

Proof. The first part of the statement is obvious, given that $D[X_{i-1}] \cong D_2[X_{i-1}]$. To see the second part of the statement, consider a cell c of $D[X_{i-1}]$. The curve e_1 in $D[X_i]$ intersects c in a (possibly empty) finite set J_1 of pairwise disjoint edge fragments that connect points on ∂c (or inside c if the endpoint of e_1 lies inside c). Analogously, the curve $\tilde{e_2}$ in $D[X_{i-1}] \cup \tilde{e_2}$ intersects c in a (possibly empty) finite set J_2 of pairwise disjoint edge fragments that connect points on ∂c (or inside c if the endpoint of $\tilde{e_2}$ lies inside c). Therefore, we can choose $\tilde{e_2}$ so that each edge fragment in J_2 crosses each edge fragment in J_1 at most once (and we also choose them this way). Handling the arcs in each cell of $D[X_{i-1}]$ accordingly results in a curve $\tilde{e_2}$ that intersects e_1 finitely many times.

In total, e_1 and $\tilde{e_2}$ each have $O(|X_{i-1}|^2)$ edge fragments, since they intersect every edge of $D[X_{i-1}]$ at most once. Further, every edge fragment of e_1 crosses every edge fragment of $\tilde{e_2}$ at most once. Hence the number of crossings between $\tilde{e_2}$ and e_1 is bounded by $O(|X_{i-1}|^4)$ as claimed.

D.5 An empty lens exists (Proof of Lemma 13)

This section is devoted to the proof of Lemma 13. It is the key lemma in our proof of Gioan's Theorem for complete multipartite graphs, as it provides a means to untangle the possibly complicated relationship between the two geometric representations of the edge xb_j under consideration, one of which stems from each of the two drawings. We start by setting up some terminology.

Given two simple drawings H_1, H_2 of G with the same ERS, we consider the subdrawings induced by the edge set X_i as defined in Appendix D.2. Assuming $H := H_1[X_{i-1}] \cong H_2[X_{i-1}]$, we consider a virtual copy $\tilde{e_2}$ of the edge xb_j in $H_1[X_i]$, which already has a curve e_1 that represents the edge xb_j . The purpose of $\tilde{e_2}$ is to mimic the role of xb_j in H_2 , and set a target state for the transformation of H_1 so as to obtain $H_1[X_i] \cong H_2[X_i]$. We argued in the proof of Theorem 1 (using Lemma 12) that we can find a simple curve $\tilde{e_2}$ such that $(1) \ H \cup \tilde{e_2} \cong H_2[X_i]$ and $(2) \ e_1$ and $\tilde{e_2}$ have finitely many intersections in $H^+ = H_1[X_i] \cup \tilde{e_2}$.

Quasi-lenses. Denote by $\Gamma = H^+[e_1 \cup \tilde{e_2}]$ the subdrawing induced by the closed curve $e_1 \cup \tilde{e_2}$. Recall that a lens of a drawing is a cell whose boundary consists of exactly two edge fragments. (For a lens of Γ one fragment must be from e_1 and the other from $\tilde{e_2}$.) We slightly generalize this concept and define a *quasi-lens* of Γ to be an open region of S^2 that (1) is bounded by a simple closed curve that is formed by a single edge fragment e of Γ together with the part of the other edge (e_1 if $e \subseteq \tilde{e_2}$ and $\tilde{e_2}$ if $e \subseteq e_1$) between the endpoints of e and that (2) does not contain either of b_j or x. Note that, in particular, a lens of Γ is also a quasi-lens, but not necessarily the other way around, as a quasi-lens need not be a cell of Γ .

Aichholzer, Chiu, Hoang, Hoffmann, Kynčl, Maus, Vogtenhuber, Weinberger

See Figure 25 for an illustration. Furthermore, each edge fragment of Γ induces at most one quasi-lens—unless b_j and x are the two endpoints of the fragment, in which case it induces two quasi-lenses (that are, in fact, lenses). We can think of lenses as (inclusion-)minimal quasi-lenses, as the following simple lemma illustrates. Note that the quasi-lens of Figure 25 contains a lens, as marked in Figure 26. We will show that this always has to be the case.



Figure 25 Different types of bigons induced by an edge fragment $\chi \phi$ of e_1 in Γ : A lens (left), a quasi-lens (middle), and an anti-lens (right).



Figure 26 A quasi-lens (shaded grey) containing one lens (marked with a purple grid).

Lemma 17. Every quasi-lens of Γ contains a lens.

Proof. Consider a quasi-lens Q of Γ , and suppose without loss of generality that Q is induced by a fragment uv of e_1 . Let e' denote the part of $\tilde{e_2}$ between u and v. We proceed by induction on the number of edge fragments that form e'. If this number is one, that is, if e' is a fragment of $\tilde{e_2}$, then Q is a lens and the statement holds. Otherwise, the edge e_1 crosses e'at some point χ . Trace e_1 starting from χ into Q. As e_1 is a simple curve and as neither endpoint of e_1 lies in Q by definition (of quasi-lens), this trace must leave Q eventually at some point $\psi \in \partial Q \cap \tilde{e_2}$, so that $\chi \psi$ is a fragment of e_1 . Then $\chi \psi$ induces a quasi-lens $Q' \subsetneq Q$ that has strictly fewer fragments of $\tilde{e_2}$ on its boundary than Q. By the inductive hypothesis, the quasi-lens Q' contains a lens, and, therefore, so does Q.

As Γ is a subdrawing of H^+ , we can classify the cells and quasi-lenses of Γ according to what parts of H^+ they contain. Specifically, a cell or quasi-lens of Γ is

- \blacksquare stabbed if it contains⁴ at least one vertex of H;
- free if it is not stabled (that is, it does not contain any vertex of H);
- normal if it contains at least one crossing but not any vertex of H;
- = empty if it contains at least one edge fragment but not any vertex or crossing of H;
- redundant if it does not contain any vertex or edge fragment of H;
- *essential* if it is not redundant (that is, it contains some vertex or edge fragment of *H*).

Anti-lenses. By definition a quasi-lens does not contain x or b_j . But we also need to work with regions that contain one of these two points. We define an *anti-lens* of Γ to be an open region of S^2 that (1) is bounded by a simple closed curve that is formed by a single edge fragment e of Γ together with the part of the other edge $(e_1 \text{ if } e \subseteq \tilde{e_2} \text{ and } \tilde{e_2} \text{ if } e \subseteq e_1)$ between the endpoints of e and that (2) contains exactly one of b_j or x. Observe that an an anti-lens is induced by two crossings between e_1 and $\tilde{e_2}$ that are consecutive along e_1 or $\tilde{e_2}$

⁴ As all cells and quasi-lenses are open sets, "contains" is equivalent to "contains in its interior".

and have the same crossing rotation. Equivalently, an edge fragment of e_1 or $\tilde{e_2}$ in Γ induces an anti-lens if and only if it connects to the two different sides of $\tilde{e_2}$ or e_1 , respectively, at its endpoints. In particular, for an anti-lens A that is induced by an edge fragment fof $c \in \{e_1, \tilde{e_2}\}$ in Γ , one of the two "stubs" that comprise $c \setminus f$ starts inside A and the other outside of A (if we consider them to be directed from A to $\{x, b_j\}$). Hence we refer to them as the *inside* and the *outside stub* of A. We have the following analogue of Lemma 17 for anti-lenses.

Lemma 18. Every anti-lens A of Γ contains a quasi-lens that is induced by an edge fragment of the inside stub of A.

Proof. Consider an anti-lens A and suppose without loss of generality that A is induced by a fragment $\chi\phi$ of e_1 and that the inside stub s of A starts at ϕ . Let ψ be the first intersection of s with $\tilde{e_2}$ (which exists because s ends on $\tilde{e_2}$). Let f denote the edge fragment $\phi\psi$ of s. Note that $f \setminus \{\phi, \psi\} \subset A$. We prove the statement by induction on the number of intersections between s and $\tilde{e_2}$.

If there is exactly one such intersection (in which case $\psi \in \{x, b_j\}$) or if f attaches to the same side of $\tilde{e_2}$ at ϕ and ψ , then f induces a quasi-lens $Q \subset A$ in Γ .

Otherwise, we know that f attaches to different sides of \tilde{e}_2 at ϕ and ψ . Then f induces an anti-lens $A' \subset A$, whose inside stub s' has at least one fewer intersection with \tilde{e}_2 than sbecause $\phi \in s \setminus s'$. By the inductive hypothesis there exists a quasi-lens $Q \subset A' \subset A$ in Γ .

Free Lenses. We are now ready to prove Lemma 13, based on another lemma whose proof we will then develop over the remainder of the section.

Lemma 13. In Γ there is a free lens, i.e., a lens that does not contain any vertex of $D[X_i]$.

Proof. As a base case suppose that e_1 and $\tilde{e_2}$ do not cross. Then Γ has only two cells, both of which are lenses. Both lenses have the same boundary, which is $e_1 \cup \tilde{e_2}$, and both x and b_j lie on this boundary. In H all blue vertices have edges to all red vertices in X_i . As i > m, there must be at least one red vertex in H (for instance, r_1). As $H_1[X_{i-1}] \cong H_2[X_{i-1}]$ and $H_1[X_{i-1}] \cup \tilde{e_2} \cong H_2[X_i]$, any edge of H either crosses both e_1 and $\tilde{e_2}$, or it crosses neither. It follows that all vertices of H lie in the same lens of Γ , and so the other lens is free.

Otherwise, there are at least two lenses in Γ . In particular, the first edge fragment of \tilde{e}_2 forms a quasi-lens, which either is or contains a lens L with $b_j \notin \partial L$. By Lemma 19 this suffices to guarantee a free lens in Γ .

▶ Lemma 19. If there is a lens L in Γ for which $b_i \notin \partial L$, then there is a free lens in Γ .

To prove Lemma 19, we combine a number of observations concerning the structure of Γ . A first such observation characterizes the placement of the red vertices of H.

▶ Lemma 20. All neighbors of b_i in H lie in a single cell U of Γ with $b_i \in \partial U$.

Proof. Consider a neighbor y of b_j in H. The edge yb_j is adjacent to xb_j and, therefore, crosses neither e_1 nor $\tilde{e_2}$ in H^+ . It follows that y lies in a cell C of Γ with $b_j \in \partial C$. As H_1 and H_2 have the same ERS, the edges e_1 and $\tilde{e_2}$ are consecutive in the rotation around b_j in H^+ . It follows that all neighbors of b_j in H^+ lie in the same cell of Γ .

Unfortunately, the situation for the blue vertices is not symmetric because they need not be connected to x or b_j in H. In the following, we consider both curves e_1 and $\tilde{e_2}$ to be oriented from x to b_j . This induces an orientation on the edge fragments around each cell of Γ . A (quasi-)lens Q is *cyclic* if the edge fragments on ∂Q form an oriented cycle; otherwise, Q is *acyclic*. As a next step, we want to show that all essential quasi-lenses in Γ are acyclic.

Lemma 21. Every quasi-lens that has x or b_j on its boundary is acyclic.

Proof. All edge fragments that are incident to x are oriented away from x. Similarly, all edge fragments that are incident to b_j are oriented towards b_j .

▶ Lemma 22. There is a free lens in Γ or every quasi-lens Q with $b_j \notin \partial Q$ is stabled by a blue vertex of H.

Proof. Assume that all lenses in Γ are stabled, which by Lemma 17 implies that all quasilenses in Γ are stabled. Let Q be a quasi-lens in Γ . By definition (of quasi-lens) Q does not contain b_j , and we have $b_j \notin \partial Q$ by assumption. Thus, by Lemma 20 we have $Q \cap R = \emptyset$. Therefore, Q is stabled by a blue vertex, as claimed.

Lemma 23. There is a free lens in Γ or every essential quasi-lens of Γ is acyclic.

Proof. Suppose for the sake of a contradiction that all lenses in Γ are stabled and there exists an essential cyclic quasi-lens Q. Without loss of generality, we make the following two assumptions about the quasi-lens Q:

- 1. Q is induced by a fragment $\beta \xi$ of e_1 in Γ such that β is closer to b_j along e_1 (and closer to x along \tilde{e}_2).
- **2.** No fragment of e_1 induces a cyclic quasi-lens along the part of $\tilde{e_2}$ between β and b_j (otherwise, we would let such a quasi-lens take the role of Q).

For ease of illustration we imagine $\tilde{e_2}$ as a horizontal line segment such that x lies to the left of b_j and e_1 as a curve that "meanders" around $\tilde{e_2}$. By Lemma 21 and Lemma 22 we know that Q contains a blue vertex v of H. Let $r \in R \setminus \{x\}$. Such a vertex exists because i > mand thus $r_1 \in R \setminus \{x\}$. As r lies in the exterior of Q, the edge vr of H crosses ∂Q . We prove the statement in two steps. First, we consider the case that Q is a lens. Then we address the extension to quasi-lenses.

So suppose for now that Q is a lens. Then we may suppose without loss of generality that vr crosses ∂Q along e_1 (otherwise, as Q is also induced by a fragment of $\tilde{e_2}$, we may exchange the roles of e_1 and $\tilde{e_2}$). As $e_1 \cong_H \tilde{e_2}$, the rotation of the crossing between vr of Hand $\tilde{e_2}$ is the same as the rotation of the crossing between vr and e_1 . There are no further crossings between vr and $\tilde{e_2}$ or e_1 . The labeling of the endpoints of $\tilde{e_2}$ also fixes which of the endpoints the two stubs of e_1 connect to, which results in the following two cases. In Case 1 (Figure 27(left)), the edge vr intersects $\tilde{e_2}$ between Q and b_j ; in Case 2 (Figure 27(right)), it intersects $\tilde{e_2}$ between Q and x. If x is blue, then also rx is an edge of H, and the roles of xand b_j in this proof are interchangeable. Therefore, we may opt to treat this situation as an instance of Case 2, which conversely allows us to assume $x \in R$ in Case 1 without loss of generality.

In both cases we also add the edge rb_j to the picture. As rb_j is adjacent in H to all other edges considered so far, it does not cross any of them. We consider the two cases in order. Removal of ∂Q splits $c \in \{e_1, \tilde{e_2}\}$ into two parts; denote the part that connects to $p \in \{x, b_j\}$ as *p*-stub of *c* with respect to Q. We consider a *p*-stub to be directed from Q to *p*.

Case 1: $x \in R$ and vr intersects $\tilde{e_2}$ between Q and b_j . Each edge of H is crossed at most once by e_1 and $\tilde{e_2}$. So there are no further crossings of e_1 or $\tilde{e_2}$ with vr. Let c_x denote the first intersection of the x-stub s_x^1 of e_1 w.r.t. Q with the x-stub s_x^2 of $\tilde{e_2}$ w.r.t. Q (well-defined



Figure 27 The two cases in the proof of Lemma 23: Case 1 (left) and Case 2 (right).

because x is such an intersection). Consider the simple closed curve C that goes from c_x along \tilde{e}_2 to ξ and then along s_x^1 back to c_x . Now consider the b_j -stub s_b^1 of e_1 w.r.t. Q. As Ccannot cross either of the edges vr and rb_j , it follows that b_j and the starting fragment of s_b^1 are on different sides of C. Therefore, in order to reach b_j , the stub s_b^1 crosses C. While s_b^1 cannot cross $C \cap e_1$, it can cross $C \cap \tilde{e}_2$, possibly many times. However, it cannot cross ∂Q because Q is a lens, and so it crosses s_x^2 . Denote by \mathcal{R}_C the region of \mathcal{S} enclosed by C that does not contain b_j .

We claim that there exists a quasi-lens $Q' \subset \mathcal{R}_C$ that is induced by a fragment of s_b^1 in Γ . To see this, consider the first intersection c_b of s_b^1 with s_x^2 . If s_b^1 crosses s_x^2 from left to right at c_b , then the claim holds because the part of s_b^1 up to c_b induces a cyclic quasi-lens with \tilde{e}_2 . Otherwise, the stub s_b^1 crosses s_x^2 from right to left at c_b , that is, the initial edge fragment of s_b^1 induces an anti-lens in Γ whose inside stub is formed by the part of s_b^1 that starts at c_b . In this case the claim follows by Lemma 18.

As Q' and b_j are separated by C, it follows by Lemma 22 that Q' contains a blue vertex uof H. See Figure 28(left) for an illustration. Next we consider the edge ur of H. It cannot cross the other edges that are incident to r in H, in particular, the edges vr and rb_j . But urcan (and actually has to) cross e_1 and $\tilde{e_2}$, but at most once each and with the same rotation. Observe that Q' and r are on different sides of C. So when tracing ur starting from u, the edge has to cross both $\partial Q'$ and C in order to reach r. As it may cross e_1 at most once, ur can leave C only by crossing s_x^2 ; let us denote this crossing by $p_u = ur \cap s_x^2$. Consequently, the edge ur also crosses s_b^1 with the same rotation. These two crossings can occur in either order, which determines the drawing of ur so that either way there is a simple closed curve C' that starts at p_u , then reaches r along ur, then continues along rv until the crossing with $\tilde{e_2}$, from where it returns back to p_u . Observe that x and v are on different sides of C'. See Figure 28(middle) for an illustration.



Figure 28 Situations in Case 1 of the proof of Lemma 23.

Finally, we consider the edge vx, which exists because we assume $x \in R$ for this case. Note that it may not be an edge of H, but still we need to be able to draw it consistently (with the same crossings and crossing rotations) in both $H \cup e_1$ and $H \cup \tilde{e_2}$, so as to be able to extend these drawings to drawings of the complete multipartite graph with the same ERS. So let us first consider the drawing of vx in $H \cup \tilde{e_2}$. Note that vx must not cross the adjacent edges $\tilde{e_2}$ and vr. As vx crosses C' it follows that it crosses ur so that when passing through this crossing from v to x, the vertex u is on the left side. So the drawing of vx in $H \cup e_1$ must cross ur in the same manner. In $H \cup e_1$, the edge vx must not cross the adjacent edges e_1 and vr. Consider the simple closed curve C'' that starts from r along ur up to the crossing with e_1 , from there continues along e_1 up to the crossing with vr, and from there returns back to r along vr. See Figure 28(right) for an illustration. Observe that v is on one side of C'' and the edge ur is on the other side and on $\partial C''$. The only edge along $\partial C''$ that vxmay cross is ur; but doing so would cross ur with the wrong crossing rotation (so that when passing through this crossing from v to x, the vertex u is on the right side). Therefore, there is no way to draw vx in $H \cup e_1$ as required, in contradiction to our assumption that both $H \cup e_1$ and $H \cup \tilde{e_2}$ can be extended to drawings of the complete multipartite graph with the same ERS. So we conclude that this case cannot occur.

Case 2: vr intersects $\tilde{e_2}$ between Q and x. As both drawings $H \cap e_1$ and $H \cap \tilde{e_2}$ are simple, there are no further crossings of vr with e_1 or $\tilde{e_2}$. Consider the closed curve C that goes from b_j along rb_j to r and then along rv until it reaches ∂Q from where it continues along $\partial Q \cap e_1$ to β , and then along $\tilde{e_2}$ back to b_j . Now consider the x-stub s_x^1 of e_1 w.r.t. Q. See Figure 29 for an illustration. As $\tilde{e_2}$ cannot cross ∂Q nor the edge rb_j and it can only cross vr once, it follows that x and the initial fragment of s_x^1 are on different sides of C. Therefore, in order to reach x, the stub s_x^1 crosses C. As s_x^1 cannot cross vr, rb_j , or ∂Q (the latter because Q is a lens), this crossing is along the b_j -stub of $\tilde{e_2}$ w.r.t. Q, at a point ϕ . Then the edge fragment of s_x^1 that ends at ϕ induces a cyclic quasi-lens Q' in Γ along the part of $\tilde{e_2}$ between β and b_j . This is a contradiction to our Assumption 2 about Q and, therefore, this case cannot occur, either.



Figure 29 The situation in Case 2 in the proof of Lemma 23.

Since we derive a contradiction in both cases, we conclude that Q is not a lens, which completes the first part of the proof. It remains to consider the case that Q is a general quasi-lens that is not a lens. Then by Lemma 17 there is a lens $L \subset Q$. See Figure 30(left) for an illustration. As L lies along the part of \tilde{e}_2 between β and b_j , by Assumption 2 it is acyclic. Furthermore, by the assumption of the lemma, the lens L is stabbed by a vertex vof H. As neither x nor b_j lies on ∂L , by Lemma 22 we know that v is blue and, therefore, connected to r by an edge in H. As $r \notin Q$, the edge vr crosses ∂Q exactly once. Given that vr also crosses both ∂L and e_1 exactly once, we conclude that it crosses $\partial Q \cap \tilde{e}_2$ at a point μ between β and ξ . Observe that vr cannot cross $\partial L \cap e_1$ because such a crossing would have a different rotation than the crossing with \tilde{e}_2 at μ . It follows that $\mu \in \partial L$. Let ν denote the vertex of L between μ and ξ .



Figure 30 Extending the statement to general quasi-lenses in Lemma 23.

Consider the simple closed curve C that goes from μ along $\tilde{e_2}$ to b_j , then along $b_j r$ to r, and then along rv back to μ . Let e' denote the part of e_1 from ν towards b_j up to its next

intersection with C (which exists, possibly at b_j). Note that e' cannot cross vr because such a crossing would have the wrong rotation compared to the crossing $vr \cap \tilde{e}_2 = \mu$. It cannot cross the adjacent edge rb_j , either, nor the part of \tilde{e}_2 between μ and ν because Lis a lens. So e' intersects $C \cap \tilde{e}_2$ somewhere between ν and b_j . Furthermore, given that vrcrosses \tilde{e}_2 , it must cross e_1 as well. Denote by f the part of e_1 from $e_1 \cap vr$ towards b_j up to the next intersection with C (which exists, possibly at b_j). To have the correct rotation at the crossing of e_1 and vr, the curve f starts on the same side of C as e'. Also, being a part of e_1 , the curve f has the same crossing restrictions as e' and, therefore, it ends on $C \cap \tilde{e}_2$, somewhere between ν and b_j . As the intersections of e' and f with $C \cap \tilde{e}_2$ are distinct, it follows that e' crosses \tilde{e}_2 properly between ν and b_j . In other words, the fragment e' of e_1 induces an acyclic quasi-lens Q' in Γ .

As $x \notin \partial Q'$ and $b_j \notin \partial Q'$, by Lemma 22 there is a blue vertex $u \in Q'$. Consider the edge ur in H^+ : It starts inside Q' and its target r lies outside of Q' and on C. However, it cannot reach r by staying on the same side of C because this would require crossing both e' and f. Hence, the edge ur has to cross C. But it cannot cross the adjacent edges vr and rb_j , nor can it cross through Q, as this would require crossing e_1 and $\tilde{e_2}$ with different rotations. So we conclude that ur crosses C along $\tilde{e_2}$, at a point λ between ξ and b_j . See Figure 30 (right) for an illustration.

Let C' denote the simple closed curve that goes from r along rv to μ , then along \tilde{e}_2 to λ , and then along ur back to r. Now consider the curves e' and f. Both are part of e_1 , so one appears before the other along e_1 . We assume that f appears before e' along e_1 . (The reasoning when e' appears before f along e_1 is analogous.) Note that both e' and f start at a crossing of e_1 with $C \cap C'$. Thus, in order to connect f to e', the edge e_1 must get from the endpoint of f, which lies on C but strictly on one side of C', to the other side of C'. In other words, somewhere between f and e', the edge e_1 crosses C'. Let ϕ denote the first such crossing. Then $\phi \notin vr$ because we already know the unique crossing $e_1 \cap vr$, which is the startpoint of f. Also $\phi \notin ur$ because then ϕ would have the wrong rotation, compared to the crossing $\lambda = ur \cap \widetilde{e_2}$. It follows that $\phi \in \widetilde{e_2} \cap C'$. Let σ denote the crossing $e_1 \cap \widetilde{e_2}$ that immediately precedes ϕ along e_1 . As e_1 cannot cross the adjacent edge rb_i , the fragment $\sigma\phi$ of e_1 attaches to the same side of $\tilde{e_2}$ at ϕ and σ . Thus, this fragment induces a quasi-lens Q''in Γ . Moreover, by definition of ϕ (as the first crossing of e_1 with C' after f) and the position of the endpoint of f (after all of $C' \cap \tilde{e_2}$ along $\tilde{e_2}$) we conclude that Q'' is cyclic. But this is a contradiction to our choice of Q, concretely to Assumption 2 above. Hence, such a cyclic quasi-lens Q does not exist to begin with, which concludes the proof of the lemma.

After these preparations we are now ready to prove Lemma 19.

▶ Lemma 19. If there is a lens L in Γ for which $b_i \notin \partial L$, then there is a free lens in Γ .

Proof. Assume for the sake of a contradiction that every lens in Γ is stabbed. Then by Lemma 17 every quasi-lens in Γ is stabbed and, therefore, essential. Let L be a lens in Γ with $b_j \notin \partial L$, which exists by assumption. As L is stabbed, it contains a vertex v of H. Moreover, as $b_j \notin \partial L$ we have $v \in B$ by Lemma 20. As L is essential, by Lemma 23 it is acyclic.

Let χ_1, \ldots, χ_q be the intersections of e_1 and $\tilde{e_2}$, ordered along $\tilde{e_2}$, with $\chi_1 = x$ and $\chi_q = b_j$. As $b_j \notin \partial L$, the lens L is defined by a fragment of $\tilde{e_2}$ between χ_k and χ_{k+1} , for some $1 \leq k \leq q-2$. For ease of illustration we imagine $\tilde{e_2}$ as a horizontal line segment such that x lies to the left of b_j and e_1 as a curve that "meanders" around $\tilde{e_2}$. We may assume without loss of generality that e_1 crosses $\tilde{e_2}$ from *left to right* at χ_{k+1} (otherwise, flip the drawing vertically). Then, as L is acyclic, the edge e_1 crosses $\tilde{e_2}$ from right to left at χ_k .

Aichholzer, Chiu, Hoang, Hoffmann, Kynčl, Maus, Vogtenhuber, Weinberger

Let $r \in R \setminus \{x\}$. Such a vertex exists because i > m and thus $r_1 \in R \setminus \{x\}$. As $L \cap R = \emptyset$, the edge vr in H crosses ∂L in exactly one of its two edge fragments. Assume without loss of generality that vr crosses ∂L along e_1 (otherwise, exchange the roles of e_1 and $\tilde{e_2}$), and denote this crossing by ψ . Then vr crosses e_1 from right to left. As vr crosses $\tilde{e_2}$ in the same direction, it must "go around" b_j or x and cross $\tilde{e_2}$ between χ_{k+1} and b_j before ending at r; see Figure 31. (In the figure, χ_k is depicted as a crossing, but we could also have $\chi_k = x$.)



Figure 31 Base situation in the proof of Lemma 19.

Let χ_{ℓ} denote the next consecutive intersection of e_1 with $\tilde{e_2}$ after χ_{k+1} (possibly $\chi_{\ell} = b_j$). We distinguish several cases depending on the relation of k and ℓ , and on the rotation of χ_{ℓ} .

Case 1: $\ell < k$ and e_1 crosses \tilde{e}_2 from right to left at χ_ℓ . Then the edge fragment of e_1 between χ_{k+1} and χ_ℓ induces a cyclic quasi-lens in Γ , in contradiction to Lemma 23. Hence this case cannot occur.

Case 2: $\ell < k$ and e_1 crosses $\tilde{e_2}$ from left to right at χ_{ℓ} . Then the edge fragment $\chi_{k+1}\chi_{\ell}$ of e_1 forms an anti-lens A in Γ whose inside stub s_A is formed by the part of e_1 from χ_{ℓ} to b_j . See Figure 32 for an illustration. By Lemma 18 there is a quasi-lens $Q \subset A$ in Γ that is induced by an edge fragment of s_A . As A separates Q from b_j , by Lemma 22 there is a blue vertex $u \in Q$. We claim that there is no way to add the edge ur to this drawing.



Figure 32 Case 2 in the proof of Lemma 19: $\ell < k$.

To see this, observe first that the edge fragment $\chi_k \chi_{k+1}$ of $\tilde{e_2}$ is not crossed by s_A because L is a lens. In particular, the closures of Q and L are disjoint. As $u \in Q$ and $r \notin Q$, as a first step an edge from u to r has to leave Q, by crossing ∂Q along one of e_1 or $\tilde{e_2}$. As $\partial Q \cap \partial L = \emptyset$ and $r \notin L$, this implies that ur cannot enter L (without crossing one of e_1 or $\tilde{e_2}$ a second time on its way to r, which is forbidden). Next observe that u and r are also separated by the simple closed curve C that starts at χ_{k+1} along $\tilde{e_2}$ towards b_j up to the crossing with the edge rb, then along rb towards b up to the crossing with ∂L , and then along ∂L back to χ_{k+1} . So the edge ur has to also cross both ∂A and C. The curve C consists of three parts:

- 1. The part along ∂L , which cannot be crossed by ur, as argued above.
- 2. The part along the edge br, which is adjacent to ur and therefore cannot be crossed by ur, either.
- 3. The part along $\tilde{e_2}$, which is the only remaining option to be crossed by ur.

The boundary ∂A consists of two parts:

- 1. The part along \tilde{e}_2 , which ur cannot cross, given that it already crosses \tilde{e}_2 along C (and the only common point χ_{k+1} is part of ∂L).
- 2. The part along e_1 , which is the only remaining option to be crossed by ur.

However, in addition the edge ur also crosses ∂Q . As $C \cap \partial Q = \emptyset$ and the only common point $\partial A \cap s_A$ is $\chi_{\ell} \in \widetilde{e_2}$, there is no way to realize all these required crossings along ur. So this case cannot occur, either.

Case 3: either $k + 1 < \ell < q$ and e_1 crosses $\tilde{e_2}$ from left to right at χ_ℓ or $\ell = q$ and hence $\chi_\ell = b_j$. Then there is no way to draw the edge vr without crossing one of e_1 or $\tilde{e_2}$ more than once, which is forbidden. See Figure 33 for an illustration. So this case cannot occur, either.



Figure 33 Case 3 in the proof of Lemma 19: $k + 1 < \ell < q$ or $\ell = q$.

Case 4: $k + 1 < \ell < q$ and e_1 crosses \tilde{e}_2 from right to left at χ_ℓ . Then the edge fragment $\chi_{k+1}\chi_\ell$ of e_1 forms a quasi-lens Q with \tilde{e}_2 , which by Lemma 22 contains a blue vertex u. The edge ur must leave Q, which involves crossing one of e_1 or \tilde{e}_2 .

Assume first that ur crosses e_1 along ∂Q . Then ur crosses e_1 from left to right. Let F be the region that contains r and is bounded by the edge e_1 between χ_{k+1} and ψ , by vr between ψ and the crossing of br with $\tilde{e_2}$, and by $\tilde{e_2}$ between that crossing and χ_{k+1} ; see the shaded area in Figure 34. The part of ur directly after crossing e_1 lies outside F. To reach r, the edge ur must enter F. However, it must not intersect vr, it must not intersect e_1 again, and it must intersect $\tilde{e_2}$ from left to right. Hence ur cannot be completed, a contradiction.

So we may assume that ur crosses \tilde{e}_2 from right to left along ∂Q . Let $\Delta \subset F$ be bounded by $b_j r$, the part of vr between r and the crossing of vr and \tilde{e}_2 , and the part of \tilde{e}_2 between that crossing and b_j . The part of ur directly after crossing \tilde{e}_2 lies in F. To reach r, the edge ur must stay in F and it must not intersect Δ , leading to the drawing depicted in Figure 34(right).



Figure 34 Case 4 of Lemma 19

Consider how e_1 continues beyond χ_ℓ :

- 1. It cannot cross ur because the rotation of $ur \cap e_1$ does not match with the rotation of $ur \cap \widetilde{e_2}$.
- **2.** It cannot cross rv because of the crossing $rv \cap e_1 = \psi$.

- 3. It cannot cross \tilde{e}_2 to the left of χ_ℓ because this would create a cyclic quasi-lens, in contradiction to Lemma 23.
- 4. Thus, it crosses \tilde{e}_2 to the right of χ_ℓ , thereby inducing a (cyclic) quasi-lens Q'.

The edge wr of H must cross $\partial Q'$ to reach r, which means it crosses one of e_1 or $\tilde{e_2}$. So it also crosses the other, with the same crossing rotation. As wr cannot cross the adjacent edges vr and ur, it has to leave F so as to cross $\tilde{e_2}$. In order to reach r, which lies inside F, the edge wr has to enter F again. But this is impossible because:

- 1. The edge wr cannot cross \tilde{e}_2 again.
- **2.** It cannot cross vr.
- **3.** It cannot reach F through L without crossing e_1 twice.

Therefore, this case cannot occur, either.

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To summarize, in each of the four cases we obtained a contradicition to the assumption that every lens in Γ is stabled. Hence, there exists a free lens in Γ .

D.6 Invertible triangles are empty (Proof of Lemma 14)

By definition of an invertible triangle, there exists a drawing D' where the crossing triangle Δ' formed by the same combinatorial edges as those for Δ has a different parity than Δ . We call Δ' the *inverted triangle* of Δ in D'. Lemma 14 is an immediate corollary of two lemmata below (see the end of the section).

▶ Lemma 24. For $m, n \ge 3$ and $m + n \ge 7$, let D and D' be two drawings of $K_{m,n}$ on the sphere with the same ERS and let Δ and Δ' be crossing triangles in D_1 and D_2 , respectively, such that Δ' is an inverted triangle of Δ . Let v be a vertex that in D lies inside Δ . Then in D', v cannot lie inside Δ' .



Figure 35 If the vertex v in drawing D (left) lies inside the triangle Δ (shaded red) then its position w.r.t. an induced $K_{2,3}$ is fixed: in the drawing D' (right) v must be in the same cell of this $K_{2,3}$ and can thus not be in Δ' .

Proof. First, observe that out of the six endpoints of the three edges forming Δ in D, exactly three belong to a bipartition class of the vertices of $K_{m,n}$, and the other three to the other class. Consider the three curves connecting these endpoints in the way as depicted as dashed lines on the left side of Figure 35. At least one of these curves connects two points from different bipartition classes of the vertices of $K_{m,n}$, and is thus a valid edge of the complete bipartite drawing. Otherwise, at least four of the endpoints would have to belong to the same bipartition class - a contradiction to the first observation.

W.l.o.g. let v_1 and v_2 be the endpoints of the edges e_1 and e_2 , respectively, which are connected this way. Consider the drawing of a $K_{2,2}$ induced by the vertices of e_1 and e_2 , see Figure 35(right). Together with v this gives a drawing of a $K_{2,3}$ and by Lemma 16 the vertex v lies in a fixed cell of this drawing. In the drawing D this is the area inside Δ , which is read shaded in Figure 35. Thus in the drawing D', the vertex v must lie in the same area of the $K_{2,3}$ induced by e_1 , e_2 (shaded blue in Figure 35) and cannot lie inside of Δ' .

▶ Lemma 25. For $m, n \ge 3$ and $m + n \ge 7$ let D and D' be two drawings of $K_{m,n}$ on the sphere with the same ERS and let Δ and Δ' be triangles in D_1 and D_2 , respectively, such that Δ' is an inverted triangle of Δ . Let v be a vertex that in D lies inside Δ . Then in D', v cannot lie outside Δ' .



Figure 36 Drawing D where the triangle Δ has a vertex v on the inside of Δ . The vertices of the two bipartition classes of vertices are drawn as circles and squares, respectively. On the left, the endvertices of the edges of Δ belong to the two bipartition classes of the vertices of $K_{m,n}$ alternatingly, while on the right, they form two consecutive blocks.

Proof. Consider the endvertices of Δ . There are two different possibilities how they belong to the two bipartition classes of the vertices of $K_{m,n}$. Either they are alternating (Case 1, see Figure 36(left)) or there are two blocks of three vertices each from the same class (Case 2, see Figure 36(right)). To simplify the description, we color the three edges that form Δ in blue (edge b), red (edge r), and green (edge g), and consider three of the edges connecting their endpoints, namely the black edges b_{rg} , b_{gb} , and b_{br} as shown in the drawings in Figure 36. Note that we do not assume anything on how the three black edges are drawn, except that they belong to a simple drawing. For example, b_{rg} may intersect b_{gb} and b.

In order to obtain a contradiction, in both cases, we assume that, in drawing D the vertex v lies on the inside of Δ , and that in drawing D', the vertex v lies on the outside of Δ' . A drawing of D' is shown in Figure 37. As in Figure 36, we do not assume anything on how the three black edges are drawn beyond simplicity (and there might be intersections not drawn in the figure).

Case 1: Alternating endpoints of the classes. Consider the drawing of $K_{2,3}$ induced by the endpoints of r and g and the vertex v and the drawing of $K_{2,2}$ induced by the endpoints of r and g. Applying Lemma 16 to the drawing of $K_{2,3}$ above, it follows that the vertex v in Δ also lies in the cell of the drawing of $K_{2,2}$ that is bounded by a part of the edge r, a part of the edge g, and the edge b_{rg} , as indicated by the dotted cycle in the drawing in Figure 36(left). We denote this bounding cycle by C_{rg} . Similarly, v also lies in the cell of the



Figure 37 Drawing D' where the vertex v must lie in the three indicated cycles outside of Δ' .

drawing of $K_{2,2}$ induced by r and b bounded by the dotted cycles C_{rb} , and that corresponding to g and b bounded by the dotted cycle C_{qb} .

Consider the subgraph induced by v and the endpoints of r and b. Since D and D' have the same ERS, v has to lie in the same cell of any drawing of this subgraph. Thus, it has to lie on the same side of the cycle C_{rb} in D and D'. Analogously, it has to lie on the same side of C_{rg} and C_{gb} in D and D'. But since the three cycles do not have a common area inside Δ' anymore, there must be some other place where the intersection of the relevant area of the three cycles (that is, the area where v lies in) is non-empty; cf Figure 38.



Figure 38 Drawing D' where the intersection of the three regions is non empty so that v can be placed there.

Let us start with the intersection of C_{rg} and C_{gb} . The possible crossings for these two cycles are $b_{rg} \cap b$, $b_{gb} \cap r$, and $b_{rg} \cap b_{gb}$. As the two cycles must have a common area, it follows by a simple parity argument, that we have precisely two crossings between these two cycles. By symmetry, we can w.l.o.g. assume that one of these crossings is $b_{rg} \cap b$. Note that there is only one possible crossing rotation for these two edges (Figure 38(left)).

Now b_{gb} cannot cross the part of r which is part of C_{rg} without also crossing b_{rg} . So the second crossing is $b_{rg} \cap b_{gb}$. Again there is only one possible rotation for this crossing. b_{gb} might cross r outside of C_{rg} (drawn as b'_{gb} in Figure 38(left)) or might not cross r outside of C_{rg} (drawn as b'_{gb} in Figure 38(left)).

This gives an area where vertex v could potentially lie; it is draw in orange in Figure 38. Thus also b_{rb} must intersect this region. As it cannot intersect b, it has to intersect both, b_{rg} and b_{gb} . If we draw b''_{gb} then this is not possible, as g can only be crossed once by b_{rb} . So b_{gb}



Figure 39 In D (left), v must lie inside the indicated cycle C_{rg} but outside the indicated cycles C_{gb} and C_{rb} . Hence, the same must hold for D' (right).

is drawn as b'_{ab} , that is, crossing r.

Then there is only one unique way to draw b_{rb} (Figure 38(middle)).

So far the drawing has already 9 crossings, the maximum for a drawing of $K_{3,3}$. So the remaining three edges have to be drawn without crossings, which completes the drawing (Figure 38(middle)).

As the edges on the boundary of this cell are disjoint from the edges on the boundary of Δ' , we can have a drawing \tilde{D} that is identical to the drawing of this $K_{3,3}$ in D', except that the triangle $\tilde{\Delta}$ corresponding to Δ' is flipped (Figure 38(right)). Observe that the ERS restricted to this $K_{3,3}$ is the same in both \tilde{D} and D'. However, now there are two cells where v can lie, inside the triangle $\tilde{\Delta}$ and the cell where v lies in D' (see shaded cells in Figure 38(right)). But this is a contradiction to Lemma 11. So it follows that in this setting, v cannot lie on the inside of Δ' .

Case 2: Blocks of three endpoints of each class. For convenience, we label the vertices and the crossings among b, r, g as in Figure 39; w.l.o.g. vertex v and the vertices with subscript 2 are in the same bipartition class. In the drawing of $K_{2,2}$ induced by the endvertices of r and g, consider the tricell enclosed by b_{rg} , r, and g, indicated by dotted cycle in the drawing in Figure 39(left). Let C_{rg} be the bounding cycle of this cell, and we will refer to this cell as the fixed side of C_{rg} . Observe that v lies in this fixed side. Similarly, in the drawing of $K_{2,2}$ induced by the endvertices of r and b (resp. g and b), we denote by C_{rb} (resp. C_{gb}) the bounding cycle of the tricell that has b_{rb} (resp. b_{gb}) on its boundary, also indicated by dotted cycle in the drawing of Δ' in the drawing in Figure 39(left). We call the cell the fixed side of C_{rb} (resp. C_{gb}), and v does not lie in this fixed side. Following a similar argument as in Case 1, in the drawing of Δ' in D', v lies in the corresponding fixed side of C_{rg} and not in those of C_{gb} and C_{rb} ; see Figure 39(right).

In D', since v lies in the fixed side of C_{rg} but not in the fixed side of C_{gb} and since the edge b_{gb} cannot cross b_{rg} due to the simple drawing assumption, b_{gb} has to cross the edge fragment $\chi_{rg}r_1$, where χ_{rg} is the crossing point of r and g. Consider the cell of the drawing of $K_{2,2}$ induced by r and b_{gb} in D' that v lies. In particular, this cell is formed by b_{rg} , b_{gb} , and r, and hence b_1 does not lie in the cell. Hence, the edge e that connects v and b_1 has to cross the boundary of the cell an odd number of times. As e cannot cross b_{gb} , it has to cross that boundary exactly once. We consider two subcases:

Case 2.1: e crosses $r_1\chi_{rg}$ in D' and does not cross b_{rg} . In D', because (i) v and b_1 lie on the same side of C_{rb} , (ii) e crosses $r_1\chi_{rg}$, and (iii) e cannot cross b, we must have (iv) e also crosses b_{rb} . Similarly, in D, because of (i), (iii), and (iv) in the previous statement, we can conclude that e has to cross $r_1\chi_{rb}$. Since v and b_1 lie in different sides of Δ in D, e has

to cross at least an edge of Δ . Since e crosses $r_1\chi_{rb}$, it cannot cross $\chi_{rg}\chi_{rb}$. Coupled with the fact that e cannot cross b, we conclude that e has to cross g. However, consider now the cycle C_{rg} in D. When we go along e from v to b_1 , we first cross g and since we cannot cross b_{rg} , we cannot cross $\chi_{rb}r_1$ with the same crossing rotation as in D'. Hence, we have a contradiction.

Case 2.2: e crosses b_{rg} in D'. We have the situation in D' as in Figure 39(left). As observed at the beginning, b_{gb} crosses r in D', and hence the same holds in D. As b_{gb} cannot cross g, there are two ways to draw b_{gb} in D. If b_{gb} crosses the edge fragment $r_2\chi_{rg}$ as in Figure 39(middle), observe that we cannot draw the edge e that does not cross b or b_{gb} and at the same time crosses b_{rg} with the same crossing rotation as in D'. Hence, b_{gb} has to cross the edge fragment $\chi_{rb}r_1$. However, as e cannot cross b_{gb} , the only way we can draw efrom v while respecting the rotation of the crossing $e \cap b_{rg}$ is as in Figure 39(right). However, we then cannot complete the drawing of edge e to b_1 .

This completes the proof of Lemma 25.

▶ Lemma 14 (Invertible triangles are empty). Let D be a simple drawing of a complete bipartite graph G, and let Δ be an invertible triangle in D. Then all vertices of D lie outside Δ .

Proof. As Δ is an invertible triangle in D, there exists an inverted triangle Δ' in another drawing D' of $K_{m,n}$ with the same ERS. Suppose that there is a vertex v that lies on the inside of Δ . Then, due to Lemma 24, in D' the vertex v cannot lie on the inside of Δ' , and, due to Lemma 25, v also cannot lie on the outside of Δ' . Hence, v cannot be drawn anywhere in D', a contradiction. Thus, all vertices of $K_{m,n}$ lie on the outside of Δ in D.

E Missing proof of Section 6: Lower bound on the flip distance

To prove Theorem 10, the lower bound on the flip distance, we show that the two drawings of K_n in Figure 40 are at flip distance $\Theta(n^6)$. The figure is the same as Figure 10 in Section 6 and repeated for readability. We further repeat the theorem before its proof for convenience.

▶ **Theorem 10.** Let G be a multipartite graph G with n vertices that contains two vertexdisjoint subgraphs each forming a $K_{m,m}$ for some $m = \Theta(n)$. Then G admits two drawings D_1 and D_2 with the same ERS that have flip distance $\Omega(n^6)$.

Proof. We first show the bound for $G = K_n$ by giving a construction of D_1 and D_2 ; see Figure 40 for a depiction. For convenience assume that n is divisible by four. Both drawings in Figure 40 have the same extended rotation system. There are $\Theta(n^2)$ black edges connecting vertices of $A = \{a_1, a_2, \ldots, a_{\frac{n}{4}}\}$ with vertices of $C = \{c_1, c_2, \ldots, c_{\frac{n}{4}}\}$, which generate $\Theta(n^4)$ crossings. All these crossings are in the gray shaded area. Thus, there are $\Theta(n^4)$ crossings between black edges inside this area. All the green edges connecting vertices of $B = \{b_1, b_2, \ldots, b_{\frac{n}{4}}\}$ with vertices of $D = \{d_1, d_2, \ldots, d_{\frac{n}{4}}\}$ cross all the black edges. In the drawing D_1 (Figure 40(left)), all crossings between a green and a black edge are between the vertices of C and the gray shaded area. In the drawing D_2 (Figure 40(right)), all crossings between a green and a black edge area. Thus, any two crossing black edges together with every green edge induce a triangle that has to be flipped to transform D_1 into D_2 . Since there are $\Theta(n^2)$ green edges, at least $\Theta(n^6)$ triangles need to be flipped.

The same arguments hold for any subdrawing which contains the black edges between A and C and the green edges between B and D, where an arbitrary subset of the gray



Figure 40 Two simple drawings of K_n with the same ERS. There are $\Theta(n^2)$ (green) edges between B and D which need to be moved over $\Theta(n^4)$ crossings of (black) edges between A and C, resulting in a total of $\Theta(n^6)$ triangle flips.

edges can be included. Thus, the lower bound holds for a more general class of complete multipartite graphs, as long as it includes complete bipartite subgraphs both between A and C and between B and D.