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On the metric upper density of Birkhoff sums for irrational rotations

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Abstract
This article examines the value distribution of $S_N(f,\alpha) := \sum_{n=1}^{N} f(n\alpha)$ for almost every α where $N \in \mathbb{N}$ is ranging over a long interval and f is a 1-periodic function with discontinuities or logarithmic singularities at rational numbers. We show that for N in a set of positive upper density, the order of $S_N(f,\alpha)$ is of Khintchine-type, unless the logarithmic singularity is symmetric. Additionally, we show the asymptotic sharpness of the Denjoy–Koksma inequality for such f, with applications in the theory of numerical integration. Our method also leads to a generalized form of the classical Borel–Bernstein Theorem that allows very general modularity conditions.

Keywords: diophantine approximation, metric theory of continued fractions, irrational circle rotation, discrepancy theory, Birkhoff sums, uniform distribution modulo 1
Mathematics Subject Classification numbers: 11K60, 37E10, 11K50

1. Introduction and main results

Let $f : \mathbb{R} \to \mathbb{R}$ be 1-periodic with $\int_{[0,1)} |f(x)|\, dx < \infty$ and $q \in \mathbb{R}$. In this article, the object of our interest is

$$S_N(f,\alpha,q) := \sum_{n=1}^{N} f(n\alpha + q) - N \int_{[0,1)} f(x)\, dx,$$

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which is known as a Birkhoff sum of the irrational circle rotation. We consider the temporal value distribution along a single orbit of $S_N(f,\alpha,q)$, that is, we fix some initial point q and a rotation parameter α, and examine the value distribution of $\{S_N(f,\alpha,q) : 1 \leqslant N \leqslant M\}$, as $M \to \infty$ for (Lebesgue-) almost every α.

Since the irrational rotation together with the Lebesgue measure is an ergodic system for all irrational α, Birkhoff's ergodic theorem implies that for 1-periodic $f \in L^1([0,1))$ and almost every q, we have $|S_N(f,\alpha,q)| = o(N)$. If the Fourier coefficients of $f \sim \sum_{n\in\mathbb{Z}} c_n e(nx)$ decay at rate $c_n = O(1/n^2)$ (which holds in particular for $f \in C^2$), then $S_N(f,\alpha,q)$ is bounded for almost every α and all $q \in \mathbb{R}$ (see [15, 20]). Thus, the interesting functions to consider are the functions that lack smoothness, in particular functions that have discontinuities or singularities.

In this article, we examine $S_N(f,\alpha,q)$ where $q \in \mathbb{Q}$ and all non-smooth points of f lie at rational numbers. The first class of functions we define is the following (compare to, e.g. [14, 15]).

Definition 1.1 (Piecewise smooth functions with rational discontinuities). We call a 1-periodic function $f : \mathbb{R} \to \mathbb{R}$ a *piecewise smooth function with rational discontinuities* if there exist $\nu \geqslant 1$ and $0 \leqslant x_1 < \ldots < x_\nu < 1$ with $x_i \in \mathbb{Q}, 1 \leqslant i \leqslant \nu$ such that the following properties hold:

- f is differentiable on $[0,1) \setminus \{x_1,\ldots,x_\nu\}$.
- $f'\big|_{[0,1)}$ extends to a function of bounded variation on $[0,1)$.
- There exists an $i \in \{1,\ldots,\nu\}$ such that $\lim_{\delta\to 0}[f(x_i-\delta) - f(x_i+\delta)] \neq 0$.

This class of functions contains several important representatives such as the sawtooth function $f(x) = \{x\} - 1/2$ and the local discrepancy functions with rational endpoints $f(x) = \mathbb{1}_{[a,b]}(\{x\}) - (b-a), a, b \in \mathbb{Q}$. These functions are not only of interest in Discrepancy theory (see [7, 8, 33]), but are closely related to the theory of 'deterministic random walks' (see, e.g. [1, 6]).

For these local discrepancy functions, a classical theorem of Kesten [25] shows that $|S_N(f,\alpha,q)|$ is unbounded, since $b-a \notin \mathbb{Z}+\alpha\mathbb{Z}$ for irrational α. In addition to considering essentially smooth f, we also examine functions with logarithmic singularities at rational numbers, a class of functions that falls in the framework considered in [13].

Definition 1.2 (Smooth functions with rational logarithmic singularity). We call a 1-periodic function $f : \mathbb{R} \to \mathbb{R}$ with $\int_{[0,1)} f(x)\,\mathrm{d}x = 0$ a *smooth function with rational logarithmic singularity* if there exist constants $c_1, c_2 \in \mathbb{R}$, a 1-periodic function $t : \mathbb{R} \to \mathbb{R}$ with bounded variation on $[0,1)$ and $x_1 \in \mathbb{Q}$ such that

$$f(x) = \begin{cases} c_1 \log(\|x-x_1\|) + c_2 \log(\{x-x_1\}) + t(x), & \text{if } x \not\equiv x_1 \pmod 1 \\ t(x), & \text{if } x \equiv x_1 \pmod 1. \end{cases}$$

Here and throughout the paper, $\{.\}$ denotes the fractional part and $\|.\|$ denotes the distance to the nearest integer (for a proper definition see section 2.1).

If $c_2 = 0$ and $c_1 \neq 0$, we call the singularity **symmetric**. If $c_2 \neq 0$, we call it **asymmetric**.

We examine the maximal and typical oscillations of $S_N(f,\alpha,q)$ for Lebesgue almost every α where $q \in \mathbb{Q}$ and f is either of the form as in definition 1.1 or definition 1.2. Our methods give rise to results in two different directions that are elaborated in detail below.

1.1. Khintchine-type upper density results

Let us recall the classical Khintchine Theorem from the metric theory of Diophantine approximation. If $(\psi(q))_{q\in\mathbb{N}}$ is a non-negative sequence, and $q\psi(q)$ is decreasing, then for almost all α, the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q\psi(q)}$$

has infinitely many integer solutions $(p,q) \in \mathbb{Z}\times\mathbb{N}$ if and only if $\sum_{k=1}^{\infty} \frac{1}{\psi(k)}$ diverges.

Such a convergence-divergence criterion, often called Khintchine-type result, appears in many different statements that deal with the metric theory of Diophantine approximation and the closely related theory of continued fractions. Classical results of this form are, among others, the Borel–Bernstein Theorem (see section 2) and another theorem of Khintchine [26] on the discrepancy of the Kronecker sequence. Recall that the discrepancy of a sequence $(x_n)_{n\in\mathbb{N}}$ in the unit interval is defined as

$$D_N((x_n)_{n\in\mathbb{N}}) := \sup_{0\leqslant a<b\leqslant 1} \left| \frac{|\{1\leqslant n\leqslant N : x_n \in [a,b)\}|}{N} - (b-a) \right|.$$

By [26], for an increasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$, one has $D_N((n\alpha)_{n\in\mathbb{N}}) \gg \psi(\log N) + \log N \log\log N$ infinitely often if and only if $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$.

Such a Khintchine-type behaviour was also discovered for a certain Birkhoff sum arising from the logarithm of the Sudler product $\prod_{r=1}^{N} 2|\sin(\pi r\alpha)|$. The analysis of this product led to many interesting developments in various areas in mathematics in the last decades, see, e.g. [2–5, 18, 28, 30]. Lubinsky [29] showed that if $f(x) = \log(2|\sin(\pi x)|)$, then, for almost every α and every monotone increasing $\psi:\mathbb{R}_+\to\mathbb{R}_+$ with $\liminf_{k\to\infty} \frac{\psi(k)}{k\log k} = \infty$, the inequalities

$$S_N(f,\alpha,0) \geqslant \psi(\log N), \quad S_N(f,\alpha,0) \leqslant -\psi(\log N), \tag{1}$$

hold for infinitely many N, if and only if $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$. This result was refined by Borda [10] in the following way: Recall that for a set of integers $A\subseteq\mathbb{N}$, we define its upper density to be $\limsup_{M\to\infty} \frac{|A_M|}{M}$ where $A_M := \{N\leqslant M : N\in A\}$. It was shown in [10] that in the diverging case, both inequalities (1) hold on a set of positive upper density. This was further improved in [19], where it was shown that the actual upper density equals 1. Note that $f(x) = \log(2|\sin(\pi x)|)$ does have a single symmetric logarithmic singularity at 0. As pointed out in [13], the behaviour of Birkhoff sums with f having a symmetric logarithmic singularity is expected to be similar to f being as in definition 1.1, since the decay of the Fourier coefficients is of the same order. Our first theorem supports this expected behaviour as we obtain a Khintchine-type result on the upper density of the same form for piecewise smooth functions with rational discontinuities.

Theorem 1. *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be a monotone increasing function, f a function as in definition 1.1, i.e. f is smooth up to finitely many discontinuities $0\leqslant x_1<\ldots<x_\nu<1$ at rationals, and $q\in\mathbb{Q}$. If $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$, then, for almost all $\alpha\in[0,1)$, both sets*

$$\{N\in\mathbb{N} : S_N(f,\alpha,q) \geqslant \psi(\log N)\}, \qquad \{N\in\mathbb{N} : S_N(f,\alpha,q) \leqslant -\psi(\log N)\} \tag{2}$$

have positive upper density. If $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} < \infty$*, there exists a constant* $c > 0$ *such that, for almost all* $\alpha \in [0,1)$*, the set*

$$\{N \in \mathbb{N} : |S_N(f,\alpha,q)| \geqslant \psi(\log N) + c \log N \log\log N\}$$

is finite.

Theorem 1 shows the following interesting behaviour: Let ψ be monotonically increasing and $\liminf_{k\to\infty} \frac{\psi(k)}{k \log k} = \infty$. As soon as there are infinitely many $N \in \mathbb{N}$ such that $|S_N(f,\alpha,q)| \geqslant \psi(\log N)$, then immediately a 'positive proportion' of all natural numbers satisfies this property.

In section 3, we prove theorem 7, which is a slightly stronger result than theorem 1. It turns out that for many classical functions that fall in the framework of definition 1.1, the sets (2) have actually upper density 1.

Corollary 2. *Let* ψ *and* q *be as in theorem* 1 *with* $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$ *. Then, for almost all* $\alpha \in [0,1)$*, the sets in* (2) *have upper density 1 for the following functions f:*

- *The classical saw-tooth function* $f(x) = \{x\} - 1/2$*.*
- *The local discrepancy functions with rational endpoints* $f(x) = \mathbb{1}_{[a,b]}(\{x\}) - (b-a)$*. In particular, this means that the set* $\{N \in \mathbb{N} : D_N((n\alpha)_{n\in\mathbb{N}}) \geqslant \psi(\log N)\}$ *has upper density 1, improving the result in* [26].
- *the* $1/2$*-discrepancy function* $f(x) = \mathbb{1}_{[0,1/2)}(\{x\}) - \mathbb{1}_{[1/2,1)}(\{x\})$*, which is intensively studied in, e.g.* [31, 32].

Naturally, the question arises whether a similar behaviour can be expected for functions with logarithmic singularities. Note that in many applications (see for example [16] and the references therein), one is not only interested in symmetric, but also asymmetric singularities. In the next statement we show that for functions f with an asymmetric singularity, there is an analogue of theorem 1 with an additional scaling factor of $\log N$.

Theorem 3. *Let f be as in definition* 1.2*. Then, for any non-decreasing* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *and for any* $q \in \mathbb{Q}$*, the following holds:*

(i) *If the logarithmic singularity is asymmetric and* $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$*, then, for almost every* $\alpha \in [0,1)$*, we have that both sets*

$$\{N \in \mathbb{N} : S_N(f,\alpha,q) \geqslant \log N \psi(\log N)\}, \qquad \{N \in \mathbb{N} : S_N(f,\alpha,q) \leqslant -\log N \psi(\log N)\} \tag{3}$$

have upper density 1. If $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} < \infty$*, then there exists a constant* $c > 0$ *such that, for almost all* $\alpha \in [0,1)$*, the set*

$$\{N \in \mathbb{N} : |S_N(f,\alpha,q)| \geqslant \log N(\psi(\log N) + c \log N \log\log N)\}$$

is finite.

(ii) *If the logarithmic singularity is symmetric, then for almost every* $\alpha \in [0,1)$*, we have*

$$|S_N(f,\alpha,q)| \ll (\log N)^2 \log\log N.$$

Theorem 3 shows the surprising fact that one should expect a completely different oscillation between the Birkhoff sums of functions with symmetric and asymmetric singularity. To see this, we note that $\psi:\mathbb{R}_+\to\mathbb{R}_+$ with $\psi(x):=x\log(x+1)\log\log(x+10)$ satisfies $\sum_{k=1}^{\infty}\frac{1}{\psi(k)}=\infty$ and thus, for a function f with an asymmetric logarithmic singularity, the sets in (3) have upper density 1. However, the same sets are finite if the underlying Birkhoff sum is generated by a function g with symmetric logarithmic singularity. This is because theorem 3 (ii) tells us that

$$|S_N(f,\alpha,q)|\ll(\log N)^2\log\log N=o(\log N\psi(\log N)).$$

Remark 1.3.

- One can clearly see from the proof of theorem 3(i) that it is possible to generalize the result to functions of the form $f(x)=f_1(x)+f_2(x)+f_3(x)$, where $f_1(x)$ is a smooth function with asymmetric singularity at a rational number x_1, f_2 is a smooth function with finitely many symmetric logarithmic singularities at rational numbers $x_2,x_3,\dots,x_n$ and f_3 is a piecewise smooth function with finitely many discontinuities.
- The actual maximal oscillation of Birkhoff sums with a symmetric logarithmic singularity for generic α remains open. Using [10, proposition 12], the result of [29] and f being a smooth 1-periodic function with its symmetric logarithmic singularity located at 0, we have that, for almost every $\alpha\in[0,1)$, $|S_N(f,\alpha,0)|\ll\psi(\log N)+\log N\log\log N$ for any monotone ψ with $\sum_{k=1}^{\infty}\frac{1}{\psi(k)}<\infty$. Most probably, the bound $(\log N)^2\log\log N$ is not sharp for logarithmic singularities that are located at arbitrary rationals. We did not aim to achieve the best possible bound, but wanted to stress the different behaviour of symmetric and asymmetric logarithmic singularities. Possibly, the upper bound for the Birkhoff sum with symmetric logarithmic singularity coincides with the Khintchine-type behaviour in theorem 1. A proof of this would probably need to make use of delicate estimates on shifted cotangent sums and is beyond the scope of this paper.

1.2. Sharpness of the Denjoy–Koksma inequality

Recall the classical Denjoy–Koksma inequality (see, e.g. [20]): Let α be fixed and let p_n/q_n denote its nth convergent. If f is a 1-periodic function of bounded variation $\mathrm{Var}(f)$ on $[0,1)$, then, for any $q\in\mathbb{R}$,

$$|S_{q_n}(f,\alpha,q)|=\left|\sum_{k=0}^{q_n-1}f(k\alpha+q)-q_n\int_0^1 f(x)\,\mathrm{d}x\right|\leqslant\mathrm{Var}(f).$$

For $N=\sum_{i=0}^{K(N)-1}b_iq_i$ being its Ostrowski expansion (see section 2.2 for details), we immediately obtain the bound

$$|S_N(f,\alpha,q)|\leqslant\mathrm{Var}(f)\sum_{i=1}^{K(N)}a_i, \tag{4}$$

where $\alpha=[0;a_1,a_2,\dots]$ is the classical continued fraction expansion and $K(N)$ denotes the integer K such that $q_{K-1}\leqslant N<q_K$. It is natural to ask whether (4) is sharp for particular functions f. This was essentially already proven in [26] for both the classical saw-tooth function $\{x\}-1/2$ and the local discrepancy functions $\mathbb{1}_{[a,b]}(\{x\})-(b-a)$: for almost every α, there are infinitely many N where (4) can be reverted up to an absolute positive constant. However,

to the best of our knowledge, all results in this direction only show that this bound is essentially obtained for infinitely many N, but there is no statement about the frequency of those N. This is shown in the following theorem.

Theorem 4. *Let f be a function with finitely many discontinuities at rationals (see definition 1.1) and let $q \in \mathbb{Q}$. For fixed α and $N \in \mathbb{N}$, let $K(N)$ denote the integer K such that $q_{K-1} \leqslant N < q_K$. Then, for almost all $\alpha \in [0,1)$, both sets*

$$\left\{ N \in \mathbb{N} \mid S_N(f,\alpha,q) \gg \sum_{i=1}^{K(N)} a_i \right\}, \qquad \left\{ N \in \mathbb{N} \mid S_N(f,\alpha,q) \ll -\sum_{i=1}^{K(N)} a_i \right\}$$

have positive upper density. The implied constants depend on α, f and q.

Analogously to corollary 2, the following stronger version of theorem 4 can be obtained.

Corollary 5. *If $f(x) = \{x\} - 1/2$ or $f(x) = \mathbb{1}_{[a,b]}(\{x\}) - (b-a)$ we have the following: For almost all $\alpha \in [0,1)$ and any $0 < r < 1$, there exists a constant $C(r) = C(r,f,q) > 0$ such that both sets*

$$\left\{ N \in \mathbb{N} \,\middle|\, S_N(f,\alpha,q) \geqslant C(r) \sum_{i=1}^{K(N)} a_i \right\}, \qquad \left\{ N \in \mathbb{N} \,\middle|\, S_N(f,\alpha,q) \leqslant -C(r) \sum_{i=1}^{K(N)} a_i \right\}$$

have upper density at least r.

Remark 1.4. Note that in contrast to corollary 2, we cannot include $r = 1$ in corollary 5 since our method of proof gives $\lim_{r \to 1} C(r) = 0$. This is due to the fact that the convergence-divergence condition in the Khintchine formulation enables to hide a sufficiently large constant in the divergence condition of ψ (since for $\tilde{\psi}(k) := c\psi(k)$, $\sum_{k=1}^{\infty} \frac{1}{\tilde{\psi}(k)}$ converges if and only if $\sum_{k=1}^{\infty} \frac{1}{\psi(k)}$ does) , which is not possible to do in this setting.

Theorem 4 has consequences in the theory of numerical integration: Assuming that there are functions f such that the bound (4) is attained only along a very sparse subsequence $(N_k)_{k \in \mathbb{N}}$, one could hope with a randomized approach to hit this sequence very rarely. Then, one could generate by some (randomized) algorithm an increasing sequence of integers $(M_j)_{j \in \mathbb{N}}$ and consider an irrational α drawn uniformly at random from the unit interval. With high probability, one would expect $\left|S_{M_j}(f,a,q)\right| = o(\sum_{i=1}^{K(M_j)} a_i)$. Theorem 4 implies that such an approach will most likely fail. It shows that, almost surely, a positive proportion of those M_j satisfies

$$|S_{M_j}(f,a,q)| \gg \sum_{i=1}^{K(M_j)} a_i \gg \log M_j \log\log M_j.$$

This implies that every low-discrepancy sequence (such as the Kronecker sequences $(\{n\alpha\})_{n \in \mathbb{N}}$ where α is badly approximable) gives a better error bound in numerical integration, regardless of the support of the function f and the chosen algorithm to generate the sequence $(M_j)_{j \in \mathbb{N}}$.

Next, assume that f has a singularity, but is still integrable. The singularity makes an application of the classical Denjoy–Koksma inequality impossible since the variation of f is not bounded. However, we can still get a nontrivial bound on $S_N(f,\alpha,q)$ provided that the orbit $\{q + n\alpha : 1 \leqslant n \leqslant N\}$ stays away from the singularity.

Proposition 1.5 (Denjoy–Koksma inequality with singularity). *Let $N \in \mathbb{N}, q \in [0,1)$ arbitrary and let $N = \sum_{i=0}^{K(N)-1} b_i q_i$ be its Ostrowski expansion. Assume that f is a 1-periodic function with a single singularity in x_1. Let $x_1 \in A_N \subseteq [0,1)$, where A_N is an interval (modulo 1) such that $\{q + n\alpha - x_1\} \notin A_N$ for all $1 \leqslant n \leqslant N$. Then,*

$$|S_N(f,\alpha,q)| \ll \sup_{x \in [0,1) \setminus A_N} |f(x)| \sum_{i=0}^{K(N)-1} b_i + N \left| \int_{A_N} f(x)\,\mathrm{d}x \right|. \tag{5}$$

In particular, we have, for $K \in \mathbb{N}$,

$$\max_{1 \leqslant N \leqslant q_K} |S_N(f,\alpha,q)| \ll \sup_{x \in [0,1) \setminus A_{q_K}} |f(x)| \sum_{i=1}^{K} a_i + q_K \left| \int_{A_{q_K}} f(x)\,\mathrm{d}x \right|.$$

This bound is not new but was used already in, e.g. [21, 22, 27]. The statement follows immediately by defining

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in [0,1) \setminus A_N, \\ 0 & \text{otherwise,} \end{cases}$$

and applying the classical Denjoy–Koksma inequality (4) to $\tilde{f}$.

The following theorem shows that for asymmetric logarithmic singularities, the estimate (5) is also sharp in the sense of theorem 4, which in particular extends the consequences for numerical integration to functions with an asymmetric logarithmic singularity.

Theorem 6. *Let f be a 1-periodic smooth function with rational asymmetric logarithmic singularity in x_1 as in definition 1.2. For fixed $\alpha \in [0,1), q \in \mathbb{Q}$ and $N \in \mathbb{N}$, we denote by K(N) the integer K such that $q_{K-1} \leqslant N < q_K$. Further, let $A_N = (x_1 - g(N), x_1 + g(N))$ be an interval with $g(N) = \min_{n \leqslant N} \|n\alpha + q - x_1\|$. Then, for almost all $\alpha \in [0,1)$ and any $0 < r < 1$, there exists a constant $C(r) = C(r,f,q) > 0$ such that both sets*

$$\left\{ N \in \mathbb{N} \mid S_N(f,\alpha,q) \geqslant C(r) \sup_{x \in [0,1) \setminus A_N} |f(x)| \sum_{i=1}^{K(N)} a_i + N \left| \int_{A_N} f(x)\,\mathrm{d}x \right| \right\},$$

$$\left\{ N \in \mathbb{N} \mid S_N(f,\alpha,q) \leqslant -C(r) \sup_{x \in [0,1) \setminus A_N} |f(x)| \sum_{i=1}^{K(N)} a_i - N \left| \int_{A_N} f(x)\,\mathrm{d}x \right| \right\}$$

have upper density of at least r.

2. Prerequisites

2.1. Notation

Given two functions $f, g : (0,\infty) \to \mathbb{R}$, we write $f(t) = O(g(t)), f \ll g$ or $g \gg f$ if $\limsup_{t \to \infty} \frac{|f(t)|}{|g(t)|} < \infty$. Any dependence of the value of the limes superior above on potential parameters is denoted by appropriate subscripts. For two sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ with $b_k \neq 0$ for all $k \in \mathbb{N}$, we write $a_k \sim b_k, k \to \infty$, if $\lim_{k \to \infty} \frac{a_k}{b_k} = 1$. Given a real number $x \in \mathbb{R}$, we write $\{x\} = x - \lfloor x \rfloor$ for the fractional part of x and $\|x\| = \min\{|x - k| : k \in \mathbb{Z}\}$ for the distance of x to its nearest integer. We denote the characteristic function of a set A by $\mathbb{1}_A$ and understand the value of empty sums as 0. We denote the cardinality of a set $A \subseteq \mathbb{N}$ as $|A|$. For shorter notation, we write $S_N(f,\alpha) := S_N(f,\alpha,0)$. Let $\sigma\big(X_1, X_2, \dots\big)$ denote the σ-algebra generated by random variables $X_1, X_2, \dots$.

2.2. Continued fractions

In this subsection, we collect all classical results from the theory of continued fractions that we need to prove our main results. Every irrational $\alpha \in \mathbb{R}$ has a unique infinite continued fraction expansion denoted by $[a_0; a_1, a_2, \dots]$ with convergents $p_k/q_k := [a_0; a_1, \dots, a_k]$ that satisfy the recursions

$$p_{k+1} = p_{k+1}(\alpha) = a_{k+1}(\alpha) p_k + p_{k-1}, \qquad q_{k+1} = q_{k+1}(\alpha) = a_{k+1}(\alpha) q_k + q_{k-1}, \quad k \in \mathbb{N},$$

with initial values $p_0 = a_0$, $p_1 = a_1 a_0 + 1$, $q_0 = 1$, $q_1 = a_1$. For the sake of brevity, we just write a_k, p_k, q_k, although these quantities depend on α. We know that p_k/q_k are good approximations for α and satisfy the inequalities

$$\frac{1}{q_{k+1} + q_k} \leqslant \delta_k := |q_k \alpha - p_k| \leqslant \frac{1}{q_{k+1}}, \quad k \in \mathbb{N}.$$

Fixing an irrational α, the Ostrowski expansion of a non-negative integer N is the unique representation

$$N = \sum_{\ell=0}^{K-1} b_\ell q_\ell,$$

where $b_{K-1} \neq 0$, $0 \leqslant b_0 < a_1$, $0 \leqslant b_\ell \leqslant a_{\ell+1}$ for $1 \leqslant \ell \leqslant K-1$ and if $b_\ell = a_\ell$, then $b_{\ell-1} = 0$.

So far all statements can be made for arbitrary irrational numbers, but since this article considers the almost sure behaviour, we make use of the well-studied area of the metric theory of continued fractions. We state several classical results below which hold for almost every $\alpha \in \mathbb{R}$. We will use them frequently in the proofs of our results.

- The Borel–Bernstein theorem ([9], see also [11] for a more modern formulation without monotonicity condition): For any non-negative function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, we have

$$|\{k \in \mathbb{N} : a_k > \psi(k)\}| \text{ is } \begin{cases} \text{infinite} & \text{if } \sum_{k=0}^{\infty} \frac{1}{\psi(k)} = \infty, \\ \text{finite} & \text{if } \sum_{k=0}^{\infty} \frac{1}{\psi(k)} < \infty. \end{cases} \tag{6}$$

- (Diamond and Vaaler [12]):

$$\sum_{\ell \leqslant K} a_\ell - \max_{\ell \leqslant K} a_\ell \sim \frac{K \log K}{\log 2}, \quad K \to \infty. \tag{7}$$

- (Khintchine and Lévy, see, e.g. [34, chapter 5, §9, theorem 1]):

$$\log q_k \sim \tfrac{\pi^2}{12 \log 2} k \text{ as } k \to \infty. \tag{8}$$

3. Functions with discontinuities

We start this section with a decomposition lemma that is of a similar form to [15, appendix A].

Lemma 3.1 (Decomposition lemma). *Let $f:\mathbb{R}\to\mathbb{R}$ be as in definition 1.1, i.e. f is piecewise smooth with (possible) discontinuities at $0\leqslant x_1<\ldots<x_\nu<1$. Let $g:\mathbb{R}\to\mathbb{R}$ be defined as*

$$g(x)=\left(\{x\}-\frac{1}{2}\right)\sum_{i=1}^{\nu}A_i+\sum_{i=1}^{\nu}c_i\left(\mathbb{1}_{[x_{i-1},x_i)}(x)-(x_i-x_{i-1})\right),$$

where $A_i:=\lim_{\delta\to 0}[f(x_i-\delta)-f(x_i+\delta)]$, $c_i:=\sum_{j=1}^{i}A_j$ and $x_0:=0$. Then, for any $N\in\mathbb{N}, q\in\mathbb{Q}$ and for almost every α, we have

$$S_N(f,\alpha,q)=S_N(g,\alpha,q)+O(1).$$

If f satisfies

$$\sum_{i=1}^{\nu}A_i\neq 0 \text{ or } \sum_{i=1}^{\nu}c_i(x_i-x_{i-1})\neq 0, \tag{9}$$

then also $f_y(x):=f(x+y)$, for any $y\in[0,1)$, satisfies (9).

Proof. We define $\varphi:\mathbb{R}\to\mathbb{R}$ as $\varphi(x):=f(x)-\sum_{i=1}^{\nu}A_i\left(\{x-x_i\}-\frac{1}{2}\right)$. Further, we see $\{x-x_i\}-\frac{1}{2}=\{x\}-\frac{1}{2}-x_i+\mathbb{1}_{[0,x_i)}(x)$ and hence

$$\begin{aligned}f(x)&=\sum_{i=1}^{\nu}A_i\left(\{x-x_i\}-\frac{1}{2}\right)+\varphi(x)\\&=\left(\{x\}-\frac{1}{2}\right)\sum_{i=1}^{\nu}A_i+\sum_{i=1}^{\nu}A_i\left(\mathbb{1}_{[0,x_i)}(x)-x_i\right)+\varphi(x)\\&=\left(\{x\}-\frac{1}{2}\right)\sum_{i=1}^{\nu}A_i+\sum_{i=1}^{\nu}c_i\left(\mathbb{1}_{[x_{i-1},x_i)}(x)-(x_i-x_{i-1})\right)+\varphi(x)\\&=g(x)+\varphi(x).\end{aligned}$$

By the choice of f, there exists a function ψ that is differentiable with ψ' being of bounded variation and $\psi\big|_{[0,1)\setminus\{x_1,\ldots,x_\nu\}}=\varphi\big|_{[0,1)\setminus\{x_1,\ldots,x_\nu\}}$. By the properties of ψ, we get $S_N(\psi,\alpha,q)=O(1)$ (see appendix A in [15]). Since α is irrational and $q\in\mathbb{Q}$, we have $\psi(n\alpha+q)=\varphi(n\alpha+q)$ for any $n\in\mathbb{N}$ and thus,

$$S_N(f,\alpha,q)=S_N(g,\alpha,q)+S_N(\psi,\alpha,q)=S_N(g,\alpha,q)+O(1),$$

which proves the first part of the statement. For the second part, one sees immediately that $\sum_{i=1}^{\nu}A_i$ is invariant under translation. By a slightly longer, but elementary calculation, one finds that under the assumption of $\sum_{i=1}^{\nu}A_i=0$, also $\sum_{i=1}^{\nu}c_i(x_i-x_{i-1})\neq 0$ is invariant under translation. □

The definition of the quantities A_i, c_i in lemma 3.1 allows us to state stronger, but more technically involved versions of theorem 1 respectively theorem 4. This refinement will also immediately imply corollaries 2 and 5.

Theorem 7. *Let* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a monotonically increasing function with* $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$ *and let* $f : \mathbb{R} \to \mathbb{R}$ *be as in definition 1.1, i.e. f is essentially smooth with (possible) discontinuities at finitely many rationals* $0 \leqslant x_1 < \ldots < x_\nu < 1$. *Let* $A_i = \lim_{\delta \to 0} [f(x_i - \delta) - f(x_i + \delta)]$, $c_i = \sum_{j=1}^{i} A_j$ *for* $i \in \{1, \ldots, \nu\}$ *and set* $x_0 = 0$. *If* $\sum_{i=1}^{\nu} A_i \neq 0$ *or* $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) \neq 0$, *for almost all* $\alpha \in [0, 1)$ *and all* $q \in \mathbb{Q}$, *both sets*

$$\{N \in \mathbb{N} \mid S_N(f, \alpha, q) \geqslant \psi(\log N)\}, \qquad \{N \in \mathbb{N} \mid S_N(f, \alpha, q) \leqslant -\psi(\log N)\} \tag{10}$$

have upper density 1.

If $\sum_{i=1}^{\nu} A_i = 0$ *and* $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) = 0$, *for almost all* $\alpha \in [0, 1)$ *and all* $q \in \mathbb{Q}$, *both sets in (10) have positive upper density.*

Theorem 8. *Let* ψ, f, ν, A_i *and* c_i *for* $i \in \{1, \ldots, \nu\}$ *be as in theorem 7. If* $\sum_{i=1}^{\nu} A_i \neq 0$ *or* $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) \neq 0$, *for almost all* $\alpha \in [0, 1)$ *and all* $q \in \mathbb{Q}$, *we have the following:*

For any $0 < r < 1$, *there exists a constant* $C(r) = C(r, f, q) > 0$ *such that both sets*

$$\left\{ N \in \mathbb{N} \,\middle|\, S_N(f, \alpha, q) \geqslant C(r) \sum_{i=1}^{K(N)} a_i \right\}, \qquad \left\{ N \in \mathbb{N} \,\middle|\, S_N(f, \alpha, q) \leqslant -C(r) \sum_{i=1}^{K(N)} a_i \right\}$$

have upper density of at least r. Here K(N) denotes the integer such that $q_{K(N)-1} \leqslant N < q_{K(N)}$.

If $\sum_{i=1}^{\nu} A_i = 0$ *and* $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) = 0$, *for almost all* $\alpha \in [0, 1)$, *there exist constants* $r_0 = r_0(f, \alpha, q) > 0$ *and* $C = C(f, \alpha, q) > 0$ *such that both sets*

$$\left\{ N \in \mathbb{N} \,\middle|\, S_N(f, \alpha, q) \geqslant C \sum_{i=1}^{K(N)} a_i \right\}, \qquad \left\{ N \in \mathbb{N} \,\middle|\, S_N(f, \alpha, q) \leqslant -C \sum_{i=1}^{K(N)} a_i \right\}$$

have upper density of at least r_0.

Naturally, the question arises whether the conditions $\sum_{i=1}^{\nu} A_i \neq 0$ or $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) \neq 0$ give an exact characterization of functions that satisfy the statements in theorems 7 and 8. We show that these assumptions are not necessary, but without any condition on the interplay of the location of the discontinuities and their jump heights, one cannot hope to achieve upper density 1. In fact, we provide two classes of functions that, in general, do not satisfy $\sum_{i=1}^{\nu} A_i \neq 0$ or $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) \neq 0$; for one class, the sets

$$\{N \in \mathbb{N} \mid S_N(f, \alpha, q) \geqslant \psi(\log N)\} \qquad \{N \in \mathbb{N} \mid S_N(f, \alpha, q) \leqslant -\psi(\log N)\}$$

both have upper density 1 for any $q \in \mathbb{Q}$. However for the other class of functions, this fails to hold. The proofs of both these statements (propositions 3.2 and 3.4) can be found in the appendix.

Proposition 3.2. *Let*

$$f(x) = \mathbb{1}_{\left[\frac{r_1}{s_1}, \frac{r_2}{s_2}\right]}(\{x\}) - \mathbb{1}_{\left[\frac{r_2}{s_2}, \frac{r_3}{s_3}\right]}(\{x\}), \quad r_i, s_i \in \mathbb{Z} \setminus \{0\}, \gcd(r_i, s_i) = 1, \quad r_1^2 + r_3^2 > 0. \tag{11}$$

Then, for every $q \in \mathbb{Q}$ *and almost all* $\alpha \in [0, 1)$, *both sets*

$$\{N \in \mathbb{N} \mid S_N(f, \alpha, q) \geqslant \psi(\log N)\}, \qquad \{N \in \mathbb{N} \mid S_N(f, \alpha, q) \leqslant -\psi(\log N)\} \tag{12}$$

have upper density 1.

Remark 3.3. Let $f(x) = \mathbb{1}_{[\frac{1}{4},\frac{1}{2}]}(\{x\}) - \mathbb{1}_{[\frac{1}{2},\frac{3}{4}]}(\{x\})$ which satisifes the assumptions of proposition 3.2. One can easily check that $\sum_{i=1}^{\nu} A_i = \sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) = 0$. Theorem 7 (and hence theorem 1) only shows that the sets in (2) have positive upper density. Proposition 3.2 shows that we have indeed upper density 1, meaning that the assumptions in theorem 7 are not sharp.

The following proposition shows that there are functions of the form as in definition 1.1 where the sets in (2) do not have upper density 1.

Proposition 3.4. *Let u, v, w be positive integers with $u < v < w/2$, $\gcd(u,w) = \gcd(v,w) = 1$ and let*

$$f(x) = \mathbb{1}_{[\frac{u}{w},\frac{v}{w}]}(\{x\}) - \mathbb{1}_{[1-\frac{v}{w},1-\frac{u}{w}]}(\{x\}).$$

Then, there exists a monotone increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$ and a constant $0 < \delta < 1$ such that, for almost every $\alpha \in [0,1)$, the set

$$\{N \in \mathbb{N} : S_N(f,\alpha) \geqslant \psi(\log N)\}$$

has upper density of at most $1 - \delta$.

Proof of theorems 1, 4 and Corollaries 2, 5. Corollary 2 follows immediately from theorem 7 since all functions considered in corollary 2 satisfy $\sum_{i=1}^{\nu} A_i \neq 0$ or $\sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) \neq 0$. Analogously, theorem 4 and corollary 5 follow directly from theorem 8. Theorem 7 implies the statement in theorem 1 for the case $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} = \infty$, so we are left with the case $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} < \infty$. By the Denjoy–Koksma inequality (4), we have

$$|S_N(f,\alpha,q)| \leqslant \mathrm{Var}(f) \left(\sum_{i=1}^{K(N)} a_i \right),$$

where $q_{K(N)-1} \leqslant N < q_{K(N)}$. We have $\sum_{i=1}^{K(N)} a_i = a_{K_0} + \sum_{i=1, i \neq K_0}^{K(N)} a_i$ with $K_0 = \arg\max_{i=1,\ldots,K(N)} a_i$. We define $\tilde{\psi}(k) := c_1 \psi(c_2 k)$ for constants $c_1, c_2 > 0$ specified later. Since ψ is monotone, it follows immediately that $\sum_{k=1}^{\infty} \frac{1}{\tilde{\psi}(k)} < \infty$. Thus, (6) implies that, for sufficiently large N, we have $a_{K_0} \leqslant \tilde{\psi}(K_0)$. Moreover, by (7) there exists an absolute constant $\tilde{c} > 0$ such that $\sum_{i=1, i \neq K_0}^{K(N)} a_i \leqslant \tilde{c} K(N) \log K(N)$. Together, we get

$$\begin{aligned} |S_N(f,\alpha,q)| &\leqslant \mathrm{Var}(f) \left(\tilde{\psi}(K_0) + \tilde{c} K(N) \log K(N) \right) \\ &\leqslant \psi(\log N) + c \log N \log\log N, \end{aligned}$$

where we choose c_1 and c_2 such that

$$\mathrm{Var}\,(f)\,\tilde{\psi}\,(K_0) \leqslant \psi\,(\log N)$$

and $c > 0$ in a way that $\tilde{c}\mathrm{Var}(f)K(N)\log K(N) \leqslant c\log N\log\log N$. Thus, there exists a constant $c > 0$ such that, for almost all $\alpha \in [0,1)$, there are only finitely many $N \in \mathbb{N}$ with

$$|S_N\,(f,\alpha,q)| \geqslant \psi\,(\log N) + c\log N\log\log N.$$

□

3.1. Heuristic of the proofs

We will briefly line out the main ideas of the proof of theorems 7 and 8. One of the core tools we are using is the well-known result in metric number theory that for almost every $\alpha \in [0,1)$, there are infinitely many $K \in \mathbb{N}$ such that a_K dominates the sum of the preceding partial quotients, that is, $\sum_{i=1}^{K-1} a_i = o(a_K)$. Here, K will always satisfy this property. For $q_{K-1} < N < q_K$ and $N = b_{K-1}q_K + N'$, $N' < q_{K-1}$, we first get rid of $S_{N'}(f,\alpha)$ by an application of the Denjoy–Koksma inequality. The rest of the proof is to show that essentially $S_{b_{K-1}q_{K-1}}(f,\alpha) \gg a_K$ for most N, which then implies morally both theorems 7 and 8 by an application of the Borel–Bernstein Theorem. As $\{n\alpha\}_{n=1}^{q_{K-1}}$ is close to being uniformly distributed in the unit interval, it is natural to analyze

$$S_{(b+1)q_{K-1}}\,(f,\alpha) - S_{bq_{K-1}}\,(f,\alpha) = \sum_{n=1}^{q_{K-1}} f(n\alpha + bq_{k-1}\alpha) - q_{K-1}\int_0^1 f(x)\,\mathrm{d}x$$

for every $b \leqslant b_{K-1} - 1$. For f of the form considered in theorems 7 and 8 and $\frac{b}{a_k} \in [c',c]$ for some $0 \leqslant c' < c < 1$, one obtains

$$(-1)^K \left(S_{(b+1)q_{K-1}}\,(f,\alpha) - S_{bq_{K-1}}\,(f,\alpha)\right) \gtrapprox d$$

with $d > 0$, provided that q_{K-1} satisfies some congruence relation that depends on the location of the discontinuities (lemma 3.8). If $\frac{b}{a_k} \in [0,d')$ we get

$$(-1)^K \left(S_{(b+1)q_{K-1}}\,(f,\alpha) - S_{bq_{K-1}}\,(f,\alpha)\right) \gtrapprox 0.$$

Thus, for $\delta > 0$ and N with $(\delta + c')a_K < b_{K-1}(N) < ca_K$, we have

$$(-1)^K S_N\,(f,\alpha) \approx S_{b_{K-1}q_{K-1}}\,(f,\alpha) \gtrapprox \delta d a_K \gg \sum_{i=1}^{K} a_i.$$

Letting $\delta \to 0$, we see that the desired inequality holds on a proportion of at least $\frac{c-c'}{c}$ many elements among $\{1,\ldots,ca_K\}$. By a refinement of the Borel–Bernstein Theorem, we ensure that there are infinitely many odd respectively even K that both satisfy $\sum_{i=1}^{K-1} a_i = o(a_K)$ and this certain congruence relation on q_{K-1}. Thus, we obtain the positive upper density by considering the subsequence ca_K where the K are chosen out of this infinite set, giving upper density of at least $\frac{c-c'}{c} > 0$.

Under the assumptions $\sum_{i=1}^{\nu} A_i \neq 0$ or $\sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) \neq 0$, it is possible to prove that we can choose in the above discussion $c' = 0$ (lemma 3.7), which leads to the result with upper density 1.

3.2. Preparatory lemmas

Before we come to the proof of theorems 7 and 8, we need a few auxiliary results. The first lemma treats the sawtooth function, which in view of lemma 3.1 is a building block for the decomposition of f.

Lemma 3.5. *Let $\alpha \in [0,1)$ be an irrational number and $f(x) = \left(\{x\} - \frac{1}{2}\right)$. If a_K is sufficiently large, then, for $0 \leqslant b \leqslant a_K$, we have*

$$S_{bq_{K-1}}(f,\alpha) = (-1)^K \frac{b}{2}\left(1 - \frac{b}{a_K}\right) + O(1).$$

Proof. We only consider the case where K is odd since the other case can be treated analogously. We thus have

$$\frac{1}{q_K + q_{K-1}} \leqslant \delta_{K-1} = q_{K-1}\alpha - p_{K-1} \leqslant \frac{1}{q_K}.$$

Since $q_K = a_K q_{K-1} + q_{K-2}$, we get the asymptotic expression $\delta_{K-1} = \frac{1}{q_{K-1}a_K(1+\varepsilon_K)}$, where $\varepsilon_K = O(a_K^{-1})$. We obtain

$$\begin{aligned}
S_{bq_{K-1}}(f,\alpha) &= \sum_{n=1}^{bq_{K-1}} \left(\{n\alpha\} - \frac{1}{2}\right) \\
&= \sum_{u=0}^{b-1}\sum_{n=1}^{q_{K-1}} \left(\{n\alpha + uq_{K-1}\alpha\} - \frac{1}{2}\right) \\
&= \sum_{u=0}^{b-1}\sum_{n=1}^{q_{K-1}} \left(\left\{n\frac{p_{K-1}}{q_{K-1}} + \frac{n}{q_{K-1}}\delta_{K-1} + u\delta_{K-1}\right\} - \frac{1}{2}\right) \\
&= \sum_{u=0}^{b-1}\sum_{n=1}^{q_{K-1}} \left(\left\{n\frac{p_{K-1}}{q_{K-1}} + O\left(\frac{1}{q_{K-1}a_K}\right) + \frac{u}{q_{K-1}a_K(1+\varepsilon_K)}\right\} - \frac{1}{2}\right) \\
&= \sum_{u=0}^{b-1}\sum_{n=0}^{q_{K-1}-1} \left(\frac{n}{q_{K-1}} + O\left(\frac{1}{q_{K-1}a_K}\right) + \frac{u}{q_{K-1}a_K(1+\varepsilon_K)}\right) - \frac{bq_{K-1}}{2}.
\end{aligned}$$

In the second last line we used that $\delta_{K-1} = \frac{1}{q_{K-1}a_K(1+\varepsilon_K)}$ and we employed $n \leqslant q_{K-1}$ in the inner summation. Further, we used that $\gcd(p_{K-1}, q_{K-1}) = 1$ and hence the remainders of np_{K-1} modulo q_{K-1} are precisely the integers $n = 0, \ldots, q_{K-1} - 1$. Finally, we omitted the fractional part $\{\cdot\}$, since $\frac{n}{q_{K-1}} + O\left(\frac{1}{q_{K-1}a_K}\right) + \frac{u}{q_{K-1}a_K(1+\varepsilon_K)} < 1$ holds for all n and u, provided a_K is sufficiently large. This leads to

$$\sum_{u=0}^{b-1}\sum_{n=0}^{q_{K-1}-1}\left(\frac{n}{q_{K-1}}+O\left(\frac{1}{q_{K-1}a_K}\right)+\frac{u}{q_{K-1}a_K(1+\varepsilon_K)}\right)-\frac{bq_{K-1}}{2}$$
$$=\frac{q_{K-1}(q_{K-1}-1)b}{2q_{K-1}}+O(1)+\frac{b(b-1)}{2a_K(1+\varepsilon_K)}-\frac{bq_{K-1}}{2}$$
$$=-\frac{b}{2}+\frac{b^2}{2a_K(1+\varepsilon_K)}+O(1)$$
$$=-\frac{b}{2}\left(1-\frac{b}{a_K}\right)+O(1),$$

where we also made use of the estimate $b\leqslant a_K$.

□

The next lemma treats the local discrepancy function, which in view of lemma 3.1 is also a building block for the decomposition of f.

Lemma 3.6. *Let $f=\mathbb{1}_{\left[\frac{r_1}{s_1},\frac{r_2}{s_2}\right]}$ with $r_i,s_i\in\mathbb{N}$ and $\gcd(r_1,s_1)=\gcd(r_2,s_2)=1$. Further, let $\alpha=[0;a_1,a_2,\ldots]$ be fixed and let both $k\in\mathbb{N}$ and a_k be sufficiently large. Moreover, let $b\in\mathbb{N}$ such that $b\leqslant\frac{a_k}{2s_1s_2}$. Then, we have*

$$S_{bq_{k-1}}(f,\alpha)=b\left(\left\{\frac{r_1}{s_1}q_{k-1}\right\}+(-1)^{k-1}\mathbb{1}_{\{s_1|q_{k-1}\}}-\left\{\frac{r_2}{s_2}q_{k-1}\right\}-(-1)^{k-1}\mathbb{1}_{\{s_2|q_{k-1}\}}\right),$$

where $\mathbb{1}_{\{s_i|q_{k-1}\}}=1$ if $s_i|q_{k-1}$ and $\mathbb{1}_{\{s_i|q_{k-1}\}}=0$ if $s_i\nmid q_{k-1}$.

Proof. We show the statement only in the case where k is odd and $s_1|q_{k-1}$ and $s_2\nmid q_{k-1}$. The other cases can be treated analogously. Thus, we need to show that

$$S_{bq_{k-1}}(f,\alpha)=b\left(1-\left\{\frac{r_2}{s_2}q_{k-1}\right\}\right).$$

We recall that $S_{bq_{k-1}}(f,\alpha)=\left|\left\{1\leqslant n\leqslant bq_{k-1}:\{n\alpha\}\in\left[\frac{r_1}{s_1},\frac{r_2}{s_2}\right]\right\}\right|-bq_{k-1}\left(\frac{r_2}{s_2}-\frac{r_1}{s_1}\right)$. In the following, we use that for odd k, we have $0\leqslant\delta_{k-1}=q_{k-1}\alpha-p_{k-1}=\frac{1}{a_kq_{k-1}(1+o_{a_k}(1))}$. We obtain

$$\left|\left\{1\leqslant n\leqslant bq_{k-1}:\{n\alpha\}\in\left[\frac{r_1}{s_1},\frac{r_2}{s_2}\right]\right\}\right|$$
$$=\sum_{j=0}^{b-1}\left|\left\{1\leqslant n\leqslant q_{k-1}:\{n\alpha+jq_{k-1}\alpha\}\in\left[\frac{r_1}{s_1},\frac{r_2}{s_2}\right]\right\}\right|$$
$$=\sum_{j=0}^{b-1}\left|\left\{1\leqslant n\leqslant q_{k-1}:\left\{n\frac{p_{k-1}}{q_{k-1}}+n\frac{\delta_{k-1}}{q_{k-1}}+j\delta_{k-1}\right\}\in\left[\frac{r_1}{s_1},\frac{r_2}{s_2}\right]\right\}\right|$$
$$=\sum_{j=0}^{b-1}\left|\left\{0\leqslant n\leqslant q_{k-1}-1:\left\{\frac{n}{q_{k-1}}+m(n)\frac{\delta_{k-1}}{q_{k-1}}+j\delta_{k-1}\right\}\in\left[\frac{r_1}{s_1},\frac{r_2}{s_2}\right]\right\}\right|,$$

where $m(n):=p_{k-1}n \mod q_{k-1}$ and we used that p_{k-1} and q_{k-1} are coprime in the last line. Let $\varepsilon_n:=m(n)\frac{\delta_{k-1}}{q_{k-1}}+j\delta_{k-1}$. Now we have to count the number of $0\leqslant n\leqslant q_{k-1}-1$

such that $\frac{n}{q_{k-1}} + \varepsilon_n \in [\frac{r_1}{s_1}, \frac{r_2}{s_2}]$. Since k is odd and we assume that $b \leqslant \frac{a_k}{2s_1s_2}$, we have $0 \leqslant \varepsilon_n \leqslant \frac{1}{a_k q_{k-1}(1+o_{a_k}(1))} + \frac{1}{2s_1s_2q_{k-1}(1+o_{a_k}(1))}$. Using $s_1 \mid q_{k-1}$ and $\varepsilon_n \geqslant 0$, we see that the smallest integer n with $0 \leqslant n \leqslant q_{k-1}-1$ such that $\frac{n}{q_{k-1}} + \varepsilon_n \geqslant \frac{r_1}{s_1}$ is $n = \frac{r_1}{s_1}q_{k-1}$. Since $s_2 \nmid q_{k-1}$ and $\varepsilon_n \leqslant \frac{1}{a_k q_{k-1}(1+o_{a_k}(1))} + \frac{1}{2s_1s_2q_{k-1}(1+o_k(1))} < \frac{1}{s_2 q_{k-1}}$ (since $a_k \geqslant 2s_1s_2$, if a_k is sufficiently large, where also $1+o_{a_k}(1)$ is close to 1), we have that the largest integer n with $0 \leqslant n \leqslant q_{k-1}-1$ such that $\frac{n}{q_{k-1}} + \varepsilon_n \leqslant \frac{s_2}{r_2}$ is $n = \lfloor \frac{r_2}{s_2}q_{k-1} \rfloor$. This means, the number of integers $0 \leqslant n \leqslant q_{k-1}-1$ such that $\frac{n}{q_{k-1}} + \varepsilon_n \in [\frac{r_1}{s_1}, \frac{r_2}{s_2}]$ is equal to $\lfloor \frac{r_2}{s_2}q_{k-1} \rfloor - \frac{r_1}{s_1}q_{k-1} + 1 = 1 - \{\frac{s_2}{r_2}q_{k-1}\} + q_{k-1}(\frac{r_2}{s_2} - \frac{r_1}{s_1})$. This leads to

$$\sum_{j=0}^{b-1} \left| \left\{ 1 \leqslant n \leqslant q_{k-1} : \left\{ n\frac{p_{k-1}}{q_{k-1}} + n\frac{\delta_{k-1}}{q_{k-1}} + j\delta_{k-1} \right\} \in \left[\frac{r_1}{s_1}, \frac{r_2}{s_2} \right] \right\} \right|$$
$$= b(1 - \left\{ \frac{s_2}{r_2} q_{k-1} \right\} + q_{k-1}\left(\frac{r_2}{s_2} - \frac{r_1}{s_1} \right).$$

This implies that $S_{bq_{k-1}}(f,\alpha) = b(1 - \{\frac{s_2}{r_2}q_{k-1}\})$, as claimed. □

Lemma 3.7. *Let f be as in Definition 1.1, and let A_i, c_i, g as in lemma 3.1. Further, assume that $\sum_{i=1}^{\nu} A_i \neq 0$ or $\sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) \neq 0$. Then, there exist constants $c, c' > 0$, and integers $\alpha_j, \beta_j, \gamma_j, \delta_j, j = 1,2$ (depending on f) such that the following holds:*

- *If $K \equiv \alpha_1 \pmod{\beta_1}$, $\quad q_{K-1} \equiv \gamma_1 \pmod{\delta_1}$, then for any integer b with $0 \leqslant b \leqslant ca_K$, we have for sufficiently large K and a_K,*

$$S_{bq_{K-1}}(g,\alpha) \geqslant bc' + O\left(\sum_{i=1}^{K-1} a_i \right). \tag{13}$$

- *If $K \equiv \alpha_2 \pmod{\beta_2}$, $\quad q_{K-1} \equiv \gamma_2 \pmod{\delta_2}$, then for any integer b with $0 \leqslant b \leqslant ca_K$, we have for sufficiently large K and a_K,*

$$S_{bq_{K-1}}(g,\alpha) \leqslant -bc' + O\left(\sum_{i=1}^{K-1} a_i \right). \tag{14}$$

Proof. We only prove (13), the inequality (14) can be shown analogously. First, assume that $\sum_{i=1}^{\nu} A_i > 0$. Let $x_i = \frac{r_i}{s_i}$ with $\gcd(r_i, s_i) = 1$. We set $c := \frac{1}{4s_1 \cdots s_\nu}$ and choose $\alpha_1 := 1, \beta_1 := 2, \gamma_1 := 0$ and $\delta_1 := s_1 \cdots s_\nu$. Assume that $K \in \mathbb{N}$ satisfies $K \equiv \alpha_1 \pmod{\beta_1}$ and $q_{K-1} \equiv \gamma_1 \pmod{\delta_1}$.

By lemma 3.1, we can write $g(x) = \left(\sum_{i=1}^{\nu} A_i\right)\left(\{x\} - \frac{1}{2}\right) + \sum_{i=1}^{\nu} c_i \left(\mathbb{1}_{[x_{i-1},x_i]}(\{x\}) - (x_i - x_{i-1})\right)$, where we set $x_0 := 0$. This leads to

$$S_{bq_{K-1}}(g,\alpha) = \left(\sum_{i=1}^{\nu} A_i \right) \sum_{n=1}^{bq_{K-1}} \left(\{n\alpha\} - \frac{1}{2} \right) + \sum_{n=1}^{bq_{K-1}} \sum_{i=1}^{\nu} c_i \left(\mathbb{1}_{[x_{i-1},x_i]}(\{n\alpha\}) - (x_i - x_{i-1}) \right).$$

By lemma 3.6, the second sum above is equal to 0. Since $K-1$ is even, we get by lemma 3.5

$$\left(\sum_{i=1}^{\nu} A_i\right) \sum_{n=1}^{bq_{K-1}} \left(\{n\alpha\} - \frac{1}{2}\right) \geqslant \left(\sum_{i=1}^{\nu} A_i\right) \frac{b}{2}\left(1 - \frac{b}{a_K}\right) + O(1)$$
$$\geqslant \left(\sum_{i=1}^{\nu} A_i\right) \frac{b}{2}(1-c) + O(1).$$

We define $c' := \frac{1}{2}\left(\sum_{i=1}^{\nu} A_i\right)(1-c)$ which is a positive constant, since $c<1$. This finishes the proof in case of $\sum_{i=1}^{\nu} A_i > 0$. The case $\sum_{i=1}^{\nu} A_i < 0$ can be handled analogously.

Now assume $\sum_{i=1}^{\nu} A_i = 0$, but $\sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) \neq 0$. This implies $\nu \geqslant 2$ and we assume without loss of generality $x_1 > 0$. Further let $\sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) > 0$, the case where $\sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) < 0$ can be treated analogously. Let $x_i = \frac{r_i}{s_i}$ with $\gcd(r_i, s_i) = 1$ for $i = 0, \ldots, \nu$. We choose $\alpha_1 = 0, \beta_1 = 2, \gamma_1 = s_1 \cdots s_\nu - 1$ and $\delta_1 = s_1 \cdots s_\nu$ (we note that $s_1 \cdots s_\nu \geqslant 2$). Let $K \in \mathbb{N}$ such that $K \equiv \alpha_1 \pmod{\beta_1}$, $q_{K-1} \equiv \gamma_1 \pmod{\delta_1}$ and define $c := \frac{1}{4 s_1 \cdots s_\nu}$. Then, for $0 \leqslant b \leqslant c a_K$, we get

$$S_{bq_{K-1}} = \sum_{n=1}^{bq_{K-1}} \sum_{i=1}^{\nu} c_i \left(\mathbb{1}_{[x_{i-1}, x_i]}(\{n\alpha\}) - (x_i - x_{i-1})\right)$$
$$= b\sum_{i=2}^{\nu} c_i \left(\left\{\frac{r_{i-1}}{s_{i-1}} q_{K-1}\right\} - \left\{\frac{r_i}{s_i} q_{K-1}\right\}\right) + bc_1\left(1 - \left\{\frac{r_1}{s_1} q_{K-1}\right\}\right).$$

In the last line, we applied lemma 3.6 and used that $s_i \nmid q_{K-1}$ for every $i = 1, \ldots, \nu$. By the choice of γ_1 and δ_1, we have $q_{K-1} \equiv -1 \pmod{s_i}$ for all $i = 1, \ldots, \nu$ and therefore

$$b\sum_{i=2}^{\nu} c_i \left(\left\{\frac{r_{i-1}}{s_{i-1}} q_{K-1}\right\} - \left\{\frac{r_i}{s_i} q_{K-1}\right\}\right) + bc_1\left(1 - \left\{\frac{r_1}{s_1} q_{K-1}\right\}\right) = b\sum_{i=1}^{\nu} c_i \left(\frac{r_i}{s_i} - \frac{r_{i-1}}{s_{i-1}}\right).$$

By now defining $c' = \sum_{i=1}^{\nu} c_i \left(\frac{r_i}{s_i} - \frac{r_{i-1}}{s_{i-1}}\right) = \sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) > 0$, the proof is finished. □

Next, we consider the analogue of the previous lemma in the case $\sum_{i=1}^{\nu} A_i = \sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) = 0$, where we aim for a positive upper density.

Lemma 3.8. *Let f be as in definition 1.1, and let A_i, c_i, g as in lemma 3.1. Further, assume that $\sum_{i=1}^{\nu} A_i = \sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) = 0$. Then, there exist constants $c, c', d > 0$ with $c' < c$ and integers $\alpha_j, \beta_j, \gamma_j, \delta_j, j = 1, 2$ (all depending on f) such that the following holds:*

- *If $K \equiv \alpha_1 \pmod{\beta_1}$, $q_{K-1} \equiv \gamma_1 \pmod{\delta_1}$, then for $c' a_K \leqslant b \leqslant c a_K$, we have for sufficiently large K and a_K*

$$S_{bq_{K-1}}(g, \alpha) \geqslant d a_K. \tag{15}$$

- *If $K \equiv \alpha_2 \pmod{\beta_2}$, $q_{K-1} \equiv \gamma_2 \pmod{\delta_2}$, then for $c' a_K \leqslant b \leqslant c a_K$, we have for sufficiently large K and a_K*

$$S_{bq_{K-1}}(g, \alpha) \leqslant -d a_K. \tag{16}$$

Proof. We only show (15), since (16) can be shown analogously. By assumption $c_1 = \sum_{i=1}^{\nu} A_i = 0$, thus there exist at least two $A_i \neq 0$. To keep notation simple we assume $A_1 \neq 0$ and thus $c_2 = \sum_{i=2}^{\nu} A_i \neq 0$. We assume $c_2 > 0$ and note that the case where $c_2 < 0$ can be handled similarly.

Let $x_i = \frac{r_i}{s_i}$ for all $i = 0, \ldots, \nu$. We choose $\alpha_1 = 1, \beta_1 = 2, \gamma_1 = 1$ and $\delta_1 = s_1 \cdots s_\nu$. We take $K \in \mathbb{N}$ with $K \equiv \alpha_1 \pmod{\beta_1}$, $q_{K-1} \equiv \gamma_1 \pmod{\delta_1}$ and $a_K > \frac{8}{(x_2 - x_1)c_2}$. Moreover, let $c' = \frac{x_1 + x_2}{2}$ and $c = x_2$, which implies $0 < c' < c$. Let $b \in \mathbb{N}$ with $c' a_K < b < c a_K$ and consider

$$\begin{aligned} S_{bq_{K-1}}(g, \alpha) &= \sum_{n=1}^{bq_{K-1}} \sum_{i=1}^{\nu} c_j \left(\mathbb{1}_{[x_{i-1}, x_i]}(\{n\alpha\}) - (x_i - x_{i-1}) \right) \\ &= \sum_{i=1}^{\nu} c_i \left(|\{1 \leqslant n \leqslant bq_{K-1} : \{n\alpha\} \in [x_{i-1}, x_i]\}| - bq_{K-1}(x_i - x_{i-1}) \right). \end{aligned}$$

Now, for every $1 \leqslant i \leqslant \nu$, we provide a suitable estimate for

$$|\{1 \leqslant n \leqslant bq_{K-1} : \{n\alpha\} \in [x_{i-1}, x_i]\}| - bq_{K-1}(x_i - x_{i-1})$$

from below. Starting with the case $i \geqslant 3$, we observe that

$$|\{1 \leqslant n \leqslant bq_{K-1} : \{n\alpha\} \in [x_{i-1}, x_i]\}| = \sum_{u=0}^{b-1} |\{1 \leqslant n \leqslant q_{K-1} : \{n\alpha + uq_{K-1}\alpha\} \in [x_{i-1}, x_i]\}|.$$

For $0 \leqslant u \leqslant b-1$, we have

$$\begin{aligned} &|\{1 \leqslant n \leqslant q_{K-1} : \{n\alpha + uq_{K-1}\alpha\} \in [x_{i-1}, x_i]\}| \\ &= \left| \left\{ 1 \leqslant n \leqslant q_{K-1} : \left\{ n\frac{p_{K-1}}{q_{K-1}} + \varepsilon_n \right\} \in [x_{i-1}, x_i] \right\} \right|, \end{aligned}$$

where $\varepsilon_n := uq_{K-1}\left(\alpha - \frac{p_{K-1}}{q_{K-1}}\right) + n\left(\alpha - \frac{p_{K-1}}{q_{K-1}}\right)$. Since $2|(K-1)$ and $u < b \leqslant x_2 a_K$, we have that $0 \leqslant \varepsilon_n \leqslant \frac{x_2}{q_{K-1}}$.Thus, we get

$$\begin{aligned} &\left| \left\{ 1 \leqslant n \leqslant q_{K-1} : \left\{ n\frac{p_{K-1}}{q_{K-1}} + \varepsilon_n \right\} \in [x_{i-1}, x_i] \right\} \right| \\ &\geqslant \left| \left\{ 0 \leqslant n \leqslant q_{K-1} - 1 : \frac{n}{q_{K-1}} \geqslant x_{i-1}, \frac{n}{q_{K-1}} + \frac{x_2}{q_{K-1}} \leqslant x_i \right\} \right| \\ &= \lfloor x_i q_{K-1} \rfloor - \lceil x_{i-1} q_{K-1} \rceil + 1, \end{aligned}$$

where we used that, for $i \geqslant 3$, we have $x_i > x_2$ and thus, the largest integer n such that $\frac{n}{q_{K-1}} + \frac{x_2}{q_{K-1}} \leqslant x_i$ is $n = \lfloor x_i q_{K-1} \rfloor$. Moreover, the smallest n such that $\frac{n}{q_{K-1}} \geqslant x_{i-1}$ is $n = \lceil x_{i-1} q_{K-1} \rceil$. Since $q_{K-1} \equiv 1 \pmod{s_1 \ldots s_\nu}$, we have $\{q_{K-1} x_i\} = x_i$ for any $1 \leqslant i \leqslant \nu$ (recall that the s_i denote the denominators of the rationals x_i) and thus, in case of $i \geqslant 3$, we have shown that

$$|\{1 \leqslant n \leqslant q_{K-1} : \{n\alpha + uq_{K-1}\alpha\} \in [x_{i-1}, x_i]\}| \geqslant (x_i - x_{i-1}) q_{K-1} - (x_i - x_{i-1})$$

holds for any $0 \leqslant u \leqslant b-1$. For the case $i = 1$, a similar argument as before shows that for any $0 \leqslant u \leqslant b-1$, we have

$$|\{1 \leqslant n \leqslant q_{K-1} : \{n\alpha + uq_{K-1}\alpha\} \in [0, x_1]\}| \geqslant x_1 q_{K-1} - x_1.$$

Now we turn our attention to the case of $i=2$, where we establish a slightly different lower bound for $|\{1\leqslant n\leqslant q_{K-1}:\{n\alpha+uq_{K-1}\alpha\}\in[x_1,x_2]\}|$. First, we consider those u with $0\leqslant u<x_1a_K(1+\varepsilon)$. We choose $\varepsilon>0$ small enough such that $x_1(1+4\varepsilon)<x_2$. In that case, we get the same lower bound as before, i.e. we establish $|\{1\leqslant n\leqslant q_{K-1}:\{n\alpha+uq_{K-1}\alpha\}\in[x_1,x_2]\}|\geqslant(x_2-x_1)q_{K-1}-(x_2-x_1)$. We are now in the case, where $x_1a_K(1+\varepsilon)\leqslant u\leqslant x_2a_K-1$ (which is a complete case distinction, since $u<b\leqslant x_2a_K$), we get

$$
\begin{aligned}
&|\{1\leqslant n\leqslant q_{K-1}:\{n\alpha+uq_{K-1}\alpha\}\in[x_1,x_2]\}|\\
&=\left|\left\{1\leqslant n\leqslant q_{K-1}:\left\{n\frac{p_{K-1}}{q_{K-1}}+\varepsilon_n\right\}\in[x_1,x_2]\right\}\right|,
\end{aligned}
$$

where we again set $\varepsilon_n=uq_{K-1}\left(\alpha-\frac{p_{K-1}}{q_{K-1}}\right)+n\left(\alpha-\frac{p_{K-1}}{q_{K-1}}\right)$. Since $2|(K-1)$ and $a_Kx_1(1+\varepsilon)\leqslant u\leqslant a_Kx_2-1$, we have that for any $1\leqslant n\leqslant q_{K-1}$, $\frac{x_1(1+\varepsilon)}{q_{K-1}(1+o_{a_K}(1))}\leqslant\varepsilon_n\leqslant\frac{x_2}{q_{K-1}}$. Further, we use that $\frac{1+\varepsilon}{1+o_{a_K}(1)}\geqslant 1$ if K is sufficiently large and hence $\frac{x_1}{q_{K-1}}\leqslant\varepsilon_n$. This gives us the estimate

$$
\begin{aligned}
&\left|\left\{1\leqslant n\leqslant q_{K-1}:\left\{n\frac{p_{K-1}}{q_{K-1}}+\varepsilon_n\right\}\in[x_1,x_2]\right\}\right|\\
&\geqslant\left|\left\{0\leqslant n\leqslant q_{K-1}-1:\frac{n}{q_{K-1}}+\frac{x_1}{q_{K-1}}\geqslant x_1,\frac{n}{q_{K-1}}+\frac{x_2}{q_{K-1}}\leqslant x_2\right\}\right|.
\end{aligned}
$$

Here we used $\gcd(p_{K-1},q_{K-1})=1$ which implies that np_{K-1} runs through all remainder classes modulo q_{K-1}. The smallest $0\leqslant n\leqslant q_{K-1}-1$ such that $\frac{n}{q_{K-1}}+\frac{x_1}{q_{K-1}}\geqslant x_1$ is $n=\lfloor x_1q_{K-1}\rfloor=x_1q_{K-1}-x_1$, which follows from the congruence relation satisfied by q_{K-1}. The largest n such that $\frac{n}{q_{K-1}}+\frac{x_2}{q_{K-1}}\leqslant x_2$ is $n=\lfloor x_2q_{K-1}\rfloor=x_2q_{K-1}-x_2$. These estimates lead to

$$
\begin{aligned}
&\left|\left\{0\leqslant n\leqslant q_{K-1}-1:\frac{n}{q_{K-1}}+\frac{x_1}{q_{K-1}}\geqslant x_1,\frac{n}{q_{K-1}}+\frac{x_2}{q_{K-1}}\leqslant x_2\right\}\right|\\
&\quad=\lfloor x_2q_{K-1}\rfloor-\lfloor x_1q_{K-1}\rfloor+1\\
&\quad=(x_2-x_1)\,q_{K-1}+(1-(x_2-x_1)).
\end{aligned}
$$

Now we can combine all the estimates we obtained before, in order to get

$$
\begin{aligned}
&\sum_{i=1}^{\nu}c_i\sum_{u=0}^{b-1}|\{1\leqslant n\leqslant q_{K-1}:\{n\alpha+uq_{K-1}\alpha\}\in[x_{i-1},x_i]\}|-(x_i-x_{i-1})\,q_{K-1}\\
&\quad\geqslant-\sum_{i=1}^{\nu}c_i\sum_{u=0}^{b-1}(x_i-x_{i-1})+c_2\,|\{0\leqslant u\leqslant b-1:a_Kx_1\,(1+\varepsilon)\leqslant u\leqslant a_Kx_2-1\}|\\
&\quad=0+c_2\,(b-\lceil a_Kx_1\,(1+\varepsilon)\rceil)\geqslant a_Kc_2\left(\frac{x_1+x_2}{2}-(1+\varepsilon)x_1\right)-2\\
&\quad\geqslant a_Kc_2\left(\frac{x_1+x_2}{2}-(1+2\varepsilon)\,x_1\right).
\end{aligned}
$$

We used the overall assumption of $\sum_{i=1}^{\nu}c_i\,(x_i-x_{i-1})=0$ and $-2\geqslant-c_2a_K\varepsilon$, since a_K is large. The proof is now finished by defining $d:=c_2\left(\frac{x_1+x_2}{2}-(1+2\varepsilon)x_1\right)>0$.

□

3.3. Refining the Borel–Bernstein theorem

We see that lemmas 3.7 and 3.8 give us lower bounds on Birkhoff sums, provided that a_K does not only dominate the sum of the preceding partial quotients, but both K and q_{K-1} additionally satisfy certain modularity conditions. Without having to satisfy these extra conditions, the existence of infinitely many such a_K for generic α could be deduced from a combination of the Borel–Bernstein Theorem and the estimate (7) on the trimmed sum of partial quotients. The aim of this section is to establish a version of the Borel–Bernstein Theorem (lemma 3.12) that allows to include additional assumptions on K and q_{K-1}. We make use of some known auxiliary results that are stated for the reader's convenience in full detail below.

Lemma 3.9 (lemma C in [17]). *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with events $(A_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty}\mathbb{P}[A_n]=\infty$ and*

$$\limsup_{N\to\infty}\frac{\left(\sum_{K=1}^{N}\mathbb{P}\left[A_K\right]\right)^2}{\sum_{K,L=1}^{N}\mathbb{P}\left[A_K\cap A_L\right]}=1.$$

Then,

$$\mathbb{P}\left[\limsup_{n\to\infty}A_n\right]=1.$$

Lemma 3.10 (lemma 2.5 in [23]). *Let $v\in\mathbb{N}$ with $v\geqslant 2$, and $0\leqslant u\leqslant v-1$ and define $k=k(v):=|\{(u_1,u_2):0\leqslant u_1,u_2<v:\gcd(u_1,u_2,v)=1\}|$. Further, let $m,p\in\mathbb{N}$, $\alpha\sim\mathrm{Unif}([0,1))$, define $\mathscr{F}:=\sigma\left(a_1(\alpha),\ldots,a_m(\alpha)\right)$ and let $E\in\sigma\left(a_{m+p+1}(\alpha),a_{m+p+2}(\alpha),\ldots\right)$. Then, there exist constants $C>0$ and $\lambda\in(0,1)$ such that*

$$\left|\mathbb{P}\left[q_{m+p}(\alpha)\equiv u \pmod v,\alpha\in E\,|\mathscr{F}\right]-\frac{1}{k}\mathbb{P}\left[\alpha\in E\right]\right|\leqslant C\lambda^{\sqrt{p}}. \tag{17}$$

Remark 3.11. In [23], the quantity k from lemma 3.10 is not given explicitly. The fact that all pairs (u_1,u_2) with $\gcd(u_1,u_2,v)=1$ are admissible and thus $k(v)$ can be defined as in Lemma 3.10 follows from [24]. The decay rate of (17) was improved in [35] to be exponential, that is of the form $C\lambda^p$.

Lemma 3.12. *Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be a monotonically increasing function such that $\sum_{K=1}^{\infty}\frac{1}{\psi(K)}=\infty$ and let $b,d\in\mathbb{N}$ with $b,d\geqslant 2$. Then, for any fixed $0\leqslant a\leqslant b-1$ and $0\leqslant c\leqslant d-1$, for almost all $\alpha\in[0,1)$, the set*

$$\left\{K\in\mathbb{N}\;\middle|\;K\equiv a \pmod b,q_{K-1}\equiv c \pmod d,\psi(K)<a_K<K^2,\sum_{i=1}^{K-1}a_i\leqslant 2K\log K\right\}$$

has infinite cardinality.

Remark 3.13. In particular, Bernstein's Theorem can be strengthened in the following way: For any monotonic increasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ and any positive integers a,b,c,d we have, for almost every $\alpha\in[0,1)$,

$$|\{K\in\mathbb{N}:a_K>\psi(K),K\equiv a \pmod b,q_{K-1}\equiv c \pmod d\}| \text{ is } \begin{cases}\text{infinite} & \text{if}\sum_{K=0}^{\infty}\frac{1}{\psi(K)}=\infty,\\ \text{finite} & \text{if}\sum_{K=0}^{\infty}\frac{1}{\psi(K)}<\infty.\end{cases}$$

The method of proof even allows replacing the condition $K \equiv a \pmod{b}$ with a condition of the form $K \in A \subseteq \mathbb{N}$, where A has positive lower density. For our purpose the given version is sufficient.

Proof. We first show that for almost every $\alpha \in [0,1)$, the set $\left\{K \in \mathbb{N} \mid K \equiv a \pmod{b}, \psi(K) < a_K < K^2\right\}$ has infinite cardinality. We can assume without loss of generality that $\liminf_{K\to\infty} \frac{\psi(K)}{K\log K} = \infty$ since the result then also follows for all slower growing ψ. Now we define

$$\tilde{\psi}(K) := \begin{cases} \psi(K) & \text{if } K \equiv a \pmod{b}, \\ K^2 & \text{else.} \end{cases}$$

Since ψ is monotone, we have that $\sum_{K=1}^{\infty} \frac{1}{\tilde{\psi}(K)} = \infty$ (since $\sum_{K=1}^{Nb} \frac{1}{\tilde{\psi}(K)} \geqslant \frac{1}{b} \sum_{K=b}^{Nb} \frac{1}{\psi(K)} \underset{N\to\infty}{\to} \infty$). By (6), there exist infinitely many K such that $a_K > \tilde{\psi}(K)$. Again by (6), there are only finitely many $K \in \mathbb{N}$ such that $a_K > K^2$ and thus, we can conclude that there are infinitely many $K \in \mathbb{N}$ with $K \equiv a \pmod{b}$ and $\psi(K) < a_K < K^2$. Now we introduce the sets

$$E_K := \left\{\alpha \in [0,1) \mid K \equiv a \pmod{b}, \psi(K) < a_K < K^2\right\},$$
$$A_K := \left\{\alpha \in [0,1) \mid K \equiv a \pmod{b}, q_{K-1} \equiv c \pmod{d}, \psi(K) < a_K < K^2\right\}.$$

In the following, we show that lemma 3.9 can be applied to the sequence of sets $(A_K)_{K\in\mathbb{N}}$. To that end, we define $k = |\{0 \leqslant u_1, u_2 \leqslant d-1 \mid \gcd(u_1,u_2,d) = 1\}|$ and we note that E_K only depends on a_K. Further, we will denote the 1-dimensional Lebesgue measure on $[0,1)$ by $\mathbb{P}$. Using lemma 3.10, we get

$$\mathbb{P}[A_K] = \frac{1}{k}\mathbb{P}[E_K] + O\left(\lambda^{\sqrt{K}}\right), \tag{18}$$

with $\lambda \in (0,1)$. This gives us $\sum_{K=1}^{\infty} \mathbb{P}[A_K] = \infty$, since there exist infinitely many $K \in \mathbb{N}$ such that $\mathbb{P}[E_K] = 1$ by the first part of this proof. This shows the first assumption in lemma 3.9, i.e. $\sum_{K=1}^{\infty} \mathbb{P}[A_K] = \infty$. Now we take $K, L \in \mathbb{N}$ with $L+1 \leqslant K$ and consider

$$\begin{aligned}
\mathbb{P}[A_K \cap A_L] &= \mathbb{P}[A_L]\,\mathbb{P}[A_K|A_L] \\
&= \mathbb{P}[A_L]\,\mathbb{P}\left[\psi(K) < a_K < K^2, q_{K-1} \equiv c \pmod{d}, K \equiv a \pmod{b} | A_L\right] \\
&= \mathbb{P}[A_L]\left(\frac{1}{k}\mathbb{P}[E_K] + O\left(\lambda^{\sqrt{K-L}}\right)\right) \\
&= \mathbb{P}[A_L]\,\mathbb{P}[A_K] + \mathbb{P}[A_L]\,O\left(\lambda^{\sqrt{K-L}}\right),
\end{aligned}$$

where we employed lemma 3.10 and used the estimate from (18) in the last line. We fix $N \in \mathbb{N}$ sufficiently large and consider

$$\begin{aligned}
\sum_{K,L=1}^{N} \mathbb{P}[A_K \cap A_L] &= \sum_{K,L=1}^{N} \mathbb{P}[A_K]\,\mathbb{P}[A_L] + 2\sum_{L+1\leqslant K\leqslant N} \mathbb{P}[A_L]\,O\left(\lambda^{\sqrt{K-L}}\right) \\
&\quad + \sum_{K=1}^{N}\left(\mathbb{P}[A_K] - \mathbb{P}[A_K]^2\right).
\end{aligned} \tag{19}$$

Next, we obtain an upper bound for two of the sums in (19). First, we get

$$2\sum_{L+1\leqslant K\leqslant N}\mathbb{P}[A_L]\,O\left(\lambda^{\sqrt{K-L}}\right)=2\sum_{L=1}^{N-1}\mathbb{P}[A_L]\sum_{K=L+1}^{N}O\left(\lambda^{\sqrt{K-L}}\right)\leqslant O(1)\sum_{L=1}^{N}\mathbb{P}[A_L],$$

where we used that $\lambda\in(0,1)$. Moreover, we get

$$\sum_{K=1}^{N}\left(\mathbb{P}[A_K]-\mathbb{P}[A_K]^2\right)\leqslant\sum_{K=1}^{N}\mathbb{P}[A_K]$$

and thus, we have

$$\sum_{L+1\leqslant K\leqslant N}\mathbb{P}[A_L]\,O\left(\lambda^{\sqrt{K-L}}\right)+\sum_{K=1}^{N}\left(\mathbb{P}[A_K]-\mathbb{P}[A_K]^2\right)\leqslant O(1)\sum_{K=1}^{N}\mathbb{P}[A_K]=o\left(\left(\sum_{K=1}^{N}\mathbb{P}[A_K]\right)^2\right).$$

For the equality in the previous equation, we used that $\sum_{K=1}^{\infty}\mathbb{P}[A_K]=\infty$. In total, we have shown that

$$\limsup_{N\to\infty}\frac{\left(\sum_{K=1}^{N}\mathbb{P}[A_K]\right)^2}{\sum_{K,L=1}^{N}\mathbb{P}[A_K\cap A_L]}=\limsup_{N\to\infty}\frac{1}{1+o_N(1)}=1.$$

Lemma 3.9 now gives us that $\mathbb{P}[\limsup_{K\to\infty}A_K]=1$ or, in other words, for almost all $\alpha\in[0,1)$, there are infinitely many $K\in\mathbb{N}$ such that $\psi(K)<a_K<K^2$, $K\equiv a \pmod b$ and $q_{K-1}\equiv c \pmod d$.

For K sufficiently large, we have $\psi(K)>\frac{K\log K}{\log(2)}$, so (7) shows that $\max_{\ell\leqslant K-1}a_\ell<a_K$ and thus, applying (7) again leads to $\sum_{i=1}^{K-1}a_i\leqslant 2K\log K$ (note that $\frac{1}{\log 2}\leqslant 2$). □

We have now all ingredients to turn our attention to the proofs of theorems 7 and 8.

3.4. *Proofs of theorems 7 and 8*

Note that the class of functions considered in both theorems 7 and 8 is closed under translation by rational numbers, and by lemma 3.1 the same holds for the condition $\sum_{i=1}^{\nu}A_i\neq 0$ or $\sum_{i=1}^{\nu}c_i(x_i-x_{i-1})\neq 0$. Thus, we can assume without loss of generality that $q=0$. Moreover, we can assume that $\lim_{K\to\infty}\frac{\psi(K)}{K\log K}=\infty$ since the result then follows also for slower growing ψ.

Assume first that $\sum_{i=1}^{\nu}A_i\neq 0$ or $\sum_{i=1}^{\nu}c_i(x_i-x_{i-1})\neq 0$. Let $c,c',\alpha_1,\beta_1,\gamma_1,\delta_1$ be as in lemma 3.7. By lemma 3.12, for almost every α, there exist infinitely many K such that

$$\left\{K\in\mathbb{N}\,\middle|\,K\equiv\alpha_1 \pmod{\beta_1},\,q_{K-1}\equiv\gamma_1 \pmod{\delta_1},\,\tilde{\psi}(K)<a_K<K^2,\sum_{i=1}^{K-1}a_i\leqslant 2K\log K\right\},$$

where $\tilde{\psi}(k)=C_1\psi(C_2k)$ with $C_1,C_2>0$ specified later. Denote by $(k_j)_{j\in\mathbb{N}}$ the increasing sequence of integers such that the above holds. Now let $N\leqslant ca_{k_j}q_{k_j-1}$ be arbitrary. Thus, we can write $N = b_{k_j-1}q_{k_j-1} + N'$ where $N' < q_{k_j-1}$ and $b_{k_j-1} \leqslant ca_{k_j}$. Defining g as in lemma 3.1 and $\tilde{g}(x) = g(x + b_{k_j-1}q_{k_j-1}\alpha)$, we have

$$\begin{aligned} S_N(f,\alpha) &= S_N(g,\alpha) + O(1) \\ &= S_{b_{k_j-1}q_{k_j-1}}(g,\alpha) + S_{N'}(\tilde{g},\alpha) + O(1) \\ &\geqslant b_{k_j-1}c' + O\left(\sum_{i=1}^{k_j-1} a_i\right), \end{aligned} \tag{20}$$

where we used lemma 3.7 and the Denjoy–Koksma inequality in the last line. Let $M_j := \lfloor ca_{k_j}q_{k_j-1} \rfloor$ and, for $\delta > 0$, we define the set $A_j^\delta := \left\{1 \leqslant N \leqslant M_j : \delta \leqslant \frac{b_{k_j-1}(N)}{a_{k_j}}\right\}$. We note that $\lim_{\delta\to 0} |A_j^\delta| = M_j$. Thus, fixing $\varepsilon > 0$, we can choose $\delta > 0$ such that $\frac{|A_j^\delta|}{M_j} \geqslant 1 - \varepsilon$ for all sufficiently large j. Let $C_1, C_2 > 0$ such that, if $a_{k_j} \geqslant \tilde{\psi}(k_j) = C_1\psi(C_2k_j)$ it follows that $b_{k_j-1}c' + O\left(\sum_{i=1}^{k_j-1} a_i\right) \geqslant \psi(\log N)$ for all $N \in A_j^\delta$. We note that C_1, C_2 only depend on $\delta > 0$, since k_j is chosen such that $\sum_{i=1}^{k_j-1} a_i \leqslant 2k_j\log(k_j) = o(\psi(k_j))$. This yields

$$\begin{aligned} \frac{|\{1 \leqslant N \leqslant M_j : S_N(f,\alpha) \geqslant \psi(\log N)\}|}{M_j} &\geqslant \frac{\left|\left\{N \in A_j^\delta : b_{k_j-1}c' + O\left(\sum_{i=1}^{k_j-1} a_i\right) \geqslant \psi(\log N)\right\}\right|}{M_j} \\ &\geqslant \frac{|\{N \in A_j^\delta : a_{k_j} \geqslant \tilde{\psi}(k_j)\}|}{M_j} \\ &= \frac{|A_j^\delta|}{M_j} \geqslant 1 - \varepsilon. \end{aligned}$$

By taking the limes inferior as $j \to \infty$ and letting $\varepsilon \to 0$, we get

$$\liminf_{j\to\infty} \frac{|\{1 \leqslant N \leqslant M_j : S_N(f,\alpha) \geqslant \psi(\log N)\}|}{M_j} = 1.$$

This shows the claimed upper density 1 in theorem 7 for the set $\{N \in \mathbb{N}\ S_N(f,\alpha) \geqslant \psi(\log N)\}$ in case of $\sum_{i=1}^{\nu} A_i \neq 0$ or $\sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) \neq 0$.

In order to prove the first part of theorem 8, let $0 < r < 1$ be fixed. Choosing $\delta = \delta(r)$ sufficiently small such that $\frac{|A_j^\delta|}{M_j} \geqslant r$, we can deduce from (20) that, for $N \in A_j^\delta$, we have

$$S_N(f,\alpha) \geqslant c'\delta a_{k_j} + O\left(\sum_{i=1}^{k_j-1} a_i\right).$$

By choosing $C(r) := \frac{c'\delta}{2}$, the first statement of theorem 8 follows, since the sequence $(k_j)_{j\in\mathbb{N}}$ is chosen such that a_{k_j} dominates $\sum_{i=1}^{k_j-1} a_i$.

Now we prove the remaining parts of theorem 7, where we need to show that if $\sum_{i=1}^{\nu} A_i = \sum_{i=1}^{\nu} c_i(x_i - x_{i-1}) = 0$, then the set $\{N \in \mathbb{N} : S_N(f,\alpha) \geqslant \psi(\log N)\}$ has positive upper density.

Let $c, c', d, \alpha_1, \beta_1, \gamma_1, \delta_1$ be as in lemma 3.8. By lemma 3.12, there are infinitely many K such that

$$\left\{K \in \mathbb{N} \middle| K \equiv \alpha_1 \pmod{\beta_1}, q_{K-1} \equiv \gamma_1 \pmod{\delta_1}, \psi(K) < a_K < K^2, \sum_{i=1}^{K-1} a_i \leqslant 2K \log K \right\}.$$

Denote by $(k_j)_{j \in \mathbb{N}}$ the increasing sequence of integers such that the above holds. Now let $N \in \mathbb{N}$ with $c' a_{k_j} q_{k_j} \leqslant N \leqslant c a_{k_j} q_{k_j}$ be arbitrary. Thus, we can write $N = b_{k_j-1} q_{k_j-1} + N'$ where $N' < q_{k_j-1}$ and $c a_{k_j} \leqslant b_{k_j-1} \leqslant a_{k_j}$. Arguing as in (20), we obtain by lemma 3.8

$$S_N(f, \alpha) \geqslant b_{k_j-1} d + O\left(\sum_{i=1}^{k_j-1} a_i\right). \tag{21}$$

Let $M_j := \lfloor c a_{k_j} q_{k_j-1} \rfloor$ and $A_j := \left\{1 \leqslant N \leqslant N_j : c' \leqslant \frac{b_{k_j-1}(N)}{a_{k_j}} \leqslant c\right\}$. We note that there exists $r_0 > 0$ such that $\frac{|A_j|}{M_j} \geqslant r_0 > 0$ for all sufficiently large $j \in \mathbb{N}$. Similar to the first part of this proof, we get

$$\begin{aligned}
\frac{|\{1 \leqslant N \leqslant M_j : S_N(f,\alpha) \geqslant \psi(\log N)\}|}{M_j} &\geqslant \frac{\left|\left\{1 \leqslant N \leqslant M_j : b_{k_j-1} d + O\left(\sum_{i=1}^{k_j-1} a_i\right) \geqslant \psi(\log N)\right\}\right|}{M_j} \\
&\geqslant \frac{\left|\left\{N \in A_j : b_{k_j-1} d + O\left(\sum_{i=1}^{k_j-1} a_i\right) \geqslant \psi(\log N)\right\}\right|}{M_j} \\
&= \frac{|A_j|}{M_j} \geqslant r_0.
\end{aligned}$$

By taking the liminf as $j \to \infty$, we obtain

$$\liminf_{j \to \infty} \frac{|1 \leqslant N \leqslant M_j : S_N(f,\alpha) \geqslant \psi(\log N)|}{M_j} \geqslant r_0 > 0.$$

This shows the claimed positive upper density in theorem 7 for the set $\{N \in \mathbb{N} \;\; S_N(f,\alpha) \geqslant \psi(\log N)\}$ in case of $\sum_{i=1}^{\nu} A_i = \sum_{i=1}^{\nu} c_i (x_i - x_{i-1}) = 0$.

To prove the second statement of theorem 8, we see that, for $N \in A_j$, by (21), we have $S_N(f,\alpha) \geqslant dc' a_{k_j} + O\left(\sum_{i=1}^{k_j-1} a_i\right)$. Since a_{k_j} dominates $\sum_{i=1}^{k_j-1} a_i$ by construction and $\frac{|A_j|}{M_j} \geqslant r_0 > 0$, the statement follows immediately.

4. Functions with logarithmic singularities

4.1. Heuristic of the proofs

We will briefly line out the main ideas of the proof of theorems 3 and 6. Again, we are using that, for almost every $\alpha \in [0,1)$, $\sum_{i=1}^{K-1} a_i = o(a_K)$ for infinitely many $K \in \mathbb{N}$. Here, K will always satisfy this property. For $q_{K-1} < N < q_K$ and $N = b_{K-1} q_K + N'$, $N' < q_{K-1}$, we get rid of $S_{N'}(\ldots)$ by an application of the Denjoy–Koksma inequality with singularity (5). We make sure to stay away from the singularity $x_1 = \frac{r}{s}$ by the fact that if $\|N\alpha - \frac{r}{s}\|$ is small, then so is $\|sN\alpha\|$ (proposition 4.2). Thus, we can morally work with the homogeneous case of Diophantine approximation and the corresponding metric theory gives sufficient estimates.

Again, we analyze $S_{(b+1)q_{K-1}}(f,\alpha) - S_{bq_{K-1}}(f,\alpha)$ for every $b \leqslant b_{K-1} - 1$ and observe that

$$\{\{n\alpha + bq_{K-1}\alpha\}\}_{n=1}^{q_{K-1}} \approx \left\{ \frac{j + \frac{b_{K-1}}{a_K}}{q_{K-1}} \right\}_{j=1}^{q_{K-1}}.$$

In the asymmetric case, we see that $f(x) = \log\{x\}$ is monotonically increasing on $[0,1)$. Comparing $f\left(\frac{j+\frac{b_{K-1}}{a_K}}{q_{K-1}}\right)$ with $\int_{j/q_{K-1}}^{(j+1)/q_K} f(x)\,\mathrm{d}x$, the value of $\frac{b_{K-1}}{a_K}$ is decisive and, for some $c,d >$ 0 and $\frac{b}{a_K} \in [0,c]$, this leads to an estimate (see lemma 4.4) of the form

$$(-1)^K \left(S_{(b+1)q_{K-1}}(f,\alpha) - S_{bq_{K-1}}(f,\alpha)\right) \gtrapprox d \log q_K \gg \log N.$$

Then the proof can be concluded similarly to the proof of theorem 7.

In the symmetric case $f(x) = \log\|x\|$, we see that, for $j \leqslant q_{K-1}/2$, we have $f\left(1 - \frac{j-1+\frac{b_{K-1}}{a_K}}{q_{K-1}}\right) = f\left(\frac{j+(1-\frac{b_{K-1}}{a_K})}{q_{K-1}}\right)$. So, the terms $f\left(\frac{j+\frac{b_{K-1}}{a_K}}{q_{K-1}}\right) - \int_{j/q_{K-1}}^{(j+1)/q_K} f(x)\,\mathrm{d}x$ and $f\left(\frac{1-(j+\frac{b_{K-1}}{a_K})}{q_{K-1}}\right) - \int_{1-(j-1)/q_{K-1}}^{1-j/q_{K-1}} f(x)\,\mathrm{d}x$ are of opposite sign and lead to some cancellation (lemma 4.5). This cancellation is responsible for the different behaviour of symmetric and asymmetric singularities.

4.2. Asymmetric logarithmic singularities

Proposition 4.1. *Let* $x_j = \frac{j+\varepsilon_j}{q_\ell}, 0 \leqslant j \leqslant q_\ell - 1$, *for* $0 < \varepsilon_j < 1$, *where* $\ell \in \mathbb{N}$. *Then, we have*

$$\sum_{j=0}^{q_\ell - 1} \log(x_j) - q_\ell \int_0^1 \log(x)\,\mathrm{d}x = \left(\sum_{j=1}^{q_\ell - 1} \frac{\varepsilon_j - 1/2}{j}\right) + \log(\varepsilon_0) + O(1)$$

with the implied constant being absolute, independent of ε_j *.*

Proof. For $j \geqslant 1$, we have

$$\begin{aligned} &q_\ell \int_{j/q_\ell}^{(j+1)/q_\ell} \log(x)\,\mathrm{d}x \\ &= (j+1)\log\left(\frac{j+1}{q_\ell}\right) - (j+1) - j\log\left(\frac{j}{q_\ell}\right) + j \\ &= \log\left(\frac{j+1}{q_\ell}\right) + j\log\left(1 + \frac{1}{j}\right) - 1. \end{aligned}$$

So, we obtain

$$\log(x_j) - q_\ell \int_{j/q_\ell}^{(j+1)/q_\ell} \log(x)\,\mathrm{d}x = \log\left(\frac{j+\varepsilon_j}{j+1}\right) - j\log\left(1+\frac{1}{j}\right) + 1.$$

By the Taylor expansion $\log(1+x) = x - x^2/2 + O(x^3)$, we get

$$\log\left(\frac{j+\varepsilon_j}{j+1}\right) - j\log\left(1+\frac{1}{j}\right) + 1 = \frac{\varepsilon_j - 1/2}{j} + O\left(\frac{1}{j^2}\right).$$

For $j = 0$, we get $q_\ell \int_0^{1/q_\ell} \log(x)\,\mathrm{d}x = -\log(q_\ell) - 1$ and thus,

$$\log(x_0) - q_\ell \int_0^{1/q_\ell} \log(x)\,\mathrm{d}x = \log(\varepsilon_0) + 1.$$

Combining the obtained estimates yields

$$\sum_{j=0}^{q_\ell - 1} \log(x_j) - q_\ell \int_0^1 \log(x)\,\mathrm{d}x = \left(\sum_{j=1}^{q_\ell - 1} \frac{\varepsilon_j - 1/2}{j} \right) + \log(\varepsilon_0) + O(1).$$

□

Proposition 4.2. *Let $\frac{r}{s} \in [0,1) \cap \mathbb{Q}$ and let $1 < N < \frac{q_K}{s}$. Then,*

$$\left\| N\alpha - \frac{r}{s} \right\| > \frac{\|q_{K+1}\alpha\|}{s}.$$

Proof. Assume to the contrary that $\left\| N\alpha - \frac{r}{s} \right\| \leqslant \frac{\|q_{K+1}\alpha\|}{s}$. Then, we have

$$\|sN\alpha\| = \left\| sN\alpha - s\frac{r}{s} \right\| = \left\| s\left(N\alpha - \frac{r}{s} \right) \right\| \leqslant s \left\| N\alpha - \frac{r}{s} \right\| \leqslant \|q_{K+1}\alpha\|.$$

Since $sN < q_{K+1}$, this is a contradiction to the best approximation property of q_{K+1}: There would exist an integer $N' < q_{K+1}$ such that $q_{K-1}\|N'\alpha\| \leqslant \|q_{K+1}\alpha\|$.

□

Proposition 4.3 (Error term estimate for a rational shift). *Let $f(x) = \log(\{x - q\})$, where $q = \frac{r}{s} \in [0,1)$ is a rational number. Let $N \in \mathbb{N}$ with $N < \frac{q_K}{s}$ and let $N = b_{K-1}q_{K-1} + N'$, where $0 \leqslant N' < q_{K-1}$. Then, we have*

$$|S_{N'}(f, \alpha, b_{K-1}q_{K-1}\alpha)| \ll \log(q_{K+1}) \sum_{\ell=1}^{K-1} a_\ell.$$

Proof. Define $A_{N'} := \left(q - \frac{\|q_K\alpha\|}{s}, q + \frac{\|q_K\alpha\|}{s} \right)$. Using proposition 4.2 we see $n\alpha + b_{K-1}q_{K-1}\alpha \notin A_{N'}$ for all $n \leqslant N'$. Thus, the Denjoy–Koksma inequality with singularity (proposition 1.5) yields

$$|S_{N'}(f, \alpha, b_{K-1}q_{K-1}\alpha)| \ll \sup_{x \in [0,1) \setminus A_{N'}} |f(x)| \sum_{i=1}^{K-1} a_i + q_{K-1} \left| \int_{A_{N'}} f(x)\,\mathrm{d}x \right|.$$

We have $\sup_{x \in [0,1) \setminus A_{N'}} |f(x)| = \log \frac{\|q_K\alpha\|}{s} \ll \log q_{K+1}$. Further, we use the estimate $q_{K-1} \left| \int_{A_{N'}} f(x)\,\mathrm{d}x \right| \ll \log(q_{K+1})$ to obtain the desired result. □

Lemma 4.4 (main term estimate for a rational shift). *Let $\delta > 0$, $q = \frac{r}{s} \in [0,1) \cap \mathbb{Q}$ and define $f(x) := \log(\{x - q\})$ for $x \in [0,1)$. Further, let $K \in \mathbb{N}$ with $q_{K-1} \equiv 0 \pmod{s}$ and choose $N \in \mathbb{N}$ with $q_{K-1} < N < q_K$ and $\delta a_K < b_{K-1}(N) < \frac{a_K}{4}$. Then, if a_K is sufficiently large, we get*

$$(-1)^K S_{b_{K-1}q_{K-1}}(f, \alpha) \gg \delta a_K \log(q_{K-1}).$$

Proof. By definition, we have

$$S_{b_{K-1}q_{K-1}}(f,\alpha) = \sum_{n=1}^{b_{K-1}q_{K-1}} \log(\{n\alpha - q\}) + b_{K-1}q_{K-1}$$
$$= \sum_{b=0}^{b_{K-1}-1} \left(\sum_{n=1}^{q_{K-1}} \log(\{n\alpha + bq_{K-1}\alpha - q\}) + q_{K-1} \right)$$
$$= \sum_{b=0}^{b_{K-1}-1} \left(\sum_{n=1}^{q_{K-1}} \log\left(\left\{n\alpha + (-1)^{K-1} b\delta_{K-1}\alpha - q\right\}\right) + q_{K-1} \right),$$

where we recall that $\delta_{K-1} = |\alpha q_{K-1} - p_{K-1}| = (-1)^{K-1}(\alpha q_{K-1} - p_{K-1})$. For $1 \leqslant n \leqslant q_{K-1}$ and $0 \leqslant b \leqslant b_{K-1} - 1$, we get

$$\left\{n\alpha - q + (-1)^{K-1}(b\delta_{K-1})\right\} = \left\{ n\frac{p_{K-1}}{q_{K-1}} - q + \frac{(-1)^{K-1}}{q_{K-1}}\delta_{K-1}(bq_{K-1}+n) \right\}$$
$$= \left\{ \frac{np_{K-1} - qq_{K-1}}{q_{K-1}} + \frac{(-1)^{K-1}}{q_{K-1}} \left(\underbrace{\delta_{K-1}(bq_{K-1}+n)}_{=:\varepsilon_{b,n}} \right) \right\}$$
$$= \left\{ \frac{np_{K-1} - n'}{q_{K-1}} + \frac{(-1)^{K-1}}{q_{K-1}}\varepsilon_{b,n} \right\}.$$

We introduced $n' := qq_{K-1}$ which is an integer since $s|q_{K-1}$ with $0 \leqslant n' \leqslant q_{K-1} - 1$. Now observe that $\delta_{K-1} > 0$ by definition and since $b_{q_{K-1}} < \frac{a_K}{4}$, we have $bq_{K-1} + n \leqslant b_{q_{K-1}}q_{K-1} \leqslant (1/2 - \delta)a_K q_{K-1}$. Thus, for any $0 \leqslant n \leqslant q_{K-1} - 1$, we get

$$0 < \varepsilon_{b,n} < \frac{a_k}{4} q_{K-1}\delta_{K-1} \leqslant 1/4,$$

where we used that $q_{K-1}\delta_{K-1} \leqslant 1/a_K$. First, we assume that K is odd implying $(-1)^{K-1} = 1$. We apply proposition 4.1 with $x_n = \left\{ \frac{n}{q_{K-1}} + \frac{\varepsilon_n}{q_{K-1}} \right\}$, where $\varepsilon_n = \varepsilon_{b,((np_{K-1}^{-1}+n') \pmod{q_{K-1}})}$ for $1 \leqslant n \leqslant q_{K-1} - 1$ and $\varepsilon_0 = \varepsilon_{b,(1-q)q_{K-1}}$. This leads to

$$\sum_{n=1}^{q_{K-1}} \log\left(\left\{n\alpha + (-1)^{K-1} b\delta_{K-1}\alpha - q\right\}\right) + q_{K-1}$$
$$= \sum_{n=1}^{q_{K-1}} \log\left(\left\{n\alpha + (-1)^{K-1} b\delta_{K-1}\alpha - q\right\}\right) - q_{K-1}\int_0^1 \log(x)\,dx$$
$$= \left(\sum_{n=1}^{q_{K-1}-1} \frac{\varepsilon_n - 1/2}{n} \right) + \log(\varepsilon_0) + O(1)$$
$$\leqslant -\frac{1}{4}\log(q_{K-1}) + \log(\varepsilon_0) + O(1) \leqslant -\frac{1}{8}\log(q_{K-1}).$$

We used that $\varepsilon_0 \leqslant \frac{1}{4} < 1$ and hence $\log(\varepsilon_0) \leqslant 0$. Moreover, we applied the rough estimate $O(1) \leqslant \frac{1}{8}\log(q_{K-1})$. By summing over all $b = 0, \ldots, b_{K-1} - 1$, we obtain

$$S_{b_{K-1}q_{K-1}}(f,\alpha) \leqslant -\frac{1}{8} b_{K-1} \log(q_{K-1}) \leqslant -\frac{\delta}{8} a_K \log(q_{K-1}) \ll -\delta a_K \log(q_{K-1}),$$

where we also used the assumption $b_{K-1} \geqslant \delta a_K$. By rewriting, we finally get

$$-S_{b_{K-1}q_{K-1}}(f,\alpha) \gg \delta a_K \log(q_{K-1}),$$

as claimed.

Now let K be even. We apply proposition 4.1 with $x_n = \left\{\frac{n}{q_{K-1}} + \frac{\varepsilon_n}{q_{K-1}}\right\}$, where $\varepsilon_n = 1 - \varepsilon_{b,(np_{K-1}^{-1}+n'+1) \pmod{q_{K-1}}}$ for $1 \leqslant n \leqslant q_{K-1} - 1$ and $\varepsilon_0 = 1 - \varepsilon_{b,qq_{K-1}}$. Similar to before, we obtain

$$\begin{aligned} S_{b_{K-1}q_{K-1}}(f,\alpha) &= S_{b_{K-1}q_{K-1}}(f,\alpha) \\ &\geqslant \frac{1}{8} b_{K-1} \log(q_{K-1}) \\ &\gg \delta a_K \log(q_{K-1}). \end{aligned}$$

This finishes the proof. □

4.3. Symmetric logarithmic singularities

Lemma 4.5. *Let $f(x) = \log\left\|x - \frac{r}{s}\right\|$, where $\frac{r}{s} \in [0,1) \cap \mathbb{Q}$. Then, for almost every $\alpha \in [0,1)$, we have*

$$|S_N(f,\alpha)| \ll (\log N)^2 \log\log N.$$

Proof. Writing $N = \sum_{\ell=0}^{K-1} b_\ell q_\ell$ in its Ostrowski expansion with $b_{K-1} \neq 0$, we obtain the decomposition

$$S_N(f,\alpha) = S_{N'}(f,\alpha) + S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha) + S_{N''}(f,\alpha,(N' + b_{K_0-1}q_{K_0-1})\alpha),$$

where $K_0 = \arg\max_{\ell=1,\ldots,K} a_\ell$, $N' = \sum_{\ell=K_0}^{K-1} b_\ell q_\ell$ and $N'' = \sum_{\ell=0}^{K_0-2} b_\ell q_\ell$. By the Denjoy–Koksma inequality with singularity (see (5) in proposition 1.5), we can bound $S_{N'}(f,\alpha)$ by

$$|S_{N'}(f,\alpha)| \ll \sup_{x\in[0,1)\setminus A_{N'}} |f(x)| \sum_{i=K_0}^{K-1} b_i + N' \left|\int_{A_{N'}} f(x)dx\right|,$$

where we choose $A_{N'} = (q - \min_{n<q_K} \|n\alpha - q\|, q + \min_{n<q_K} \|n\alpha - q\|)$. This ensures $\{n\alpha - q\} \notin A_{N'}$ and we have

$$\begin{aligned}
\sup_{x\in[0,1)\setminus A_{N'}} |\log(\|x-q\|)| &\leqslant \left|\log\left(\min_{n<q_K}\|n\alpha-q\|\right)\right| \\
&\leqslant \left|\log\left(\min_{n<q_{K+1}/s}\|n\alpha-q\|\right)\right| \\
&\leqslant \left|\log\left(\frac{\|q_{K+s+1}\alpha\|}{s}\right)\right| \\
&\ll \log(q_{K+s+2}) \\
&\ll K, \qquad (22)
\end{aligned}$$

where we have used that $\log(q_K) \ll K$ by (8) and we used proposition 4.2. A simple calculation reveals

$$\begin{aligned}
\left|\int_{A_{N'}} f(x)\,dx\right| &= 2\left|\min_{n<q_K}\|n\alpha-q\|\left(\log\left(\min_{n<q_K}\|n\alpha-q\|\right)\right)\right| \\
&\ll \frac{K}{q_K},
\end{aligned}$$

where we used that $\min_{n<q_K}\|n\alpha-q\| \leqslant \frac{1}{q_K}$ and $|\log(\min_{n\leqslant q_K}\|n\alpha-q\|)| \ll K$, as shown before. In total, we get

$$\begin{aligned}
|S_{N'}(f,\alpha)| &\ll K\sum_{i=K_0}^{K-1} b_i + N'\frac{K}{q_K} \\
&\ll K\sum_{i=K_0+1}^{K} a_i \\
&\ll K^2\log K,
\end{aligned}$$

where the estimate in the last line uses (7). Analogously, one obtains the same bound for $S_{N''}(f,\alpha,(N'+b_{K_0-1}q_{K_0-1})\alpha)$, i.e. we get

$$|S_{N''}(f,\alpha,(N'+b_{K_0-1}q_{K_0-1})\alpha)| \ll K^2\log K.$$

We are now left with $S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha)$, where we will show that

$$\left|S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha)\right| \ll K^2\log K.$$

By definition, we have

$$\begin{aligned}
S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha) &= \sum_{n=1}^{b_{K_0-1}q_{K_0-1}} \log\left(\left\|n\alpha-\frac{r}{s}+N'\alpha\right\|\right) + b_{K_0-1}q_{K_0-1} \\
&= \sum_{b=0}^{b_{K_0-1}-1}\left(\sum_{n=1}^{q_{K_0-1}} \log\left(\left\|n\alpha-\frac{r}{s}+N'\alpha+bq_{K_0-1}\alpha\right\|\right)+q_{K_0-1}\right) \\
&= \sum_{b=0}^{b_{K_0-1}-1}\left(\sum_{n=1}^{q_{K_0-1}} \log\left(\left\|n\alpha-\frac{r}{s}+N'\alpha+b(-1)^{K_0-1}\delta_{K_0-1}\right\|\right)+q_{K_0-1}\right),
\end{aligned}$$

where we recall that $\delta_{K_0-1} = (-1)^{K_0-1}(q_{K_0-1}\alpha - p_{K_0-1})$.

We first assume that K_0 is odd. We can write $N'\alpha - \frac{r}{s} = \frac{m}{q_{K_0-1}} + \frac{r'}{q_{K_0-1}}$ where $m \in \mathbb{Z}$ and $0 \leqslant r' < 1$. We observe that, for any $0 \leqslant b \leqslant b_{K_0-1}$, we have $0 \leqslant bq_{K_0-1}\delta_{K_0-1} + r' < 2$. For the following analysis, we define the quantity $d_b := bq_{K_0-1}\delta_{K_0-1} + r'$ and the sets B_1, B_2, B_3 as

$$\begin{aligned}
B_1 &:= \{0 \leqslant b \leqslant b_{K_0-1} - 1 : d_{b+1} < 1\}, \\
B_2 &:= \{0 \leqslant b \leqslant b_{K_0-1} - 1 : d_b > 1\}, \\
B_3 &:= \{0, \ldots, b_{K_0-1} - 1 : d_b < 1 < d_{b+1}\}.
\end{aligned}$$

Since d_b is irrational for all $0 \leqslant b \leqslant b_{K_0-1} - 1$, the sets B_1, B_2, B_3 form a partition of $\{0, \ldots, b_{K_0-1} - 1\}$. We first assume $b \in B_1$, i.e. $d_{b+1} < 1$. We see that, for all $n = 1, \ldots, q_{K_0-1}$, we have

$$\left\{(n + bq_{K_0-1})\,\alpha + N'\alpha - \frac{r}{s}\right\} = \left\{\frac{np_{K_0-1} + m + d_b + n\delta_{K_0-1}}{q_{K_0-1}}\right\}.$$

Since p_{K_0-1} and q_{K_0-1} are coprime, the map $j(n) = np_{K_0-1} + m \pmod{q_{K_0-1}}$ is bijective with inverse $n(j)$. Thus, we can introduce the quantities

$$y_j := \frac{j + d_b + n\,(j)\,\delta_{K_0-1}}{q_{K_0-1}}, \quad j = 0, \ldots, q_{K_0-1} - 1.$$

Since $d_b + n(j)\delta_{K_0-1} \leqslant d_{b+1} < 1$ by assumption, it holds that $0 \leqslant y_j < 1$ for all $j = 0, \ldots, q_{K_0-1} - 1$. Further, we have the following equality of sets

$$\{y_j : j = 0, \ldots, q_{K_0-1} - 1\} = \left\{\left\{\frac{np_{K_0-1} + m + d_b + n\delta_{K_0-1}}{q_{K_0-1}}\right\} : n = 1, \ldots, q_{K_0-1}\right\}.$$

The previous arguments reveal that, for $b \in B_1$, we can write

$$\begin{aligned}
\sum_{n=1}^{q_{K_0-1}} \log\left(\left\|n\alpha - \frac{r}{s} + N'\alpha + b\,(-1)^{K_0-1}\,\delta_{K_0-1}\right\|\right) + q_{K_0-1} &= \sum_{j=0}^{q_{K_0-1}-1} \log\|y_j\| + q_{K_0-1} \\
&= \sum_{j=0}^{q_{K_0-1}-1} \left(\log\|y_j\| - I_j\right),
\end{aligned}$$

where we set $I_j := q_{K_0-1}\int_{j/q_{K_0-1}}^{(j+1)/q_{K_0-1}} \log(x)\,\mathrm{d}x$ for $j = 0, \ldots, q_{K_0-1} - 1$. In the following, we will compare the value of $\log\|y_j\|$ to the value of I_j for all $j = 0, \ldots, q_{K_0-1} - 1$. We start with the case where $1 \leqslant j \leqslant \lfloor q_{K_0-1}/2 \rfloor - 1$. Then, we have $\|y_j\| = y_j$ and $\|y_{q_{K_0-1}-j-1}\| = 1 - y_{q_{K_0-1}-j-1}$. This leads to

$$\begin{aligned}\log\|y_j\| - I_j &= \log(y_j) - I_j \\ &= \log\left(\frac{j + d_b + n(j)\,\delta_{K_0-1}}{j+1}\right) - j\log\left(1 + \frac{1}{j}\right) + 1.\end{aligned}$$

By the Taylor expansion $\log(1+x) = x - x^2/2 + O(x^3)$, we have

$$\log\left(\frac{j + d_b + n(j)\,\delta_{K_0-1}}{j+1}\right) - j\log\left(1+\frac{1}{j}\right) + 1 = \frac{d_b + n(j)\,\delta_{K_0-1} - 1/2}{j} + O\left(\frac{1}{j^2}\right).$$

By the same arguments, we obtain

$$\log\|y_{q_{K_0-1}-j-1}\| - q_{K_0-1} I_j = \frac{1 - d_b + n(q_{K_0-1} - j - 1)\,\delta_{K_0-1} - 1/2}{j} + O\left(\frac{1}{j^2}\right).$$

So, by combining the two previous estimates, we get

$$\begin{aligned}\left|\log\|y_j\| + \log\|y_{q_{K_0-1}-j-1}\| - 2I_j\right| &\leqslant \left|\frac{(n(j) - n(q_{K_0-1} - j - 1))\,\delta_{K_0-1}}{j}\right| + O\left(\frac{1}{j^2}\right) \\ &\leqslant \frac{1}{j\,a_{K_0}} + O\left(\frac{1}{j^2}\right).\end{aligned}$$

In the last line, we used the estimate $q_{K_0-1}\delta_{K_0-1} \leqslant \frac{1}{a_{K_0}}$. It is easy to see that

$$\sum_{j=\lfloor q_{K_0-1}/2\rfloor}^{q_{K_0-1}-\lfloor q_{K_0-1}/2\rfloor} (\log(\|y_j\|) - 2I_j) = O(1),$$

since the number of summands on the left-hand side is bounded by a constant and the y_j are bounded away from 0 and 1. Thus, we have shown that

$$\begin{aligned}\left|\sum_{j=1}^{q_{K_0-1}-2} \log\|y_j\| - I_j\right| &\ll \frac{1}{a_{K_0}} \sum_{j=1}^{\lfloor q_{K_0-1}/2\rfloor - 1} \frac{1}{j} + \sum_{j=1}^{\lfloor q_{K_0-1}/2\rfloor - 1} \frac{1}{j^2} + O(1) \\ &\ll \frac{\log q_{K_0-1}}{a_{K_0}} + O(1).\end{aligned}$$

We are now left with the cases $j = 0$ and $j = q_{K_0-1} - 1$, where we get

$$\begin{aligned}\left|\log\|y_0\| + \log\|y_{q_{K_0-1}-1}\| - 2I_0\right| &\leqslant |\log\|y_0\| - I_0| + \left|\log\|y_{q_{K_0-1}-1}\| - I_0\right| \\ &\leqslant |\log(d_b + n(0)\,\delta_{K_0-1})| \\ &\quad + |\log(1 - d_b - n(q_{K_0-1} - 1)\,\delta_{K_0-1})| + 2.\end{aligned}$$

We discuss here the first term $|\log(d_b + n(0)\delta_{K_0-1})|$ in detail, the second term can be treated analogously. Observe that $d_b + n(0)\delta_{K_0-1} = q_{K_0-1}\|(N' + bq_{K_0-1} + n(0))\alpha - \frac{r}{s})\|$ by construction of y_0.

We claim that there exists at most one $b' \in B_1$ such that

$$\left\|(N' + b'q_{K_0-1} + n(0))\,\alpha - \frac{r}{s}\right\| \leqslant \frac{1}{4q_{K_0}}.$$

Assume to the contrary that there are two integers $b', b'' \in B_1$ with $b' \neq b''$ such that both satisfy this estimate. Then, we get

$$\|(b' - b'')\, q_{K_0-1}\alpha\| \leqslant \frac{1}{2q_{K_0}}$$

by the triangle inequality, which is an immediate contradiction to the best approximation property of q_{K_0-1}. Thus, for the only possible $b' \in B_1$, we get

$$\begin{aligned} q_{K_0-1}\left\|\left(N' + b' q_{K_0-1} + n(0)\right)\alpha - \frac{r}{s}\right\| &\geqslant q_{K_0-1}\min_{n<q_K}\left\|n\alpha - \frac{r}{s}\right\| \\ &\geqslant \frac{q_{K_0-1}}{q_{K+s+1}} \\ &\geqslant \frac{1}{q_{K+s+1}}, \end{aligned}$$

where the estimate in the second last step can be argued analogously to (22). For all $b \in B_1$ with $b \neq b'$, we have

$$\begin{aligned} q_{K_0-1}\left\|\left(N' + b q_{K_0-1} + n(0)\right)\alpha - \frac{r}{s}\right\| &\geqslant \frac{q_{K_0-1}}{4q_{K_0}} \\ &\geqslant \frac{1}{4\left(a_{K_0}+1\right)}. \end{aligned}$$

Thus, by combining the estimates we obtained, we get

$$\begin{aligned} &\left|\sum_{b\in B_1}\sum_{n=1}^{q_{K_0-1}} f(n\alpha + bq_{K_0-1}\alpha + N'\alpha)\right| \\ &= \left|\sum_{b\in B_1}\sum_{j=0}^{q_{K_0-1}} (\log\|y_j\| - I_j)\right| \\ &\leqslant \left|\sum_{b\in B_1}\sum_{j=1}^{q_{K_0-2}} (\log\|y_j\| - I_j)\right| + \left|\sum_{b\in B_1}\sum_{j\in\{0,q_{K_0-1}\}} (\log\|y_j\| - I_j)\right| \\ &\ll \log q_{K_0-1}a_{K_0} + \left|\sum_{b\in B_1, b\neq b'}\sum_{j\in\{0,q_{K_0-1}\}} (\log\|y_j\| - I_j)\right| + \left|\sum_{j\in\{0,q_{K_0-1}\}} (\log\|y_j(b')\| - I_j)\right| \\ &\ll \log q_{K_0-1}a_{K_0} + \sum_{b\in B_1, b\neq b'} \log a_{K_0} + \log q_{K+s+1} \\ &\ll a_{K_0}\left(\log a_{K_0} + \log q_{K_0-1}\right) \\ &\ll K^2 \log(K). \end{aligned}$$

Here we used that $|B_1| \leqslant b_{K_0-1} \leqslant a_{K_0}$, $\log q_{K+s+1} \ll K$ by (8) and by (6), $a_{K_0} \leqslant K^2$ if K is sufficiently large. For the set B_2, a similar analysis leads to the same asymptotic bound, i.e. we get

$$\left|\sum_{b\in B_2}\sum_{n=1}^{q_{K_0-1}} f(n\alpha + bq_{K_0-1}\alpha + N'\alpha)\right| \ll K^2 \log K.$$

The set B_3 contains at most 1 element $\bar{b} \in \{0,\ldots,b_{K_0-1}-1\}$ and thus, we can write

$$\left|\sum_{b\in B_3}\sum_{n=1}^{q_{K_0-1}} f(n\alpha + bq_{K_0-1}\alpha + N'\alpha)\right| \leqslant \left|\sum_{n=1}^{q_{K_0-1}} \log\left(\|n\alpha + x\|\right)\right|,$$

where $x := \bar{b}q_{K_0-1}\alpha + N'\alpha$. Applying the Denjoy–Koksma inequality with singularity in the form of (5), we obtain

$$\left|\sum_{n=1}^{q_{K_0-1}} \log\left(\|n\alpha + x\|\right)\right| \ll K^2 \log K,$$

as we did for $S_{N'}(f,\alpha)$ at the beginning of this proof. Combining the estimates for B_1, B_2, B_3, we can deduce that

$$\left|S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha)\right| \ll K^2 \log K,$$

which finishes the proof for odd K_0. The case where K_0 is even can be handled under minor modifications. In total, we have shown that, for $q_{K-1} \leqslant N < q_K$

$$\begin{aligned}|S_N(f,\alpha)| &\ll K^2 \log K \\ &\ll (\log N)^2 \log\log N.\end{aligned}$$

□

4.4. *Proof of theorems 3 and 6*

We start by proving (ii) of theorem 3, where the Birkhoff sum $S_N(f,\alpha,q)$ is generated by a function $f:\mathbb{R}\to\mathbb{R}$ with symmetric logarithmic singularity at a rational, i.e. f is of the form $f(x) = c\log\|x-x_1\| + t(x)$, where $c \neq 0$, $x_1 = \frac{r}{s} \in [0,1)\cap\mathbb{Q}$ and t is of bounded variation. Without loss of generality, we can assume that $q=0$ because otherwise we just set $\tilde{x}_1 := x_1 - q$. Let $N\in\mathbb{N}$ with Ostrowski expansion $N = \sum_{i=1}^{K-1} b_i q_i$. By the Denjoy–Koksma inequality, we obtain $S_N(t,\alpha) \leqslant \mathrm{Var}(t)\sum_{i=1}^{K} a_i \ll K^2$. Thus by lemma 4.5, we get

$$|S_N(f,\alpha)| \ll (\log N)^2 \log\log N,$$

implying statement (ii) of theorem 3.

Next, we consider the asymmetric case, i.e. where the Birkhoff sum is generated by a function f of the form $f(x) = \underbrace{c_1\log(\{x-x_1\})}_{=:f_1(x)} + \underbrace{c_2\log\|x-x_1\| + t(x)}_{=:f_2(x)}$, where $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq 0$ and $x_1 = \frac{r}{s} \in [0,1)\cap\mathbb{Q}$ and t is of bounded variation. Again, without loss of generality, it suffices to consider the case where $q=0$.

We start with the case where $\sum_{k=1}^{\infty}\frac{1}{\psi(k)} = \infty$ where we show that the set $\{N\in\mathbb{N} : S_N(f,\alpha) \geqslant \log N\psi(\log(N))\}$ has upper density 1. Without loss of generality, we can assume that $\lim_{K\to\infty}\frac{\psi(K)}{K\log K} = \infty$ since the result then follows also for slower growing ψ. Note that $S_N(f,\alpha) = S_N(f_1,\alpha) + S_N(f_2,\alpha)$. By the first part of this proof, we have $|S_N(f_2,\alpha)| \ll (\log N)^2\log\log N$ and since $\log N\psi(\log N)$ dominates $(\log N)^2\log\log N$, it suffices to show that $\{N\in\mathbb{N} : S_N(f_1,\alpha) \geqslant \log N\psi(\log N)\}$ has upper density 1.

First, assume that $c_1 > 0$. By lemma 3.12, for almost every $\alpha \in [0,1)$, the set

$$\left\{ K \in \mathbb{N} \middle| K \equiv 0 \pmod 2, q_{K-1} \equiv 0 \pmod s, \tilde{\psi}(K) < a_K < K^2, \sum_{i=1}^{K-1} a_i \leqslant 2K \log K \right\}$$

has infinite cardinality, where $\tilde{\psi}(k) = C_1\psi(C_2 k)$ with $C_1, C_2 > 0$ specified later. Denote by $(k_j)_{j\in\mathbb{N}}$ the increasing sequence of integers such that the above holds. Define $M_j := \left\lfloor \frac{a_{k_j}}{4s} q_{k_j-1} \right\rfloor$ and note that, for any $1 \leqslant N \leqslant M_j$, we can write $N = b_{k_j-1}(N) q_{k_j-1} + N'$ where $N' < q_{k_j-1}$. Moreover, for $\delta > 0$, let $A_j^\delta := \left\{ 1 \leqslant N \leqslant M_j : \delta < \frac{b_{k_j-1}(N)}{a_{k_j}} \leqslant 1 \right\}$. We note that for any $N \in A_j^\delta$, the assumptions of lemma 4.4 are satisfied, and hence

$$S_{b_{k_j-1} q_{k_j-1}}(f_1, \alpha) \geqslant c(\delta) a_{k_j} \log\left(q_{k_j-1}\right),$$

where $c(\delta)$ is a positive constant only depending on δ. Moreover, by proposition 4.3, we have

$$|S_{N'}\left(f_1, \alpha, b_{k_j-1} q_{k_j-1} \alpha\right)| \leqslant D \log\left(q_{k_j+1}\right) \sum_{\ell=1}^{k_j-1} a_\ell,$$

where D is a positive absolute constant. For j sufficiently large, this leads to

$$\begin{aligned} S_N(f_1, \alpha) &= S_{b_{k_j-1} q_{k_j-1}}(f_1, \alpha) + S_{N'}\left(f_1, \alpha, b_{k_j-1} q_{k_j-1} \alpha\right) \\ &\geqslant c(\delta) a_{k_j} \log\left(q_{k_j-1}\right) - D \sum_{\ell=1}^{k_j-1} a_\ell \\ &\geqslant \frac{c(\delta)}{2} a_{k_j} \log\left(q_{k_j-1}\right), \end{aligned} \tag{23}$$

where the inequality in the last line holds since a_{k_j} dominates $\sum_{i=1}^{k_j-1} a_i$ by construction. Moreover, we have used that c_1 in the definition of f_1 is positive by assumption and we employed $\log q_{k_j-1} \gg \log q_{k_j+1}$ which holds by (8). We note that $\lim_{\delta\to 0} |A_j^\delta| = M_j$ and thus, fixing $\varepsilon > 0$, we can choose $\delta > 0$ such that $\frac{|A_j^\delta|}{M_j} \geqslant 1 - \varepsilon$ for all sufficiently large j. Let $C_1, C_2 > 0$ such that, if $a_{k_j} \geqslant \tilde{\psi}(k_j) = C_1\psi(C_2 k_j)$, it follows that $\frac{c(\delta)}{2} a_{k_j} \log(q_{k_j-1}) \geqslant \log N \psi(\log N)$ for all $N \in A_j^\delta$. We note that C_1, C_2 only depend on $\delta > 0$, since k_j is chosen such that $\sum_{i=1}^{k_j-1} a_i \leqslant 2k_j \log(k_j) = o(\psi(k_j))$. This yields

$$\begin{aligned} \frac{\left|\left\{1 \leqslant N \leqslant M_j : S_N(f_1, \alpha) \geqslant \log N \psi(\log N)\right\}\right|}{M_j} &\geqslant \frac{\left|\left\{N \in A_j^\delta : \frac{c(\delta)}{2} a_{k_j} \log\left(q_{k_j-1}\right) \geqslant \log N \psi(\log N)\right\}\right|}{M_j} \\ &\geqslant \frac{\left|\left\{N \in A_j^\delta : a_{k_j} \geqslant \tilde{\psi}(k_j)\right\}\right|}{M_j} \\ &= \frac{|A_j^\delta|}{M_j} \geqslant 1 - \varepsilon. \end{aligned} \tag{24}$$

By taking the liminf as $j \to \infty$ and letting $\varepsilon \to 0$, we get

$$\liminf_{j\to\infty} \frac{|\{1 \leqslant N \leqslant M_j : S_N(f_1,\alpha) \geqslant \log N \psi(\log N)\}|}{M_j} = 1.$$

The case where c_1 from the definition of f_1 is negative can be handled under minor modifications. Analogously, one can show that the set $\{N \in \mathbb{N} : S_N(f,\alpha) \leqslant -\log(N)\psi(\log(N))\}$ has upper density 1.

The case where $\sum_{k=1}^{\infty} \frac{1}{\psi(k)} < \infty$ can be treated analogously to the proof of theorem 1 by using the Denjoy–Koksma inequality with singularity (proposition 1.5).

To prove theorem 6, we start with a few general estimates. Observe that, for $x_1 = \frac{r}{s}$ and $q_{K-1} < N < q_K$, we have

$$\begin{aligned}
\sup_{x\in[0,1)\setminus A_N} |f_1(x) + f_2(x)| &\ll \left|\log\left(\min_{n<q_K} \|n\alpha - x_1\|\right)\right| \\
&\leqslant \left|\log\left(\min_{n<q_{K+1}/s} \|n\alpha - x_1\|\right)\right| \\
&\leqslant \left|\log\left(\frac{\|q_{K+s+1}\alpha\|}{s}\right)\right| \\
&\ll \log\left(q_{K+s+2}\right) \\
&\ll \log q_K,
\end{aligned}$$

where we have used (8) and proposition 4.2. This shows that

$$\sup_{x\in[0,1)\setminus A_N} |f(x)| \sum_{i=1}^{K} a_i \ll \log q_K \sum_{i=1}^{K} a_i.$$

Further, we get

$$\begin{aligned}
N\left|\int_{A_N} f(x)\, dx\right| &\ll N\left|\min_{n<q_K} \|n\alpha - x_1\| \left(\log\left(\min_{n<q_K} \|n\alpha - x_1\|\right) - 1\right)\right| \\
&\ll \frac{NK}{q_K} \\
&\ll \log q_K = o\left(a_K\right),
\end{aligned}$$

where we used that $\min_{n<q_K} \|n\alpha - x_1\| \leqslant \frac{1}{q_K}$ and $|\log(\min_{n\leqslant q_K} \|n\alpha - x_1\|)| \ll K$ by the previous calculation. Now let $0 < r < 1$. We will show that there exists a constant $C(r) > 0$ such that the set

$$\left\{N \in \mathbb{N} \mid S_N(f,\alpha,q) \geqslant C(r) \sup_{x\in[0,1)\setminus A_N} |f(x)| \sum_{i=1}^{K(N)} a_i + N\left|\int_{A_N} f(x)\, \mathrm{d}x\right|\right\}$$

has upper density of at least r. To that end, let $(k_j)_{j\in\mathbb{N}}$ be the sequence of integers from the first part of this proof. Let $A_j^{\delta'} = \left\{1 \leqslant N \leqslant M_j : \delta' < \frac{b_{k_j-1}(N)}{a_{k_j}} \leqslant 1\right\}$ and $M_j = \left\lfloor \frac{a_{k_j}}{4s} q_{k_j-1} \right\rfloor$ be as in the first part of this proof, where we choose $\delta' = \delta'(r) > 0$ sufficiently small such that $\frac{|A_j^{\delta'}|}{M_j} \geqslant r$. Using (23) we get, for $N \in A_j^{\delta'}$,

$$S_N(f_1, \alpha) \geqslant \frac{c(\delta')}{2} a_{k_j} \log q_{k_j - 1}.$$

Since $|S_N(f_2, \alpha)| \ll k_j^2 \log k_j = o(a_{k_j} \log q_{k_j-1})$, we obtain

$$S_N(f,\alpha) \geqslant \frac{c(\delta')}{4} a_{k_j} \log_{q_{k_j-1}} \gg \sup_{x\in[0,1)\setminus A_N} |f(x)| \sum_{i=1}^{k_j} a_i + N \left| \int_{A_N} f(x)\,\mathrm{d}x \right|.$$

Thus, there exists a $C(\delta') = C(r)$ such that

$$S_N(f,\alpha) \geqslant C(r) \sup_{x\in[0,1)\setminus A_N} |f(x)| \sum_{i=1}^{k_j} a_i + N \left| \int_{A_N} f(x)\,\mathrm{d}x \right|$$

holds for all $N \in A_j^{\delta'}$. Since $\frac{|A_j^{\delta'}|}{M_j} \geqslant r$, the statement of theorem 6 follows by an analogous argument as in (24). The second case in theorem 6, where we deal with the set

$$\left\{ N \in \mathbb{N} \mid S_N(f,\alpha,q) \leqslant -C(r) \sup_{x\in[0,1)\setminus A_N} |f(x)| \sum_{i=1}^{K(N)} a_i - N \left| \int_{A_N} f(x)\,\mathrm{d}x \right| \right\}$$

can be handled similarly.

Data availability statement

No new data were created or analysed in this study.

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Appendix

In the appendix, we provide the proofs of propositions 3.2 and 3.4.

Proof of proposition 3.2. One can easily check that the condition of f being as in (11) is invariant under rational translation, thus it suffices to prove the statement for $q = 0$. We show that the set $\{N \in \mathbb{N} : S_N(f,\alpha) \geqslant \psi(\log N)\}$ has upper density 1. We can write

$$S_N(f,\alpha) = S_{b_{K-1}q_{K-1}}(f,\alpha) + S_{N'}(f,\alpha, b_{K-1}q_{K-1}\alpha),$$

where $N = b_{K-1}q_{K-1} + N'$ with $N' < q_{K-1}$. By the Denjoy–Koksma inequality in the form of (4), we have that

$$S_{N'}(f,\alpha,b_{K-1}q_{K-1}\alpha) \ll \sum_{i=1}^{K-1} a_i.$$

We analyze the dominating term $S_{b_{K-1}q_{K-1}}(f,\alpha)$ for certain b_{K-1}. By lemma 3.6, we have

$$\begin{aligned} &S_{b_{K-1}q_{K-1}}(f,\alpha) \\ &= b_{K-1}\left(\left\{\frac{q_{K-1}r_1}{s_1}\right\} + (-1)^{K-1}\mathbb{1}_{\{s_1|q_{K-1}\}} - 2\left\{\frac{q_{K-1}r_2}{s_2}\right\} - 2(-1)^{K-1}\mathbb{1}_{\{s_2|q_{K-1}\}}\right. \\ &\quad \left. + \left\{\frac{q_{K-1}r_3}{s_3}\right\} + (-1)^{K-1}\mathbb{1}_{\{s_3|q_{K-1}\}}\right), \end{aligned} \tag{25}$$

provided $b_{K-1} \leqslant \frac{1}{2s_1s_2s_3}a_K$ and a_K is sufficiently large. By lemma 3.12, for almost all $\alpha \in [0,1)$ and for any pair of integers (a,b), the sets

$$\left\{K \in \mathbb{N} \mid a_K > \psi(K), 2 \nmid (K-1), q_{K-1} \equiv a \pmod{b}, \sum_{i=1}^{K-1} a_i \leqslant 2K\log K\right\},$$
$$\left\{K \in \mathbb{N} \mid a_K > \psi(K), 2 \mid (K-1), q_{K-1} \equiv a \pmod{b}, \sum_{i=1}^{K-1} a_i \leqslant 2K\log K\right\}$$

both contain infinitely many integers K. In the upcoming case distinction we will consider different choices of a and b.

Case 1: $s_1 \nmid s_2$: The congruence relation $q_{K-1} \equiv s_2 \mod s_1s_2$ ensures that $s_2 \mid q_{K-1}$ and $s_1 \nmid q_{K-1}$ since $s_1 \nmid s_2$ by assumption. Thus, (25) gives us

$$\begin{aligned} S_{b_{K-1}q_{K-1}}(f,\alpha) &= b_{K-1}\left(\underbrace{\left\{\frac{q_{K-1}r_1}{s_1}\right\}}_{\geqslant 0} + 2 + \underbrace{\left\{\frac{q_{K-1}r_3}{s_3}\right\} - \mathbb{1}_{\{s_3|q_{K-1}\}}}_{\geqslant -1}\right) \\ &\geqslant b_{K-1}. \end{aligned}$$

Case 2: We assume $s_3 \nmid s_2$. Under the congruence conditions $q_{K-1} \equiv s_2 \mod s_2s_3$ and $2 \nmid (K-1)$, we obtain

$$\begin{aligned} S_{b_{K-1}q_{K-1}}(f,\alpha) &= b_{K-1}\left(\left\{\frac{q_{K-1}r_1}{s_1}\right\} - \mathbb{1}_{\{s_1|q_{K-1}\}} + 2 + \left\{\frac{q_{K-1}r_3}{s_3}\right\}\right) \\ &\geqslant b_{K-1}. \end{aligned}$$

Case 3: If $s = s_1 = s_2 = s_3$, then we use the congruence conditions $2 \nmid (K-1)$ and $q_{K-1} \equiv (r_2)^{-1} \pmod{s}$ to show

$$S_{b_{K-1}q_{K-1}}(f,\alpha) = b_{K-1}\left(\left\{\frac{q_{K-1}r_1}{s}\right\} - 2\left\{\frac{q_{K-1}r_2}{s}\right\} + \left\{\frac{q_{K-1}r_3}{s}\right\}\right)$$
$$= b_{K-1}\left(\left\{\frac{q_{K-1}r_1}{s}\right\} - \frac{2}{s} + \left\{\frac{q_{K-1}r_3}{s}\right\}\right)$$
$$\geqslant \frac{2}{s}b_{K-1}.$$

The latter inequality holds since r_1, r_3 are distinct from r_2, and thus $\left\{\frac{q_{K-1}r_1}{s}\right\} + \left\{\frac{q_{K-1}r_3}{s}\right\} \geqslant \frac{4}{s}$.

Case 4: $s_1 \mid s_2$ and $s_3 \mid s_2$, but $s_1 \neq s_2$ or $s_3 \neq s_2$: Without loss of generality, we assume $s_1 \neq s_2$. We use the congruence conditions $2 \mid (K-1)$ and $q_{K-1} \equiv as_1 \pmod{s_2}$ with $a \equiv r_2^{-1} \mod \frac{s_2}{s_1}$ (which is possible since $\gcd(r_2, s_2) = 1$). We obtain

$$S_{b_{K-1}q_{K-1}}(f,\alpha) = b_{K-1}\left(1 - 2\left\{\frac{q_{K-1}r_2}{s_2}\right\} + \left\{\frac{q_{K-1}r_3}{s_3}\right\} + \mathbb{1}_{\{s_3 \mid q_{K-1}\}}\right)$$
$$= b_{K-1}\left(1 - 2\left\{\frac{s_1}{s_2}\right\} + \left\{\frac{q_{K-1}r_3}{s_3}\right\} + \mathbb{1}_{\{s_3 \mid q_{K-1}\}}\right)$$
$$\geqslant b_{K-1}\left(\left\{\frac{q_{K-1}r_3}{s_3}\right\} + \mathbb{1}_{\{s_3 \mid q_{K-1}\}}\right)$$
$$\geqslant \frac{1}{s_3}b_{K-1}.$$

The second last inequality follows from the congruence relation $q_{K-1} \equiv as_1 \pmod{s_2}$ and $2\left\{\frac{s_1}{s_2}\right\} \leqslant 1$, where the latter holds since $s_1 \mid s_2$ and $s_1 \neq s_2$. In the last line we used that if $\mathbb{1}_{\{s_3 \mid q_{K-1}\}} = 0$, then $\left\{\frac{q_{K-1}r_3}{s_3}\right\} \geqslant \frac{1}{s_3}$.

Thus, in either case, there exists a constant $C > 0$ such that, for $N \in \mathbb{N}$ with $b_{K-1} \leqslant \frac{1}{2s_1s_2s_3}a_K$, we have

$$S_N(f,\alpha) \geqslant Cb_{K-1} + O\left(\sum_{i=1}^{K-1} a_i\right).$$

The remaining part of the proof can be argued in the same way as it is done in the proof of theorem 7. The set $\{N \in \mathbb{N} : S_N(f,\alpha) \leqslant -\psi(\log N)\}$ can be handled analogously. □

Proof of proposition 3.4. We will show that for almost every $\alpha \in [0,1)$, there exists a $\delta > 0$ with

$$\liminf_{M\to\infty} \frac{|\{1 \leqslant N \leqslant M : |S_N(f,\alpha)| \ll \log N \log\log N\}|}{M} \geqslant \delta.$$

By choosing $\psi(k) := k \log k \log\log(k+10)$, this implies that

$$\limsup_{M\to\infty} \frac{|\{1 \leqslant N \leqslant M : |S_N(f,\alpha)| \geqslant \psi(\log N)\}|}{M} \leqslant 1 - \delta.$$

Fixing $M \in \mathbb{N}$, there is exactly one $K \in \mathbb{N}$ such that $q_{K-1} \leqslant M < q_K$. Let $K_0 = \arg\max_{k \leqslant K} a_k$ (if the maximum is not unique, we can choose an arbitrary one among the maximizers). We define

$$A_M^\delta := \{\sqrt{q_{K-1}} \leqslant N \leqslant M : b_{K_0-1}(N) \leqslant \delta a_{K_0}\},$$

where $\delta > 0$ is a small constant specified later. In the following, we will show that for any $N \in A_M^\delta$, we have $|S_N(f,\alpha)| \ll K\log K$. Writing $N = \sum_{\ell=0}^{K-1} b_\ell q_\ell$ in its Ostrowski expansion, we obtain the decomposition

$$S_N(f,\alpha) = S_{N'}(f,\alpha) + S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha) + S_{N''}(f,\alpha,(N' + b_{K_0-1}q_{K_0-1})\alpha),$$

with $N' = \sum_{\ell=K_0}^{K-1} b_\ell q_\ell$ and $N'' = \sum_{\ell=0}^{K_0-2} b_\ell q_\ell$. By the Denjoy–Koksma inequality (4), we can bound $S_{N'}(f,\alpha)$ by

$$\begin{aligned}|S_{N'}(f,\alpha)| &\ll \sum_{i=K_0+1}^{K} a_i \\ &\ll K\log K,\end{aligned}$$

where we used (7) in the second line. Analogously, one obtains the same bound for $S_{N''}(f,\alpha,(N' + b_{K_0-1}q_{K_0-1})\alpha)$, i.e. we get

$$|S_{N''}(f,\alpha,(N' + b_{K_0-1}q_{K_0-1})\alpha)| \ll K\log K.$$

We now turn our attention to $S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha)$, where we will show $S_{b_{K_0-1}q_{K_0-1}}(f,\alpha,N'\alpha) = 0$. Indeed, an analogous analysis to the proof of lemma 3.6 shows that there exists a $\delta > 0$ such that, for any $b_{K_0-1} \leqslant \delta a_{k_0}$, it holds that

$$S_{b_{K_0-1}q_{K_0-1}}(f,\alpha) = \left(\left\{\frac{q_{K_0-1}u}{w}\right\} + \left\{\frac{q_{K_0-1}(1-u)}{w}\right\}\right) - \left(\left\{\frac{q_{K_0-1}v}{w}\right\} + \left\{\frac{q_{K_0-1}(1-v)}{w}\right\}\right).$$

Regardless of the congruence class of q_{K_0-1} modulo w, the expression above equals 0. In total, we have shown that, for all $N \in A_M^\delta$, we get the asymptotic bound

$$|S_N(f,\alpha)| \ll K\log K \ll \log N \log\log N,$$

where the last estimate uses that $N \geqslant \sqrt{q_{K-1}}$, which holds by the definition of A_M^δ. This leads to

$$\begin{aligned}&\liminf_{M\to\infty} \frac{|\{1 \leqslant N \leqslant M : |S_N(f,\alpha)| \ll \log N \log\log N\}|}{M} \\ &\quad\geqslant \liminf_{M\to\infty} \frac{\left|\{N \in A_M^\delta : |S_N(f,\alpha)| \ll \log N \log\log N\}\right|}{M} \\ &\quad\geqslant \liminf_{M\to\infty} \frac{|A_M^\delta|}{M} \\ &\quad\geqslant \delta.\end{aligned}$$

This finishes the proof. □

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