Shooting Stars in Simple Drawings of $K_{m,n}$[*]

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Abstract. Simple drawings are drawings of graphs in which two edges have at most one common point (either a common endpoint, or a proper crossing). It has been an open question whether every simple drawing of a complete bipartite graph $K_{m,n}$ contains a plane spanning tree as a subdrawing. We answer this question to the positive by showing that for every simple drawing of $K_{m,n}$ and for every vertex v in that drawing, the drawing contains a *shooting star rooted at v*, that is, a plane spanning tree containing all edges incident to v.

Keywords: Simple drawing · Simple topological graph · Complete bipartite graph · Plane spanning tree · Shooting star

1 Introduction

A *simple drawing* is a drawing of a graph on the sphere S^2 or, equivalently, in the Euclidean plane where (1) the vertices are distinct points in the plane, (2) the edges are non-self-intersecting continuous curves connecting their incident points, (3) no edge passes through vertices other than its incident vertices, (4) and every

pair of edges intersects at most once, either in a common endpoint, or in the relative interior of both edges, forming a proper crossing. Simple drawings are also called *good drawings* [4,6] or *(simple) topological graphs* [10,11]. In *star-simple drawings*, the last requirement is softened so that edges without common endpoints are allowed to cross several times. Note that in any simple or star-simple drawing, there are no tangencies between edges and incident edges do not cross. If a drawing does not contain any crossing at all, it is called *plane*.

The search for plane subdrawings of a given drawing has been a widely considered topic for simple drawings of the complete graph K_n which still holds tantalizing open problems. For example, Rafla [13] conjectured that every simple drawing of K_n contains a plane Hamiltonian cycle, a statement which is by now known to be true for $n \leq 9$ [1] and several classes of simple drawings (e.g., 2-page book drawings, monotone drawings, cylindrical drawings), but still remains open in general. A related question concerns the least number of pairwise disjoint edges in any simple drawing of K_n. The currently best lower bound is $\Omega(n^{1/2})$[2], which is improving over several previous bounds [7,8,9,11,12,14,15], while the trivial upper bound of $n/2$ would be implied by a positive answer to Rafla's conjecture. A structural result of Fulek and Ruiz-Vargas [9] implies that every simple drawing of K_n contains a plane sub-drawing with at least $2n - 3$ edges.

We will focus on plane trees. Pach et al. [11] proved that every simple drawing of K_n contains a plane drawing of any fixed tree with at most $c \log^{1/6} n$ vertices. For paths specifically, every simple drawing of K_n contains a plane path of length $\Omega(\frac{\log n}{\log \log n})$ [2,16]. Further, it is trivial that simple drawings and star-simple drawings of K_n contain a plane spanning tree, because every vertex is incident to all other vertices and adjacent edges do not cross. Thus, the vertices together with all edges incident to one vertex form a plane spanning tree. We call this subdrawing the *star* of that vertex.

In this work, we consider the search for plane spanning trees in drawings of complete bipartite graphs. Finding plane spanning trees there is more involved than for K_n. In fact, not every star-simple drawing of a complete bipartite graph contains a plane spanning tree; see Fig. 1.
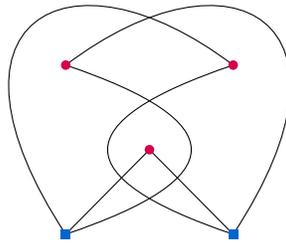

Fig. 1. Star-simple drawing of $K_{2,3}$ that does not contain a plane spanning tree.
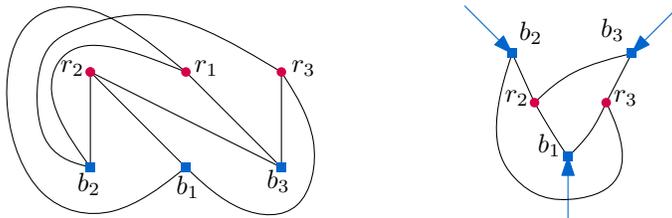
Fig. 2. Left: Simple drawing of $K_{3,3}$. Right: its stereographic projection from r_1.

It is not hard to see that straight-line drawings of complete bipartite graphs always contain plane spanning trees. Consider the star of an arbitrary vertex v. The prolongation of these edges creates a set of rays originating at v that partitions the plane into wedges, which we divide into two parts using the angle bisectors. We connect the vertices in each part of a wedge to the point on the ray that bounds it. These connections together with the star of v form a plane spanning tree of a special type called *shooting star*. A *shooting star rooted at v* is a plane spanning tree with root v that has height 2 and contains the star of vertex v. Aichholzer et al. showed in [3] that simple drawings of $K_{2,n}$ and $K_{3,n}$, as well as so-called *outer drawings* of $K_{m,n}$, always contain shooting stars. *Outer drawings of $K_{m,n}$* [5] are simple drawings in which all vertices of one bipartition class lie on the outer boundary.

Results.   We show in Section 2 that every simple drawing of $K_{m,n}$ contains shooting stars rooted at an arbitrary vertex of $K_{m,n}$. The tightness of the conditions is shown in Section 3 and in Section 4 we discuss algorithmic aspects.

2   Existence of Shooting Stars

In this section, we prove our main result, the existence of shooting stars:

Theorem 1. *Let D be a simple drawing of $K_{m,n}$ and let r be an arbitrary vertex of $K_{m,n}$. Then D contains a shooting star rooted at r.*

Proof. We can assume that D is drawn on a point set $P = R \cup B$, $R = \{r_1, \ldots, r_m\}$, $B = \{b_1, \ldots, b_n\}$, in which the points in the two bipartition classes R and B are colored red and blue, respectively. Without loss of generality let $r = r_1$.

To simplify the figures, we consider the drawing D on the sphere and apply a stereographic projection from r onto a plane. In that way, the edges in the star of r are represented as (not necessarily straight-line) infinite rays; see Fig. 2. We will depict them in blue. In the following, we consider all edges oriented from their red to their blue endpoint. To specify how two edges cross each other, we introduce some notation. Consider two crossing edges $e_1 = r_i b_k$ and $e_2 = r_j b_l$ and let x be their crossing point. Consider the arcs xr_i and xb_k on e_1 and xr_j and xb_l on e_2. We say that e_2 crosses e_1 in *clockwise direction* if the clockwise cyclic
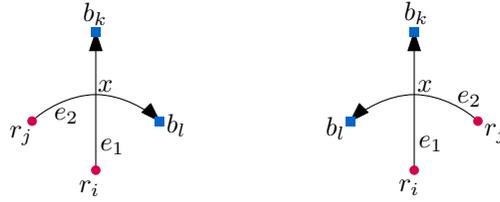
Fig. 3. Left: e_2 crosses e_1 in clockwise direction. Right: e_2 crosses e_1 in counterclockwise direction.

order of these arcs around the crossing x is xr_i, xr_j, xb_k, and xb_l. Otherwise, we say that e_2 crosses e_1 in *counterclockwise direction*; see Fig. 3.

We prove Theorem 1 by induction on n. For $n = 1$ and any $m \geq 1$, the whole drawing D is a shooting star rooted at any vertex, and in particular at r.

Assume that the existence of shooting stars rooted at any vertex has been proven for any simple drawing of $K_{m,n'}$ with $n' < n$. By the induction hypothesis, the subdrawing of D obtained by deleting the blue vertex b_1 and its incident edges contains at least one shooting star rooted at r. Of all such shooting stars, let S be one whose edges have the minimum number of crossings with rb_1, and let M be the set of edges of S that are not incident to r. We will show that $S \cup \{rb_1\}$ is plane and hence forms the desired shooting star. Note that it suffices to show that $M \cup \{rb_1\}$ is plane, since rb_1 cannot cross any edges of $\{\bigcup_{j=2}^{n} rb_j\}$ in any simple drawing.

Assume for a contradiction that rb_1 crosses at least one edge in M. When traversing rb_1 from b_1 to r, let x be the first crossing point of rb_1 with an edge $r_k b_t$ in M. W.l.o.g., when orienting rb_1 from r to b_1 and $r_k b_t$ from r_k to b_t, $r_k b_t$ crosses rb_1 in counterclockwise direction (otherwise we can mirror the drawing).

Suppose first that the arc $r_k x$ (on $r_k b_t$ and oriented from r_k to x) is crossed in counterclockwise direction by an edge incident to b_1 (and oriented from the red endpoint to b_1). Let $e = r_l b_1$ be such an edge whose crossing with $r_k x$ at a point y is the closest to x. Otherwise, let e be the edge $r_k b_1$ and y be the point r_k. In the remaining figures, we represent in blue the edges of the star of r, in red the edges in M, and in black the edge e.

We distinguish two cases depending on whether e crosses an edge of the star of r. The idea in both cases is to define a region Γ and, inside it, redefine the connections between red and blue points to reach a contradiction.

Case 1: e does not cross any edge of the star of r. Let Γ be the closed region of the plane bounded by the arcs yb_1 (on e), $b_1 x$ (on rb_1), and xy (on $r_k b_t$); see Fig. 4. Observe that all the blue points b_j lie outside the region Γ and that for all the red points r_i inside region Γ, the edge $r_i b_1$ must be in Γ. Let M_Γ denote the set of edges $r_i b_1$ with $r_i \in \Gamma$ and note that $r_k b_1 \in M_\Gamma$. Consider the set M' of red edges obtained from M by replacing, for each red point $r_i \in \Gamma$, the (unique) edge incident to r_i in M by the edge $r_i b_1$ in M_Γ, and keeping the other edges in M unchanged. In particular, the edge $r_k b_t$ has been replaced by the
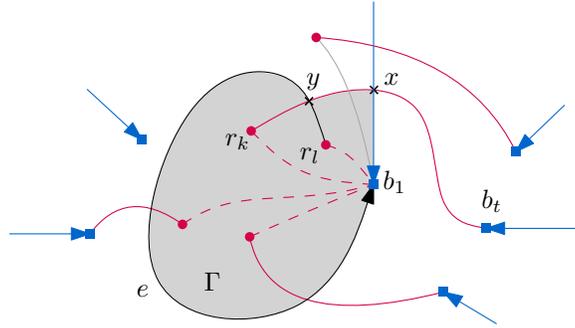
Fig. 4. Illustration of Case 1.

edge $r_k b_1$. The edges in M_Γ neither cross each other nor cross any of the blue edges $r b_j$. Moreover, we now show that the non-replaced edges in M must lie completely outside Γ. These edges can neither cross $r_k b_t$ (by definition of M) nor the arc $b_1 x$ (on $r b_1$). Thus, if they are incident to b_1, they cannot cross the boundary of Γ; If they are not incident to b_1, both their endpoints lie outside Γ and they can only cross the boundary of Γ at most once (namely, on the arc $b_1 y$). Therefore, M' satisfies that $M' \cup \{\bigcup_{j=2}^{n} r b_j\}$ is plane and has fewer crossings with $r b_1$ than M, since at least the crossing x has been eliminated and no new crossings have been added. This contradicts the definition of M as the one with the minimum number of crossings with $r b_1$.

Case 2: e crosses the star of r. When traversing e from r_k or r_l (depending on the definition of e) to b_1, let $I = \{\alpha, \beta, \ldots, \rho\}$ be the indices of the edges of the star of r in the order as they are crossed by e and let $y_\alpha, \ldots, y_\rho$ be the corresponding crossing points on e. Note that, when orienting e from r_k or r_l to b_1, the edges $r b_\xi, \xi \in I$, oriented from r to b_ξ, cross e in counterclockwise direction, since they can neither cross $r_k b_t$ (by definition of M) nor $r b_1$.

    The three arcs $r y_\alpha$ (on $r b_\alpha$), $y_\alpha b_1$ (on e), and $b_1 r$ divide the plane into two (closed) regions, Π_{left}, containing vertex r_k, and Π_{right}, containing vertex b_t. For each $\xi \in I$, let M_ξ be the set of red edges of M incident to some red point in Π_{right} and to b_ξ. Note that all the edges in M_ξ (if any) must cross the edge e. When traversing e from r_k or r_l to b_1, we denote by x_ξ, z_ξ the first and the last crossing points of e with the edges of $M_\xi \cup r b_\xi$, respectively; see Fig. 5 for an illustration. We remark that both x_ξ and z_ξ might coincide with y_ξ and, in particular, if $M_\xi = \emptyset$ then $x_\xi = y_\xi = z_\xi$.

    We now define some regions in the drawing D. Suppose first that there are edges in M (oriented from the red to the blue point) that cross $r b_1$ (oriented from r to b_1) in clockwise direction. Let $r_s b_\eta$ be the edge in M whose clockwise crossing with $r b_1$ at a point x' is the closest one to x (recall that the arc $b_1 x$ on $r b_1$ is not crossed by edges in M). Then, if $\eta \notin I$, we denote by W_η the region bounded by the arcs $r x'$ (on $r b_1$), $x' b_\eta$ (on $r_s b_\eta$), and $r b_\eta$ and not containing b_1; see Fig. 5 (left). If $\eta \in I$, we define W_η as the region bounded by the arcs $r x'$ (on

Fig. 5. Illustration of Case 2. Region Π_{right} is striped in gray, region Γ is shaded in blue, and regions in $\bigcup_{\xi \in I} W_\xi \cup W_\eta$ are shaded in yellow. Left: b_η does not cross e ($\eta \notin I$). Right: b_η crosses e ($\eta \in I$).

rb_1), $x'b_\eta$ (on $r_s b_\eta$), $b_\eta z_\eta$, $z_\eta y_\eta$ (on e), and $y_\eta r$ (on rb_η) and not containing b_1; see Fig. 5 (right). If no edges in M cross rb_1 in clockwise direction, then η is undefined and we set $W_\eta = \emptyset$ for convenience. Moreover, for each $\xi \in I \setminus \{\eta\}$, we define W_ξ as the region bounded by the arcs $x_\xi b_\xi$, $b_\xi z_\xi$, and $z_\xi x_\xi$ (and not containing b_1); see again Fig. 5.

We can finally define the region Γ for Case 2, which is the region obtained from Π_{left} by removing the interior of all the regions W_ξ, $\xi \in I$ plus region W_η if $\eta \notin I$ (otherwise it is already contained in $\bigcup_{\xi \in I} W_\xi$). Now consider the set of red and blue vertices contained in Γ. Let J denote the set of indices such that for all $j \in J$, the blue point b_j lies in Γ (note that $1 \in J$). Since b_t is not in Γ, we can apply the induction hypothesis to the subdrawing of D induced by the vertices in Γ plus r. Hence there exists a set of edges M_Γ connecting each red point in Γ with a blue point b_j, $j \in J$ such that $M_\Gamma \cup \{\bigcup_{j \in J} rb_j\}$ is plane. Moreover, all the edges in M_Γ lie entirely in Γ: An edge in M_Γ cannot cross any of the edges rb_j, with $j \in J$. Thus, it cannot leave Π_{left}, as otherwise it would cross e twice. Further, if it entered one of the regions in $\bigcup_{\xi \in I} W_\xi \cup W_\eta$, it would have to leave it crossing e, and then it could not re-enter Γ.

Consider the set M' of red edges obtained from M by replacing, for each red point $r_i \in \Gamma$, the edge $r_i b_\xi$ in M by the edge $r_i b_j$, $j \in J$, in M_Γ, and keeping the other edges in M unchanged. In particular, the edge $r_k b_t$ has been replaced by some edge $r_k b_j$, $j \in J$. The edges in M_Γ neither cross each other nor cross any of the blue edges rb_j, $j \in J$ nor any of the other ones, lying completely outside Γ. Moreover, the non-replaced edges in M cannot enter Γ since the only boundary part of Γ that they can cross are arcs on e. Therefore, M' satisfies that $M' \cup \{\bigcup_{j=2}^n rb_j\}$ is plane and has fewer crossings with rb_1 than M. This contradicts the definition of M as the one with the minimum number of crossings with rb_1. □
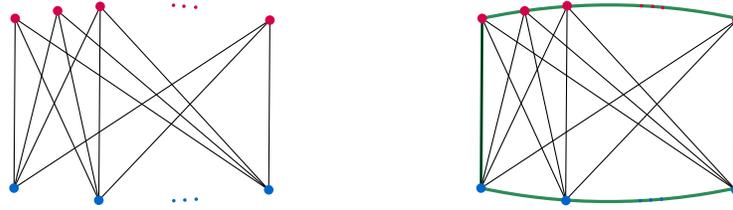
Fig. 6. Left: A simple drawing of $K_{m,n}$ where no plane subdrawing has more edges than a shooting star. Right: convex $(n + m)$-gon on the convex hull (green).

3   Some Observations on Tightness

There exist simple drawings of $K_{m,n}$ in which every plane subdrawing has at most as many edges as a shooting star. For example, consider a straight-line drawing of $K_{m,n}$ where all vertices are in convex position such that all red points are next to each other in the convex hull; see Fig. 6 (left). The convex hull is an $(m + n)$-gon which shares only two edges with the drawing of $K_{m,n}$; see Fig. 6 (right). All other edges of the drawing of $K_{m,n}$ are diagonals of the polygon. As there can be at most $(m + n) - 3$ pairwise non-crossing diagonals in a convex $(m + n)$-gon, any plane subdrawing of this drawing of $K_{m,n}$ contains at most $m + n - 1$ edges.

Furthermore, both requirements from Theorem 1—the drawing being simple and containing a complete bipartite graph—are in fact necessary: As mentioned in the introduction, not all star-simple drawings of $K_{m,n}$ contain a plane spanning tree. Further, if in the example in Fig. 6 (left), we delete one of the two edges of $K_{m,n}$ on the boundary of the convex hull, then any plane subdrawing has at most $m + n - 2$ edges and hence it cannot contain any plane spanning tree.

4   Computing Shooting Stars

The proof of Theorem 1 contains an algorithm with which we can find shooting stars in given simple drawings. We start with constructing the shooting star for a subdrawing that is a $K_{m,1}$ and then inductively add more vertices. Every time we are adding a new vertex, the shooting star of the step before is a set fulfilling all requirements of $M_1 \cup \{\bigcup_{j=2}^{n} rb_j\}$ in the proof. By replacing edges as described in the proof, we obtain a new set with the same properties and fewer crossings. We continue replacing edges until we obtain a set of edges (M in the proof) that form a shooting star for the extended vertex set. We remark that the runtime of this algorithm might be exponential, as finding the edges of M_Γ might require solving the problem for the subgraph induced by Γ. However, we believe that there exists a polynomial-time algorithm for this task.

Open Problem 1 *Given a simple drawing of $K_{m,n}$, is there a polynomial-time algorithm to find a plane spanning tree contained in the drawing?*

For some relevant classes of simple drawings of $K_{m,n}$ we can efficiently compute shooting stars. This is the case of outer drawings. In [3] it was shown that these drawings contain shooting stars and this existential proof leads directly to a polynomial-time algorithm to find shooting stars in outer drawings. In Appendix A we show that *monotone drawings* of $K_{m,n}$, which are simple drawings in which all edges are x-monotone curves, admit an efficient algorithm for computing a shooting star. Fig. 7 shows an illustration. The idea is as follows. Let the sides of the bipartition be R and B and let v be the leftmost vertex (without loss of generality assume $r \in R$). We first consider the star of r, which we denote by T. For each vertex $w \in R$ not in T we shoot two vertical rays, one up and one down. If only one of those vertical rays intersects T we connect w with the endpoint in B of the first intercepted edge. If both vertical rays intersect T we consider the endpoints in B of the first edge intercepted by the upwards and the downwards ray. We connect w with the horizontally closest one of the two. If neither of the rays intersects T we connect w with the horizontally closest vertex in B. In Appendix A we prove that this indeed constructs a shooting star and we show how to efficiently compute it.
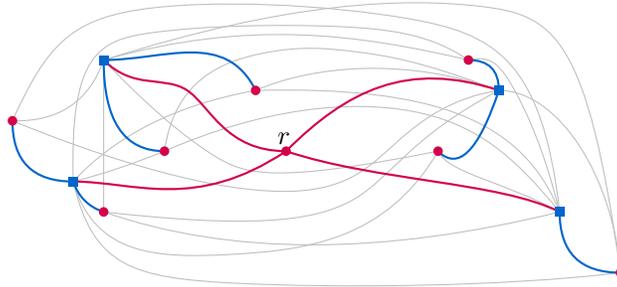

Fig. 7. An example of a shooting star rooted at r in a monotone drawing of $K_{9,5}$.

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A   Monotone Drawings

In this section we consider monotone drawings of $K_{m,n}$. We assume the information about the drawing is given as the rotation system (clockwise cyclic order of the edges around each vertex) together with the crossings sorted along each edge and the vertices sorted by x-coordinate. The *linear separator* of a vertex v in an x-monotone drawing is the vertical line going through the vertex. It separates the edges incident to v into the set of edges going to vertices left of v and the set of edges going to the vertices right of v. Note that the linear separators are implicitly given.

Theorem 2. *Given a monotone drawing of the complete bipartite graph $K_{m,n}$ we can compute a shooting star in linear time in the size of the input.*

Proof. Let the sides of the bipartition be R (red vertices) and B (blue vertices), and without loss of generality assume the leftmost vertex is $r \in R$. We denote the star of r by T. We construct a set M of edges such that $T \cup M$ forms a shooting star rooted at r. For each red vertex $w \in R$ not in T, we shoot two vertical rays, one up $(r\uparrow)$ and one down $(r\downarrow)$. There are three possible cases (see Fig. 8 for an illustration):
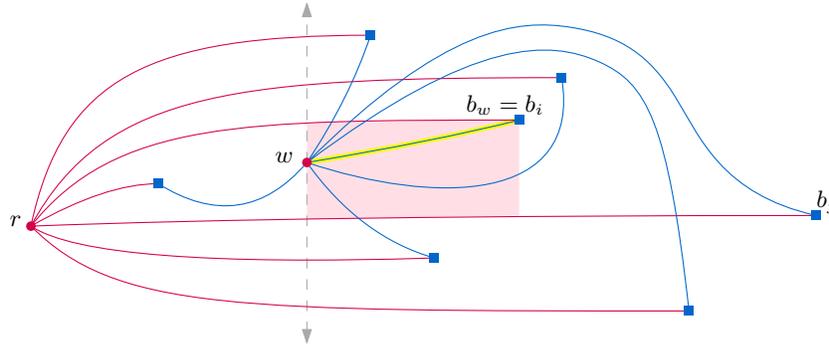

Fig. 8. Construction of the shooting star in a monotone drawing. The star of r (T) is shown in red. The edge incident to w picked by the algorithm is highlighted.

(i) If only one of those vertical rays intersects T, let rb_i be the edge in T producing the intersection that is closest to w, that is, b_i is the endpoint in B of the first intercepted edge in T. We define $b_w := b_i$ and add wb_w to M.

(ii) If both vertical rays intersect T we consider the endpoints b_i and b_j in B of the first edge intercepted by $r\uparrow$ and by $r\downarrow$, respectively. More precisely, rb_i (rb_j) is the edge in T producing the intersection with the upwards (resp. downwards) ray that is closest to w. Let $b_w \in \{b_i, b_j\}$ be the point that is horizontally closest to w. We add wb_w to M.

(iii) If neither of the rays intersects T, let b_i be the horizontally closest vertex in B. We define $b_w := b_i$ and add wb_w to M.

We next prove that $T \cup M$ is indeed a shooting star and we show how to efficiently compute it. The first part relies on the following claim:

Claim. Let $w \in R$ be a red vertex not in T. The edge wb_w does not cross T.

Proof. The proof for case (i) follows from the monotonicity property and the fact that the uncrossed ray, the part of the crossed ray until the first intersection, and the edge rb_w cannot be crossed by any edge in T. For case (ii), note that wb_w must be contained in the region bounded by rb_i, rb_j, and the vertical lines through w and b_w. The boundary of this region cannot be crossed by any edge in T; see the shaded region in Fig. 8 for an illustration. For case (iii), since r is the leftmost vertex, b_i is horizontally closer to w than r. Thus, the horizontally closest point to w in T is $b_w = b_i$ and the statement follows immediately from monotonicity. $\square$

To prove that $T \cup M$ is indeed a shooting star it remains to show that no two edges in M cross each other. Consider a red vertex $w \in R$ in T and the edge wb_w. By construction, the first intersection of a vertical ray up (or vertical ray down) from any point in the edge wb_w with edges of T is with the same edge if any. This means that shooting a ray from any point in the edge wb_w we find the same situation as the one that defines the case: If w falls under case (i), the vertically closest edge in T above or below wb_w is, at any point, rb_i and no other edge from T lies on the opposite side; if w falls under case (ii), the vertically closest edges in T above and below wb_w are, at any point, rb_i and rb_j, respectively; and if w falls under case (iii), no edge from T is above or below wb_w at any point. This implies that no two edges in M can cross, since by the definition of simple drawings incident edges do not cross.

For the algorithmic part, note that if w fall under case (iii) this is easy to detect and wb_w is easy to compute just using the horizontal sorting of the points (and the existence of crossings with T). Otherwise, we consider the edges incident to w on the right side of the linear separator sorted clockwise around w (starting the sweeping from a vertical up direction). The edge wb_w is either the first or the last such edge that does not cross any edge from T. More precisely, among those it is the one with the leftmost blue endpoint. This allows to efficiently compute M and therefore the shooting star. $\square$